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Quasilocal thermodynamics of dilaton gravity coupled to gauge fields

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We consider an Einstein-Hilbert-dilaton action for gravity coupled to various types of Abelian and non-Abelian gauge fields in a spatially finite system. These include Yang-Mills fields and Abelian gauge fields with three-form and four-form field strengths. We obtain various quasilocal quantities associated with these fields, including their energy and angular momentum, and develop methods for calculating conserved charges when a solution possesses sufficient symmetry. For stationary black holes, we find an expression for the entropy from the microcanonical form of the action. We also find a form of the first law of black hole thermodynamics for black holes with the gauge fields of the type considered here.

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I. INTRODUCTION

The relationship between the Euclidean-action formulation of quantum gravity and the thermodynamics of the gravitational field has been a subject of increasing interest in recent years. Fundamental connections between the partition function of the grand canonical ensemble and the Euclidean-action path integral were first pointed out by Gibbons and Hawking [1], who argued that the Euclidean gravitational action is equal to the grand canonical free energy times the reciprocal of the temperature associated with a black hole (or cosmological) event horizon [2].

More recently Brown and York have extended this work by considering the formulation of the partition function for gravitating systems of finite spatial extent [3,4]. Virtually all systems with which we have any experience have a finite spatial boundary; indeed, one of the central concepts in thermodynamics is that of a system and a reservoir that are separated by a partition. A physical realization of these concepts is needed in order to apply thermodynamics in a sensible way.[1] When we use thermodynamics to describe self-gravitating systems (including objects such as black holes), we must divide our space-time into a region which contains the system (the black hole) and the remainder of the space-time which can be treated as the reservoir. The traditional practice is to study black hole thermodynamics at spacelike infinity, assuming reasonable asymptotic conditions without any boundary. However, this approach has a number of deficiencies. First, it requires that a space-time which is a solution to the field equations (and often the accompanying matter fields) possess appropriate asymptotic behavior, typically asymptotic flatness. However asymptotic flatness is never satisfied in reality, and is not always an appropriate theoretical idealization: many black hole space-times exist that are solutions to the (dilatonic) gravitational field equations that do not possess asymptotic flatness. Some have been found recently that are not even asymptotic to de Sitter or anti–de Sitter space-time [5]. Furthermore, in the study of pair creation of black holes one is forced to consider nonasymptotically flat space-times with an acceleration horizon [6], necessitating a more careful consideration of boundary terms in the formulation of the Hamiltonian [7]. Second, construction of a partition function (which is central in the study of statistical mechanics) requires the stability of the system, which is only realized when a finite size is imposed [4], a point also noted by Hayward and Wong [8]. For example, the heat capacity for a Schwarzschild black hole is negative [9] if one fixes the temperature at infinity, and the formal expression for the partition function is not logically consistent [10]. However if the temperature is fixed at a finite spatial boundary, there is no inconsistency in the black hole partition function and the heat capacity is positive [11]. This approach may also be extended to black holes in anti–de Sitter space-times [12], where analogous results are obtained. Finally, it seems to us that, on physical grounds, one should be able to define thermodynamics appropriate to observers who are at a finite distance from the black hole.

It is important, therefore, to construct thermodynamic quantities appropriate to observers at the (finite) boundary of a system. The quasilocal formalism, which has been developed extensively by Brown and York [13] pro-

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[1] Another central concept of thermodynamics is, of course, thermodynamic equilibrium. The realization of this is the stationarity of a system, where we say a system is stationary if there exists a timelike (Killing) vector field for which the Lie derivative of all the fields considered vanish, and the boundary of the system is chosen to contain the orbits of these vectors. We shall assume that our systems are stationary.

vides us with a means. This formalism is based on a Hamilton-Jacobi principle wherein the boundary terms of the action functional for a compact region give rise to the quasilocal quantities such as energy and angular momentum. Brown and York [3] have also developed a paradigm for understanding the relationship between the classical mechanics, the statistical mechanics, and the thermodynamics of gravitating systems, based on the boundary conditions of the system. Here, the fields that are held fixed on the boundary in deriving the classical mechanics equations of motion from an action principle determine the statistical ensemble of the corresponding statistical mechanics, and thus the type of thermodynamic partition that must be imposed. There is a parallelism between the Legendre transformation, a canonical transformation of the boundary terms of the action, and the Laplace transformation, which changes the type of ensemble, of the path integral for the statistical mechanics. This parallelism may be used to identify a "microcanonical" action that has boundary terms appropriate to a microcanonical statistical ensemble [3], and from it an expression for the entropy of a thermodynamic system may be obtained.

In this paper we extend this formalism to include the most general action of gauge fields coupled to dilaton gravity that has at most two derivatives in any term. This class of actions includes the low-energy limit to string theories [14] and is also of interest as an empirical foil for testing general relativity [15,16]. We consider non-Abelian gauge theories coupled to gravity, the dilaton, and an axion field, as well as Abelian two-form and three-form gauge potentials with three-form and four-form field strengths, respectively, also coupled to a dilaton. (We note that recent research in non-Abelian gauge fields coupled to gravity has led to new black hole solutions that refute much of the folklore about black holes, such as the "no hair" conjecture [17].) We will find that a form of the "first law" of black hole thermodynamics can be reconstructed for the theory considered which is quite reminiscent of the conventional one for the Einstein-Maxwell theory. Although we restrict ourselves to four-dimensional space-times, higher dimensional generalizations of our work are straightforward. For each sector of the action, we obtain expressions for the quasilocal thermodynamic quantities that will appear in the first law of thermodynamics. In addition, we discuss the role of conserved quantities associated with the gauge and gravitational fields. In general, the presence of a conserved quantity depends on the type of solution. Such quantities always require the presence of some sort of symmetry in the solution. (Solutions that satisfy certain asymptotic conditions often also possess an asymptotic symmetry that can be used to define conserved quantities. Here, the solution must at least possess exact symmetries on some finite boundary in order to admit conserved quantities for finite-sized systems.) We also give a brief summary of the construction of a statistical mechanics for the class of theories considered, and we obtain an expression for the entropy and a form of the first law of thermodynamics for systems possessing an event horizon. We note that a version of the first law of black hole thermodynamics for a finite system has been obtained for vacuum general relativity as well as vacuum Einstein-Maxwell [18]; we recover these results in the case of an Abelian gauge theory with vanishing dilaton and axion fields.

Closely related to the work of Brown and York, as well as to our present work, is that of Wald's Noether charge formalism [19]. Wald obtains an expression for the entropy of a space-time solution of a very general class of theories that are required to be derivable via an action principle from a Lagrangian density that is covariant under diffeomorphisms. Here, the entropy is identified as the Noether charge associated with this covariance. Recently, Iyer and Wald [20] have extended their treatment to systems with finite boundaries, and they have shown how to formally recover some of the results of Brown and York [13]. The present work can be viewed, therefore, as complementary to this technique. We restrict our considerations to actions with at most two derivatives in every term, and we explicitly evaluate quantities which are only implicitly defined in Ref. [20] for specific types of boundary conditions. We justify the choices of boundary conditions by appealing to the role they play in the connection of the classical mechanics, the statistical mechanics, and the thermodynamics discussed above.

Therefore, our agenda is the following: In Sec. II, we study the Einstein-Hilbert-dilaton sector, which we consider as our gravitational theory in the absence of additional fields. In Sec. III, we consider a Yang-Mills field that couples to both the metric and the dilaton, extending this to include axion couplings in Sec. IV. In Sec. V, we turn to an Abelian gauge theory involving a four-form field strength (also coupled to the dilaton), and briefly consider its relation to a cosmological constant. In Sec. VI we complete our survey of matter fields with an Abelian three-form field strength gauge theory coupled to the dilaton. Section VII is a review of statistical mechanics based on path integral techniques. Here we adopt a "general form" for the gauge fields rather than treating each case separately. In Sec. VIII we apply these results to obtain an expression for the entropy of a system containing a black hole, and a form of the first law of black hole thermodynamics. Concluding remarks follow.

In this paper, we adopt the conventions of Wald [21]. Units are chosen so that the speed of light, Newton's constant, and the rationalized Planck constant are all taken to be unity. A summary of the notation of the manifolds, fields, and related quantities considered in this paper is given in the Appendix.

II. EINSTEIN-HILBERT-DILATON SECTOR

The theory of gravity that we consider is one which is based on the usual action of general relativity, but with the addition of a scalar function that couples to the curvature called the dilaton. In general, it is possible to redefine the metric via a conformal transformation so that the dilaton does not couple to the new curvature. However, as we shall see, such a transformation affects physical quantities such as the entropy associated with black holes, so we consider the more general case.

As indicated, our theory of gravitation will be based on an action principle, and is closely related to the usual Einstein-Hilbert action. The gravitational field equations may be deduced using Hamilton's principle when considering variations in the geometry. It is now known that serious restrictions must be placed on the boundary of the space-time region in order that such a variational principle be well defined. In particular, both the variation of the induced metric and its derivatives must be held fixed on the boundary. Alternately, the action may be supplemented with additional boundary terms such that we need only fix the induced metric on the boundary. We take the latter approach; from these boundary terms many useful quantities can be defined, as we will show.

In this section, we look at the gravitational sector of the theory, and we defer consideration of various types of matter to later sections. Insofar as we restrict ourselves to dilatonic gravity, we will consider a very general action. The coupling to the curvature, the kinetic energy of the dilaton, and the potential energy of the dilaton will all be arbitrary functions of the dilaton (alone). This practice will be continued in the next section when we consider gauge fields with arbitrary couplings to the dilaton.

We will restrict our considerations to a spatially finite region of a four-dimensional manifold, $\mathcal{M}$, which has a topology of $\Sigma \times \mathbb{R}$ where Σ is a spacelike hypersurface and $\mathbb{R}$ is a real interval. On this manifold, we define a metric $g_{\mu\nu}$ and its compatible derivative operator ∇_μ. Objects with Greek indices represent tensor quantities on the four-dimensional manifold. We will consider two parts of the boundary of this manifold: the outer boundary $\mathcal{T}_{\text{outer}} = \partial\Sigma \times \mathcal{I}$, and the initial spacelike hypersurface Σ_{initial}. The addition of an inner boundary and a final spacelike hypersurface are trivial extensions of the present analysis that will be important in following sections. Here we shall refer to them simply as $\mathcal{T}$ and Σ, respectively.

Associated with $\mathcal{T}$ is an outward directed spacelike normal vector n^μ. We can construct the first and second fundamental forms $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $\Theta_{\mu\nu} = -\frac{1}{2}\pounds_n\gamma_{\mu\nu}$. These are to be viewed as tensors on $\mathcal{T}$, and the indices i, j, etc., will denote tensor quantities on $\mathcal{T}$. However, we will use $\hat{\gamma}_i^\mu$, defined in an analogous manner to $\gamma_{\mu\nu}$, to denote the projection operator from $\mathcal{M}$ onto $\mathcal{T}$. The derivative operator compatible with γ_{ij} is $\triangle_i$.

We can foliate $\mathcal{M}$ into spacelike leaves Σ_t via the parameter t which is a coördinate along $\mathbb{R}$; associate with it the vector field $t^\mu = (\partial/\partial t)^\mu$. View the initial hypersurface as one of these leaves, say, the $t = 0$ one. Little confusion arises from dropping the t index and considering an arbitrary leaf, Σ, in the foliation. There is a future directed timelike normal vector u^μ to Σ. We may define the fundamental forms $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and $K_{\mu\nu} = -\frac{1}{2}\pounds_u h_{\mu\nu}$—these are viewed as tensors on Σ, and such tensors will be shown with indices $\bar{\imath}$, $\bar{\jmath}$, etc.—and projection operator $\hat{h}_{\bar{\imath}}^\mu$ from $\mathcal{M}$ onto Σ. We may also define the lapse and shift of the foliation by $N = -t^\mu u_\mu$ and $N^{\bar{\imath}} = \hat{h}_\mu^{\bar{\imath}} t^\mu$, so that $t^\mu = N u^\mu + N^\mu$. The derivative operator on Σ compatible with $h_{\bar{\imath}\bar{\jmath}}$ is given by $\nabla_{\bar{\imath}}$, and this can be used to define the Ricci scalar on Σ: $R[h]$. It is also important to consider the boundary, $\partial\Sigma$ of Σ. We require that, on the intersection of Σ and $\mathcal{T}$, $u^\mu n_\mu = 0$. The first and second fundamental forms on $\partial\Sigma$ are $\sigma_{\bar{\imath}\bar{\jmath}} = h_{\bar{\imath}\bar{\jmath}} - n_{\bar{\imath}} n_{\bar{\jmath}}$ and $k_{\bar{\imath}\bar{\jmath}} = -\frac{1}{2}\pounds_n \sigma_{\bar{\imath}\bar{\jmath}}$. The Latin indices a, b, etc., are used to denote tensors on $\partial\Sigma$. The projection operator from $\mathcal{T}$ onto $\partial\Sigma$, $\hat{\sigma}_a^i$, is obtained from $\sigma_{ij} = \gamma_{ij} + u_i u_j$.

A zero vorticity observer is one for whom the vorticity, $\boldsymbol{\varpi} = *(\boldsymbol{v} \wedge \boldsymbol{dv})$, is zero, where $\boldsymbol{v}$ is the velocity of the observer. It can be seen that observers who are comoving with a given foliation on the boundary $\mathcal{T}$ are zero vorticity observers. For such an observer, the velocity is just the normal vector u^μ, and the acceleration of the normal vector is given by $a^\mu = u^\nu \nabla_\nu u^\mu = N^{-1} h^{\mu\nu} \nabla_\nu N$.

A summary of the notation described above is given in Table I of the Appendix.

A. Variation of the action

In accordance with the above considerations, we choose the action for the gravitational sector to be

$$
\begin{aligned}
S_{\text{EHD}} = \int_{\mathcal{M}} d^4x\, \sqrt{-g}\, & [f_{\text{EH}}(\Psi) R[g] \\
& + f_{\text{KE}}(\Psi)(\nabla\Psi)^2 + f_{\text{PE}}(\Psi)] \\
& - 2\int_{\mathcal{T}} d^3x\, \sqrt{-\gamma}\, f_{\text{EH}}(\Psi)\,\text{tr}(\Theta) \\
& - 2\int_{\Sigma} d^3x\, \sqrt{h}\, f_{\text{EH}}(\Psi)\,\text{tr}(K),
\end{aligned} \tag{2.1}
$$

where the dilaton field is given by Ψ, and $f_{\text{EH}}(\Psi)$, $f_{\text{KE}}(\Psi)$, and $f_{\text{PE}}(\Psi)$ are model-dependent functions of the dilaton. They are arbitrary insofar as they contain no fields other than the dilaton and no derivatives of the dilaton. Notice the presence of the two boundary terms: these are just what is needed to make a well defined variational principle on the initial spacelike hypersurface and outer boundary as we shall see below.

Now we vary the geometry and dilaton field configurations. The geometry is varied subject to the (gauge) restriction that the leaves of foliation remain orthogonal to the boundary. (That is, we hold the boundary fixed in that variations of the normal dual vectors to the boundaries are proportional to the normal dual vectors.) For convenience, we vary the inverse metric, except on the boundaries where we can write the variation in terms of the covariant induced metrics. The induced variation in the action is given by

$$
\begin{aligned}
\delta S_{\text{EHD}} = \int_{\mathcal{M}} d^4x\, \sqrt{-g}\, & [(\Xi_{\text{geom}})_{\mu\nu}\delta g^{\mu\nu} + \Xi_{\text{dil}}\delta\Psi] \\
& + \int_{\mathcal{T}} d^3x\, (\pi^{ij}\delta\gamma_{ij} + \Pi_{\text{dil}}\delta\Psi) \\
& - \int_{\Sigma} d^3x\, (p^{\bar{\imath}\bar{\jmath}}\delta h_{\bar{\imath}\bar{\jmath}} + P_{\text{dil}}\delta\Psi).
\end{aligned} \tag{2.2}
$$

Here

$$(\Xi_{\rm geom})_{\mu\nu} = f_{\rm EH}(\Psi)G_{\mu\nu}[g] + \tfrac{1}{2}f_{\rm KE}(\Psi)\left[g_{\mu\nu}(\nabla\Psi)^2 - 2(\nabla_\mu\Psi)(\nabla_\nu\Psi)\right] + \tfrac{1}{2}g_{\mu\nu}f_{\rm PE}(\Psi) \quad (2.3)$$

(with $G_{\mu\nu}[g] = R_{\mu\nu}[g] - \frac{1}{2}g_{\mu\nu}R[g]$) and

$$\Xi_{\rm dil} = \frac{df_{\rm EH}(\Psi)}{d\Psi}R[g] + \frac{df_{\rm KE}(\Psi)}{d\Psi}(\nabla\Psi)^2 - 2\nabla^\mu\left[f_{\rm KE}(\Psi)\nabla_\mu\Psi\right] + \frac{df_{\rm PE}(\Psi)}{d\Psi} \quad (2.4)$$

can be considered as equations of motion in the following way: under variations of the geometry and dilaton field that leave the geometry and dilaton configurations on the boundary fixed, $(\Xi_{\rm geom})_{\mu\nu}$ and $\Xi_{\rm dil}$ will be zero at an extremum of the action. (In general, there will be contributions to these equations of motion from any additional matter present.)

Alternately, we could consider variations of the geometry and dilaton configurations in which the equations of motion are held fixed, but the boundary geometry and dilaton configurations are varied. Under these variations, we see from Eq. (2.2) that we can define the momenta conjugate to the boundary geometry of the initial and outer hypersurfaces as $\pi^{ij} = (\delta S_{\rm EHD}/\delta\gamma_{ij})_{\rm CL}$ and $p^{\bar{i}\bar{j}} = -(\delta S_{\rm EHD}/\delta h_{\bar{i}\bar{j}})_{\rm CL}$, respectively. (The subscripted "CL" emphasizes that the variations are among field configurations satisfying the equations of motion.) Explicitly, these are

$$\pi^{ij} = \sqrt{-\gamma}\left\{\gamma^{ij}\left[n^\mu\partial_\mu f_{\rm EH}(\Psi)\right] + f_{\rm EH}(\Psi)\left[\Theta^{ij} - \gamma^{ij}\,{\rm tr}(\Theta)\right]\right\} \quad (2.5a)$$

and

$$p^{\bar{i}\bar{j}} = -\sqrt{h}\left\{(h^{\bar{i}\bar{j}}\left[u^\mu\partial_\mu f_{\rm EH}(\Psi)\right]) + f_{\rm EH}(\Psi)\left[K^{\bar{i}\bar{j}} - h^{\bar{i}\bar{j}}\,{\rm tr}(K)\right]\right\}. \quad (2.5b)$$

Similarly, there are momenta conjugate to the dilaton field on the initial and outer hypersurfaces defined in an analogous way. These quantities are

$$\Pi_{\rm dil} = \left(\frac{\delta S_{\rm EHD}}{\delta\Psi}\right)_{\rm CL} = \sqrt{-\gamma}\left(2f_{\rm KE}(\Psi)n^\mu\partial_\mu\Psi - 2\frac{df_{\rm EH}(\Psi)}{d\Psi}\,{\rm tr}(\Theta)\right) \quad (2.6a)$$

and

$$P_{\rm dil} = -\left(\frac{\delta S_{\rm EHD}}{\delta\Psi}\right)_{\rm CL} = -\sqrt{h}\left(2f_{\rm KE}(\Psi)u^\mu\partial_\mu\Psi - 2\frac{df_{\rm EH}(\Psi)}{d\Psi}\,{\rm tr}(K)\right). \quad (2.6b)$$

B. Quasilocal quantities

The $\mathcal{T}$ boundary momentum, π^{ij}, contains useful information about the energy and momentum densities of the gravitational field held within this boundary. To identify these quantities, it is useful to decompose the variation of the boundary metric, γ_{ij}, into the various projections normal and onto the foliation. Thus we write

$$\delta\gamma_{ij} = \hat{\sigma}^a_i\hat{\sigma}^b_j\delta\sigma_{ab} - \frac{2}{N}\delta N u_i u_j - \frac{2}{N}u_{(i}\hat{\sigma}_{j)a}\delta N^a. \quad (2.7)$$

The corresponding decomposition of the boundary momentum, π^{ij}, conjugate to γ_{ij} leads us to define the surface energy density, the surface momentum density, and the surface stress density as

$$\mathcal{E} = \sqrt{\sigma}\,u_i u_j\tau^{ij}, \quad (2.8a)$$

$$(\mathcal{J}_{\rm EHD})^a = -\sqrt{\sigma}\,u_i\hat{\sigma}^a_j\tau^{ij}, \quad (2.8b)$$

and

$$\mathcal{S}^{ab} = \sqrt{\sigma}\,\hat{\sigma}^a_i\hat{\sigma}^b_j\tau^{ij}, \quad (2.8c)$$

respectively, where $\tau^{ij} = 2\pi^{ij}/\sqrt{-\gamma}$ is the surface stress energy momentum on $\mathcal{T}$.

We next use Eq. (2.5a), and the relationship

$$\Theta_{ij} = k_{ij} + u_i u_j(n^\mu a_\mu) + u_{(i}\hat{\sigma}^{\bar{i}}_{j)}n^{\bar{j}}K_{\bar{i}\bar{j}} \quad (2.9)$$

(recall that $a^\mu = u^\nu\nabla_\nu u^\mu$ is the acceleration of the normal u^μ) to obtain expressions for $\mathcal{E}$, $(\mathcal{J}_{\rm EHD})^a$, and $\mathcal{S}^{ab}$. An immediate consequence of Eq. (2.9) is that ${\rm tr}(\Theta) = {\rm tr}(k) - n^\mu a_\mu$. Then, we have

$$\mathcal{E} = -2\sqrt{\sigma}\left[n^\mu\partial_\mu f_{\rm EH}(\Psi) - f_{\rm EH}(\Psi)\,{\rm tr}(k)\right] - \mathcal{E}_0, \quad (2.10a)$$

$$(\mathcal{J}_{\rm EHD})^a = 2\sqrt{\sigma}\,f_{\rm EH}(\Psi)n_{\bar{i}}\hat{\sigma}^a_{\bar{j}}K^{\bar{i}\bar{j}} - (\mathcal{J}_{\rm EHD,0})^a = -\frac{2\sqrt{\sigma}}{\sqrt{h}}n_{\bar{i}}\hat{\sigma}^a_{\bar{j}}p^{\bar{i}\bar{j}} - (\mathcal{J}_{\rm EHD,0})^a\,, \quad (2.10b)$$

and

$$\mathcal{S}^{ab} = 2\sqrt{\sigma}\,(\sigma^{ab}n^\mu\partial_\mu f_{\rm EH}(\Psi) + f_{\rm EH}(\Psi)\{k^{ab} - \sigma^{ab}[{\rm tr}(k) - n^\mu a_\mu]\}) - (\mathcal{S}_0)^{ab}, \quad (2.10c)$$

where $\mathcal{E}_0$, $(\mathcal{J}_{\rm EHD,0})^a$, and $(\mathcal{S}_0)^{ab}$ are additional contributions that arise in supplementing the action $S_{\rm EHD}$ with an additional (reference) functional of the boundary fields, as we will discuss in the following subsection.

We can construct a quantity $\mathcal{Y}$ from the momentum $\Pi_{\rm dil}$ conjugate to the dilaton configuration on $\mathcal{T}$: $\mathcal{Y} = N^{-1}\Pi_{\rm dil} - \mathcal{Y}_0$ where $\mathcal{Y}_0$ is an arbitrary background contribution that will be discussed below. Thus we have

$$\mathcal{Y} = \sqrt{\sigma}\left(2f_{\rm KE}(\Psi)n^\mu\partial_\mu\Psi - 2\frac{df_{\rm EH}(\Psi)}{d\Psi}\left[{\rm tr}(k) - n^\mu a_\mu\right]\right) - \mathcal{Y}_0. \quad (2.11)$$

Note that $\mathcal{Y}$ is also a scalar density on the two-surface $\partial\Sigma$. Using the definitions in Eqs. (2.10) and (2.11), we can write the $\mathcal{T}$ boundary terms in the variation of the action as

$$\delta S_{\rm EHD}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x\,[-\mathcal{E}\delta N + (\mathcal{J}_{\rm EHD})_a\delta N^a + N(\tfrac{1}{2}\mathcal{S}^{ab}\delta\sigma_{ab} + \mathcal{Y}\delta\Psi)]. \quad (2.12)$$

It is worth noting that $\mathcal{E}$, $(\mathcal{J}_{\rm EHD})_a$, σ_{ab}, and Ψ are all *extensive variables* because they can be constructed out of the phase space variables $(p^{\bar{i}\bar{j}}, h_{\bar{i}\bar{j}})$ and $(P_{\rm dil}, \Psi)$ on Σ. However the lapse and shift cannot be constructed out of this information, and such quantities are called *intensive variables*. We see that, in Eq. (2.12), the first two terms of the integrand involve variations of intensive variables (with extensive variables as coefficients), while the last three terms involve variations of extensive variables (with coefficients that are intensive variables due to the lapse function). We will consider the implications of this below.

Define the quasilocal energy to be the integral over the two-surface $\partial\Sigma$ of the quasilocal energy density:

$$\mathbb{E} = \int_{\partial\Sigma} d^2x\,\mathcal{E}. \quad (2.13)$$

This quantity has useful properties such as additivity [13], although it is not necessarily positive definite. The quasilocal energy is observer dependent, i.e., even if we are given a natural choice of boundary $\mathcal{T}$, the above definition of the quasilocal energy will still depend on the foliation; in particular, how the leaves Σ intersect with $\mathcal{T}$. However, we would expect that we should be able to do better than this when the space-time is stationary. The construction of conserved charges resulting from space-times with symmetry will be addressed below.

C. Reference space-time

The complete gravitational action will involve the action $S_{\rm EHD}$ given in Eq. (2.1), plus some additional functional of the boundary metric and dilaton configuration. This additional piece, S_0, does not contribute to the equations of motion because it contributes only to the boundary. It does, however, contribute to the momenta conjugate to the gravitational and dilaton fields and thus to the quasilocal quantities discussed above. It will be sufficient in the present analysis to consider S_0 to be a functional on the boundary $\mathcal{T}$ alone. Furthermore, we will assume that it is a functional of the metric γ_{ij}, and the $\mathcal{T}$ boundary dilaton configuration, but we will not consider the general case in which it is also a functional of boundary configurations of matter fields (though the inclusion of these is straightforward).

The specification of the functional S_0 is akin to the specification of some reference space-time. Although there are restrictions on the types of reference space-time allowed (which have to do with the embedding of the two-surfaces $\partial\Sigma$ in the reference space-time), we will assume that a natural reference space-time can be found. Often, this can be achieved simply by setting constants of integration of a particular solution to some special value that then specifies the reference. Our restriction that the functional S_0 not be a functional of matter fields (other than the dilaton) is just a specification of a vacuum reference space-time.

The primary advantage of the above interpretation of S_0 is that it ensures that the action is a linear functional of lapse and shift and, as we shall see, this means that the extensive variables defined above continue to be extensive. (This is the principal restriction that must be placed on S_0.) We shall thus write the additional piece in the suggestive form:

$$S_0 = \int_{\mathcal{T}} d^3x\,[N\mathcal{E}_0 - N^a(\mathcal{J}_{\rm EHD,0})_a]. \quad (2.14)$$

Here, $\mathcal{E}_0$ and $(\mathcal{J}_{\rm EHD,0})_a$ can be constructed out of the phase-space variables alone (and are thus extensive), and can be identified by the functional derivative of S_0 with respect to the lapse and the (negative) shift, respectively. The surface-stress and dilaton density of the reference space-time can be obtained from the relation

$$\int_{\mathcal{T}} d^3x\,N[\tfrac{1}{2}(\mathcal{S}_0)^{ab}\delta\sigma_{ab} + \mathcal{Y}_0\delta\Psi] = -\int_{\mathcal{T}} d^3x\,[N\delta\mathcal{E}_0 - N^a\delta(\mathcal{J}_{\rm EHD,0})_a] \quad (2.15)$$

which allows one to determine $(\mathcal{S}_0)^{ab}$ and $\mathcal{Y}_0$ when the forms of $\mathcal{E}_0$ and $(\mathcal{J}_{\rm EHD,0})_a$ are known in terms of σ_{ab} and Ψ. The net result of the addition of S_0 to $S_{\rm EHD}$ is the inclusion of the extra terms in Eqs. (2.10a)–(2.10c) as well as in the definition of $\mathcal{Y}$. Of course, one possible choice is always $S_0 = 0$ in which case these terms would be absent.

D. Conserved charges

Given a set of observers, all having histories on $\mathcal{T}$, we wish to identify quantities related to the geometry of $\mathcal{M}$ that are the same for all the observers. We will require the presence of a Killing vector, ξ^i, on $\mathcal{T}$. Our first step is to consider the equation of motion for the geometry. We have been deferring a detailed discussion of matter to a later section; however, here we will allow for the presence of matter so that the equation of motion reads

$$(\Xi_{\rm geom})_{\mu\nu} = \tfrac{1}{2}T_{\mu\nu}, \quad (2.16)$$

where $T_{\mu\nu}$ is the stress-energy-momentum tensor of the matter and $(\Xi_{\rm geom})_{\mu\nu}$ is given by Eq. (2.3). Computing $n^\mu\hat{\gamma}_i^\nu(\Xi_{\rm geom})_{\mu\nu}$ with the aid of the Gauss-Codacci relationship

$$n^\mu\hat{\gamma}^{i\nu}R_{\mu\nu}[g] = -\triangle_j\,[\Theta^{ij} - \gamma^{ij}\,{\rm tr}(\Theta)] \quad (2.17)$$

we obtain

$$\sqrt{-\gamma}\,\triangle_j\,\tau^{ij} = \Pi_{\rm dil}\,\triangle^i\,\Psi - \sqrt{-\gamma}n^\mu\hat{\gamma}^{i\nu}T_{\mu\nu} \quad (2.18)$$

which shows that the surface stress-energy-momentum tensor is not divergenceless: it has source terms arising from the presence of the dilaton and the matter.

However, we can contract Eq. (2.18) with the Killing vector. In addition to the usual requirements of a Killing vector, we require that the dilaton is constant on orbits of the Killing vector. That is, we assume that ξ^i satisfies both the usual Killing equation, $\pounds_\xi \gamma_{ij} = 0$, for a Killing vector on $\mathcal{T}$, as well as $\pounds_\xi \Psi = 0$. The left-hand side of Eq. (2.18) becomes a total divergence due to the Killing equation and the symmetry of τ^{ij}. Integrating over $\mathcal{T}$, we find

$$-\int_{\mathcal{T}\cap\Sigma_{\rm initial}}^{\mathcal{T}\cap\Sigma_{\rm final}} d^2x\,\sqrt{\sigma}\,\xi_i u_j \tau^{ij} = \int_{\mathcal{T}} d^3x\,\sqrt{-\gamma}\,\xi^i T_{i\mu}n^\mu. \tag{2.19}$$

In the event that the right-hand side vanishes for arbitrary $\Sigma_{\rm final}$, Eq. (2.19) expresses a conservation law for a geometric charge:

$$\Bbbk[\xi] = -\int_{\partial\Sigma} d^2x\sqrt{\sigma}\xi_i u_j \tau^{ij}. \tag{2.20}$$

Contingent upon the type of matter present, there are many reasons why this may be the case. First, it may be that $\mathcal{T}$ is positioned such that there is little matter in its vicinity, so $T_{\mu\nu}$ is negligible. Second, it may be the case that the particular projection $\xi^i T_{i\mu}n^\mu$ vanishes. Alternately, if $T_{\mu\nu}$ is conserved, that is $\nabla_\mu T^{\mu\nu} = 0$, and if the Killing vector field on $\mathcal{T}$ can be promoted to a Killing field over $\mathcal{M}$ (that is, if ξ^μ is a Killing vector field on $\mathcal{M}$, satisfying the additional requirements above, and $\mathcal{T}$ contains the orbits of these vectors), then we can rewrite Eq. (2.19) in the form

$$\Bbbk(\mathcal{T}\cap\Sigma_{\rm initial}) - \Bbbk(\mathcal{T}\cap\Sigma_{\rm final}) = \int_{\Sigma_{\rm initial}}^{\Sigma_{\rm final}} d^3x\sqrt{h}u^\mu\xi^\nu T_{\mu\nu} \tag{2.21}$$

and the integrand on the right-hand side of this equation may vanish.

Suppose φ^i is a spacelike azimuthal Killing vector. Then, we can define an angular momentum as $\mathbb{J} = \Bbbk[\varphi]$. If the surface $\partial\Sigma$ is taken so that it contains the orbits of the Killing vector, then we can write

$$\mathbb{J} = \int_{\partial\Sigma} d^2x\,(\mathcal{J}_{\rm EHD})_a\varphi^a. \tag{2.22}$$

When ξ^i is timelike, we can define a mass as $\mathbb{M} = -\Bbbk[\xi]$. If the space-time is also static, that is, ξ^i is surface forming, then we can choose a two-surface $\partial\Sigma$ for which the Killing vector is proportional to the timelike normal. In this case, the mass can be written in the form

$$\mathbb{M} = \int_{\partial\Sigma} d^2x\,N\mathcal{E}. \tag{2.23}$$

Note that, in general, the quasilocal energy will *not* agree with the conserved mass of the space-time unless u^i is a Killing vector on $\mathcal{T}$. However, in asymptotically flat, static space-times for which $\mathcal{T}$ is taken at spatial infinity, a foliation in which u^i approaches the timelike Killing vector is usually adopted, and then the definitions of mass and quasilocal energy given here will agree.

E. Canonical form of the action

We now turn to the canonical decomposition of the action of Eq. (2.1). The Hamiltonian density of the Einstein-Hilbert-dilaton sector is defined as

$$H_{\rm EHD} = p^{\bar\imath\bar\jmath}\pounds_t h_{\bar\imath\bar\jmath} + P_{\rm dil}\pounds_t\Psi - L_{\rm EHD}, \tag{2.24}$$

where $L_{\rm EHD}$ is the Lagrangian density which, when integrated over $\mathcal{M}$ yields the action of Eq. (2.1). To evaluate the first term, we use the definition of the second fundamental form, and the relationship between u^μ and t^μ. We find that

$$p^{\bar\imath\bar\jmath}\pounds_t h_{\bar\imath\bar\jmath} = -2Np^{\bar\imath\bar\jmath}K_{\bar\imath\bar\jmath} - 2N_{\bar\imath}\nabla_{\bar\jmath}p^{\bar\imath\bar\jmath} + 2\nabla_{\bar\jmath}(N_{\bar\imath}p^{\bar\imath\bar\jmath}). \tag{2.25}$$

Similarly, the second term is

$$P_{\rm dil}\pounds_t\Psi = NP_{\rm dil}\mathring{\Psi} + N^{\bar\imath}(P_{\rm dil}\nabla_{\bar\imath}\Psi), \tag{2.26}$$

where the symbol $\mathring{\Psi}$ is shorthand for $u^\mu\partial_\mu\Psi$. The Lagrangian density can be canonically decomposed using the Gauss-Codacci relationship

$$R[g] = R[h] + K^{\bar\imath\bar\jmath}K_{\bar\imath\bar\jmath} - [\mathrm{tr}(K)]^2 - 2\nabla_\mu[u^\mu\,\mathrm{tr}(K) + a^\mu] \tag{2.27}$$

as well as

$$(\nabla\Psi)^2 = (\nabla\Psi)^2 - \mathring{\Psi}^2. \tag{2.28}$$

We also use the relationship $a_{\bar\imath} = N^{-1}\nabla_{\bar\imath}N$ which is appropriate for zero-vorticity observers. The Hamiltonian can then be written as

$$\begin{aligned}\mathbb{H}_{\rm EHD} &= \int_\Sigma d^3x\,H_{\rm EHD}\\ &= \int_\Sigma d^3x\,[\mathcal{H}_{\rm EHD}N + (\mathcal{H}_{\rm EHD})_{\bar\imath}N^{\bar\imath}]\\ &\quad + \int_{\partial\Sigma} d^2x\,[\mathcal{E}N - (\mathcal{J}_{\rm EHD})_a N^a],\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}\mathcal{H}_{\rm EHD} = &-2p^{\bar\imath\bar\jmath}K_{\bar\imath\bar\jmath} + P_{\rm dil}\mathring{\Psi}\\ &-\sqrt{h}\Bigg(f_{\rm EH}(\Psi)\{R[h] + K^{\bar\imath\bar\jmath}K_{\bar\imath\bar\jmath} - [\mathrm{tr}(K)]^2\}\\ &+2\,\mathrm{tr}(K)\frac{df_{\rm EH}(\Psi)}{d\Psi}\mathring{\Psi} - 2\nabla^2 f_{\rm EH}(\Psi)\\ &+f_{\rm KE}(\Psi)(\nabla\Psi)^2 - f_{\rm KE}(\Psi)\mathring{\Psi}^2 + f_{\rm PE}(\Psi)\Bigg)\end{aligned} \tag{2.30}$$

is the Hamiltonian constraint and

$$(\mathcal{H}_{\rm EHD})_{\bar\imath} = -2\nabla_{\bar\jmath}p^{\bar\jmath}_{\bar\imath} + P_{\rm dil}\nabla_{\bar\imath}\Psi \tag{2.31}$$

is the momentum constraint. When the constraint equations hold, the Hamiltonian is purely a boundary term.

The action, Eq. (2.1), can now be rewritten in canonical form:

$$S_{\rm EHD} = \int_{\mathcal{M}} d^4x \left(p^{\bar{\imath}\bar{\jmath}}\pounds_t h_{\bar{\imath}\bar{\jmath}} + P_{\rm dil}\pounds_t\Psi - H_{\rm EHD}\right)$$
$$= \int dt \left(\int_{\Sigma} d^3x \left[p^{\bar{\imath}\bar{\jmath}}\pounds_t h_{\bar{\imath}\bar{\jmath}} + P_{\rm dil}\pounds_t\Psi - \mathcal{H}_{\rm EHD}N - (\mathcal{H}_{\rm EHD})_{\bar{\imath}}N^{\bar{\imath}}\right] + \int_{\partial\Sigma} d^2x \left[-\mathcal{E}N + (\mathcal{J}_{\rm EHD})_a N^a\right]\right). \quad (2.32)$$

This form of the action will be useful later in the study of the thermodynamics of gravitational systems.

III. YANG-MILLS SECTOR

Here we consider non-Abelian gauge fields with an arbitrary gauge group $\mathfrak{G}$. The case of electromagnetism is, of course, just a simplification of the general results of this section when the gauge group is U(1). We shall refer to the internal degrees of freedom as color degrees of freedom, and the associated gauge charges as color charges.

In what follows, quantities that possess color are represented with Fraktur characters; color indices, when needed, will be given by lower case Fraktur characters, and the adjoint representation will be assumed. The gauge covariant derivative (both gauge covariant and covariant on the manifold $\mathcal{M}$) is given by $(\mathfrak{D}_\mu)^{\mathfrak{a}}{}_{\mathfrak{b}} = \nabla_\mu\delta^{\mathfrak{a}}{}_{\mathfrak{b}} + \mathfrak{f}^{\mathfrak{a}}{}_{\mathfrak{bc}}\mathfrak{A}_\mu{}^{\mathfrak{c}}$ where $\mathfrak{A}_\mu{}^{\mathfrak{a}}$ is the connection and $\mathfrak{f}^{\mathfrak{a}}{}_{\mathfrak{bc}}$ are the structure constants of the group. The curvature of this gauge covariant derivative operator is the field tensor: $\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}} = 2\nabla_{[\mu}\mathfrak{A}_{\nu]}{}^{\mathfrak{a}} + \mathfrak{f}^{\mathfrak{a}}{}_{\mathfrak{bc}}\mathfrak{A}_\mu{}^{\mathfrak{b}}\mathfrak{A}_\nu{}^{\mathfrak{c}}$. This field tensor is *co*variant under gauge transformations of the form $\mathfrak{A}_\mu{}^{\mathfrak{a}}[\mathfrak{x}] = \mathfrak{A}_\mu{}^{\mathfrak{a}}[0]+(\mathfrak{D}_\mu)^{\mathfrak{a}}{}_{\mathfrak{b}}\mathfrak{x}^{\mathfrak{b}}$. We partially restrict the gauge freedom of the potential so that the components in an orthonormal frame are finite everywhere in $\mathcal{M}$ (except, perhaps, at truly pathological points such as curvature singularities).

On a spacelike hypersurface Σ it is possible to decompose the above quantities. The electric field as seen by an observer who is stationary with respect to the foliation is given by $\mathfrak{E}_{\bar{\imath}}{}^{\mathfrak{a}} = \hat{h}^\mu_{\bar{\imath}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}}u^\nu$. The magnetic field is given by $\mathfrak{B}_{\bar{\imath}}{}^{\mathfrak{a}} = -\frac{1}{2}\hat{h}^\mu_{\bar{\imath}}\epsilon_{\mu\nu}{}^{\rho\sigma}\mathfrak{F}_{\rho\sigma}{}^{\mathfrak{a}}u^\nu$. The gauge covariant derivative that is compatible with the metric $h_{\bar{\imath}\bar{\jmath}}$ is $(\mathfrak{d}_{\bar{\imath}})^{\mathfrak{a}}{}_{\mathfrak{b}}$.

A. The dilaton-Yang-Mills action and variations

The action is

$$S_{\rm DYM} = -\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\frac{1}{4}f_{\rm YM}(\Psi)(\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}}). \quad (3.1)$$

The color indices are raised and lowered by the Killing metric $\mathfrak{g}_{\mathfrak{ab}} = \frac{1}{2}\mathfrak{f}^{\mathfrak{c}}{}_{\mathfrak{da}}\mathfrak{f}^{\mathfrak{d}}{}_{\mathfrak{cb}}$. The function, $f_{\rm YM}(\Psi)$, is a function of the dilaton alone, and contains no derivatives of the dilaton.

Variation of the action of (3.1) yields source terms for the Einstein-Hilbert field equations as well as for the dilaton field equation. In addition, variation with respect to the gauge potential $\mathfrak{A}_\mu{}^{\mathfrak{a}}$ gives a source-free field equation for the gauge fields. Note that the variation, on the boundary, of the potential is the same as the variation of the potential projected onto the boundary. The induced variation in the dilaton-Yang-Mills sector of the action is

$$\delta S_{\rm DYM} = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left[-\tfrac{1}{2}(T_{\rm DYM})_{\mu\nu}\delta g^{\mu\nu} -\tfrac{1}{2}\Upsilon_{\rm DYM}\delta\Psi + \mathfrak{X}^\mu{}_{\mathfrak{a}}\delta\mathfrak{A}_\mu{}^{\mathfrak{a}}\right] + \int_{\mathcal{T}} d^3x\,(\Pi_{\rm DYM})^i{}_{\mathfrak{a}}\delta\mathfrak{A}_i{}^{\mathfrak{a}} - \int_{\Sigma} d^3x\,(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\delta\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}}, \quad (3.2)$$

where

$$\mathfrak{X}^\nu{}_{\mathfrak{a}} = (\mathfrak{D}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}\left[f_{\rm YM}(\Psi)\mathfrak{F}^{\mu\nu}{}_{\mathfrak{b}}\right]. \quad (3.3)$$

This yields the dilaton-Yang-Mills vacuum equation of motion, $\mathfrak{X}^\nu{}_{\mathfrak{a}} = 0$ when the boundary terms vanish provided that there are no Yang-Mills source terms. The dilaton-Yang-Mills stress energy and dilaton source are

$$(T_{\rm DYM})_{\mu\nu} = f_{\rm YM}(\Psi)(\mathfrak{F}_{\mu\alpha}{}^{\mathfrak{a}}\mathfrak{F}_\nu{}^\alpha{}_{\mathfrak{a}} - \tfrac{1}{4}g_{\mu\nu}\mathfrak{F}_{\alpha\beta}{}^{\mathfrak{a}}\mathfrak{F}^{\alpha\beta}{}_{\mathfrak{a}}) \quad (3.4)$$

and

$$\Upsilon_{\rm DYM} = \frac{1}{2}\,\frac{df_{\rm YM}(\Psi)}{d\Psi}\,\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}}\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}, \quad (3.5)$$

respectively.

In addition we have momenta conjugate to the variation of the gauge fields on the boundaries $\mathcal{T}$ and Σ. These are

$$(\Pi_{\rm DYM})^i{}_{\mathfrak{a}} = \sqrt{-\gamma}\,f_{\rm YM}(\Psi)\hat{\gamma}^i_\mu\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}n_\nu \quad (3.6a)$$

and

$$(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}} = -\sqrt{h}\,f_{\rm YM}(\Psi)\hat{h}^{\bar{\imath}}_\mu\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}u_\nu = -\sqrt{h}\,f_{\rm YM}(\Psi)\mathfrak{E}^{\bar{\imath}}{}_{\mathfrak{a}}, \quad (3.6b)$$

respectively.

On the boundary $\mathcal{T}$, we can decompose the variation of $\mathfrak{A}_i{}^{\mathfrak{a}}$ into pieces normal and tangential to the foliation. Let $\mathfrak{V}^{\mathfrak{a}} = -u^i\mathfrak{A}_i{}^{\mathfrak{a}}$ and $\mathfrak{W}_a{}^{\mathfrak{a}} = \hat{\sigma}^i_a\mathfrak{A}_i{}^{\mathfrak{a}}$. Then,

$$\delta\mathfrak{A}_i{}^{\mathfrak{a}} = N^{-1}u_i\left[\delta(N\mathfrak{V}^{\mathfrak{a}}) - \mathfrak{W}_a{}^{\mathfrak{a}}\delta N^a\right] + \hat{\sigma}^a_i\delta\mathfrak{W}_a{}^{\mathfrak{a}}. \quad (3.7)$$

A similar decomposition of $(\Pi_{\rm DYM})^i_{\mathfrak{a}}$ leads us to define a surface Yang-Mills charge density:

$$(\mathcal{Q}_{\rm DYM})_{\mathfrak{a}} = \sqrt{\sigma}\,f_{\rm YM}(\Psi)n_{\bar{\imath}}\mathfrak{E}^{\bar{\imath}}{}_{\mathfrak{a}} \quad (3.8)$$

and a surface Yang-Mills momentum density

$$(\mathcal{J}_{\rm DYM})_a = (\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}\mathfrak{W}_a{}^{\mathfrak{a}}. \quad (3.9)$$

Also, define a surface Yang-Mills current

$$\mathfrak{I}_a{}^{\mathfrak{a}} = \sqrt{\sigma}\, f_{\rm YM}(\Psi)\epsilon_a{}^{\bar{\imath}\bar{\jmath}} n_{\bar{\imath}} \mathfrak{B}_{\bar{\jmath}}{}^{\mathfrak{a}}. \tag{3.10}$$

Then, the variation of $S_{\rm DYM}$ on $\mathcal{T}$ is given by

$$\delta S_{\rm DYM}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x \left[(\mathcal{J}_{\rm DYM})_a \delta N^a - (\mathcal{Q}_{\rm DYM})_{\mathfrak{a}} \delta(N\mathfrak{V}^{\mathfrak{a}}) + N\mathfrak{I}^a{}_{\mathfrak{a}} \delta\mathfrak{W}_a{}^{\mathfrak{a}} \right]. \tag{3.11}$$

We will consider the interpretation of the surface charge density in the following.

B. Conserved Yang-Mills charges

In the presence of a source, $\mathfrak{J}^\mu{}_{\mathfrak{a}}$, the equations of motion for the Yang-Mills field (with dilaton coupling) are

$$(\mathfrak{D}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}[f_{\rm YM}(\Psi)\mathfrak{F}^{\mu\nu}{}_{\mathfrak{b}}] = \mathfrak{J}^\nu{}_{\mathfrak{a}}. \tag{3.12}$$

It can be seen that the source respects the identity $(\mathfrak{D}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}\mathfrak{J}^\mu{}_{\mathfrak{b}} = 0$. In an Abelian gauge theory (such as electromagnetism) this quantity is gauge invariant and can be used to define a conserved charge. However, in a general Yang-Mills theory, the identity is gauge covariant, and the separation of the color contained in the charge and the color contained in the field it produces depends on the gauge choice.

Yet, it is still possible to construct a conserved color charge if we require the solution to the field equations to have certain properties [22,23]. Suppose that the solution possesses a gauge Killing scalar $\mathfrak{k}^{\mathfrak{a}}$, that is, a Lie-algebra-valued scalar field on $\mathcal{M}$ that is covariantly constant: $(\mathfrak{D}_\mu)^{\mathfrak{a}}{}_{\mathfrak{b}}\mathfrak{k}^{\mathfrak{b}} = 0$. Then the quantity $\mathfrak{k}^{\mathfrak{a}}\mathfrak{J}^\mu{}_{\mathfrak{a}}$ is gauge invariant and divergenceless: $\nabla_\mu(\mathfrak{k}^{\mathfrak{a}}\mathfrak{J}^\mu{}_{\mathfrak{a}}) = 0$. We can then define a charge

$$\mathbb{Q}_{\rm DYM}[\mathfrak{k}] = -\int_\Sigma d^3x\, \sqrt{h}\, u_\mu \mathfrak{k}^{\mathfrak{a}}\mathfrak{J}^\mu{}_{\mathfrak{a}} \tag{3.13}$$

that is conserved provided that the source $\mathfrak{k}^{\mathfrak{a}}\mathfrak{J}^\mu{}_{\mathfrak{a}}$ vanishes in the vicinity of $\mathcal{T}$, that is, the charge of Eq. (3.13) has the same value regardless of the volume Σ chosen for the integration.

Recall now the equation of motion (3.12). Contracting both sides with the gauge Killing vector, one can take $\mathfrak{k}^{\mathfrak{a}}$ through the gauge covariant derivative to produce a gauge scalar as its argument. Further contracting both sides by u_μ and recalling that u_μ is proportional to a gradient, $u_\mu = -N\partial_\mu t$, we can write $u_\nu(\mathfrak{D}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}[f_{\rm YM}(\Psi)\mathfrak{F}^{\mu\nu}{}_{\mathfrak{b}}]$ as $(\mathfrak{d}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}[f_{\rm YM}(\Psi)\mathfrak{F}^{\mu\nu}{}_{\mathfrak{b}}u_\nu]$, due to the antisymmetry of the field tensor $\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}$ in μ and ν. Integrating over the spacelike hypersurface Σ, we obtain

$$\mathbb{Q}_{\rm DYM}[\mathfrak{k}] = \int_{\partial\Sigma} d^2x\, (\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}\mathfrak{k}^{\mathfrak{a}}. \tag{3.14}$$

This alternate expression for the conserved gauge charge gives us our interpretation of $(\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}$ as a surface gauge charge density.

C. Canonical form of the dilaton-Yang-Mills action

Finally, we turn to the task of writing the dilaton-Yang-Mills action of Eq. (3.1) in canonical form. A straightforward calculation shows that $\pounds_t\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} = -\mathfrak{F}_{\bar{\imath}\bar{\jmath}}{}^{\mathfrak{a}}t^{\bar{\jmath}} - (\mathfrak{d}_{\bar{\imath}})^{\mathfrak{a}}{}_{\mathfrak{b}}\mathfrak{A}_t{}^{\mathfrak{b}}$ where $\mathfrak{A}_t{}^{\mathfrak{a}} = -t^\mu\mathfrak{A}_\mu{}^{\mathfrak{a}}$. Using the decomposition of t^μ into u^μ and N^μ, we have $\mathfrak{A}_t{}^{\mathfrak{a}} = N\mathfrak{V}^{\mathfrak{a}} - N^a\mathfrak{W}_a{}^{\mathfrak{a}}$. Then, we can show that

$$\begin{aligned}(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} = {} & -N(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{E}_{\bar{\imath}}{}^{\mathfrak{a}} \\ & -(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{F}_{\bar{\imath}\bar{\jmath}}{}^{\mathfrak{a}}N^{\bar{\jmath}} \\ & +\mathfrak{A}_t{}^{\mathfrak{a}}(\mathfrak{d}_{\bar{\imath}})^{\mathfrak{b}}{}_{\mathfrak{a}}(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{b}} \\ & -\nabla_{\bar{\imath}}[\mathfrak{A}_t{}^{\mathfrak{a}}(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}].\end{aligned} \tag{3.15}$$

We also use the decomposition

$$\tfrac{1}{4}\mathfrak{F}^{\mu\nu}{}_{\mathfrak{a}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}} = \tfrac{1}{2}(\mathfrak{B}^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{B}_{\bar{\imath}}{}^{\mathfrak{a}} - \mathfrak{E}^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{E}_{\bar{\imath}}{}^{\mathfrak{a}}). \tag{3.16}$$

The Hamiltonian density is given by $H_{\rm DYM} = (P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} - L_{\rm DYM}$ where $L_{\rm DYM}$ is the Lagrangian density [the integrand of Eq. (3.1)]. The Hamiltonian is, then,

$$\begin{aligned}\mathbb{H}_{\rm DYM} &= \int_\Sigma d^3x\, H_{\rm DYM} \\ &= \int_\Sigma d^3x\, [\mathcal{H}_{\rm DYM}N + (\mathcal{H}_{\rm DYM})_{\bar{\imath}}N^{\bar{\imath}} + (\mathcal{G}_{\rm DYM})_{\mathfrak{a}}\mathfrak{A}_t{}^{\mathfrak{a}}] \\ &\quad + \int_{\partial\Sigma} d^2x\, [(\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}\mathfrak{V}^{\mathfrak{a}}N - (\mathcal{J}_{\rm DYM})_aN^a],\end{aligned} \tag{3.17}$$

where

$$\mathcal{H}_{\rm DYM} = -(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{E}_{\bar{\imath}}{}^{\mathfrak{a}} + \tfrac{1}{2}\sqrt{h}\, f_{\rm YM}(\Psi)(\mathfrak{B}^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{B}_{\bar{\imath}}{}^{\mathfrak{a}} - \mathfrak{E}^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{E}_{\bar{\imath}}{}^{\mathfrak{a}}) \tag{3.18}$$

is the contribution to the Hamiltonian constraint from the dilaton-Yang-Mills sector,

$$(\mathcal{H}_{\rm DYM})_{\bar{\jmath}} = -(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\mathfrak{F}_{\bar{\imath}\bar{\jmath}}{}^{\mathfrak{a}} \tag{3.19}$$

is the contribution to the momentum constraint from the dilaton-Yang-Mills sector, and

$$(\mathcal{G}_{\rm DYM})_{\mathfrak{a}} = (\mathfrak{d}_{\bar{\imath}})^{\mathfrak{b}}{}_{\mathfrak{a}}(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{b}} \tag{3.20}$$

is the Gauss constraint for the Yang-Mills field.

The action in canonical form is simply

$$\begin{aligned}S_{\rm DYM} &= \int_{\mathcal{M}} d^4x\, [(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} - H_{\rm DYM}] \\ &= \int dt \left(\int_\Sigma d^3x\, [(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar{\imath}}^{\mathfrak{a}} - \mathcal{H}_{\rm DYM}N - (\mathcal{H}_{\rm DYM})_{\bar{\imath}}N^{\bar{\imath}} - (\mathcal{G}_{\rm DYM})_{\mathfrak{a}}\mathfrak{A}_t{}^{\mathfrak{a}}] \right. \\ &\quad \left. + \int_{\partial\Sigma} d^2x\, [-(\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}\mathfrak{V}^{\mathfrak{a}}N + (\mathcal{J}_{\rm DYM})_aN^a] \right).\end{aligned} \tag{3.21}$$

IV. AXION SECTOR

In this section we consider the coupling of an axion field to the Yang-Mills field strength. Such couplings provide an interesting counterexample [15] to Schiff's conjecture[2] [24] and, in the case of an Abelian gauge theory, imply interesting new tests of the equivalence principle [25,26].

Many of the derivations in this section are similar to those in the previous section on the Yang-Mills field. Define the dual to the Yang-Mills field tensor by $(*\mathfrak{F})_{\mu\nu}{}^{\mathfrak{a}} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\mathfrak{F}_{\rho\sigma}{}^{\mathfrak{a}}$. Note that there is the identity $(\mathfrak{D}_\mu)^{\mathfrak{b}}{}_{\mathfrak{a}}(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{b}} = 0$.

A. The axion-Yang-Mills action and its variation

We take the action for the axionic sector to be

$$S_{\text{AYM}} = \int_{\mathcal{M}} d^4x \sqrt{-g}\,\big[\tfrac{1}{4}\vartheta_{\text{YM}}(\theta)(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}} + \vartheta_{\text{KE}}(\theta)(\nabla\theta)^2 + \vartheta_{\text{PE}}(\theta)\big], \tag{4.1}$$

where θ is the axion field and $\vartheta_{\text{YM}}(\theta)$, $\vartheta_{\text{KE}}(\theta)$, and $\vartheta_{\text{PE}}(\theta)$ are functions of the axion (but not its derivatives) that are the couplings to the Yang-Mills fields, the kinetic energy, and the potential energy, respectively.

Varying the action with respect to the geometry, the gauge field, and the axion yields

$$\begin{aligned}\delta S_{\text{AYM}} = &\int_{\mathcal{M}} d^4x \sqrt{-g}\,\big[-\tfrac{1}{2}(T_{\text{AYM}})_{\mu\nu}\delta g^{\mu\nu} + \mathfrak{Y}^{\mu}{}_{\mathfrak{a}}\delta\mathfrak{A}_\mu{}^{\mathfrak{a}} + \Xi_{\text{axi}}\delta\theta\big]\\ &+\int_{\mathcal{T}} d^3x\,\big[(\Pi_{\text{AYM}})^{i}{}_{\mathfrak{a}}\delta\mathfrak{A}_i{}^{\mathfrak{a}} + \Pi_{\text{axi}}\delta\theta\big]\\ &-\int_{\Sigma} d^3x\,\big[(P_{\text{AYM}})^{\bar{\imath}}{}_{\mathfrak{a}}\delta\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} + P_{\text{axi}}\delta\theta\big],\end{aligned} \tag{4.2}$$

where the stress-energy-momentum contribution from this sector is

$$(T_{\text{AYM}})_{\mu\nu} = -2\vartheta_{\text{KE}}(\theta)(\nabla_\mu\theta)(\nabla_\nu\theta) + g_{\mu\nu}\vartheta_{\text{KE}}(\theta)(\nabla\theta)^2 + g_{\mu\nu}\vartheta_{\text{PE}}(\theta). \tag{4.3}$$

The equation of motion for the gauge field, when boundary variations vanish, is $\mathfrak{Y}^{\mu}{}_{\mathfrak{a}} = 0$ (vacuum case) where

$$\mathfrak{Y}^{\mu}{}_{\mathfrak{a}} = (*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}}\nabla_\nu\vartheta_{\text{YM}}(\theta) \tag{4.4}$$

while the equation of motion for the axion itself is $\Xi_{\text{axi}} = 0$ with

$$\Xi_{\text{axi}} = \frac{1}{4}\frac{d\vartheta_{\text{YM}}(\theta)}{d\theta}(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}} + \frac{d\vartheta_{\text{KE}}(\theta)}{d\theta}\big[(\nabla\theta)^2 - 2\nabla^2\theta\big] + \frac{d\vartheta_{\text{PE}}(\theta)}{d\theta}. \tag{4.5}$$

Similarly, there are momenta conjugate to the gauge field configurations on $\mathcal{T}$ and Σ:

$$(\Pi_{\text{AYM}})^{i}{}_{\mathfrak{a}} = \sqrt{-\gamma}\,\vartheta_{\text{YM}}(\theta)n_\mu\hat{\gamma}^i_\nu(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}} \tag{4.6a}$$

and

$$(P_{\text{AYM}})^{\bar{\imath}}{}_{\mathfrak{a}} = -\sqrt{h}\,\vartheta_{\text{YM}}(\theta)u_\mu\hat{h}^{\bar{\imath}}_\nu(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}}, \tag{4.6b}$$

respectively. Also, there are momenta on $\mathcal{T}$ and Σ conjugate to the axionic field configurations:

$$\Pi_{\text{axi}} = 2\sqrt{-\gamma}\,\vartheta_{\text{KE}}(\theta)n^\mu\partial_\mu\theta \tag{4.7a}$$

and

$$P_{\text{axi}} = -2\sqrt{h}\,\vartheta_{\text{KE}}(\theta)u^\mu\partial_\mu\theta, \tag{4.7b}$$

respectively.

Recall that the variation of the gauge field on the boundary $\mathcal{T}$ can be decomposed as in Eq. (3.7). We perform a similar decomposition of the momentum $(\Pi_{\text{AYM}})^{i}{}_{\mathfrak{a}}$. Define a surface axion-Yang-Mills charge density,

$$(\mathcal{Q}_{\text{AYM}})_{\mathfrak{a}} = \sqrt{\sigma}\,\vartheta_{\text{YM}}(\theta)n_{\bar{\imath}}\mathfrak{B}^{\bar{\imath}}{}_{\mathfrak{a}}, \tag{4.8}$$

a surface axion-Yang-Mills momentum density,

$$(\mathcal{J}_{\text{AYM}})_a = (\mathcal{Q}_{\text{AYM}})_{\mathfrak{a}}\mathfrak{W}_a{}^{\mathfrak{a}}, \tag{4.9}$$

and a surface axion-Yang-Mills current,

$$\mathfrak{K}_a{}^{\mathfrak{a}} = \sqrt{\sigma}\,\vartheta_{\text{YM}}(\theta)\epsilon_a{}^{\bar{\imath}\bar{\jmath}}n_{\bar{\imath}}\mathfrak{E}_{\bar{\jmath}}{}^{\mathfrak{a}}. \tag{4.10}$$

Also define the scalar density

$$\mathcal{A} = N^{-1}\Pi_{\text{axi}}. \tag{4.11}$$

Then, the $\mathcal{T}$ portion of the variation of S_{AYM} is decomposed into

$$\delta S_{\text{AYM}}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x\,\big[(\mathcal{J}_{\text{AYM}})_a\delta N^a - (\mathcal{Q}_{\text{AYM}})_{\mathfrak{a}}\delta(N\mathfrak{V}^{\mathfrak{a}}) + N(\mathfrak{K}^a{}_{\mathfrak{a}}\delta\mathfrak{W}_a{}^{\mathfrak{a}} + \mathcal{A}\delta\theta)\big]. \tag{4.12}$$

In analogy with the discussion of conserved charges in the previous section, we can interpret the quantity $(\mathcal{Q}_{\text{AYM}})_{\mathfrak{a}}$ to be some sort of magnetic charge surface density that will yield a magnetic charge

$$\mathbb{Q}_{\text{AYM}}[\mathfrak{k}] = \int_{\partial\Sigma} d^2x\,(\mathcal{Q}_{\text{AYM}})_{\mathfrak{a}}\mathfrak{k}^{\mathfrak{a}} \tag{4.13}$$

when some gauge Killing scalar $\mathfrak{k}^{\mathfrak{a}}$ is present.

B. Canonical form of the axion-Yang-Mills action

The Hamiltonian density of the axion-Yang-Mills sector is given by $H_{\text{AYM}} = (P_{\text{AYM}})^{\bar{\imath}}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar{\imath}}{}^{\mathfrak{a}} + P_{\text{axi}}\pounds_t\theta - L_{\text{AYM}}$. Denote by $\overset{\circ}{\theta}$ the quantity $u^\mu\partial_\mu\theta$. The first term in the Hamiltonian density looks just like Eq. (3.15) but with $(P_{\text{AYM}})^{\bar{\imath}}{}_{\mathfrak{a}}$ replacing $(P_{\text{DYM}})^{\bar{\imath}}{}_{\mathfrak{a}}$ everywhere. The second term is just like Eq. (2.26), but here we replace P_{dil} and Ψ with P_{axi} and θ. In addition, we can decompose

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[2] Schiff's conjecture states that any self-consistent theory of gravity that obeys the weak equivalence principle necessarily obeys the Einstein equivalence principle.

the terms in the Lagrangian density, $L_{\rm AYM}$, which is just the integrand of Eq. (4.1). Note that

$$\tfrac{1}{4}\vartheta_{\rm YM}(\theta)(*\mathfrak{F})^{\mu\nu}{}_{\mathfrak{a}}\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}} = \tfrac{1}{2}\vartheta_{\rm YM}(\theta)\mathfrak{B}^{\bar\imath}{}_{\mathfrak{a}}\mathfrak{E}_{\bar\imath}{}^{\mathfrak{a}}. \qquad (4.14)$$

Furthermore, the kinetic term of the axion can be decomposed just as the kinetic term of the dilaton was decomposed in Eq. (2.28). Thus the Hamiltonian is

$$\begin{aligned}\mathbb{H}_{\rm AYM} &= \int_\Sigma d^3x\, H_{\rm AYM}\\ &= \int_\Sigma d^3x\,[\mathcal{H}_{\rm AYM}N + (\mathcal{H}_{\rm AYM})_{\bar\imath}N^{\bar\imath} + (\mathcal{G}_{\rm AYM})_{\mathfrak{a}}\mathfrak{A}_t{}^{\mathfrak{a}}]\\ &\quad + \int_\Sigma d^3x\,[(\mathcal{Q}_{\rm AYM})_{\mathfrak{a}}\mathfrak{V}^{\mathfrak{a}}N - (\mathcal{J}_{\rm AYM})_a N^a], \qquad (4.15)\end{aligned}$$

where

$$\mathcal{H}_{\rm AYM} = -(P_{\rm AYM})^{\bar\imath}{}_{\mathfrak{a}}\mathfrak{E}_{\bar\imath}{}^{\mathfrak{a}} + P_{\rm axi}\overset{\circ}{\theta} - \sqrt{h}\,\{\tfrac{1}{2}\vartheta_{\rm YM}(\theta)\mathfrak{B}^{\bar\imath}{}_{\mathfrak{a}}\mathfrak{E}_{\bar\imath}{}^{\mathfrak{a}} + \vartheta_{\rm KE}(\theta)[(\nabla\theta)^2 - \overset{\circ}{\theta}{}^2] + \vartheta_{\rm PE}(\theta)\} \qquad (4.16)$$

is the axion-Yang-Mills sector contribution to the Hamiltonian constraint,

$$(\mathcal{H}_{\rm AYM})_{\bar\jmath} = -(P_{\rm AYM})^{\bar\imath}{}_{\mathfrak{a}}\mathfrak{F}_{\bar\imath\bar\jmath}{}^{\mathfrak{a}} + P_{\rm axi}\nabla_{\bar\jmath}\theta \qquad (4.17)$$

is the axion-Yang-Mills sector contribution to the momentum constraint, and

$$(\mathcal{G}_{\rm AYM})_{\mathfrak{a}} = (\mathfrak{D}_{\bar\imath})^{\mathfrak{b}}{}_{\mathfrak{a}}(P_{\rm AYM})^{\bar\imath}{}_{\mathfrak{b}} \qquad (4.18)$$

is the axion-Yang-Mills sector contribution to the Yang-Mills Gauss constraint.

The action of Eq. (4.1) may be written in canonical form as

$$\begin{aligned}S_{\rm AYM} &= \int_{\mathcal{M}} d^4x\,[(P_{\rm AYM})^{\bar\imath}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar\imath}{}^{\mathfrak{a}} + P_{\rm axi}\pounds_t\theta - H_{\rm AYM}]\\ &= \int dt\left[\int_\Sigma d^3x\,[(P_{\rm AYM})^{\bar\imath}{}_{\mathfrak{a}}\pounds_t\mathfrak{A}_{\bar\imath}{}^{\mathfrak{a}} + P_{\rm axi}\pounds_t\theta - \mathcal{H}_{\rm AYM}N - (\mathcal{H}_{\rm AYM})_{\bar\imath}N^{\bar\imath} - (\mathcal{G}_{\rm AYM})_{\mathfrak{a}}\mathfrak{A}_t{}^{\mathfrak{a}}]\right.\\ &\qquad\left. + \int_{\mathcal{T}} d^3x\,[-(\mathcal{Q}_{\rm AYM})_{\mathfrak{a}}\mathfrak{V}^{\mathfrak{a}}N + (\mathcal{J}_{\rm AYM})_a N^a]\right). \qquad (4.19)\end{aligned}$$

V. DILATON-FOUR-FORM SECTOR

It is possible to treat a cosmological constant as a constant of motion arising from a four-form field rather than as a fundamental constant. Doing so allows one to consider space-times in which the cosmological constant takes different values. In this section, we will consider such a four-form field, but here we will also allow possible couplings to the dilaton.

The four-form field strength will be given by $\Lambda_{\mu\nu\rho\sigma} = 4!\,\nabla_{[\mu}A_{\nu\rho\sigma]}$ where $A_{\lambda\mu\nu}$ is a three-form potential. The field strength is invariant under gauge transformations of the potential of the form $A_{\lambda\mu\nu}[\chi] = A_{\lambda\mu\nu}[0] + 3!\,\nabla_{[\lambda}\chi_{\mu\nu]}$ where $\chi_{\mu\nu}$ is an arbitrary two-form. We partially restrict this gauge invariance in requiring the components in an orthonormal basis be finite everywhere in $\mathcal{M}$. It will be useful to introduce an "electric" field, $E_{\hbar\bar\imath\bar\jmath} = \hat h^\mu_{\hbar}\hat h^\nu_{\bar\imath}\hat h^\rho_{\bar\jmath}\Lambda_{\mu\nu\rho\sigma}u^\sigma$. One can show that $\Lambda^{\mu\nu\rho\sigma}\Lambda_{\mu\nu\rho\sigma} = -4E^{\hbar\bar\imath\bar\jmath}E_{\hbar\bar\imath\bar\jmath}$.

A. The dilaton-four-form action

Let $f_{\rm FF}(\Psi)$ be a function of the dilaton (that contains no derivatives of the dilaton) that couples to the four-form field as follows:

$$S_{\rm DFF} = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\frac{1}{2\cdot 4!}\,f_{\rm FF}(\Psi)\Lambda^{\lambda\mu\nu\rho}\Lambda_{\lambda\mu\nu\rho}. \qquad (5.1)$$

We could replace the four-form field strength $\Lambda_{\lambda\mu\nu\rho}$ with its dual $*\Lambda$. The result would be a cosmological constant term with dilaton couplings. We will not do so here as it is more useful to consider variations of the three-form potential. We will discuss the relationship between the four-form field strength and the cosmological constant below.

Under variations of the three-form potential, the geometry, and the dilaton field configurations, the induced variation in the action of Eq. (5.1) is

$$\begin{aligned}\delta S_{\rm DFF} &= \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,[-\tfrac{1}{2}(T_{\rm DFF})_{\mu\nu}\delta g^{\mu\nu}\\ &\quad -\tfrac{1}{2}\Upsilon_{\rm DFF}\delta\Psi + (\Xi_{\rm DFF})^{\lambda\mu\nu}\delta A_{\lambda\mu\nu}]\\ &\quad + \int_{\mathcal{T}} d^3x\,(\Pi_{\rm DFF})^{ijk}\delta A_{ijk}\\ &\quad - \int_\Sigma d^3x\,(P_{\rm DFF})^{\hbar\bar\imath\bar\jmath}\delta A_{\hbar\bar\imath\bar\jmath}. \qquad (5.2)\end{aligned}$$

Note that the projection of the variation of the potential onto the boundary elements is the same as the variation of the projection of the potential onto the boundary elements. Here, the stress energy momentum of the four-form field is given by

$$(T_{\rm DFF})_{\mu\nu} = \frac{1}{3!}f_{\rm FF}(\Psi)\Big(\frac{1}{8}\,g_{\mu\nu}\Lambda^{\alpha\beta\gamma\delta}\Lambda_{\alpha\beta\gamma\delta} - \Lambda_\mu{}^{\alpha\beta\gamma}\Lambda_{\nu\alpha\beta\gamma}\Big) \qquad (5.3)$$

and the dilaton source is

$$\Upsilon_{\text{DFF}} = -\frac{1}{4!}\frac{df_{\text{FF}}(\Psi)}{d\Psi}\Lambda^{\mu\nu\rho\sigma}\Lambda_{\mu\nu\rho\sigma}. \tag{5.4}$$

The equations of motion for the four-form fields are

$$(\Xi_{\text{DFF}})^{\lambda\mu\nu} = -\nabla_\alpha\left[f_{\text{FF}}(\Psi)\Lambda^{\alpha\lambda\mu\nu}\right]. \tag{5.5}$$

In addition, the momentum conjugate to the three-form potential on the boundary $\mathcal{T}$ is given by

$$(\Pi_{\text{DFF}})^{ijk} = \sqrt{-\gamma}\, f_{\text{FF}}(\Psi)n_\alpha\Lambda^{\alpha\lambda\mu\nu}\hat{\gamma}^i_\lambda\hat{\gamma}^j_\mu\hat{\gamma}^k_\nu, \tag{5.6a}$$

while the momentum conjugate on the boundary Σ is

$$\begin{aligned}(P_{\text{DFF}})^{\hbar\bar{\imath}\bar{\jmath}} &= -\sqrt{h}\, f_{\text{FF}}(\Psi)u_\alpha\Lambda^{\alpha\lambda\mu\nu}\hat{h}^{\hbar}_\lambda\hat{h}^{\bar{\imath}}_\mu\hat{h}^{\bar{\jmath}}_\nu \\ &= \sqrt{h}\, f_{\text{FF}}(\Psi)E^{\hbar\bar{\imath}\bar{\jmath}}.\end{aligned} \tag{5.6b}$$

Since A_{ijk} is proportional to the volume element ϵ_{ijk} on $\mathcal{T}$, the quantity $V_{ij} = A_{ijk}u^k$ will be a tensor on $\partial\Sigma$, that is, $V_{ij} = \hat{h}^a_i\hat{h}^b_j V_{ab}$. With this in mind, we see that the variation of A_{ijk} can be decomposed as follows:

$$\delta A_{ijk} = -3N^{-1}\delta N u_{[i}V_{jk]} - 3u_{[i}\hat{\sigma}^a_j\hat{\sigma}^b_{k]}\delta V_{ab}. \tag{5.7}$$

With the definition

$$(\mathcal{Q}_{\text{DFF}})^{ab} = 3\sqrt{\sigma}\, f_{\text{FF}}(\Psi)n_{\bar{\imath}}E^{\bar{\imath}ab} \tag{5.8}$$

for the four-form surface charge density, the component of the variation of the action S_{DFF} on the boundary $\mathcal{T}$ can be written

$$\delta S_{\text{DFF}}|_{\mathcal{T}} = -\int_{\mathcal{T}} d^3x\, (\mathcal{Q}_{\text{DFF}})^{ab}\delta(NV_{ab}).$$

As usual, $(\mathcal{Q}_{\text{DFF}})^{ab}$ is related to a surface charge density.

B. The conserved charge of the four-form field

Recall the equation of motion for the four-form field strength given by Eq. (5.5). In the presence of a source, $J^{\lambda\mu\nu}$, we have

$$-\nabla_\alpha\left[f_{\text{FF}}(\Psi)\Lambda^{\alpha\lambda\mu\nu}\right] = J^{\lambda\mu\nu} \tag{5.9}$$

and the identity $\nabla_\lambda J^{\lambda\mu\nu}$ follows. Let the quantity $\epsilon_{\mu\nu}$ be the volume element on the surface $\partial\Sigma$. Then, $\nabla_\lambda(J^{\lambda\mu\nu}\epsilon_{\mu\nu}) = 0$. If there is no source present in the vicinity of $\mathcal{T}$, then we see that

$$\mathbb{Q}_{\text{DFF}}[\epsilon] = \int_\Sigma d^3x\,\sqrt{h}\,\rho \tag{5.10}$$

is a conserved charge of the four-form field: its value is independent of the choice of leaf Σ.[3] Here, $\rho = -3\epsilon_{ab}J^{ab\mu}u_\mu$ is the charge density.

Using our definition of ρ, we can express it in terms of the four-form field strength by means of the equation of motion, (5.9). Because the unit normal to Σ is proportional to a gradient (recall $u_\mu = -N\partial_\mu t$), it can be pulled inside the derivative operator due to the antisymmetry of the field strength; in doing so, the derivative operator on $\mathcal{M}$ becomes the one on Σ. We find that $\sqrt{h}\,\rho = 3\sqrt{h}\,\nabla_{\bar{\imath}}\left[f_{\text{FF}}(\Psi)E^{\bar{\imath}ab}\epsilon_{ab}\right]$ and thus

$$\mathbb{Q}_{\text{DFF}}[\epsilon] = \int_{\partial\Sigma} d^2x\,(\mathcal{Q}_{\text{DFF}})^{ab}\epsilon_{ab}. \tag{5.11}$$

Equation (5.11) justifies our interpretation of $(\mathcal{Q}_{\text{DFF}})^{ab}$ as a surface density of four-form charge.

C. Canonical decomposition of the dilaton-four-form action

To obtain the Hamiltonian density, $H_{\text{DFF}} = (P_{\text{DFF}})^{\hbar\bar{\imath}\bar{\jmath}}\pounds_t A_{\hbar\bar{\imath}\bar{\jmath}} - L_{\text{DFF}}$, we compute

$$\begin{aligned}(P_{\text{DFF}})^{\hbar\bar{\imath}\bar{\jmath}}\pounds_t A_{\hbar\bar{\imath}\bar{\jmath}} = N\Big(-\frac{1}{3!}\sqrt{h}\, f_{\text{FF}}(\Psi)E^{\hbar\bar{\imath}\bar{\jmath}}E_{\hbar\bar{\imath}\bar{\jmath}}\Big) \\ -3A_{\bar{\imath}\bar{\jmath}t}\nabla_{\hbar}(P_{\text{DFF}})^{\hbar\bar{\imath}\bar{\jmath}} \\ +\nabla_{\hbar}\left[3NV_{ab}(P_{\text{DFF}})^{\hbar ab}\right].\end{aligned} \tag{5.12}$$

Here, $A_{\bar{\imath}\bar{\jmath}t} = A_{\bar{\imath}\bar{\jmath}\mu}t^\mu$. We have already seen that the Lagrangian density is purely electric:

$$L_{\text{DFF}} = -N\sqrt{h}\,\frac{1}{2\times 3!}f_{\text{FF}}(\Psi)E^{\hbar\bar{\imath}\bar{\jmath}}E_{\hbar\bar{\imath}\bar{\jmath}}. \tag{5.13}$$

Thus the Hamiltonian is

$$\begin{aligned}\mathbb{H}_{\text{DFF}} &= \int_\Sigma d^3x\, H_{\text{DFF}} \\ &= \int_\Sigma d^3x\,\left[\mathcal{H}_{\text{DFF}}N + (\mathcal{G}_{\text{DFF}})^{\bar{\imath}\bar{\jmath}}A_{\bar{\imath}\bar{\jmath}t}\right] \\ &\quad + \int_{\partial\Sigma} d^2x\,(\mathcal{Q}_{\text{DFF}})^{ab}NV_{ab},\end{aligned} \tag{5.14}$$

where the four-form contribution to the Hamiltonian constraint is

$$\mathcal{H}_{\text{DFF}} = -\sqrt{h}\,\frac{1}{2\times 3!}f_{\text{FF}}(\Psi)E^{\hbar\bar{\imath}\bar{\jmath}}E_{\hbar\bar{\imath}\bar{\jmath}}, \tag{5.15}$$

and the four-form Gauss constraint is

$$(\mathcal{G}_{\text{DFF}})^{\bar{\imath}\bar{\jmath}} = -3\nabla_{\hbar}(P_{\text{DFF}})^{\hbar\bar{\imath}\bar{\jmath}}. \tag{5.16}$$

The action in canonical form is

[3]That is, given a two-form, $\epsilon_{\mu\nu}$, the associated charge is independent of the foliation. However, the interpretation of this two-form as the volume element of the boundary of the leaves of the foliation is, of course, foliation dependent.

$$S_{\rm DFF} = \int_{\mathcal{M}} d^4x\,\left[(P_{\rm DFF})^{\bar{h}\bar{\imath}\bar{\jmath}}\pounds_t A_{\bar{h}\bar{\imath}\bar{\jmath}} - H_{\rm DFF}\right]$$
$$= \int dt\left(\int_{\Sigma} d^3x\,\left[(P_{\rm DFF})^{\bar{h}\bar{\imath}\bar{\jmath}}\pounds_t A_{\bar{h}\bar{\imath}\bar{\jmath}} - \mathcal{H}_{\rm DFF}N - (\mathcal{G}_{\rm DFF})^{\bar{\imath}\bar{\jmath}}A_{\bar{\imath}\bar{\jmath}t}\right] - \int_{\partial\Sigma} d^2x\,(\mathcal{Q}_{\rm DFF})^{ab}NV_{ab}\right). \tag{5.17}$$

D. Relation to a cosmological constant

Recall the Lagrangian density

$$L_{\rm DFF} = \sqrt{-g}\,(2\times 4!)^{-1}f_{\rm FF}(\Psi)\Lambda^{\mu\nu\rho\sigma}\Lambda_{\mu\nu\rho\sigma}$$
$$= -\tfrac{1}{2}\sqrt{-g}\,f_{\rm FF}(\Psi)(*\Lambda)^2. \tag{5.18}$$

In the latter form, it appears like a dilaton coupled to a cosmological constant $\lambda = -\frac{1}{4}(*\Lambda)^2$, except that, where a cosmological constant is fixed in the theory, $*\Lambda$ is a scalar field. The stress-energy-momentum tensor can be evaluated and we find that $(T_{\rm DFF})_{\mu\nu} = \frac{1}{2}f_{\rm FF}(\Psi)(*\Lambda)^2 g_{\mu\nu}$ which is consistent with this interpretation. When the Gauss constraint equation holds, however, we find that the quantity $f_{\rm FF}(\Psi)E_{\bar{h}\bar{\imath}\bar{\jmath}}$ must be equal to the tensor $\epsilon_{\bar{h}\bar{\imath}\bar{\jmath}}$ up to some constant $2C$. Thus we find that $f_{\rm FF}(\Psi)(*\Lambda) = 2C$. The Lagrangian density is now written $L_{\rm DFF} = -2\sqrt{-g}\,[1/f_{\rm FF}(\Psi)]C^2$. Thus C^2 can be interpreted as a (negative) cosmological constant when $f_{\rm FF}(\Psi)$ is a constant. When $f_{\rm FF}(\Psi)$ is not a constant, $L_{\rm DFF}$ is essentially a contribution to the dilaton potential proportional to C^2 times the reciprocal of the coupling of the four-form field strength with the dilaton. Although C is a constant (rather than a scalar field), it is a constant of integration rather than an imposed fundamental constant.

When the Gauss constraint holds, we can obtain an explicit expression for the charge $\mathbb{Q}_{\rm DFF}[\epsilon]$. We see that $(\mathcal{Q}_{\rm DFF})^{ab} = 6C\sqrt{\sigma}\,\epsilon^{ab}$, so the charge $\mathbb{Q}_{\rm DFF}[\epsilon]$ is just $12C$ times the area of the surface $\partial\Sigma$ whose volume element is ϵ_{ab}.

VI. DILATON-THREE-FORM SECTOR

In low energy string theory, one encounters a term in the action that is a three-form field strength squared with a coupling to a dilaton. Here, we analyze such a gauge field with a somewhat more general coupling. The three-form field strength will be written as $H_{\lambda\mu\nu} = 3!\,\nabla_{[\lambda}A_{\mu\nu]}$ where $A_{\mu\nu}$ is a two-form potential field. As always, there are gauge transformations of this potential, $A_{\mu\nu}[\chi] = A_{\mu\nu}[0] + 2!\,\nabla_{[\mu}\chi_{\nu]}$, under which the three-form field strength is invariant. As usual, we can use this gauge freedom to guarantee that the components of the potential in the orthonormal frame are finite everywhere in $\mathcal{M}$. It is useful to decompose the three-form field strength into "electric" and "magnetic" pieces on a spacelike hypersurface, $E_{\bar{\imath}\bar{\jmath}} = \hat{h}^{\lambda}_{\bar{\imath}}\hat{h}^{\mu}_{\bar{\jmath}}H_{\lambda\mu\nu}u^{\nu}$ and $B = -(3!)^{-1}\epsilon^{\lambda\mu\nu\alpha}H_{\lambda\mu\nu}u_{\alpha}$, respectively. It can then be shown that $H^{\lambda\mu\nu}H_{\lambda\mu\nu} = 6B^2 - 3E^{\bar{\imath}\bar{\jmath}}E_{\bar{\imath}\bar{\jmath}}$.

A. The dilaton-three-form action

As usual, the dilaton-three-form action will be taken as the three-form field strength squared coupled to some function of the dilaton. Let this function be given by $f_{\rm TF}(\Psi)$, and make the usual assumptions about its form, i.e., that it contain only the dilaton, but no derivatives of the dilaton. Explicitly, we write the action

$$S_{\rm DTF} = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\frac{1}{2\times 3!}\,f_{\rm TF}(\Psi)H^{\lambda\mu\nu}H_{\lambda\mu\nu}. \tag{6.1}$$

If we were to perform a duality rotation of the three-form field strength, we would be left with a kinetic energy for a scalar field. However, we are not interested in this case here. (In a sense, we have already considered a scalar field in the context of the dilaton in which there is no coupling to the Ricci scalar, though we have not considered a dilaton coupled to a scalar field.)

The action of Eq. (6.1) can be subjected to variations in the geometry, the dilaton, and the two-form potential. The response in the action is given by

$$\delta S_{\rm DTF} = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left[-\tfrac{1}{2}(T_{\rm DTF})_{\mu\nu}\delta g^{\mu\nu} - \tfrac{1}{2}\Upsilon_{\rm DTF}\delta\Psi + (\Xi_{\rm DTF})^{\mu\nu}\delta A_{\mu\nu}\right]$$
$$+ \int_{\mathcal{T}} d^3x\,(\Pi_{\rm DTF})^{ij}\delta A_{ij} - \int_{\Sigma} d^3x\,(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}}\delta A_{\bar{\imath}\bar{\jmath}}. \tag{6.2}$$

As usual, the projection onto the boundary elements of the variation of the potential are the same as the variation of the projection of the potential onto the boundary elements. When the variation of the two-form potential on the boundary is held fixed, there is a contribution to the stress-energy-momentum tensor,

$$(T_{\rm DTF})_{\mu\nu} = f_{\rm TF}(\Psi)\left[(2\times 3!)^{-1}g_{\mu\nu}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} - \tfrac{1}{2}H_{\mu}{}^{\alpha\beta}H_{\nu\alpha\beta}\right], \tag{6.3}$$

as well as a dilaton source:

$$\Upsilon_{\rm DTF} = -\frac{1}{6}\,\frac{df_{\rm TF}(\Psi)}{d\Psi}\,H^{\lambda\mu\nu}H_{\lambda\mu\nu}. \tag{6.4}$$

In addition, the equation of motion for the three-form field, in the absence of a source, is the vanishing of

$$(\Xi_{\rm DTF})^{\mu\nu} = -\nabla_{\lambda}\left[f_{\rm TF}(\Psi)H^{\lambda\mu\nu}\right]. \tag{6.5}$$

The momenta conjugate to the two-form potential configuration on the boundaries $\mathcal{T}$ and Σ are, respectively,

$$(\Pi_{\rm DTF})^{ij} = \sqrt{-\gamma}\, f_{\rm TF}(\Psi) n_\lambda H^{\lambda\mu\nu} \hat{\gamma}^i_\mu \hat{\gamma}^j_\nu \tag{6.6a}$$

and

$$(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} = -\sqrt{h}\, f_{\rm TF}(\Psi) u_\lambda H^{\lambda\mu\nu} \hat{h}^{\bar{\imath}}_\mu \hat{h}^{\bar{\jmath}}_\nu = -\sqrt{h}\, f_{\rm TF}(\Psi) E^{\bar{\imath}\bar{\jmath}}. \tag{6.6b}$$

The potential A_{ij} on the boundary $\mathcal{T}$ can be written in terms of the quantities $V_a = \hat{\sigma}^i_a A_{ij} u^j$ and $W_{ab} = \hat{\sigma}^i_a \hat{\sigma}^j_b A_{ij}$ on the two-surface $\partial\Sigma$ as

$$A_{ij} = 2u_{[i}\hat{\sigma}^a_{j]} V_a + \hat{\sigma}^a_{[i}\hat{\sigma}^b_{j]} W_{ab}. \tag{6.7}$$

Note that since W_{ab} is a two-form defined on the two-surface $\partial\Sigma$, we could express it as $W_{ab} = w\epsilon_{ab}$ where w is a scalar function on the two-surface. Similarly, the variations of A_{ij} can be decomposed:

$$\delta A_{\bar{\imath}\bar{\jmath}} = 2u_{[i}\hat{\sigma}^a_{j]} N^{-1}[\delta(V_a N) - W_{ab}\delta N^b] + \hat{\sigma}^a_{[i}\hat{\sigma}^b_{j]}\delta W_{ab}. \tag{6.8}$$

This leads to a like decomposition of the momentum conjugate to the $\mathcal{T}$ boundary two-form potential. We construct a surface three-form charge density:

$$(\mathcal{Q}_{\rm DTF})^a = 2\sqrt{\sigma}\, f_{\rm TF}(\Psi) n_\mu E^{\mu a} \tag{6.9}$$

which is a vector on $\partial\Sigma$. Note that $(\mathcal{Q}_{\rm DTF})^a$ is an extensive quantity. It can be used to define conserved charges as will be seen below. Another extensive quantity is the surface three-form momentum density:

$$(\mathcal{J}_{\rm DTF})_b = (\mathcal{Q}_{\rm DTF})^a W_{ab}. \tag{6.10}$$

Also, there is a surface three-form current density, defined as

$$(\mathcal{I}_{\rm DTF})^{ab} = \sqrt{\sigma}\, f_{\rm TF}(\Psi) n_\mu H^{\mu ab}. \tag{6.11}$$

Using these, we can write the $\mathcal{T}$ boundary contribution to the variation of the dilaton three-form action as

$$\delta S_{\rm DTF}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x\, \big[(\mathcal{J}_{\rm DTF})_a \delta N^a - (\mathcal{Q}_{\rm DTF})^a \delta(N V_a) + N(\mathcal{I}_{\rm DTF})^{ab}\delta W_{ab}\big]. \tag{6.12}$$

Here, the first two terms in the integrand involve variations of intensive variables with extensive coefficients while the last term is a variation of an extensive variable with an intensive coefficient.

B. Conserved charges of the three-form field

Suppose that suitable matter is present such that there is a source term, $J^{\mu\nu}$, included in the equation of motion of the three-form field. Then we have, from Eq. (6.5),

$$-\nabla_\lambda\big[f_{\rm TF}(\Psi) H^{\lambda\mu\nu}\big] = J^{\mu\nu}. \tag{6.13}$$

However, we see from this equation that the source must be divergenceless, $\nabla_\mu J^{\mu\nu} = 0$. Thus we can construct conserved charges for it in the following way. Let $f_\mu = \nabla_\mu \phi$ be an exact one-form. (ϕ is a scalar function.) Because of the antisymmetry of $J^{\mu\nu}$ in μ and ν, we have $0 = f_\nu \nabla_\mu J^{\mu\nu} = \nabla_\mu(J^{\mu\nu} f_\nu)$. If we integrate this expression over $\mathcal{M}$, we have contributions from the initial and final hypersurface as well as the boundary $\mathcal{T}$. Supposing that the source vanishes in the vicinity of $\mathcal{T}$, we find that the quantity

$$\mathbb{Q}_{\rm DTF}[f] = -\int_\Sigma d^3x\, \sqrt{h}\, u_\mu J^{\mu\nu} f_\nu \tag{6.14}$$

is invariant upon the surface, Σ, of evaluation. Thus it is a conserved charge.

Using Eq. (6.13), we can reexpress the conserved charge of Eq. (6.14) in terms of the two-surface three-form charge density $(\mathcal{Q}_{\rm DTF})^a$. Recall that $u_\mu = -N\nabla_\mu t$. Then, the integrand of Eq. (6.14) can be written in the form $\nabla_\mu\big[f_{\rm TF}(\Psi) E^{\mu a} f_a\big]$. (Notice that we have restricted the gradient f_a to be a one-form on the dual tangent space of $\partial\Sigma$. Natural choices for the scalar ϕ are the coordinates of this hypersurface.) Therefore, we have

$$\mathbb{Q}_{\rm DTF}[f] = \int_{\partial\Sigma} d^2x\, (\mathcal{Q}_{\rm DTF})^a f_a, \tag{6.15}$$

where $f_a = \partial_a \phi$ and ϕ is an arbitrary scalar function on $\partial\Sigma$.

C. Canonical form of the dilaton-three-form action

The canonical decomposition of the action $S_{\rm DTF}$ is obtained through the construction of the Hamiltonian density $H_{\rm DTF} = (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \pounds_t A_{\bar{\imath}\bar{\jmath}} - L_{\rm DTF}$. A straightforward calculation of the first term yields

$$\begin{aligned}(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \pounds_t A_{\bar{\imath}\bar{\jmath}} = {} & \tfrac{1}{2} N (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} E_{\bar{\imath}\bar{\jmath}} + \tfrac{1}{2} N^{\bar{h}} H_{\bar{h}\bar{\imath}\bar{\jmath}} (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \\ & -2A_{t\bar{\jmath}} \nabla_{\bar{\imath}} (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \\ & +2\nabla_{\bar{\imath}}\big[A_{t\bar{\jmath}} (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}}\big],\end{aligned} \tag{6.16}$$

where $A_{t\bar{\jmath}} = t^\mu A_{\mu\bar{\jmath}}$ will act as a Lagrange multiplier. Also, the second term in the Hamiltonian density can be written as

$$L_{\rm DTF} = N\sqrt{h}\, f_{\rm TF}(\Psi)(\tfrac{1}{2}B^2 - \tfrac{1}{4}E^{\bar{\imath}\bar{\jmath}} E_{\bar{\imath}\bar{\jmath}}). \tag{6.17}$$

With these expressions, we can write the Hamiltonian

$$\begin{aligned}\mathbb{H}_{\rm DTF} &= \int_\Sigma d^3x\, H_{\rm DTF} \\ &= \int_\Sigma d^3x\, \big[\mathcal{H}_{\rm DTF} N + (\mathcal{H}_{\rm DTF})_{\bar{\imath}} N^{\bar{\imath}} + A_{t\bar{\jmath}} (\mathcal{G}_{\rm DTF})^{\bar{\jmath}}\big] \\ &\quad + \int_{\partial\Sigma} d^2x\, \big[(\mathcal{Q}_{\rm DTF})^a N V_a - (\mathcal{J}_{\rm DTF})_a N^a\big].\end{aligned} \tag{6.18}$$

Here the contribution from the dilaton-three-form sector to the Hamiltonian constraint is given by

$$\mathcal{H}_{\rm DTF} = -\sqrt{h} f_{\rm TF}(\Psi)(\tfrac{1}{2}B^2 + \tfrac{1}{4}E^{\bar{\imath}\bar{\jmath}} E_{\bar{\imath}\bar{\jmath}}), \tag{6.19}$$

the contribution to the momentum constraint is

$$(\mathcal{H}_{\rm DTF})_{\bar{k}} = \tfrac{1}{2} H_{\bar{k}\bar{\imath}\bar{\jmath}} (P_{\rm DTF})^{\bar{\imath}\bar{\jmath}}, \tag{6.20}$$

and the Gauss constraint for the three-form field is

$$(\mathcal{G}_{\rm DTF})^{\bar{\jmath}} = 2\nabla_{\bar{\imath}}(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}}. \tag{6.21}$$

Also, the action of Eq. (6.1) can be reexpressed in the canonical form. It is

$$\begin{aligned} S_{\rm DTF} &= \int_{\mathcal{M}} d^4x \left[(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \pounds_t A_{\bar{\imath}\bar{\jmath}} - L_{\rm DTF} \right] \\ &= \int dt \left(\int_{\Sigma} d^3x \left[(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}} \pounds_t A_{\bar{\imath}\bar{\jmath}} - \mathcal{H}_{\rm DTF} N - (\mathcal{H}_{\rm DTF})_{\bar{\imath}} N^{\bar{\imath}} - (\mathcal{G}_{\rm DTF})^{\bar{\imath}} A_{t\bar{\imath}} \right] \right. \\ &\qquad \left. + \int_{\partial\Sigma} d^2x \left[(\mathcal{J}_{\rm DTF})_a N^a - (\mathcal{Q}_{\rm DTF})^a N V_a \right] \right). \end{aligned} \tag{6.22}$$

VII. STATISTICAL MECHANICS

Here we address, at a formal level, the construction of statistics of the quantum-mechanical theory based on the actions we have constructed. The mechanical theory is entirely described in terms of path integrals involving our actions. These yield quantum-mechanical density matrices. The statistics of primary interest are the partition functions, which can be thought of as functional integrals over the density matrices with some periodic identification (with the period related to the temperature).

A. The general action and canonical ensembles

We will consider a general form of action that possesses all of the features of the actions we have considered thus far. Let $A_\mu{}^A$ denote the gauge field potential, where the upper-case Latin index labels the various gauge fields that may be present. The field strength associated with the potential is given by $F^{\mu\nu}{}_A$, and the gauge field action functional $S_{\rm GF}$ involves the square of the field strength along with a possible coupling to the dilaton, $f_{\rm GF}(\Psi)$. Note that for the three-form and four-form field strength gauge fields, the field index will actually be a set of space-time indices, while for the Yang-Mills fields, it is a color index. The total action of the theory is given by $S_{\rm total} = S_{\rm EHD} + S_{\rm GF} + S_0$. Variations of this action yield the usual equation of motion terms as well as boundary terms. We will primarily be concerned with the latter. The $\mathcal{T}$ portion of the variation of the total action will take the form

$$\begin{aligned} \delta S_{\rm total}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x \, &\left[-\mathcal{E}\delta N + \mathcal{J}_a \delta N^a - \mathcal{Q}_A \delta(N V^A) \right. \\ &\left. + N\left(\tfrac{1}{2}\mathcal{S}^{ab}\delta\sigma_{ab} + \mathcal{Y}\delta\Psi + \mathcal{I}^a{}_A \delta W_a{}^A\right)\right]. \end{aligned} \tag{7.1}$$

Here, $\mathcal{Q}_A$ are the surface charge densities of the gauge fields, $\mathcal{I}^a{}_A$ are the surface currents of the gauge fields, and $\mathcal{J}_a = (\mathcal{J}_{\rm EHD})_a + \mathcal{Q}_A W_a{}^A$ is the total surface momentum density. Also, V^A are the projections of the $\mathcal{T}$ boundary potentials of the gauge fields along u^i, while $W_a{}^A$ are the portions of these fields restricted to $\partial\Sigma$. Note that all of the gauge field actions we have considered can be expressed in this way. The action can be written in canonical form as

$$\begin{aligned} S_{\rm total} = \int dt &\left(\int_{\Sigma} d^3x \left(p^{\bar{\imath}\bar{\jmath}} \pounds_t h_{\bar{\imath}\bar{\jmath}} + P_{\rm dil} \pounds_t \Psi + P^{\bar{\imath}}{}_A \pounds_t A_{\bar{\imath}}{}^A - \mathcal{H}N - \mathcal{H}_{\bar{\imath}} N^{\bar{\imath}} - \mathcal{G}_A A_t{}^A \right) \right. \\ &\left. + \int_{\partial\Sigma} d^2x \left(-\mathcal{E}N + \mathcal{J}_a N^a - \mathcal{Q}_A N V^A \right) \right). \end{aligned} \tag{7.2}$$

Here, $A_t{}^A$ are the projections of the gauge potentials along the time vector. $\mathcal{H}$ is the total Hamiltonian constraint, $\mathcal{H}_a$ is the total momentum constraint, and $\mathcal{G}_A$ are the Gauss constraints of the gauge fields.

The first three terms in the integrand of Eq. (7.1) involve variation of an intensive variable with an extensive coefficient. However, the remaining terms involve variations of extensive variables with intensive coefficients. Define the microcanonical action to be

$$S_{\rm micro} = S_{\rm total} + \int_{\partial\Sigma} d^2x \left(\mathcal{E}N - \mathcal{J}_a N^a + \mathcal{Q}_A N V^A \right). \tag{7.3}$$

Since this action differs from the original by boundary terms alone, the equations of motion are unaffected. However, such a transformation changes the $\mathcal{T}$ boundary component of the variation of the action. We now have

$$\begin{aligned} \delta S_{\rm micro}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x \, N &\left(\delta\mathcal{E} - \omega^a \delta\mathcal{J}_a + V^A \delta\mathcal{Q}_A \right. \\ &\left. + \tfrac{1}{2}\mathcal{S}^{ab}\delta\sigma_{ab} + \mathcal{Y}\delta\Psi + \mathcal{I}^a{}_A \delta W_a{}^A \right), \end{aligned} \tag{7.4}$$

where we define $N\omega^a = N^a$. We will interpret ω^a as the angular velocity of observers comoving with the foliation on $\mathcal{T}$. Each term in the integrand of Eq. (7.4) is of the form of a variation of an extensive variable with an intensive coefficient. We see that the boundary conditions imposed in order to obtain the equations of motion are those in which the extensive variables are held constant. It is for this reason that the term "microcanonical" has been adopted. Also, note that the Hamiltonian for the microcanonical action contains only the constraints (i.e., there are no additional boundary terms), so the canonical form of the microcanonical action has no $\partial\Sigma$ integral. The importance of this observation is that the microcanonical action vanishes when stationarity of all the fields and the constraint equations are imposed.

Grand-canonical boundary conditions involve the fixation of all *intensive* variables on the boundary. In analogy with the procedure above, we define a grand-canonical action by

$$S_{\rm grand} = S_{\rm total} - \int_{\mathcal{T}} d^3x\, N(\tfrac{1}{2}\mathcal{S}^{ab}\sigma_{ab} + \mathcal{Y}\Psi + \mathcal{I}^a{}_A W_a{}^A). \tag{7.5}$$

Under variations of this action, the $\mathcal{T}$ boundary component becomes

$$\begin{aligned}\delta S_{\rm grand}|_{\mathcal{T}} = -\int_{\mathcal{T}} d^3x\, \big[&\mathcal{E}\delta N - \mathcal{J}_a\delta(N\omega^a) + \mathcal{Q}_A\delta(NV^A) \\ &+\sigma_{ab}\delta(N\mathcal{S}^{ab}/2) + \Psi\delta(N\mathcal{Y}) \\ &+W_a{}^A\delta(N\mathcal{I}^a{}_A)\big],\end{aligned} \tag{7.6}$$

so it is indeed the intensive boundary variables that must be fixed in order to obtain the equations of motion. In canonical form, the $\mathcal{T}$ contribution to the grand-canonical action is given by

$$\begin{aligned}S_{\rm grand}|_{\mathcal{T}} = -\int_{\mathcal{T}} d^3x\, N\big[&\mathcal{Y} - \mathcal{J}_a\omega^a + \mathcal{Q}_A V^A \\ &+\tfrac{1}{2}\,{\rm tr}(\mathcal{S}) + \mathcal{Y}\Psi + \mathcal{I}^a{}_A W_a{}^A\big].\end{aligned} \tag{7.7}$$

It will be useful to have a covariant form of the microcanonical action. To obtain this, we recall the definition of the microcanonical action, Eq. (7.3), as well as the initial covariant form of the total action, Eq. (2.1) plus matter terms. Since the difference between the microcanonical action and the total action amounts to just a difference on the boundary $\mathcal{T}$, we will just attend to this. Thus

$$\begin{aligned}S_{\rm micro}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x\, \big[&-2\sqrt{-\gamma}\, f_{\rm EH}(\Psi)\,{\rm tr}(\Theta) \\ &-N\mathcal{E} + N^a\mathcal{J}_a - NV^A\mathcal{Q}_A\big].\end{aligned} \tag{7.8}$$

To cast this in a covariant form, we employ the definitions of $\mathcal{E}$, $\mathcal{J}_a$, and $\mathcal{Q}_A$, as well as the decomposition given in Eq. (2.9). Then Eq. (7.8) can be rewritten in the covariant form

$$\begin{aligned}S_{\rm micro}|_{\mathcal{T}} = \int_{\mathcal{T}} d^3x\, \sqrt{-\gamma}\, \big[&f_{\rm EH}(\Psi)t_\mu\Theta^{\mu\nu}\partial_\nu t - 2n^\mu\partial_\mu f_{\rm EH}(\Psi) \\ &-f_{\rm GF}(\Psi)(n_\mu F^{\mu\nu}{}_A\partial_\nu t)(t^\mu A^A_\mu)\big].\end{aligned} \tag{7.9}$$

B. "Euclidean" notation

Until now, the metrics we have considered have been of Lorentzian signature. Such metrics can be reexpressed in a "Euclidean" notation by redefining the (real) metric functions in terms of new (complex) ones. Our prescription is the following: We first rewrite the volume elements as $\sqrt{g} = i\sqrt{-g}$ and $\sqrt{\gamma} = i\sqrt{-\gamma}$ and redefine the lapse and shift functions: $\bar{N} = iN$ and $\bar{N}^{\bar{\imath}} = iN^{\bar{\imath}}$. The former leads to a new unit normal $\bar{u}_\mu = -iu_\mu$ which satisfies $\bar{u}\cdot\bar{u} = +1$. In addition to the lapse and the shift, we adopt a new notation for all the Lagrange multipliers in the Hamiltonian. Thus $\bar{A}_t{}^A = iA_t{}^A$. Thus the Hamiltonian as a functional of the Lagrange multipliers is $\mathbb{H}[\bar{N},\bar{N}^{\bar{\imath}},\bar{A}_t{}^A] = i\mathbb{H}[N,N^{\bar{\imath}},A_t{}^A]$. Finally, we define a new action functional $I[g,\Psi,A] = -iS[g,\Psi,A]$ such that the phase in the path integral is written $\exp(-I)$.

An important feature of this prescription is the following. Suppose that a complex metric $g_{\rm C}$ is obtained from a real, Lorentzian metric $g_{\rm L}$ via the transformation $t \to -it$. Then, the Lagrange multipliers, N, $N^{\bar{\imath}}$, and $A_t{}^A$ become imaginary, or, equivalently, $\bar{N}$, $\bar{N}^{\bar{\imath}}$, and $\bar{A}_t{}^A$ become real. However, the canonical data, that is $p^{\bar{\imath}\bar{\jmath}}$, $h_{\bar{\imath}\bar{\jmath}}$, $P_{\rm dil}$, Ψ, $P^{\bar{\imath}}{}_A$, and $A_{\bar{\imath}}{}^A$ all are invariant. Therefore, *the extensive variables are all invariant under the Wick rotation* since these variables are constructed out of the canonical data. In particular, the values of the extensive variables of the complex metric that extremize the path integral are the same as the values of these variables on the corresponding Lorentzian metric.

We can write the microcanonical and the grand-canonical actions using the Euclidean notation in the canonical form. They are

$$\begin{aligned}I_{\rm micro} = \int dt \int_\Sigma d^3x\, \big(&-ip^{\bar{\imath}\bar{\jmath}}\pounds_t h_{\bar{\imath}\bar{\jmath}} - iP^{\bar{\imath}}{}_A\pounds_t A_{\bar{\imath}}{}^A - iP_{\rm dil}\pounds_t\Psi \\ &+\mathcal{H}\bar{N} + \mathcal{H}_{\bar{\imath}}\bar{N}^{\bar{\imath}} + \mathcal{G}_A\bar{A}_t{}^A\big)\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}I_{\rm grand} = I_{\rm micro} + \int dt \int_{\partial\Sigma} d^2x\, \bar{N}\big[&\mathcal{E} - \mathcal{J}_a\omega^a + \mathcal{Q}_A V^A \\ &+\tfrac{1}{2}\,{\rm tr}(\mathcal{S}) + \mathcal{Y}\Psi + \mathcal{I}^a{}_A W_a{}^A\big].\end{aligned} \tag{7.11}$$

In addition, we can write out the variations of these actions. For example, the $\mathcal{T}$ boundary contribution of the variation of the Euclidean form of the microcanonical action can be constructed from Eq. (7.4). It is

$$\begin{aligned}\delta I_{\rm micro}|_{\mathcal{T}} = -\int dt \int_{\partial\Sigma} d^2x\, \bar{N}\big(&\delta\mathcal{E} - \omega^a\delta\mathcal{J}_a + V^A\delta\mathcal{Q}_A \\ &+\tfrac{1}{2}\mathcal{S}^{ab}\delta\sigma_{ab} + \mathcal{Y}\delta\Psi + \mathcal{I}^a_A\delta W^A_a\big).\end{aligned} \tag{7.12}$$

C. Functional integrals

It is well known that the classical equations of motion are obtained from an action I, by setting variations of this action to zero. As argued earlier, the variations are of a restricted class which fix certain quantities on the boundary, and the quantities that must be fixed are determined by the choice of ensemble. For example, it is the extensive boundary variables that must be fixed in obtaining the equations of motion from the microcanonical action.

This method of obtaining classical equations of motion is best understood as a (classical) limit of the quantum-mechanical density matrix obtained from a path integral. Formally, we can write this density matrix as

$$\varrho = \int d[g, \Psi, A]\, e^{-I[g,\Psi,A]}, \tag{7.13}$$

where the integral is taken over all possible field configurations of the metric, the dilaton, and the matter fields, from an initial and a final spacelike surface. The field configurations are fixed on the initial and the final spacelike surfaces, $\Sigma_{\rm initial}$ and $\Sigma_{\rm final}$; so the density matrix is a functional of these quantities. In addition there will be quantities on $\mathcal{T}$ joining the boundaries of these two spacelike surfaces that must also be fixed. These quantities depend on the ensemble chosen for the action. The density matrix will be a functional of these variables as well. The density matrix gives us all the quantum-mechanical information we need to solve most problems. Furthermore, the classical limit can be understood in the following way. Given an initial configuration, there is a final configuration that is most "likely" in a quantum-mechanical sense: it is the one for which the density matrix has the largest value. In the classical limit, the action is always much larger than unity (in units of Planck's constant). Thus the final configuration that extremizes the density matrix is the one for which $\delta I = 0$, that is, the one for which the classical field equations hold. (Note that I is a complex quantity—we have not yet made a Wick rotation—so the last statement follows from stationary phase arguments.)

Much information about the system can be obtained by taking statistics of the density matrix. The most important of these is known as the partition function. It is obtained by "tracing over" the initial and final states. In terms of path integrals, this can be realized by identifying the initial and the final configurations, and integrating over all possible configurations. The partition function then depends upon the information fixed on the boundary $\mathcal{T}$ alone. Note that, in identifying the initial and the final spacelike surfaces, we have effectively changed the topology of $\mathcal{M}$ from $\Sigma \times \mathcal{I}$ to $\Sigma \times S^1$.

As an example, consider the microcanonical ensemble. The objects that are held fixed when this action is varied are the metrics and field configurations on the initial and final surfaces as well as the extensive variables on the boundary $\mathcal{T}$. Thus the functional dependence of the density matrix is

$$\varrho = \varrho[h_i, \Psi_i, A_i; h_f, \Psi_f, A_f; \mathcal{E}, \mathcal{J}, \sigma, \psi, \mathcal{Q}, W]$$
$$= \int d[g, \Psi, A]\, e^{-I_{\rm micro}[g,\Psi,A]}. \tag{7.14}$$

The subscripts i and f refer to the initial and final spacelike hypersurfaces, $\Sigma_{\rm initial}$ and $\Sigma_{\rm final}$, respectively, and ψ is the dilaton configuration on $\mathcal{T}$. The partition function is also known as the density of states. It is given by

$$\nu[\mathcal{E}, \mathcal{J}, \sigma, \psi, \mathcal{Q}, W]$$
$$= \int d[h, \Psi_\Sigma, A_\Sigma] \times \varrho[h, \Psi_\Sigma, A_\Sigma; h, \Psi_\Sigma, A_\Sigma; \mathcal{E}, \mathcal{J}, \sigma, \psi, \mathcal{Q}, W]. \tag{7.15}$$

Note that we have explicitly indicated the periodic identification of the fields on the initial and final spacelike boundaries in the density matrix.

In a similar manner, we can construct a grand-canonical density matrix by making a Laplace transform of Eq. (7.14). This density matrix would be a functional of β, ω^a, $\mathcal{S}_{ab}$, $\mathcal{Y}$, V^A, and $\mathcal{I}_A$ on the boundary $\mathcal{T}$. Here, the reciprocal temperature, β, is defined by

$$\beta = \oint dt\, \bar{N}|_{\partial\Sigma}, \tag{7.16}$$

where we have used $\oint$ as a reminder that the initial and final spacelike hypersurfaces have been periodically identified; β is the gauge invariant measure of this period. The grand-canonical partition function is then $\mathcal{Z}[\beta, \omega, \mathcal{S}, \mathcal{Y}, V, \mathcal{I}]$.

VIII. BLACK HOLE THERMODYNAMICS

The thermodynamics of systems containing black holes can be obtained using the statistical techniques of the previous section. In particular, we wish to obtain a thermodynamic "first law" which relates the entropy change of a system with changes in extensive quantities of the system. Clearly, such a result should be obtained from the microcanonical action. The entropy is normally defined as the logarithm of the density of states. (We recall that the latter is just the partition function for the microcanonical ensemble.) We will only consider black hole solutions that are stationary in the sense that the Lie derivative of all fields (including the metric, dilaton, and matter fields) vanishes. Also, we continue to use the "generalized gauge fields" introduced in the previous section. Any of the gauge fields discussed in this paper will have a similar form.

We will only construct the "zeroth order" contribution to the entropy from the path integral for the density of states. In this approximation, the density of states is written

$$\nu[\mathcal{E}, \mathcal{J}, \sigma, \psi, \mathcal{Q}, W] \approx e^{-I_{\rm micro}}, \tag{8.1}$$

where the microcanonical action is evaluated at its (complex) extremum value. Clearly, the entropy is just

$$\mathbb{S}[\mathcal{E}, \mathcal{J}, \sigma, \psi, \mathcal{Q}, W] \approx -I_{\rm micro}|_{\rm CL}. \tag{8.2}$$

Thus to evaluate the entropy is to evaluate the microcanonical action at its (complex) extremum value. however, the presence of an event horizon introduces some interpretational problems. These we address in the rest of this section.

A. Regularizations at the event horizon

As usual, we turn to the Euclidean form of the action. The system will be defined as the interior to some closed two-surface, $\mathcal{B}_{\rm outer}$, on which the microcanonical boundary data are specified. This interior will include an event horizon; the foliation will become degenerate on this horizon. In addition to the usual foliation in the timelike direction, we will suppose that it is possible to foliate from the event horizon to the outer boundary. After periodic identification, the complex manifold has the form of a cone$\times S^2$, where the azimuthal direction of the cone is the timelike direction and the "radial" direction corresponds to the spacelike vector along which the spacelike foliation was constructed. The outer edge of the cone is the two-surface $\mathcal{B}_{\rm outer}$, and the "point" of the cone is where the timelike foliation becomes degenerate: the event horizon.

It is common to impose regularity conditions on the cone to cause it to become a disk. Near the event horizon, we *assume* that it is possible to write the metric in the Euclidean form

$$ds^2_{\rm E} \simeq \bar{N}^2 dt^2 + M^2 dr^2 + \sigma_{ab} dx^a dx^b. \tag{8.3}$$

Here, r is the coordinate of foliation in the spacelike direction, and M is the analog of the lapse function defined as $n_\mu = M\partial_\mu r$. The conical singularity is not present in the metric of Eq. (8.3) if the "circumference" of the circles of constant t has the value of 2π times the proper radius near the event horizon. Let Δt be the period of the identification of the coordinate t. Then the circumference of the circles of constant t is given by $\bar{N}\Delta t$, and the regularity condition is $\Delta t(n^{\bar{\imath}}\partial_{\bar{\imath}}\bar{N}) = 2\pi$ at the event horizon. Choose the period Δt to satisfy this condition. Then it takes the value $\Delta t = 2\pi/\kappa_{\rm H}$ where $\kappa^2_{\rm H} = -h^{\bar{\imath}\bar{\jmath}}(\partial_{\bar{\imath}}\bar{N})(\partial_{\bar{\jmath}}\bar{N})$ (evaluated on the event horizon) is the surface gravity of the event horizon.

B. Evaluation of the microcanonical action

Although the event horizon is in no way distinctive to an infalling observer, nevertheless, for a system observer (i.e., one who lives on the system boundary $\mathcal{B}_{\rm outer}$) it represents a one-way membrane onto which information can approach, but from which nothing can come (at least classically). For such an observer, it would be inappropriate to attempt to calculate an action that included the event horizon. So let us remove an open set surrounding the event horizon infinitesimally close. The remaining portion of the manifold now contains a new "boundary." However, this is *not* a system boundary on which thermodynamic data must be specified—it is just a tool that will represent the information lost in "throwing away" the event horizon from the system. With this understanding, we shall denote this surface as $\mathcal{B}_{\rm inner}$.

Let us recall the covariant form of the microcanonical action. It is the usual action but with the $\mathcal{T}_{\rm outer}$ boundary term given by Eq. (7.9). There is no supplemented boundary term for the inner boundary. Using the usual techniques to perform a canonical decomposition of this action, and keeping track of the terms appearing on the new boundary $\mathcal{B}_{\rm inner}$, we find that

$$I_* = I_{\rm micro} + \int dt \int_{\mathcal{B}_{\rm inner}} d^2x \big[2\sqrt{\sigma}\, f_{\rm EH}(\Psi) n^{\bar{\imath}}\partial_{\bar{\imath}}\bar{N} - 2\sqrt{\sigma}\,\bar{N} n^{\bar{\imath}}\partial_{\bar{\imath}} f_{\rm EH}(\Psi) - \bar{N}^a \mathcal{J}_a + \bar{N}\mathcal{Q}_A V^A\big], \tag{8.4}$$

in the "Euclidean" notation. Here, I_* is the thermodynamic action which differs from the microcanonical action, $I_{\rm micro}$, in that the event horizon has been removed from the microcanonical system. In obtaining Eq. (8.4), we have used the relation $a_{\bar{\imath}} = -\bar{N}^{-1}\partial_{\bar{\imath}}\bar{N}$. The action $I_{\rm micro}$ was given in Eq. (7.10). Thus we notice that the only boundary contribution to I_* is on the boundary $\mathcal{B}_{\rm inner}$.

We wish to evaluate the action I_* on the solution in which the field equations hold—the extremal solution. The contribution from the action $I_{\rm micro}$ of Eq. (7.10) vanishes: the terms involving Lie derivatives of the fields vanish due to the stationarity of the solution, and the remaining terms vanish because the constraint equations must hold on a classical solution. Thus the sole contribution to the action I_* comes from the boundary contribution $\mathcal{B}_{\rm inner}$ of Eq. (8.4) which arises due to the discarding of the event horizon from the system. In evaluating this term, we invoke the regularity conditions of the previous subsection: $\bar{N} = 0$, $\bar{N}^a = 0$, and $\int n^{\bar{\imath}}\partial_{\bar{\imath}}\bar{N} dt = -2\pi$. (The shift approaches zero as we approach the event horizon since we have chosen a zero-vorticity observer. The negative sign in the third equation arises since the normal vector is now *inward* directed, rather than outward as it was in the previous subsection.) Thus we find that

$$I_*|_{\rm CL} = -4\pi \int_{\mathcal{B}_{\rm inner}} d^2x \sqrt{\sigma}\, f_{\rm EH}(\Psi). \tag{8.5}$$

The entropy (neglecting higher-order quantum corrections) is just the negative of this quantity according to Eq. (8.2) (using I_* rather than $I_{\rm micro}$). The usual value for the entropy with no dilaton field, one quarter of the event horizon area, is recovered when we set $f_{\rm EH}(\Psi) = (16\pi)^{-1}$. (This is the appropriate value for this coefficient for the usual Einstein-Hilbert action in units where Newton's constant is unity.)

C. The first law of thermodynamics

Since the inner boundary, $\mathcal{B}_{\rm inner}$, is not really a boundary, we will view I_* as a functional of the same extensive variables on $\mathcal{B}_{\rm outer}$ as $I_{\rm micro}$. The first law of thermodynamics is obtained by varying the entropy given by Eq. (8.2) with the aid of Eq. (7.12) which applies equally to I_*. These variations are understood to be among those that preserve the classical equations of motion, so the boundary contribution is the only one present. (Recall that we have closed the manifold with respect to the initial and final hypersurfaces, so there are no longer any initial and final spacelike boundaries.) Thus

$$\delta\mathbb{S} = \int_{\mathcal{B}_{\text{outer}}} d^2x\,\beta\big(\delta\mathcal{E} - \omega^a\delta\mathcal{J}_a + V^A\delta\mathcal{Q}_A + \tfrac{1}{2}\mathcal{S}^{ab}\delta\sigma_{ab} + \mathcal{Y}\delta\Psi + \mathcal{I}^a{}_A\delta W_a{}^A\big) \tag{8.6}$$

is our formulation of the first law of thermodynamics. Recall that β is the reciprocal temperature of the system [Eq. (7.16)]. Note that it is not necessarily constant over the system boundary, so we have left it within the integral. When the boundary is chosen to be an isothermal surface, then we can integrate the first term in the integrand of Eq. (8.6) to obtain the usual "$\beta\delta\mathbb{E}$" term. [Recall that $\mathbb{E}$ is the quasilocal energy of Eq. (2.13).] However, the angular velocity ω^a will not generally be a constant on the system boundary simultaneously, so the usual expression for the first law does not generally hold in the case of a finite system. In the case of non-Abelian fields, if a surface can be chosen such that βV^A is constant and proportional to the gauge Killing scalar (if such exists), then the contribution of this term in the first law is of the form $\beta V\delta\mathbb{Q}_{\text{DYM}}$ where the latter quantity is defined in (3.13). We note that the formulation (8.6) of the first law differs from that considered previously for non-Abelian gauge fields [27], in which certain asymptotic properties of the gauge and gravitational fields were assumed in order to define a color charge—the resultant formulation of the first law is therefore valid only in this asymptotic region. Our formulation (8.6) of the first law recovers this result when the gauge and gravitational fields have the aforementioned falloff properties.

It is useful to divide the variations of the metric on $\mathcal{B}_{\text{outer}}$ into a "shape" preserving piece and a "volume" (which is, of course, an area on a two-surface) preserving piece as follows: $\delta\sigma_{ab} = \varsigma_{ab}\delta\sqrt{\sigma} + \sqrt{\sigma}\,\delta\varsigma_{ab}$, where $\varsigma_{ab} = \sigma_{ab}/\sqrt{\sigma}$. If this is done, then the $\frac{1}{2}\beta\mathcal{S}^{ab}\delta\sigma_{ab}$ term in the integrand of Eq. (8.6) can be rewritten as $\beta(\mathcal{P}\delta\mathcal{V} + \lambda^{ab}\delta\varsigma_{ab})$. Here, $\mathcal{V} = \sqrt{\sigma}$ can be thought of as a measure of the volume of the system (by which we really mean the area of $\mathcal{B}_{\text{outer}}$), and thus $\mathcal{P} = \frac{1}{2}\,\text{tr}(\mathcal{S})/\sqrt{\sigma}$ is interpreted as the pressure on the system. The quantity $\lambda^{ab} = \frac{1}{2}\sqrt{\sigma}\,\mathcal{S}^{ab}$ is thermodynamically conjugate to shape changes of the system boundary.

IX. CONCLUDING REMARKS

The formulation of the first law of thermodynamics as given in (8.6) generalizes previous formulations [18,28] to include the most general couplings of gauge fields to dilatonic gravity in four space-time dimensions that have at most two derivatives in any term in the action. We close by commenting on possible extensions of our work.

General arguments from string theory suggest that the action considered in this paper receives corrections from terms that have more than two derivatives in the metric and matter fields, i.e., terms which are at least quadratic in the curvature and/or field strengths. Although the general form for the Noether charge for such terms has been evaluated [20], the detailed manner in which such terms contribute to the first law of thermodynamics in the context of the quasilocal formalism considered in this paper remains to be worked out.

A related problem of interest concerns the role of topological fields. These are fields which couple to the connection but not to the metric. It has recently been shown that interesting black hole solutions exist for a model topological field theory in 2 + 1 dimensions [29]. Their thermodynamical properties are considerably different from the usual case [12,30] and are not fully understood. The generalization of the quasilocal formalism to such actions represents an interesting problem since the first derivatives of the metric will play a markedly different role in the boundary terms.

TABLE I. Manifold variables.

	Manifold			
	$\mathcal{M}$	$\mathcal{T}$	Σ	$\partial\Sigma$
Indices	$\{\mu,\nu,\ldots\}$	$\{i,j,k\}$	$\{\hbar,\bar\imath,\bar\jmath\}$	$\{a,b\}$
Normal vector		n_μ	u_μ	u_i
Metric	$g_{\mu\nu}$	γ_{ij}	$h_{\bar\imath\bar\jmath}$	σ_{ab}
Compatible derivative	∇_μ	$\triangle_i$	$\nabla_{\bar\imath}$	
Intrinsic curvature scalar	$R[g]$		$R[h]$	
Extrinsic curvature		Θ_{ij}	$K_{\bar\imath\bar\jmath}$	k_{ab}
Geometric momentum		π^{ij}	$p^{\bar\imath\bar\jmath}$	–

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APPENDIX: NOTATION

This is a summary of the notation used in this paper. The manifold $\mathcal{M}$ is topologically $\Sigma\times\mathcal{I}$ where Σ is a spacelike hypersurface and $\mathcal{I}$ is a timelike interval. Foliation of $\mathcal{M}$ along $\mathcal{I}$ allows us to define the lapse, N, and the shift, N^μ. The various manifolds we will consider, and some of the tensors defined on them, are summa-

TABLE II. Field variables.

	Field strengths			Potential fields		
	Full	Electric	Magnetic	Full	Along u	On $\partial\Sigma$
Dilaton		$\mathring{\Psi}$			Ψ	
Axion		$\mathring{\theta}$			θ	
Yang-Mills	$\mathfrak{F}_{\mu\nu}{}^{\mathfrak{a}}$	$\mathfrak{E}_i{}^{\mathfrak{a}}$	$\mathfrak{B}_i{}^{\mathfrak{a}}$	$\mathfrak{A}_\mu{}^{\mathfrak{a}}$	$\mathfrak{V}^{\mathfrak{a}}$	$\mathfrak{W}_a{}^{\mathfrak{a}}$
Four-form	$\Lambda_{\mu\nu\rho\sigma}$	$E_{\hbar\bar\imath\bar\jmath}$		$A_{\lambda\mu\nu}$	V_{ab}	
Three-form	$H_{\lambda\mu\nu}$	$E_{\bar\imath\bar\jmath}$	B	$A_{\mu\nu}$	V_a	W_{ab}
General field	$F_{\mu\nu}{}^A$			$A_\mu{}^A$	V^A	$W_a{}^A$

TABLE III. Conjugate momenta.

	Conjugate momenta		Surface density		
	On Σ	On $\mathcal{T}$	Charge	Momentum	Current
Dilaton	$P_{\rm dil}$	$\Pi_{\rm dil}$		$\mathcal{Y}$	
Axion	$P_{\rm axi}$	$\Pi_{\rm axi}$		$\mathcal{A}$	
Dilaton-Yang-Mills	$(P_{\rm DYM})^{\bar{\imath}}{}_{\mathfrak{a}}$	$(\Pi_{\rm DYM})^{i}{}_{\mathfrak{a}}$	$(\mathcal{Q}_{\rm DYM})_{\mathfrak{a}}$	$(\mathcal{J}_{\rm DYM})_a$	$\mathfrak{J}^a{}_{\mathfrak{a}}$
Axion-Yang-Mills	$(P_{\rm AYM})^{\bar{\imath}}{}_{\mathfrak{a}}$	$(\Pi_{\rm AYM})^{i}{}_{\mathfrak{a}}$	$(\mathcal{Q}_{\rm AYM})_{\mathfrak{a}}$	$(\mathcal{J}_{\rm AYM})_a$	$\mathfrak{K}^a{}_{\mathfrak{a}}$
Dilaton-four-form	$(P_{\rm DFF})^{\bar{h}\bar{\imath}\bar{\jmath}}$	$(\Pi_{\rm DFF})^{ijk}$	$(\mathcal{Q}_{\rm DFF})^{ab}$		
Dilaton-three-form	$(P_{\rm DTF})^{\bar{\imath}\bar{\jmath}}$	$(\Pi_{\rm DTF})^{ij}$	$(\mathcal{Q}_{\rm DTF})^{a}$	$(\mathcal{J}_{\rm DTF})_a$	$(\mathcal{I}_{\rm DTF})^{ab}$
General field	$P^{\bar{\imath}}{}_A$		$\mathcal{Q}_A$		$\mathcal{I}^a{}_A$

rized in Table I.

We can construct the following densities on $\partial\Sigma$ out of projections of π^{ij}: the surface energy density, $\mathcal{E}$, the surface geometric momentum density, $(\mathcal{J}_{\rm EHD})_a$, and the surface stress density $\mathcal{S}^{ab}$.

We consider various forms of matter. These are summarized in Tables II and III. Table II contains the field variables used in the various sectors. The momenta corresponding to these fields (and the related densities constructed on $\partial\Sigma$) are given in Table III.

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