Solutions in self-dual gravity constructed via chiral equations

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The chiral model for self-dual gravity given by Husain in the context of the chiral equations approach is discussed. A Lie algebra corresponding to a finite dimensional subgroup of the group of symplectic diffeomorphisms is found, and then used for expanding the Lie-algebra-valued connections associated with the chiral model. The self-dual metric can be explicitly given in terms of harmonic maps and in terms of a basis of this subalgebra.

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I. INTRODUCTION

Since the introduction of Ashtekar's variables in general relativity [1], they were quickly applied to self-dual gravity. Later, Ashtekar, Jacobson, and Smolin (AJS) considered a new formulation of half-flat solutions to Einstein's equations. To be more precise, making a decomposition of a real four-manifold $\mathcal{M}^4$ into $\mathbf{R}\times\Sigma$, with Σ an arbitrary three-manifold ($\mathcal{M}^4$ has local coordinates $\{x_0,x_1,x_2,x_3\}$), the problem of finding all self-dual metrics was reduced to solving one constraint and one "evolution" equation on a field of triads V_i^a on Σ, that is,

$$\begin{aligned} &\mathrm{Div}V_i^a=0,\\ &\frac{\partial V_i^a}{\partial t}=\frac{1}{2}\epsilon_{ijk}[V_j,V_k]^a, \end{aligned} \tag{1}$$

where $i,j,k=1,2,3$ [2]. Thus, all self-dual metrics can be described in terms of the triad just as

$$g^{ab}=(\det\, V)^{-1}[V_i^aV_j^b\delta^{ij}+V_0^aV_0^b], \tag{2}$$

where V_0^a is the vector field used in the $3+1$ decomposition.

Several authors [3,4], beginning with the AJS formulation, made contact with the Plebański approach to self-dual gravity [5]. In [3] Grant has shown that Eqs. (1) are related in a very close way with the first heavenly equation of Ref. [5]. It was quickly recognized that the relation was only a Legendre transformation on a convenient coordinate chart [6]. Here the heavenly equation was brought into a Cauchy-Kovalevski evolution form.

On the other hand in [4] Husain gives a chiral formulation for the self-dual gravity. He has shown how self-dual gravity can be derived from a two-dimensional chiral model the gauge group of which corresponds to the group of symplectic diffeomorphisms[1] [area-preserving diffeomorphisms of a two-surface $\mathcal{N}^2$, SDiff($\mathcal{N}^2$)]. Similarly to Grant, starting from Eqs. (1) but using another choice of the set of vector fields V_i^a, Husain derived also the first heavenly equation. However, although the choice of vector fields is different, both formulations are equivalent to that one from Plebański. Thus, we have a class of equations (and therefore the corresponding class of solutions) which will be equivalent. This class of equations we call the Grant-Husain-Plebański (GHP) class, and they can be seen as equivalent to AJS equations. This because they are only different formulations of the same full theory.

Here, we briefly review the Husain chiral model for self-dual gravity. It is well known that Eqs. (1) lead to the set

$$\begin{aligned} &[\mathcal{T},\mathcal{X}]=[\mathcal{U},\mathcal{V}]=0,\\ &[\mathcal{T},\mathcal{U}]+[\mathcal{X},\mathcal{V}]=0, \end{aligned} \tag{3}$$

where $\mathcal{T}:=V_0+iV_1$, $\mathcal{U}:=V_0-iV_1$, $\mathcal{X}:=V_2-iV_3$, $\mathcal{V}:=V_2+iV_3$. The vector fields $\mathcal{X}$ and $\mathcal{T}$ can be fixed to be

$$\mathcal{T}=\frac{\partial}{\partial\bar z},\qquad \mathcal{X}=\frac{\partial}{\partial z}, \tag{4}$$

where the $\bar z=x_0+ix_1$, $z=x_2-ix_3$, $u=x_0-ix_1$, and $v=x_2+ix_3$. The bar does not stands for complex conjugation. The choice of vector fields enables four possibilities.

(i) The first *Husain* model [4] (see also [7]). We take

$$\begin{aligned} &\mathcal{U}=-\Omega_{,zq}\frac{\partial}{\partial p}+\Omega_{,zp}\frac{\partial}{\partial q},\\ &\mathcal{V}=\Omega_{,\bar zq}\frac{\partial}{\partial p}-\Omega_{,\bar zp}\frac{\partial}{\partial q}, \end{aligned} \tag{5}$$

where Ω is a holomorphic function of its arguments and p,q are local coordinates on the two-manifold $\mathcal{N}^2$. Equations (3) lead directly to the first heavenly equation as usual [5]:

$$\Omega_{,zp}\Omega_{,\bar zq}-\Omega_{,zq}\Omega_{,\bar zp}=1, \tag{6}$$

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[1] As we make only local considerations we assume the space $\mathcal{N}^2$ to be a two-dimensional simply connected manifold with local coordinates $\{p,q\}$. This space has a natural local symplectic structure given by the local area form $\omega=dp\wedge dq$. The group SDiff($\mathcal{N}^2$) is precisely the group of diffeomorphisms on $\mathcal{N}^2$ preserving the symplectic structure ω; i.e., for all $g\in$ SDiff($\mathcal{N}^2$), $g^*(\omega)=\omega$.

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where $\Omega_{,zp} = \frac{\partial^2 \Omega}{\partial z \partial p}$, etc.

(ii) The *Grant* model [3]. The difference with respect to Husain's formalism is just the way in which the vector fields $\mathcal{U}$ and $\mathcal{V}$ are chosen. Grant takes

$$\mathcal{U} = \frac{\partial}{\partial \bar{z}} - h_{,zq}\frac{\partial}{\partial p} + h_{,zp}\frac{\partial}{\partial q},$$
$$\mathcal{V} = h_{,\bar{z}q}\frac{\partial}{\partial p} - h_{,\bar{z}p}\frac{\partial}{\partial q}. \tag{7}$$

Equations (3) lead to the Grant evolution equation, which is of the Cauchy-Kovalevski form

$$h_{,\bar{z}\bar{z}} + h_{,zp}h_{,\bar{z}q} - h_{,zq}h_{,\bar{z}p} = 0, \tag{8}$$

where the corresponding metric is

$$g = d\bar{z} \otimes (h_{,\bar{z}q}dq + h_{,\bar{z}p}dp) + dz \otimes (h_{,zq}dq + h_{,zp}dp) + \frac{1}{h_{,\bar{z}\bar{z}}}(h_{,\bar{z}q}dq + h_{,\bar{z}p}dp)^2. \tag{9}$$

After a Legendre transformation on the variable $\bar{z}$ we recover the first heavenly equation as usual [6].

(iii) A variant of the *Grant* model [3]. This choice leads to a formulation similar to that of Grant. Choosing the vector fields as

$$\mathcal{U} = -h_{,zq}\frac{\partial}{\partial p} + h_{,zp}\frac{\partial}{\partial q},$$
$$\mathcal{V} = \frac{\partial}{\partial z} + h_{,\bar{z}q}\frac{\partial}{\partial p} - h_{,\bar{z}p}\frac{\partial}{\partial q}, \tag{10}$$

and using once again Eqs. (3), one arrives at the Grant evolution equation, which is of the Cauchy-Kovalevski form

$$h_{,zz} + h_{,\bar{z}q}h_{,zp} - h_{,\bar{z}p}h_{,zq} = 0. \tag{11}$$

The corresponding metric is, of course,

$$g = d\bar{z} \otimes (h_{,\bar{z}q}dq + h_{,\bar{z}p}dp) + dz \otimes (h_{,zq}dq + h_{,zp}dp) + \frac{1}{h_{,zz}}(h_{,zq}dq + h_{,zp}dp)^2. \tag{12}$$

And, as before, the first heavenly equation is recovered after a Legendre transformation.

(iv) The second *Husain* model [4]. For the self-dual equations (1) there exists another possibility for an appropriate selection of the vector fields. This choice leads to the chiral equations, which appear to be nonequivalent to that of the GHP class of equations. However, they might be related to them.

Introducing now two functions $\mathcal{B}_1(\bar{z}, z, p, q)$ and $\mathcal{B}_2(\bar{z}, z, p, q)$, the vector fields $\mathcal{U}$ and $\mathcal{V}$ can be written in a completely general form in terms of these functions as

$$\mathcal{U} = \frac{\partial}{\partial \bar{z}} + \alpha^b\partial_b\mathcal{B}_1, \quad \mathcal{V} = \frac{\partial}{\partial z} + \alpha^b\partial_b\mathcal{B}_2, \tag{13}$$

where $\alpha^{ab} = (\frac{\partial}{\partial p})^{[a} \otimes (\frac{\partial}{\partial q})^{b]}$.

Using Eqs. (3), the above choice of vector fields leads directly to the set of equations

$$\mathcal{B}_{2,\bar{z}} - \mathcal{B}_{1,z} + \{\mathcal{B}_1, \mathcal{B}_2\} = \mathcal{F}_{,\bar{z}}(\bar{z}, z) + \mathcal{G}_{,z}(\bar{z}, z),$$
$$\mathcal{B}_{1,\bar{z}} + \mathcal{B}_{2,z} = \mathcal{F}_{,z}(\bar{z}, z) - \mathcal{G}_{,\bar{z}}(\bar{z}, z), \tag{14}$$

for the arbitrary functions $\mathcal{F}$ and $\mathcal{G}$. In the above equation $\{,\}$ means the Poisson brackets in the coordinates p and q.

Redefining

$$A_1(\bar{z}, z, p, q) = \mathcal{B}_1 + \mathcal{G}$$

and

$$A_2(\bar{z}, z, p, q) = \mathcal{B}_2 - \mathcal{F}, \tag{15}$$

(14) transforms into a two-dimensional chiral model on a two-manifold $\mathcal{M}^2$ with local coordinates $\{\bar{z}, z\}$, having as gauge group the group of area preserving diffeomorphisms of the two-dimensional manifold $\mathcal{N}^2$. This two-dimensional chiral model is

$$F = A_{2,\bar{z}} - A_{1,z} + \{A_1, A_2\} = 0. \tag{16}$$

Vanishing curvature $F = 0$ implies that the gauge potentials A_1 and A_2 are *pure gauge*. Thus, we can write the potentials as

$$A_1 = (\partial_{\bar{z}}g)g^{-1}, \qquad A_2 = (\partial_z g)g^{-1}, \tag{17}$$

where $g : \mathcal{M}^2 \times \mathcal{N}^2 \to \mathrm{SDiff}(\mathcal{N}^2)$ given by $g(\bar{z}, z, p, q) \in \mathrm{SDiff}(\mathcal{N}^2)$. These potentials satisfy

$$A_{1,\bar{z}} + A_{2,z} = 0. \tag{18}$$

In this paper we work with the chiral formulation for self-dual gravity as given by Husain. In Sec. II, using the formalism of chiral equations approach to Einstein equations we discuss the chiral equations of the Husain model as harmonic maps in a philosophy similar to [8,9]. In Sec. III we find a finite dimensional subalgebra of the Lie algebra of $\mathrm{SDiff}(\mathcal{N}^2)$, and then we use this reduction to find solutions. We also find that the system induced by the Husain formalism is completely integrable at least for this subalgebra. Finally in Sec. IV we give our final remarks.

II. CHIRAL EQUATIONS AS HARMONIC MAPS

In this section we shall outline the method of harmonic maps for solving the chiral equations. This method consists in applying the harmonic maps ansatz to the chiral equations. Let us explain it.

First we enunciate the following theorem.

Theorem. Let $g \in G$ satisfy the chiral equations. The submanifold of solutions of the chiral equations $S \subset G$ is a symmetric manifold (the Riemann tensor of S is covariantly constant, i.e., $\nabla \mathbf{R}_S = 0$) with the metric

$$l_S = \mathrm{tr}(dg\ g^{-1} \otimes dg\ g^{-1}), \tag{19}$$

where $\otimes$ denotes, the symmetric tensor product. For the proof see Refs. [9, 10].

The ansatz consists in supposing that g can be written in terms of harmonic maps. Let V_Q be a Q-dimensional Riemannian space with an isometry group $H \subset G = \mathrm{SDiff}(\mathcal{N}^2)$. Suppose that $\{\lambda^i\}$ are local coordinates of V_Q. Let $\{\phi_s\}$, $s = 1,\ldots, d = \dim H \leq \dim G = \infty$, and $\phi_s = \phi_s^i \frac{\partial}{\partial \lambda^i}$ be a basis of the Killing vector space of V_Q and $\{\xi^s\}$ the dual basis of $\{\phi_s\}$. We suppose that

$$g = g(\lambda^i, p, q), \quad i = 1, ..., Q, \tag{20}$$

where $\lambda^i(z, \bar{z})$ are affine parameters of the minimal surfaces of G: i.e.,

$$\lambda^i_{,z\bar{z}} + \Gamma^i_{jk}\lambda^j_{,z}\lambda^k_{,\bar{z}} = 0, \quad i, j, k = 1, ..., Q. \tag{21}$$

The sdiff($\mathcal{N}^2$)-valued connection one-form on the two-manifold $\mathcal{M}^2$ in the basis $\{d\lambda^i\}$ can be written as (see Ref. [11])

$$A = a_i(z, \bar{z}, p, q)d\lambda^i = A_1(z, \bar{z}, p, q)dz + A_2(z, \bar{z}, p, q)d\bar{z}, \tag{22}$$

where $A_1(z, \bar{z}, p, q) = A_1(\lambda^i, p, q) = a_i(\lambda^i, p, q)\lambda^i_{,z}$ and $A_2(z, \bar{z}, p, q) = A_2(\lambda^i, p, q) = a_i(\lambda^i, p, q)\lambda^i_{,\bar{z}}$. The functions $a_i(\lambda^i, p, q)$ can be expanded in terms of a basis of a finite dimensional Lie subalgebra H of sdiff($\mathcal{N}^2$), $\{\sigma_j\}$, $j = 1, 2, ..., d$; that is,

$$a_i(\lambda^i, p, q) = \xi^s_i(\lambda^i)\sigma_s(p, q) \tag{23}$$

(for details of this method see Refs. [8, 9]).

Theorem. The potentials $A_1(\lambda^i, p, q) = a_i(\lambda^i, p, q)\lambda^i_{,\bar{z}}$ and $A_2(\lambda^i, p, q) = a_i(\lambda^i, p, q)\lambda^i_{,z}$ are solutions of the chiral equations (16) and (18).

Proof. Using (21), Eq. (18) implies that the quantities $\xi^s_i(\lambda^i)$ are the components of the Killing vectors of V_Q

$$A_{1,\bar{z}} + A_{2,z} = (\xi^s_{i;j} + \xi^s_{j;i})\sigma_s\lambda^i_z\lambda^j_{\bar{z}} = 0,$$

where a semicolon means a covariant derivative in V_Q. Equation (16) implies that $\{\sigma_s\}$ are the corresponding Hamiltonian functions of the simplectic form $\omega = dp \wedge dq$ on $\mathcal{N}^2$: i.e.,

$$\{\sigma_s, \sigma_t\} = C^r_{st}(p, q)\sigma_r, \tag{24}$$

where C^r_{st} are functions of p and q only. □

We shall now use the above approach to Einstein's equations [8, 9], in order to apply them to self-dual gravity. We show that it is possible to translate all relevant tools of the AJS formalism in terms of harmonic maps.

For instance the vector fields $\mathcal{U}$ and $\mathcal{V}$ are

$$\mathcal{U} = \frac{\partial}{\partial \bar{z}} + \xi^s_i\lambda^i_{,\bar{z}}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right),$$

$$\mathcal{V} = \frac{\partial}{\partial z} + \xi^s_i\lambda^i_{,z}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right). \tag{25}$$

The vectors on $\mathbf{R} \times \Sigma^3$ are, therefore,

$$V_0 = \frac{\partial}{\partial \bar{z}} + \frac{1}{2}\xi^s_i\lambda^i_{,\bar{z}}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right),$$

$$V_1 = \frac{i}{2}\xi^s_i\lambda^i_{,\bar{z}}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right),$$

$$V_2 = \frac{\partial}{\partial z} + \frac{1}{2}\xi^s_i\lambda^i_{,z}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right),$$

$$V_3 = -\frac{i}{2}\xi^s_i\lambda^i_{,z}\left(\frac{\partial\sigma_s}{\partial p}\frac{\partial}{\partial q} - \frac{\partial\sigma_s}{\partial q}\frac{\partial}{\partial p}\right). \tag{26}$$

The self-dual metric (2) can be expressed in terms of harmonic maps,

$$g^{-1} = \frac{4}{\xi^n_i\xi^m_j\lambda^i_{,\bar{z}}\lambda^j_{,z}\{\sigma_m, \sigma_n\}}\left[d\bar{z}\otimes d\bar{z} + dz\otimes dz + \xi^s_k\left(\frac{\partial\sigma_s}{\partial p}dp - \frac{\partial\sigma_s}{\partial q}dq\right)\otimes d\lambda^k\right]. \tag{27}$$

Here it can be observed that similarly to the metric (3.4) of Ref. [4], it also appears a singularity for null Poisson brackets (Abelian algebra).

For completeness we can write also the inverse of (27):

$$g = \left\{\xi^s_k(\lambda^k_{,z}d\bar{z} - \lambda^k_{,\bar{z}}dz) - \frac{(\xi^s_k)^2[(\lambda^k_{,\bar{z}})^2 + (\lambda^k_{,z})^2]}{\xi^m_i\xi^n_j\lambda^i_{,\bar{z}}\lambda^j_{,z}\{\sigma_m, \sigma_n\}}\left(\frac{\partial\sigma_s}{\partial p}dq + \frac{\partial\sigma_s}{\partial q}dp\right)\right\}\otimes\left(\frac{\partial\sigma_s}{\partial p}dq + \frac{\partial\sigma_s}{\partial q}dp\right). \tag{28}$$

From the metric (27) and (28) it is now clear that C^r_{mn} cannot vanish; thus, it is not possible to take an Abelian algebra in (24).

III. TWO-DIMENSIONAL SUBSPACES

From the metric (28) we conclude that it is not possible to take one-dimensional subspaces V_1 since all one-dimensional Riemannian spaces contain only Abelian groups of motion. We consider a two-dimensional Riemannian space V_2. In [9] it was shown that the chiral equations imply that V_2 must be a symmetric space. All two-dimensional Riemannian space is conformally flat. So the metric of V_2 can be written as [12]

$$i^*(l_S) = ds^2_2 = \frac{d\lambda d\tau}{(1 + k\lambda\tau)^2}, \tag{29}$$

where $i : V_2 \to S$. The symmetry of V_2 implies that $k = k(p,q)$ only, i.e., $k_{,\lambda} = k_{,\tau} = 0$ (since all two-dimensional symmetric space possesses constant curvature). Thus, V_2 contains a three-dimensional isometry group H. Three independent Killing vectors of V_2 are

$$\xi^1 = \frac{1}{2V^2}[(k\tau^2+1)d\lambda + (k\lambda^2+1)d\tau],$$

$$\xi^2 = \frac{1}{V^2}[-\tau d\lambda + \lambda d\tau], \qquad V = 1 + k\lambda\tau,$$

$$\xi^3 = \frac{1}{2V^2}[(k\tau^2-1)d\lambda + (1-k\lambda^2)d\tau]. \tag{30}$$

The three Hamiltonian functions σ_s satisfy the algebra

$$\{\sigma_1,\sigma_2\} = -4k\sigma_3,$$

$$\{\sigma_2,\sigma_3\} = 4k\sigma_1,$$

$$\{\sigma_3,\sigma_1\} = -4\sigma_2, \tag{31}$$

in order to have compatibility with the Killing vectors (30). These Poisson brackets can be seen as three differential equations for the three functions σ_s and the function k, and so we can take one of them arbitrarily and determine the other three ones by integration. Knowing the functions σ_s we can determine the potentials A_1 and A_2 by means of the formulas

$$A_1 = \xi_i^s \sigma_s \lambda^i_{,\bar z}, \qquad A_2 = \xi_i^s \sigma_s \lambda^i_{,z},$$

in terms of the harmonic maps λ^i. The harmonic map Eq. (21) transforms in this case into

$$\lambda_{,z\bar z} - \frac{2k\tau}{1+k\lambda\tau}\lambda_{,z}\lambda_{,\bar z} = 0,$$

$$\tau_{,z\bar z} - \frac{2k\lambda}{1+k\lambda\tau}\tau_{,z}\tau_{,\bar z} = 0. \tag{32}$$

In what follows we will solve Eqs. (31). Let us write Eq. (31) in terms of two new variables $s = s(p,q)$ and $t = t(p,q)$ and without loss of generality we can suppose that $\sigma_2 = s$. The commutation relations (31) transform into

$$\text{(i)} \quad \frac{\partial\sigma_1}{\partial t}\{s,t\} = 4k\sigma_3,$$

$$\text{(ii)} \quad \frac{\partial\sigma_3}{\partial t}\{s,t\} = 4k\sigma_1,$$

$$\text{(iii)} \quad \left(\frac{\partial\sigma_3}{\partial s}\frac{\partial\sigma_1}{\partial t} - \frac{\partial\sigma_1}{\partial s}\frac{\partial\sigma_3}{\partial t}\right)\{s,t\} = -4\sigma_2. \tag{33}$$

If we substitute (i) and (ii) into (iii), we arrive at

$$k\frac{\partial(\sigma_3^2-\sigma_1^2)}{\partial s} = -2s, \tag{34}$$

and by combining (i) and (ii) we conclude that

$$\frac{\partial(\sigma_3^2-\sigma_1^2)}{\partial t} = 0, \tag{35}$$

which imply that k does not depend on t, which means $k = k(s)$. Deriving Eqs. (i) and (ii) with respect to t we find differential equations only for σ_1 and σ_3:

$$\frac{\partial^2\sigma_s}{\partial t^2}l^2 + \frac{1}{2}\frac{\partial\sigma_s}{\partial t}\frac{\partial l^2}{\partial t} - 16k^2\sigma_s = 0, \qquad s = 1,3, \tag{36}$$

where we have defined $l = \{s,t\}$. The solution to Eq. (36) is

$$\sigma_s = \left[a_s\, e^{L(t)} + b_s\, e^{-L(t)}\right], \tag{37}$$

where

$$L(t) = 4k\int\frac{dt}{l}.$$

From (35) we find that $a_1 = a_2 = c_1$ and $b_1 = -b_2 = c_2$, such that

$$\sigma_3^2 - \sigma_1^2 = 4\, c_1c_2.$$

The no dependence on t of $\sigma_3^2 - \sigma_1^2$ implies that $c_1 = c_1(s)$, $c_2 = c_2(s)$ where $2k\frac{\partial c_1c_2}{\partial s} = -s$. So we obtain

$$\sigma_1 = \left[c_1(s)\, e^{L(t)} + c_2(s)\, e^{-L(t)}\right],$$

$$\sigma_2 = s,$$

$$\sigma_3 = \left[c_1(s)\, e^{L(t)} - c_2(s)\, e^{-L(t)}\right],$$

$$L(t) = 4k\int\frac{dt}{l}, \qquad 2k\frac{\partial c_1c_2}{\partial s} = -s, \qquad \{s,t\} = l. \tag{38}$$

Observe that $k(s)$, $c_1(s)$, and $c_2(s)$ are subjected to only one restriction; therefore, two of them are arbitrary. So we have three arbitrary functions of p,q in general.

IV. FINAL REMARKS

In this paper we found an explicit exact class of solutions to self-dual gravity [4]. We used the chiral equations approach in order to obtain explicit solutions. Solving the chiral equations with the harmonic maps method we find that the harmonic maps ansatz can be applied to the chiral equations derived from self-dual gravity. The difference with previous applications of this method is that here we have Poisson brackets in place of matrix brackets in a similar spirit as in [11]. Nevertheless, we can solve the corresponding Poisson algebra by making a coordinate transformation and finding the corresponding Hamiltonian functions by solving the Poisson algebra as differential equations. We find that there exists a class of such solutions in terms of two arbitrary functions (s and t) of two variables (p and q). The coordinate transformation can be taken also arbitrary, but in the case when the new coordinates are canonical the solution becomes very simple.

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