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Renormalization group approach to relativistic cosmology

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We discuss the averaging hypothesis tacitly assumed in standard cosmology. Our approach is implemented in a "3+1" formalism and invokes the coarse-graining arguments, provided and supported by the real-space renormalization group (RG) methods, in parallel with lattice models of statistical mechanics. Block variables are introduced and the recursion relations written down explicitly enabling us to characterize the corresponding RG flow. To leading order, the RG flow is provided by the Ricci-Hamilton equations studied in connection with the geometry of three-manifolds. The possible relevance of the Ricci-Hamilton flow in implementing the averaging in cosmology has been previously advocated, but the physical motivations behind this suggestion were not clear. The RG interpretation provides us with such physical motivations. The properties of the Ricci-Hamilton flow make it possible to study a critical behavior of cosmological models. This criticality is discussed and it is argued that it may be related to the formation of sheetlike structures in the Universe. We provide an explicit expression for the renormalized Hubble constant and for the scale dependence of the matter distribution. It is shown that the Hubble constant is affected by nontrivial scale-dependent shear terms, while the spatial anisotropy of the metric influences significantly the scale dependence of the matter distribution.

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I. INTRODUCTION

A successful physical theory usually enables us to isolate some limited range of length scales, or select a not too big set of variables, to render the problem tractable and at the same time preserve its essence. Fortunately, in many circumstances it is not necessary to resolve the details associated with each scale, since generally phenomena of each size can be treated independently. For example, in hydrodynamics there is no need to specify the motion of each water molecule and yet waves can be described as a disturbance of a continuous fluid.

There are also problems that have many various scales of length, that is to say, phenomena or processes where each length scales contribution is of equal importance. To handle them one has to take into account the entire spectrum of length scales, dealing with fluctuations of almost any wavelength and consequently many coupled degrees of freedom. For example, critical phenomena, turbulent flow, the internal structure of elementary particles, and confinement in QCD belong to the above class of problems [1]. We argue here that in gravitational physics we encounter a problem of a similar nature, namely, the so-called averaging problem in relativistic cosmology.

Relativistic cosmologies rely on some form of cosmological principle. The latter is usually a smoothing-out hypothesis imposed *a priori* on the distribution of matter in the Universe. A well-known example is provided by the Friedman-Lemaître-Robertson-Walker (FLRW) metric which is assumed to describe the real Universe.

There are at least two reasons why one should be very careful in doing so. Firstly, observations of the distribution of matter have shown that the scale at which the background homogeneity is reached,[1] is probably of the order of hundreds of Mpc, meaning that locally, i.e., at least up to this scale, the Universe is quite lumpy. Consequently, the local geometry is very complex and its nature not very illuminating from the point of view of cosmology. Seen this way, the approach usually taken is to average out all the matter, i.e., to redistribute them in the form of a homogeneous perfect fluid and use continuous functions (e.g., matter density, pressure) in modeling the Universe, assuming that they represent "volume averages" of the corresponding fine scale quantities. In doing so, one tacitly assumes that such a smoothed-out universe and the real locally inhomogeneous one behave identically under their own gravitation. However, it has been stressed in [2] that the above assumption, although usually taken for granted, is by no means justified. First of all, Einstein's equations are highly nonlinear, which is why any averaging process on them[2] is far from trivial in general.

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[1] If this scale has been reached at all is still a matter of controversy.

[2] Moreover, additional care is required since a volume average of a tensor is not a covariant quantity, unlike scalars.

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Secondly, and this is a remark of a philosophical flavor, most of the observational data are theory dependent; i.e., their meaning can be interpreted only by assuming a particular theoretical explanation.

Thus it is of great importance that the very foundations of a cosmological model should be as sound as possible. In particular, they should be free from assumptions which may not be warranted.

II. COARSE GRAINING IN COSMOLOGY

A possible solution to the averaging problem would be to explicitly construct a procedure for carrying out the smoothing process in the full theory. Almost all existing attempts were concerned with the linearized theory, with a possible exception of [3] (see also [4]); for a review see [5].

In [3] a covariant smoothing-out procedure was put forward for the space-times associated with gravitational configurations which may be considered near to the standard ones, generating closed FLRW universes. The procedure makes use of Hamilton's theorem about smooth deformations of three-metrics and is adapted for smoothing out an initial data set for cosmological solutions to the Einstein equations. While interesting in its own right, this approach seemed rather *ad hoc* and not yet capable of resolving the issues of actual limits of validity of the FLRW models in cosmology.

Now, our hope is that there is a clear and simpler way to the heart of the problem, borrowing from the known theories and methods of statistical mechanics, based on the real-space renormalization group (RG) approach for studying critical phenomena in lattice models [6,7]. Although the usual renormalization transformations, invoking averaging over square blocks, are designed mostly with ferromagnetic systems in mind, there are many more problems suitable for RG methods. These are difficult problems where the reductionist approach fails and where the effective degrees of freedom of a physical system are scale dependent. The difficulty of this kind of issue can be traced to a multiplicity of scales and, moreover, there can presumably be a gross mismatch between the largest and smallest scales in the problem. The averaging problem in cosmology can be looked at and studied as belonging to precisely this kind of problem. This is our main objective in this paper.

Some form of RG is active on any system where there are fluctuations present (they by no means have to be quantum). This is so since one can integrate the fluctuations out of the physical quantities of interest, e.g., the partition function, and depending on the "scale" up to which one is integrating the same emerging quantities are different. The functional relations between them provide recursion relations between the physical parameters, the coupling constants, which characterize the physics at each scale, and this is precisely what RG is all about.

Often a major step consists in finding a way of looking at things. Therefore we stress that the problem we face with the averaged description in cosmology is effectively a question of how a system behaves under changes of "scales." As such it is most naturally addressed using the RG approach, understood here rather as a general strategy to handle problems of multiple length scales[3] enabling us to extract the long-distance behavior of the system by making the scale successively coarser. In cosmology, we have curvature inhomogeneities and to consistently tackle this problem we will have to consider a procedure operating on the metric, not only on or apart from the matter present.

To provide even more support in favor of the presented above idea, that RG arguments might be applicable to the problem of the inhomogeneous universe, let us note that one can also be guided by scaling ideas. Scaling is exhibited (approximately or exactly) by many natural phenomena and mathematical models. The universe, namely, the distribution of matter in it, does exhibit certain scaling properties [8,9], of which the power law behavior of the two-point correlation function for galaxies, clusters, and quasars, is a fair example. Scaling, on its own right, is deeply understood within the underlying mathematical scaffold which is RG. These are hints therefore that one can regard the universe as a gravitational dynamical system not far from criticality (understood intuitively by analogy with, e.g., a ferromagnet). Later one can also try to qualify the precise nature of the critical behavior within the phase transition context, but this will not be of our concern in the present paper (apart from a simplified example in Sec. IV C).

In passing, let us note that interesting results were obtained in [10] on spherical scalar field collapse and in [11] on axisymmetric gravitational waves collapse, which show a surprising scaling and critical behavior reminiscent of that found in many (second-order) phase-transition phenomena in condensed matter. They could be described, we believe, using RG approach again, treating the Einstein equations as generating a RG flow on the space of initial data.

The real-space renormalization techniques are mostly applicable to discretized models, based on a lattice. Therefore, we now turn to describing a suitably discretized manifold model we are going to work with.

A. Discretized manifold model

The approach taken is that of a 3+1 formulation of general relativity (GR) [12]. Let us assume that we have a differentiable, compact Riemannian three-manifold without a boundary Σ, to be thought of as a particular hypersurface in a four-dimensional space-time. Generally, we will always assume that the three-manifolds we consider posses certain natural constraints on their diameter, volume, and curvature (diameter bounded above, volume and sectional curvature bounded below). The point of this requirement is that such manifolds of *bounded ge-*

[3] Although the renormalization procedure might seem purely formal there are important physical ideas behind it, namely, that of *scaling* and *universality*.

ometry, or more precisely the corresponding Riemannian structures, can then be classified according to how they can be covered by small metric balls (to be defined later). Moreover, this set of Riemannian structures has some remarkable compactness properties. This is a classical result obtained by Gromov [13], related to the possibility of introducing a distance function, which roughly speaking enables one to say how close Riemannian manifolds are to each other. The explicit definition of this distance function is not needed for our purposes, but of particular relevance to us is the fact that nearby Riemannian manifolds (in the sense of Gromov distance) can be covered with metric balls arranged in similar packing configurations [14].

In order to define such covering [15], let us parametrize the geodesics by arc length, and for any point $p \in \Sigma$ let $d_\Sigma(x,p)$ denote the distance function of the generic point x from the chosen one p. Then for any given $\epsilon > 0$ it is always possible to find an ordered set of points $\{p_1, \ldots, p_N\}$ in Σ, so that [15] (i) the open metrix balls (the geodesic balls) $B_\Sigma(p_i,\epsilon) = \{x \in \Sigma | d_\Sigma(x,p_i) < \epsilon\}, i = 1, \ldots, N$, cover Σ, in other words, the collection $\{p_1, \ldots, p_N\}$ is an ϵ-net in Σ, and (ii) the open balls $B_\Sigma(p_i, \epsilon/2), i = 1, \ldots, N$ are disjoint, i.e., $\{p_1, \ldots, p_N\}$ is a *minimal* ϵ-net in Σ (see Fig. 1).

It is fair to say that as a consequence of the compactness properties of the set of Riemannian structures that we consider, for each "length scale ϵ" there exists a finite number of "model" geometries which describe, with an ϵ approximation, any given Riemannian manifold of bounded geometry. Namely, given a ball of radius $> \epsilon$ in any Riemannian manifold of bounded geometry there exists a ball metrically similar (up to an ϵ scale) in one of the model geometries which does not retain the details of the original manifold on scales smaller than ϵ. Roughly speaking, ϵ is a measure of the typical curvature inhomogeneity with respect to the model background. Let us stress that this is a highly nontrivial result, in the sense that the metrical properties of the manifolds from an infinite dimensional set are, up to an ϵ scale, described by the metrical properties of just a finite number of model Riemannian manifolds.

The ϵ-nets underlying the ball coverings precisely provide the discretized manifold model. This coarse graining of a manifold according to Gromov is the most natural coarse graining one can think of, pertinent for manifolds with bounded geometry. This assumption does not limit the generality of our analysis, which is basically motivated by a concrete physical problem, whose nature allows us to deal from the beginning with manifolds that are already in a certain sense quasihomogeneous (cf. comments on the solvability of the Ricci-Hamilton flow, later on).

In what follows, when speaking of balls we will always mean geodesic balls here.

B. An empirical averaging procedure

On the three-manifold Σ, the average of a scalar function $f : \Sigma \to \mathbb{R}$ is given as

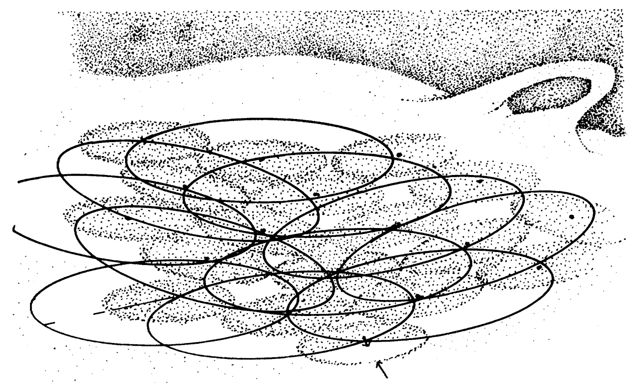

FIG. 1. A portion of a minimal geodesic balls covering. The dotted disks are the $\epsilon/2$ balls which pack a given region. The larger, undotted disks, represent the ϵ balls which provide the covering.

$$\langle f \rangle_{\Sigma(g)} = \frac{\int \Sigma f d\mu_g}{\mathrm{vol}(\Sigma, g)} , \qquad (1)$$

where $\mathrm{vol}(\Sigma, g) = \int \Sigma d\mu_g$ and μ_g is the Riemannian measure associated with the three-metric g of Σ. Since at this early stage we simply wish to put forward a few elementary geometrical considerations, we do not specify yet the choice of the hypersurface Σ and we do not attribute any particular physical meaning to the function f.

If the geometry of Σ is not known on a large scale, we cannot take Eq. (1) as an operational way of defining the average of f. From a more pragmatic point of view, supposing that we can only experience geometry in sufficiently small neighborhoods of a finite set of instantaneous observers, it makes much more sense to replace Eq. (1) with a suitable average based on the geometrical information available on the length scale of such observers.

For simplicity, given a finite set of instantaneous observers, located at the points $x_1, \ldots, x_N \in \Sigma$, we may assume that these susceptible to observation regions are suitably small geodesic balls of radius ϵ, scattered over the hypersurface Σ so as to cover it. In other words, we assume that $\{x_1, \ldots, x_N\}$ is a minimal ϵ-net in Σ. Further, we denote by U_ϵ the corresponding set of geodesic balls $\{B_\Sigma(x_i, \epsilon)\}, i = 1, \ldots, N$. It is important to stress that N is finite and that it can be uniformly bounded above in terms of ϵ, of the diameter and of the lower bound on the Ricci curvature of Σ [15].

Let us consider a partition of unity $\{\xi_i\}_{i=1,\ldots,N}$ subordinated to the covering generated by the balls $\{B_\Sigma(x_i, \epsilon)\}$. Namely, a set of smooth functions, ξ_i, such that $0 \le \xi_i \le 1$, for each $i = 1, \ldots, N$; the support of each ξ_i is contained in the corresponding $B_\Sigma(x_i, \epsilon)$; and $\sum_i^N \xi_i(p) = 1$, for all $p \in \Sigma$. Then, according to the definition of integration over a compact manifold, we can write (1) explicitly in terms of the geodesic ball covering as

$$\langle f \rangle_{\Sigma(g)} \equiv \frac{\sum_i \int_{(B_i,\epsilon)} \xi_i f d\mu_g}{\sum_j \int_{(B_j,\epsilon)} \xi_i d\mu_g} . \qquad (2)$$

In order to replace (2) with a more manageable expression, we employ a preferred system of coordinates on the set of balls $\{B_i\}$ given by the local diffeomorphism

$$\exp_x : T_x\Sigma \to \Sigma \ , \tag{3}$$

i.e., we make use of the exponential mapping

$$\varphi_i \equiv \exp|_{\exp^{-1} B_i \equiv D_i} : D_i \to B_i \ , \tag{4}$$

where $D_i = D(x_i, \epsilon)$ is the ball in[4] $T_{x_i}\Sigma$.

On D_i we use polar coordinates and pull back the Riemannian measure accordingly: namely,

$$\varphi_i^*(\mu_g) = \theta(t, x_i)dt \otimes dx_i \ , \tag{5}$$

where dx_i denotes the canonical measure (Euclidean volume form) on the unit sphere $D(x_i, 1) = S_1^2 \subset T_{x_i}\Sigma$ and where dt is the Lebesgue measure on $\mathbb{R}(t \geq 0)$.

For t small enough one can prove Puiseux's formula

$$\theta(t, x) = t^{n-1}[1 - \tfrac{1}{3}r(x)t^2 + o(t^2)] \ , \tag{6}$$

where $n = \dim\Sigma$ and $r(x)$ is the Ricci curvature Ric(g) (at the point x). Notice that we can choose the partition of unity $\{\xi_i\}_{i=1,\ldots,N}$ in such a way that for any $p \subset \Sigma$, we have

$$(\xi_i \cdot \exp\cdot)(p) = \frac{\lambda_i(p)}{\sum_i^N \lambda_i(p)} \tag{7}$$

and where $\lambda_i : R^3 \to R$, is a smooth bump function with the properties $0 \leq \lambda_i(x) \leq 1$, for all $x \in R^3, \lambda_i(x) = 1$ if and only if $d_e(0, x) \leq \frac{\epsilon}{2}, \lambda_i(x) = 0$ if and only if $d_e(0, x) \geq \epsilon$ [$d_e(0, x)$ denotes the Euclidean distance in R^3]. For ease of notation, we shall still denote by ξ_i, the function $\xi_i \cdot \exp\cdot$.

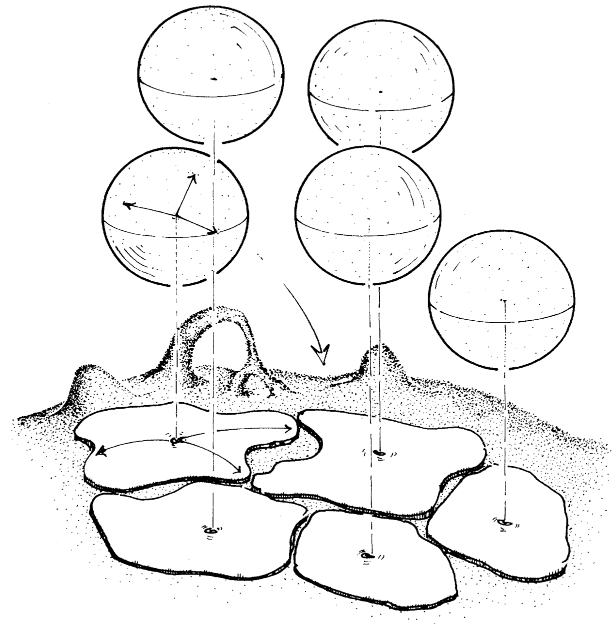

FIG. 2. We can approximate the average over Σ by averaging over Euclidean balls whose Lebesgue measure has been locally weighted through Puiseux's formula. In this drawing, the $\epsilon/2$ geodesic balls are represented by curved disks on Σ, while the corresponding Euclidean balls are correctly depicted as three dimensional.

Using these results we have (see Fig. 2)

$$\int_{B(x_i,t)} \xi_i f d\mu_g = \int_{S_1^2} \xi_i f \theta(t, x) dx_i dt \ . \tag{8}$$

Let us consider the asymptotic expansion with respect to t [17]

$$\int_{B(x_i,t)} \xi_i f d\mu_g = \omega_n t^n \left[(\xi_i f)(x_i) + \frac{t^2}{2(n+2)} \left(\Delta((\xi_i f))(x_i) - \frac{R(x_i)}{3}(\xi_i f)(x_i) \right) + o(t^2) \right] \ , \tag{9}$$

where ω_n is the volume of the unit ball of $\mathbb{R}^n$, R is the scalar curvature at the center of the ball, and Δ the Laplacian operator relative to the manifold. Since for the chosen partition of unity $\lambda_k(x_i) = 1$, the above expression reduces to

$$\int_{B(x_i,t)} \xi_i f \, d\mu_g = \frac{\omega_n t^n}{\sum_k \lambda_k(x_i)} \left[f(x_i) + \frac{t^2}{2(n+2)} \left((\Delta f)(x_i) - \frac{R(x_i)}{3} f(x_i) \right) + o(t^2) \right] \ . \tag{10}$$

Substituting $f = 1$ in the above formula we get

$$\int_{B(x_i,t)} \xi_i d\mu_g = \frac{\omega_n t^n}{\sum_k \lambda_k(x_i)} \times \left(1 - \frac{t^2}{6(n+2)} R(x_i) + o(t^2) \right) \ , \tag{11}$$

which yields for the asymptotic expansion of the volume of a geodesic ball,

$$\mathrm{vol}(B(x_i, t)) = \omega_n t^n \left(1 - \frac{R(x_i)}{6(n+2)} t^2 + o(t^2) \right) \ . \tag{12}$$

[4]The transition from (Σ, g), at a given moment of time t_0, to a tangent space parallels the prescription in Riemannian geometry for measuring [16]. Let U be a given neighborhood of an instantaneous observer in Σ and suppose it is so small that there is a neighborhood A of $0 \in T_x\Sigma$ such that $\exp_x : A(\subset T_x\Sigma) \to U(\subset \Sigma)$ is a diffeomorphism. One can then replace the considerations in (U, g) by those in A (with the Riemannian measure pulled back) via $\exp_x^{-1}$. Namely, we can say that an instantaneous observer in (Σ, g) observes the universe with the help of the exponential mapping, which just means projecting structures from an open neighborhood $U \subset \Sigma$ of x by $\exp_x^{-1}$ and treating them as structures on $T_x\Sigma$.

According to these results we can write, to leading order

$$\langle f\rangle_{\Sigma(g)} \simeq \langle f\rangle_\epsilon \equiv \frac{\sum_i [f_i + \left(\frac{\Delta f_i - R_i f_i/3}{2(n+2)}\right)\epsilon^2]}{\sum_i [1 - \frac{R_i}{6(n+2)}\epsilon^2]} \;, \tag{13}$$

where $f_i \equiv f(x_i)$. This expression suggests the consideration of $\langle f\rangle_\epsilon$ as a suitable scale-dependent approximation $\langle f\rangle_{\Sigma(g)}$.

There are certain problems lurking that we have to clear up. Obviously, there are "unwanted" details affecting this averaging over the manifold associated with a collection of geodesic balls. The important question to ask is what happens to the average $\langle f\rangle_\epsilon$ when we change the length scale represented by the radius of the balls. Depending on whether we are actually increasing or decreasing it, respectively, less or more detail of the underlying geometry will be felt by the average values. The natural philosophy is that over scales big enough, no details should be discerned since the homogeneity and isotropy prevails. This is the reason why on constant curvature spaces averaging is well defined, since there one can move the balls freely and deform them, but by so doing no new geometric details that measure the inhomogeneities will be felt in the averaged values of quantities we are interested in.

A natural question to ask now is how the geometry, specifically curvature inhomogeneities, should depend on scale so that the average $\langle f\rangle_\epsilon$ over the balls is scale independent, or equivalently, how do we have to deform the geometry in order to achieve the scaling limit when size of the balls matters no more?

Since we are interested in discussing how $\langle f\rangle_\epsilon$ behaves upon changing the radius of the balls $\{B(x_i,\epsilon)\}$, let us consider the average $\langle f\rangle_{\epsilon_0+\eta}$, with η a positive number with $\eta/\epsilon_0 \ll 1$. Upon expanding $\langle f\rangle_{\epsilon_0+\eta}$ in η, we get to leading order $[o(\epsilon_0^4)$ in ϵ_0, and $o((\eta/\epsilon_0)^2)$ in $\eta/\epsilon_0]$,

$$\langle f\rangle_{\epsilon_0+\eta} \simeq \langle f\rangle_{\epsilon_0} + \frac{1}{n+2}[\langle \Delta f\rangle_{\epsilon_0}]\epsilon_0^2\frac{\eta}{\epsilon_0} \tag{14}$$

$$+\frac{1}{3(n+2)}[\langle R\rangle_{\epsilon_0}\langle f\rangle_{\epsilon_0} - \langle Rf\rangle_{\epsilon_0}]\epsilon_0^2\frac{\eta}{\epsilon_0} \;, \tag{15}$$

where $\langle f\rangle_{\epsilon_0}$ is the average of the function f over the set of N instantaneous observers U_{ϵ_0}, (with similar expressions for $\langle R\rangle_{\epsilon_0}$, $\langle Rf\rangle\epsilon_0$, and $\langle \Delta f\rangle_{\epsilon_0}$). Thus, under a change of the cutoff we can write

$$\epsilon_0\frac{d}{d\eta}\langle f\rangle_{\epsilon_0+\eta|\eta/\epsilon_0=0} = \frac{1}{n+2}[\langle \Delta f\rangle_{\epsilon_0}] \tag{16}$$

$$+\frac{1}{3(n+2)}[\langle R\rangle_{\epsilon_0}\langle f\rangle_{\epsilon_0} - \langle Rf\rangle_{\epsilon_0}] \;, \tag{17}$$

to leading order. In the following sections we will discuss the consequences of these formulas and the connection of our averaging procedure with the Ricci-Hamilton flow (see Fig. 3).

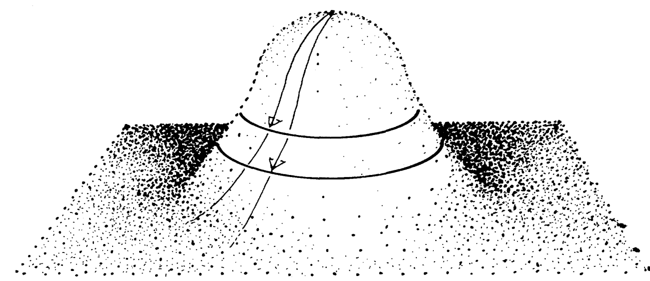

FIG. 3. The local average of a function f over a geodesic ball $B(x,\epsilon_0)$ feels the underlying curvature of the manifold through Puiseux's formula. In particular, in passing from $B(x,\epsilon_0)$ to the larger ball $B(x,\epsilon_0+\eta)$ we get correction terms which depend on the fluctuations in curvature.

III. THE RENORMALIZATION GROUP VIEW

A. Block variables and recursion relations

The real-space RG technique is based on the recursive introduction of *block variables*. A method of "blocking" is trivial to introduce on regular lattices. In particular, in the case of the Ising model (two-dimensional) it consists of a subdivision of the spin system into cells, which supposedly interact in a similar way as the original spins. The can be done by introducing the block-spin variables via the majority rule or decimation procedure [18]. The behavior of blocked lattice on large scales is equivalent to the behavior of the original lattice corresponding to a different temperature. The slopes of the parameter's surface close to the *critical fixed point* determine the macroscopic characteristics of the model (see, e.g., [18]).

In the context of the problem we are considering, the application of RG method forces us to invest some analog of Kadanoff's blocking (block-spin transformation) applied to the geometry itself. This appears to be a difficult problem since in a general case when the geometry is curved the "lattice" itself takes on a dynamical role. Moreover, since the manifold Σ we are dealing with is compact (closed, without boundary), we must implement a renormalization group strategy in a finite geometry, and thus the relevant phenomena are here related to *finite size scaling.* Roughly speaking, the size of our manifold is characterized by a length scale, say L (actually the volume, the diameter, and bounds on curvatures), which is large in terms of the microscopic scale (the radius of the typical geodesic ball coverings we shall use in the blocking procedure). A continuous theory, describing the (universal) properties of the field f on Σ, arises when the correlation length associated with the distribution of f is large (of the order of L, or bigger). This being the underlying rationale, let us proceed and be guided by the formula Eq. (17).

Let us consider our system as nearly infinite, i.e., the manifold Σ divided by the collection of the geodesic balls of radius $\epsilon_m \equiv (m+1)\epsilon_0$ for $m = 0,1,2,\ldots$, and where ϵ_0, the chosen cutoff, is much smaller than the typical length scale associated with Σ (this length scale can be

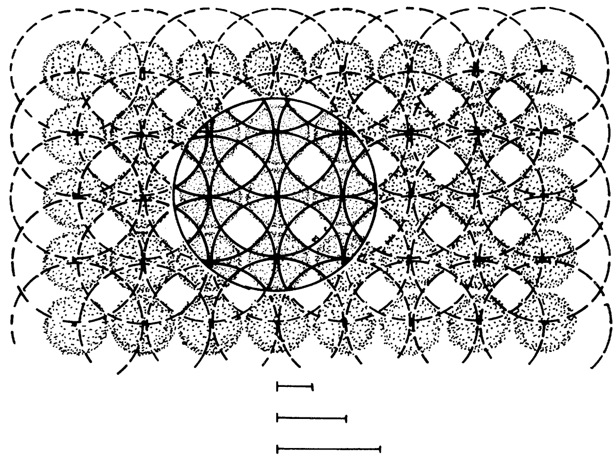

FIG. 4. The intricate but symmetrical intersection pattern of an array of $\epsilon/2$ balls in the 2D plane. Here, the $\epsilon/2$ balls are dotted while the overlapping ϵ balls, providing the covering, are the dashed circles. The solid circles and the arcs of circles describe the intersection pattern of a $\frac{3}{2}\epsilon$ ball. On a curved manifold of bounded geometry, the pattern is more complicated and not symmetrical at all. Nonetheless, one may easily obtain a recursive definition of blocking by exploiting a partition of unity argument, explained in the text.

identified with the injectivity radius of the manifold) (see Fig. 4). Each ball will be labeled by k in the sequel and the original geodesic packing covering is for $m = 0$. We can introduce this way a convenient notation for the integral of the generic function f over the ball $B(x_k, \epsilon_m)$ as

$$\psi_m(k; f) \equiv \int_{B(x_k,\epsilon_m)} f \, d\mu_g \ , \tag{18}$$

which can be seen as the block variable since it allows us to eliminate from the distribution of the field f all fluctuations on scales smaller than the cutoff distance ϵ_m. We wish to emphasize that if the geometry of the ball $B(x_k, \epsilon_m)$ is not flat, then the definition of $\psi_m(k; f)$ can be interpreted as that of a *weighted* sum over a flat ball, namely,

$$\psi_m(k; f) \equiv \int_{\exp^{-1} B(\epsilon_m)} f\theta(t, x) dt \otimes dx \ , \tag{19}$$

where the weight $\theta(t, x)$ is provided by Puiseux's formula Eq. (6).

If we consider the covering of $B(x_k, \epsilon_{m+1})$ induced by the geodesic balls $\{B(x_j, \epsilon_m)\}$, i.e., the collection of $N_k(m, m+1)$ open sets $\{B(x_k, \epsilon_{m+1})\} \cap \{B(x_j, \epsilon_m)\}$, then the above definition of block variables can be written recursively in terms of the values the function f takes correspondingly to balls of larger and larger radii. To this end, let us consider again the partition of unity $\{\xi_h\}_{h=1,\ldots,N_k(0,m+1)}$, but now subordinated to the covering of the generic enlarged ball $B(x_k, \epsilon_{m+1})$, induced by the geodesic balls $\{B(x_h, \epsilon_0)\}$. Namely, a set of smooth functions such that: $0 \leq \xi_h \leq 1$, for each h; the support of each ξ_h is contained in the corresponding $B(x_h, \epsilon_0)$; and $\sum_h \xi_h(p) = 1$, for all $p \in B(x_k, \epsilon_{m+1})$.

Under such assumptions, the block variables $\psi_m(k; f)$ can be written recursively as

$$\psi_0(k; f) = \int_{B(x_k,\epsilon_0)} \xi_k f \, d\mu_g \ ,$$
$$\psi_{m+1}(h; f) = \sum_k^{N_h(m,m+1)} \psi_m(k; f) \ . \tag{20}$$

Indeed, we have

$$\begin{aligned} \psi_1(h; f) &= \int_{B(y_h,\epsilon_1)} f \, d\mu_g \\ &= \sum_k^{N_h(0,1)} \int_{B(x_k,\epsilon_0)} \xi_k f \, d\mu_g \\ &= \sum_k^{N_h(0,1)} \psi_0(k; f) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \psi_2(j; f) &= \int_{B(z_j,\epsilon_2)} f \, d\mu_g \\ &= \sum_h^{N_j(1,2)} \int_{B(y_h,\epsilon_1)} \xi_h f \, d\mu_g \\ &= \sum_h^{N_j(1,2)} \sum_k^{N_h(0,1)} \int_{B(x_k,\epsilon_0)} \xi_k \xi_h f \, d\mu_g \\ &= \sum_h^{N_h(1,2)} \psi_1(h; f) \ , \end{aligned} \tag{22}$$

where we have exploited the fact that the functions $\{\xi_k \xi_i\}(p) = 0$, except for a finite number of indices (k, i), and $\sum_k \sum_i \xi_k \xi_i(p) = 1$, for all $p \in \Sigma$. The above expressions readily generalized for every m (see Fig. 5).

More explicitly, we can write these block variables as

$$\begin{aligned} \psi_{m+1}(k; f) &= \int_{B(x_k,\epsilon_{m+1})} f \, d\mu_g \\ &= \frac{\omega_n \epsilon_m^n}{\sum_j \lambda_j} \sum_h^{N_k(m,m+1)} \left[f(h) + \left(\frac{\Delta(f)(h) - R(h) f(h)/3}{2(n+2)} \right) \epsilon_m^2 + o(\epsilon_m^4) \right] \ . \end{aligned} \tag{23}$$

Notice that in terms of the block variables $\psi_m(k; f)$ we can rewrite the empirical averaging Eq. (2) as

$$\langle f\rangle_{\epsilon_m}(\psi_m)=\frac{\sum_k^{N(\epsilon_m)}\psi_m(k;f)}{\sum_k^{N(\epsilon_m)}\mathrm{vol}[B(x_k,\epsilon_m)]}\ , \tag{24}$$

where $N(\epsilon_m)$ denotes the number of distinct $B(x_k,\epsilon_m)$ balls providing a minimal ϵ_m covering of the manifold Σ. Thus, when m is sufficiently large, the variation in $\langle f\rangle_{\epsilon_m}(\psi_m)$ under a block transformation $\psi_m(k;f)\to\psi_{m+1}(h;f)$ is given (to leading order) by

$$\langle f\rangle_{\epsilon_{m+1}}(\psi_{m+1})-\langle f\rangle_{\epsilon_m}(\psi_m)\simeq\frac{1}{n+2}\langle\Delta f\rangle_{\epsilon_m}\epsilon_m^2\frac{1}{m+1}+\frac{1}{3(n+2)}[\langle R\rangle_{\epsilon_m}(f)_{\epsilon_m}-\langle Rf\rangle_{\epsilon_m}]\epsilon_m^2\frac{1}{m+1}\ . \tag{25}$$

The above choice of block variables brings out the coupling between averaging a scalar field over a manifold and the presence of fluctuations in the curvature of the underlying geometry.

In order to be more precise, let us assume that the variables $f(k)$ are randomly distributed according to some probability law $P(\{f(k)\})$ (later on we shall come back to this point with a definite prescription). Upon blocking the system and thus renormalization the variables $f(k)$ by increasing the scale size, the probability distribution $P(\{f(k)\})$ induces a corresponding probability distribution on the variables ψ_m, viz., $P(\{\psi_m(k;f)\})$.

From Eq. (20) it is clear that if the geometrical properties of any two balls, $B(x_i,\epsilon_{\bar m})$ and $B(x_j,\epsilon_{\bar m})$ [with $B(x_i,\epsilon_{\bar m})\cap B(x_j,\epsilon_{\bar m})=\emptyset$] are not correlated, then the corresponding block variables $\psi_m(i;f)$ and $\psi_{\bar m}(j;f)$ are uncorrelated. Such length scale $L\equiv\epsilon_{\bar m}$ characterizes the correlation (or persistence) length of the manifold (Σ,g). It is a measure of the typical linear dimension of the largest ball exhibiting a correlated spatial structures. This correlation length can be seen in close analogy with the usual correlation length in condensed matter systems. It depends there upon the coupling constants, in particular upon temperature, and diverges to infinity at the phase transition point.

Since g plays here the role of a running coupling, or if you prefer, of "temperature," the existence of a finite correlation length corresponds to a rather "irregular," crumpled geometry (as seen on scales of the order of L), or, equivalently, a *high temperature phase* of our system.

According to the central limit theorem it follows, for m large enough ($\epsilon_m\gg L$), that the block variables $\psi_m(k;f)$, being the sum of uncorrelated variables, are normally distributed (let us say around zero, for simplicity) with a variance

$$E_P(\psi_m^2(i;f))=N^{(\epsilon_0,m)}\tilde\chi\ , \tag{26}$$

where $E_P(\cdots)$ denotes the expectation according to the probability law $P(\{\psi_m\})$, $\tilde\chi$ is related to the variance of the variables $f(k),R(k)$; $N^{(\epsilon_0,m)}$ denotes the number of ϵ_0 balls in the m ball (x_i,ϵ_m).

Thus, irrespective of the details of the local distribution of the random variables $f(k)$ and $R(k)$, we can write, for the distribution of $\{\psi_m(k;f)\}_1^N$ over (Σ,g),

$$dP(\{\psi_m\})=\prod_k\{d\psi_m(k;f)(2\pi N^{(\epsilon_0,m)}\tilde\chi)^{-1/2}\times\exp[-\psi_m^2(k;f)/2N^{(\epsilon_0,m)}\tilde\chi]\}\ . \tag{27}$$

This shows that by rescaling the block variables $\psi_m(k;f)$ according to

$$\phi_m(k;f)\equiv[N^{(\epsilon_0,m)}]^{-1/2}\psi_m(k;f)\ , \tag{28}$$

we get new block variables with a finite variance as $m\to\infty$, and for *random* metrics (Σ,g) we can write (see Fig. 6)

$$dP(\{\phi_m\})=\prod_k\{d\phi_m(k;f)(2\pi\tilde\chi)^{-1/2}\times\exp[-\phi_m^2(k;f)/2\tilde\chi]\}\ . \tag{29}$$

The above remarks, paradigmatic of the real space renormalization group philosophy, show that the definition of a sensible blocking procedure, in our geometrical setting, consists of a transformation increasing the scale size, realized by passing from the variables $f(k)$ to the variables $\psi_m(k;f)$ (namely, by taking the average over all values of f in a larger and larger ball), followed by a rescaling obtained by dividing $\psi_m(k;f)$ by a suitable power of the number $N^{(\epsilon_0,m)}$ of elementary ϵ_0 balls contained in the ϵ_m ball considered (for random geometries this power is 1/2).

Following standard usage, and in order to arrive at an interesting geometrical notion of blocking, we assume that for a generic metric g this rescaling follows by dividing $\psi_m(k;f)$ by $[N^{(\epsilon_0,m)}]^{\omega_m}$, where ω_m will in general depend on m. Thus, the rescaled blocked variables of relevance are

$$\phi_m(k;f)=[N^{(\epsilon_0,m)}]^{-\omega_m}\psi_m(k;f)\ . \tag{30}$$

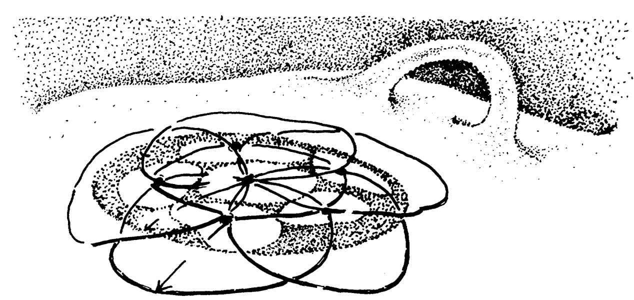

FIG. 5. The intersection pattern of a 2ϵ ball $B(x_j,2\epsilon)$ with five ϵ balls $B(x_h,\epsilon)$. The 2ϵ ball is represented by a dotted disk, while the ϵ balls are represented by solid circles. The white disks represent the packing of $B(x_j,2\epsilon)$ with $\epsilon/2$ balls. The standard representation of the integral over $B(x_j,2\epsilon)$ through the partition of unity subordinated to the $B(x_h,\epsilon)$ covering, yields the recursive relation $\psi_{m+1}(j)=\sum_h\psi_m(h)$.

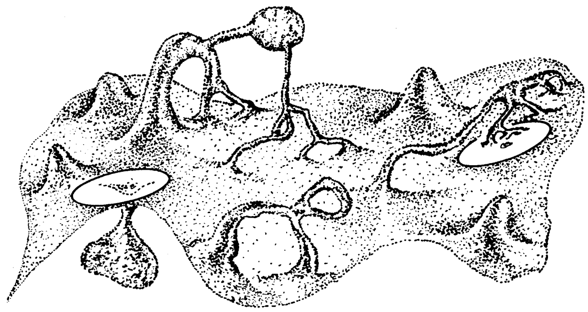

FIG. 6. Even if the function f is constant over Σ, its average values over the geodesic balls, $B(x_1,\epsilon)$ and $B(x_2,\epsilon)$, are generally uncorrelated owing to the random fluctuations in the geometry of the balls.

The value of ω_m will be fixed by the requirement that, as $m\to\infty$, and for some (critical) metric $g_{\rm crit}$ (in general for an open set of such metrics), such normalized large scale block variables have a limiting probability distributions with a finite variance. Namely,

$$\lim_{m\to\infty} P_m(\{\phi_m(k;f)\}) = P_\infty(\{\phi_\infty(f)\}) ,$$
$$E_{P\infty}(\phi^2_\infty(f)) = 1 . \qquad (31)$$

Notice that if (Σ,g) is a *nice* manifold,e.g., a constant curvature simply connected three-manifold, we are obviously expecting that the corresponding ω_m is

$$\omega_m(\Sigma,g)=1 . \qquad (32)$$

In general, we can assume that there is a set of critical metrics and a corresponding ω, such that the above requirements Eq. (31) are satisfied. Such critical metrics are not necessarily constant curvature metrics and the corresponding ω is not necessarily 1 (one may conjecture that $1/2\le\omega\le 1$, as happens in the renormalization group analysis of many magnetic systems). At this stage, this is only a tentative assumption in order to arrive at an interesting concept of geometric renormalization group in our setting. Later on, we shall see that such assumptions are justified by exhibiting examples of such nontrivial metrics.

IV. AVERAGING MATTER AND GEOMETRY

Until now, our discussion has addressed mainly geometrical issues and the function f entering the cutoff dependent averaging $\langle f\rangle_\epsilon$ was not specified. Now we wish to apply the obtained results to the averaging of the matter sources, namely, the matter density ρ, the spatial stress tensor s_{ab}, and the momentum density J_a, entering in the phenomenological description of the matter energy-momentum tensor with respect to the instantaneous observers comoving with Σ, viz.,

$$T_{ab}=\rho u_a u_b + J_a u_b + J_b u_a + s_{ab} , \qquad (33)$$

where $\mathbf{u}$ is a unit, future directed normal to the slice $\Sigma(u^a u_a=-1)$.

In fact, since in the present epoch the universe is mainly matter dominated, i.e., the pressure can be safely neglected, it would be justified to start our analysis with

$$T_{ab}=\rho u_a u_b + J_a u_b + J_b u_a , \qquad (34)$$

for the stress-energy tensor. Even the terms involving the momentum density can be eliminated if we pick up a sensible slicing Σ_t (the comoving frame), provided that the cosmological matter fluid is an irrotational fluid in equilibrium. In any case, wishing to maintain the following discussion to a sufficient degree of generality, without particular restrictive assumptions on the matter sources, we assume in our analysis that the matter energy-momentum tensor has the perfect fluid form

$$T_{ab}=\rho u_a u_b + J_a u_b + J_b u_a + g_{ab}p , \qquad (35)$$

with a pressure p which is *a priori* not vanishing, and where g_{ab} is the three-metric of Σ, namely, $g_{ab}=g^{(4)}_{ab}+n_a n_b, g^{(4)}_{ab}$ denoting the space-time metric.[5]

As argued in the preceding section, smoothing out the matter sources as described by a set of instantaneous observers (represented by the three-dimensional hypersurface Σ) means eliminating from the distribution of such sources on Σ all fluctuations on scales smaller than the cutoff distance ϵ, leaving an effective probability distribution of fluctuations for the remaining degrees of freedom. The underlying philosophy is that this effective distribution has the same properties as the original one at distances much larger than ϵ (i.e., for fluctuations with wavelengths much larger than ϵ).

In order to implement this idea along the geometrical lines discussed in the preceding section, we first need to specify better what we mean by the assignment of a collection of instantaneous observers (endowed with clocks) on Σ. Since we are adopting a Hamiltonian point of view, such observers are specified by the assignment, on the (abstract) three-manifold Σ, of the lapse function α and the shift vector field α^i. The former provides the local rate of the coordinate clocks of such observers, while the latter is the three-velocity vector of the observers with respect to the set of instantaneous observers at rest on Σ.

The macroscopic variables of interest characterizing the matter sources are in the present framework, the matter density ρ, and the momentum density $\mathbf{J}$. Actually, it would make more sense to consider the cosmological fluid phenomenologically, described by a pressure p, a baryon number density n_b, energy density ρ, specific entropy s, and temperature T; such variables being related by

[5]The fact that we are adopting a stress tensor with a nonvanishing energy flux can be traced back to the observation that even in an exact $k=0$ or $k=+1$ FLRW universe there can be a nonzero particle flux tilted with respect to the frame of the fundamental observers, even though T_{ab} has the perfect fluid form. This is possible if the macroscopic quantities (e.g., a particle flux) are derived from kinetic theory with anisotropic distribution function $f(x^i,p^a)$ [19].

$$dp = n_b dh - n_b T\, ds \tag{36}$$

and

$$h = \frac{p+\rho}{n_b}\ , \tag{37}$$

where h is the specific enthalpy. However, for simplicity, we shall in the sequel consider a barotropic fluid.

Thus, the field f characterizing the matter sources, as described by the instantaneous observers on Σ, is given by

$$f \equiv \alpha\rho + \alpha^i J_i\ . \tag{38}$$

Notice that $2\alpha p(\rho)$ is the Hamiltonian density of the fluid in the Hamiltonian formulation of Taub's variational principle for relativistic perfect fluids [20]; we also assume that the *dominant energy condition* holds (see Fig. 7).

The averaging of the matter sources along the lines described in the preceding section would then require considering a finite set of instantaneous observers, $\{x, \ldots, x_N\}$, located on Σ and setting a standard for the cutoff distance ϵ_0 over which the (experimental) distribution of matter sources [i.e., the probability that the matter variables, $\rho(i)$ and $\mathbf{J}(i)$, conform to a given distribution $\rho(i)d\rho, \mathbf{J}(i)d\mathbf{J}$] is determined. Then we proceed with the blocking prescription for eliminating unwanted degrees of freedom and consider the behavior as the averaging regions become larger and larger.

We may decide to treat the Riemannian geometry of Σ as uncoupled with the matter sources if we are simply interested in smoothing out the sources, or if we wish to consider fluctuations of the matter as essentially uncoupled to the fluctuations in geometry. Namely, the curvature fluctuations appearing in the definitions of the block variables, $\psi_m(k; f)$ and $\phi_m(k; f)$, can be thought of, under appropriate circumstances, as independent random variables with a given distribution.

In general, however, the sources are coupled to the gravitational field, and we ought to treat the full dynamical system, the cosmological fluid plus the geometry of Σ, in the procedure of blocking. Moreover, we should bear in mind that without taking explicitly into account

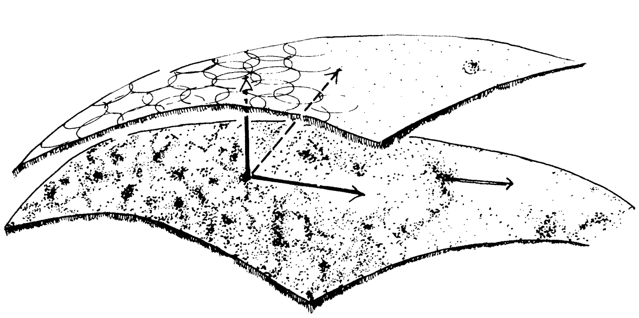

FIG. 7. The matter distribution $\rho, \mathbf{J}$, the lapse function α, and the shift vector field α associated with the instantaneous observers on Σ. Here, $\mathbf{n}$ denotes the four-dimensional unit, future-pointing, normal to Σ. The vector field $\alpha\mathbf{n}+\alpha$ connects the corresponding points in the two infinitesimally close slices Σ and $\Sigma_{\delta t}$.

the backreaction of the geometry, one cannot really provide a reasonable averaging procedure for the sources.

In order to do so, we consider for a given cutoff distance ϵ_0, the variables $\rho(k)$ and $\mathbf{J}(k)$ associated with a minimal geodesic ball covering $\{B(x_k, \epsilon_0)\}_{k=1}^N$. The change of the cutoff is naturally realized by considering balls $\{B(x_l, \epsilon_m = (m+1)\epsilon_0)\}$ with $m = 0, 1, \ldots$. The block variables, $\psi_0(k; \rho, \mathbf{J})$ and $\psi_m(k; \rho, \mathbf{J})$, are defined to Eq. (20) with f given by Eq. (38). This transformation can be seen as thinning out the degrees of freedom which is at the heart of any coarse graining.

The original cutoff ϵ_0 is chosen to set the scale over which general relativity is experimentally verified and it can be taken as, say, the scale of planetary systems. We can thus safely assume that the Einstein field equations hold on that scale. It is, however, rather impossible to provide a mathematical model of the distribution of matter in the universe going down to such fine scales; besides this task would be impractical. What one does instead is to use continuous functions assuming that they represent appropriately "volume averages." The results of such an averaging in an inhomogeneous medium obviously depend on the scale. The point is that if the Einstein equations hold on the scale where they have been verified (here taken to be that of the planetary scale), then they do not seem to hold *a priori* on larger, cosmological, scales that require averaging. To see this, note that the Einstein tensor $\tilde{G}_{\mu\nu}$, calculated from an "averaged" metric $\bar{g}_{\mu\nu}$, cannot be equal to the Einstein tensor $\bar{G}_{\mu\nu}$ which was first calculated from the fine-scale metric $g_{\mu\nu}$ and then averaged. This is so due to the noncommutativity of "averaging the metric" with calculating the Einstein tensor being strongly nonlinear in the metric components.

Below we shall assume a Hamiltonian point of view. Then the probability $P(\{\rho(k), \mathbf{J}(k)\})$ that the matter variables, according to the records of the instantaneous observers $\{B(x_k, \epsilon); \alpha(x_k), \alpha^i(x_k)\}$ in Σ, take on some particular set of values $\{\bar{\rho}(k), \bar{\mathbf{J}}(k)\}$ is given by the equation (we set $c = 1$, and numerical factors are arranged to get Einstein's gravitational field equations in the form $R^{(4)}_{ab} - \frac{1}{2} g^4_{ab} R^{(4)} = 8\pi G T_{ab}$ where G is Newton's constant),

$$P(\{\psi_0(k; \rho, \mathbf{J})\}_{1,\ldots,N}) = \frac{1}{Z} \exp[-H(\{\rho(k), \mathbf{J}(k)\})]\ , \tag{39}$$

where Z is a normalization factor, and $H(\{\rho(k), J(k)\})$ is the Hamiltonian associated with the matter variables ρ and $\mathbf{J}$. Namely,

$$P(\{\psi_0(k; \rho, \mathbf{J})\}_{1,\ldots,N}) = \frac{\exp[-\langle \alpha\rho + \alpha^h J_h \rangle_{\epsilon_0}]}{\sum_{\rho,J} \exp[-\langle \alpha\rho + \alpha^h J_h \rangle_{\epsilon_0}]}\ , \tag{40}$$

where $\sum_{\rho,J} \cdots$ denotes summation over all possible configurations of the matter variables ρ and $\mathbf{J}$ that can be experienced by the instantaneous observers, $\{B(x_k, \epsilon); \alpha(x_k),\ \alpha^i(x_k)\}$ in Σ.

The validity of general relativity at this scale implies that this Hamiltonian is a part of $\mathcal{H}_{\rm ADM}(\{\rho(k),\mathbf{J}(k)\})$, the Arnowitt-Deser-Misner (ADM) Hamiltonian, associated with the data of the three-geometry g_{ab} of Σ, its conjugate momentum π_{ab} and of the matter variables ρ and $\mathbf{J}$, evaluated in correspondence with the $\{B(x_k,\epsilon)\}$ approximation associated with the net of points $\{x_k\}$. Namely,

$$\mathcal{H}_{\rm ADM}(\{\rho(k),\mathbf{J}(k)\})|_\epsilon = \langle \alpha\mathcal{H}(g,\pi,G,\rho)+\alpha^h\mathcal{H}_h(g,\pi,G,\mathbf{J})\rangle_\epsilon \sum_i \mathrm{vol}(B(x_i,\epsilon)) \ , \tag{41}$$

where

$$\mathcal{H}(g,\pi,G,\rho)=[\det(g)]^{-1/2}[\pi^{ab}\pi_{ab}-\tfrac{1}{2}(\pi^a_a)^2]-[\det(g)]^{1/2}R(g)+8\pi[\det(g)]^{1/2}G\rho \tag{42}$$

and

$$\mathcal{H}_a(g,\pi,G,\mathbf{J})=-2\pi^b_{a;b}+16\pi[\det(g)]^{1/2}GJ_a \ , \tag{43}$$

and where the momentum conjugate to the three-metric g_{ab} is given in terms of the second fundamental form K_{ab} of the embedding of Σ in the resulting space-time, as

$$\pi^{ab}=[\det(g)]^{1/2}(K^{ab}-K^c_c g^{ab}) \ . \tag{44}$$

We wish to recall that the lapse and the shift appear in the Hamiltonian as arbitrary Lagrange multipliers (their evolution is not specified by the equations of motion), and as such they enforce the constraints

$$\mathcal{H}(g,\pi,G,\rho)=[\det(g)]^{-1/2}\left[\pi^{ab}\pi_{ab}-\frac{1}{2}(\pi^a_a)^2\right]-[\det(g)]^{1/2}R(g)+8\pi[\det(g)]^{1/2}G\rho=0 \tag{45}$$

and

$$\mathcal{H}_a(g,\pi,G,\mathbf{J})=-2\pi^b_{a;b}+16\pi(\det(g))^{1/2}GJ_a=0 \ . \tag{46}$$

As is well known, these constraints are related to the invariance of the theory under the (four-dimensional) diffeomorphisms group of the space-time resulting from the evolution of the initial data satisfying them. The momentum constraint $\mathcal{H}_a(g,\pi,G,\mathbf{J})=0$ generates the (spatial) diffeomorphisms into Σ, while the Hamiltonian constraint $\mathcal{H}(g,\pi,G,\rho)=0$ generates the deformation of the manifold Σ in the resulting space-time, i.e., the dynamics. (Note that such deformations can be interpreted as four-dimensional diffeomorphisms only after space-time has been actually constructed.)

The Hamiltonian $\mathcal{H}_{\rm ADM}(\{\rho(k),\mathbf{J}(k)\})$, apart from the lapse and the shift, depends on the three-metric g_{ab} on Σ and its conjugate momentum π_{ab}, the gravitational coupling G, and on the configuration which the matter variables, $\rho(k)$ and $\mathbf{J}(k)$, take on the set of instantaneous observers, $\{B(x_k,\epsilon);\alpha(x_k),\alpha^i(x_k)\}$, chosen to describe the distribution of matter at the given length scale ϵ.

The basic question to understand is how the block transformations

$$\{\phi_m(k;\rho,\mathbf{J})\}\to\{\psi_{m+1}(k;\rho,\mathbf{J})\} \ , \tag{47}$$

followed by rescaling, affect the Hamiltonian associated with the matter variables ρ and $\mathbf{J}$, and then discuss how this in turn affects the full Hamiltonian $\mathcal{H}_{\rm ADM}$.

Starting from the probability distribution $P(\{\psi_0(k;\rho,\mathbf{J})\}_{1,\ldots,N})$ we can inductively define the probability distribution $P^{(m+1)}(\{\psi_{m+1}(k;\rho,\mathbf{J})\})$ of the block variables $\{\psi_{m+1}(k;\rho,\mathbf{J})\}$ (and of the corresponding rescaled variables $\{\psi_{m+1}(k;\rho,\mathbf{J})\}$). Since the block variables $\{\psi_{m+1}(k;\rho,\mathbf{J})\}$ are recursively obtained from the knowledge of the block variables at the mth stage, $\{\psi_m(k;\rho,\mathbf{J})\}$, the distribution $P^{(m+1)}(\{\psi_{m+1}(k;\rho,\mathbf{J})\})$ only depends on the knowledge of $P^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})$. We can formally write

$$P^{m+1}(\{\psi_{m+1}(k;\rho,\mathbf{J})\})=\sum_{\{\psi_m(k;\rho,\mathbf{J})}P^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\}) \ , \tag{48}$$

where the sum is over the probabilities of all the configurations $\{\psi_m(k;\rho,\mathbf{J})\}$ consistent with the configuration $\{\psi_{m+1}(k;\rho,\mathbf{J})\}$ of the block variables.

As usual, this allows us to define the effective Hamiltonian for the matter variables after m iterations of the block transformation, according to

$$P^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})\equiv\frac{1}{Z^{(m)}}\exp[-H^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})], \tag{49}$$

where

$$Z^{(m)}\equiv\sum_{\{\psi_m(k;\rho,\mathbf{J})\}}\exp[-H^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})] \ . \tag{50}$$

Such $H^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})$ are defined up to an additive constant term (e.g., [18]). If we also stipulate, as is standard usage in the renormalization group approach, that the effective Hamiltonian $H^{(m+1)}(\{\psi_{m+1}(k;\rho,\mathbf{J})\})$ takes the same functional form as $H^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})$, i.e., if

$$P^{m+1}(\{\psi_{m+1}(k;\rho,\mathbf{J})\}_{1,\ldots,N})$$
$$= \frac{\exp[-\langle\alpha\rho^{(m+1)} + \alpha^h J_h^{(m+1)}\rangle_{\epsilon_{m+1}}}{\sum_{\rho,J}\exp[-\langle\alpha\rho^{(m+1)} + \alpha^h J_h^{(m+1)}\rangle_{\epsilon_{m+1}}}\ , \quad (51)$$

then this indeterminacy can be transferred to the *effective* matter variables $\rho^{(m+1)}$ and $\mathbf{J}^{(m+1)}$, in terms of which $H^{(m+1)}(\{\psi_{m+1}(k;\rho,\mathbf{J})\})$ is defined.

It is immediately checked that such effective matter variables are defined by Eq. (49) up to the transformations

$$\rho^{(m+1)} \to \rho^{(m+1)} + h\ ,$$
$$\mathbf{J}^{(m+1)} \to \mathbf{J}^{(m+1)} + \mathbf{v}\ , \quad (52)$$

where h and $\mathbf{v}$ are, respectively, a scalar function and a vector field defined on Σ.

Among the various normalization conditions that we may adopt, in order to avoid the indeterminacy connected with Eq. (52), the natural one comes about *by requiring the divergence and the Hamiltonian constraints to hold at each stage of the renormalization.* This being the case, upon coupling matter to the geometry, the lapse α and the shift α^i maintain their role of Lagrange multipliers enforcing the (four-dimensional) diffeomorphism invariance of the theory.

In this way, the renormalization is set up so that the constraints, if initially satisfied, will remain satisfied at every renormalization step. The three-metric g (and the second fundamental form), in its role of *running coupling* describing the interaction between matter and geometry, is then governed through (Coulomb-like) effects that are expressed by these constraint equations. In this sense, the Coulomb-like part of the gravitational interaction is driving the renormalization mechanism, and such requirements imply that the full effective Hamiltonian (matter plus geometry) takes on the standard ADM form pertaining to gravity interacting with a barotropic fluid at every stage of the renormalization (see Fig. 8).

According to the remarks above, we can consider as an independent parameter in the (effective) fluid Hamiltonian $H^{(m)}(\{\psi_m(k;\rho,\mathbf{J})\})$ the three-metric $g_{ab}^{(m)}$ of the three-manifold Σ, whereas the matter density $\rho^{(m)}$ and the current density $\mathbf{J}^{(m)}$ are at each stage connected to $g_{ab}^{(m)}$ and $K_{ab}^{(m)}$ by the Hamiltonian and divergence constraints that hold at each stage. Then the effect of the renormalization induced by the blocking procedure $\{\psi_m(k;\rho,\mathbf{J})\} \to \{\psi_{m+1}(k;\rho,\mathbf{J})\}$ and the corresponding rescaling can be symbolized as a nonlinear operation acting on the metric $(g_{ab}^{(m)})$ so as to produce the metric $(g_{(ab)}^{(m+1)})$, i.e.,

$$(g_{(ab)}^{(m+1)}) = \mathcal{R}(g_{(ab)}^{(m)})\ , \quad (53)$$

whereas the renormalization of the second fundamental form $K_{(ab)}$ is generated by the *linearization* of Eq. (53), as will be discussed in Sec. V.

This way, this normalization transformation $\mathcal{R}$ defines a trajectory in the space of Riemannian metrics of Σ.

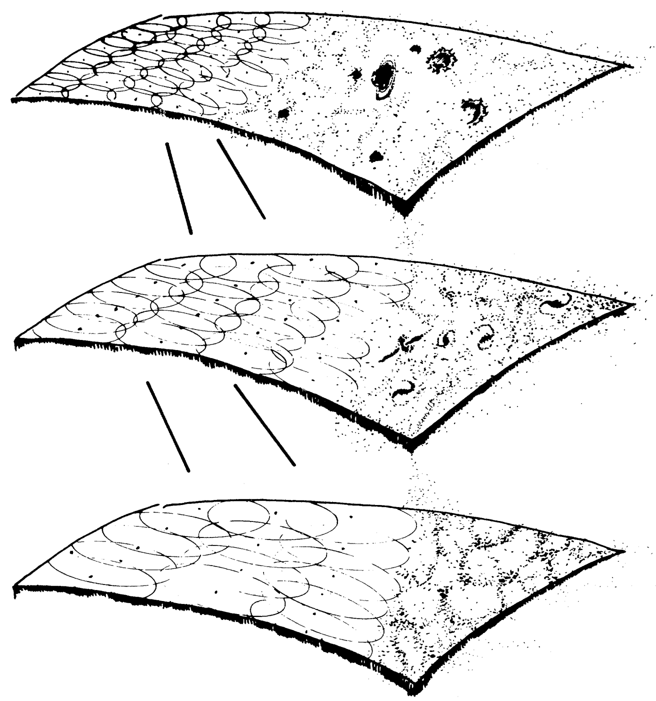

FIG. 8. Given a probability law, according to which matter is distributed at a given length scale (say, the planetary scale), we can get the corresponding probability distribution obtained by averaging the matter variables over regions of ever increasing scales. The resulting *effective* Hamiltonians are defined up to additive constants which affect the renormalized mass density and the renormalized momentum density. Such indeterminacy can be naturally removed by enforcing the Hamiltonian and divergence constraints at each step of the renormalization procedure.

One moves on such trajectory by discrete jumps. However, in what follows we shall replace such discrete dynamical system by a smooth dynamical system, describing renormalization of the parameters (g_{ab}) and (K_{ab}).

The deformation of the initial data set for the Einstein field equations, symbolically denoted by $\mathcal{R}$ above in Eq. (53), realizes in fact a *formal* (at least at this stage) mapping between the initial data sets for the field equations. As we have seen above, the renormalization acts in such a way that at each step the deformed data satisfy the constraints. The time evolution, in turn, of any such data set generates a one-parameter family of solutions to the field equations which interpolates between the initial inhomogeneous space-time to be averaged-out and its renormalized (deformed) counterpart.

A. The Ricci-Hamilton flow

In order to replace the discrete $\mathcal{R}$, describing the effect of renormalization of g_{ab}, with a continuous flow we can start by discussing some geometrical implications of Eq. (17). According to the renormalization group analysis of the preceding section, they follow by considering the average $\langle f \rangle_\epsilon$ as a functional of the metric and thinking of the metric g_{ab} as a running coupling constant, depending on the cutoff. In this connection, it can be verified that we can equivalently interpret Eq. (17) as obtained by considering the variation of $\langle f \rangle_\epsilon$ under a suitable smooth deformation of the background metric g_{ab}, rather than by deforming the (Euclidean) radius of the balls $\{B(x_i, \epsilon)\}$ (see Fig. 9).

As a matter of fact, we can equivalently rewrite the second term on the right hand side of Eq. (17) as

$$\langle R \rangle_\epsilon \langle f \rangle_\epsilon - \langle R f \rangle_\epsilon = -D \langle f \rangle_\epsilon \frac{\partial g_{(ab)}}{\partial \eta} , \tag{54}$$

where $D\langle f \rangle_\epsilon \partial g_{ab}/\partial \eta$ denotes the formal linearization of the functional $\langle f \rangle_\epsilon$ in the direction of the symmetric two-tensor $\partial g_{ab}/\partial \eta$, and where

$$\frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} \langle R(\eta) \rangle g_{ab}(\eta) - 2R_{ab}(\eta) , \tag{55}$$

$R_{ab}(\eta)$ being the components of the Ricci tensor $\mathrm{Ric}(g(\eta))$, and $\langle R(\eta) \rangle$ is the average scalar curvature given by

$$\langle R(\eta) \rangle = \frac{1}{\mathrm{vol}(\Sigma)} \int_\Sigma R(\eta) d\mu_\eta . \tag{56}$$

Indeed, the linearization of $\langle f \rangle_\epsilon$ in the direction of the generic two-tensor $\partial g_{ab}/\partial \eta$ is provided by

$$D\langle f \rangle_\epsilon \frac{\partial g_{ab}}{\partial \eta} = \frac{1}{2} \left\langle f g^{ab} \frac{\partial}{\partial \eta} g_{ab} \right\rangle_\epsilon - \frac{1}{2} \langle f \rangle_\epsilon \left\langle g^{ab} \frac{\partial}{\partial \eta} g_{ab} \right\rangle_\epsilon , \tag{57}$$

so Eq. (54) follows, given the expression Eq. (55) for $\partial g_{ab}/\partial \eta$.

According to what has been said in the previous paragraphs, the effective distribution of matter sources, according to a set of instantaneous observers $\{B(x_i, \epsilon)\}$, is characterized by the underlying three-geometry thought of as an effective parameter depending on the cutoff ϵ. Since the Hamiltonian and the divergence constraints hold at each stage of the renormalization procedure (they fix the effective Hamiltonian which otherwise is undetermined up to a constant factor), the renormalization of the matter fields is intrinsically tied with the renormalization of the three-metric.

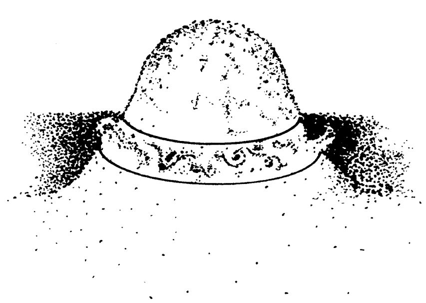

FIG. 9. Upon enlarging the ball, from ϵ to $e^\gamma \epsilon$, we alter the average value of the matter variables ρ and $\mathbf{J}$ (even if ρ and $\mathbf{J}$ do not vary appreciably), since there can be strong curvature fluctuations in the annulus $B(x, e^\gamma \epsilon) \backslash B(x, \epsilon)$.

The invariance of the long distance properties of the matter distribution, under simultaneous change of the cutoff ϵ and the parameter g_{ab}, can be expressed as a differential equation for the effective Hamiltonian $\mathcal{H}(\rho, \mathbf{J})$ (actually for the partition function associated with this effective Hamiltonian), namely,

$$\left(-\epsilon \frac{\partial}{\partial \epsilon} + \beta_{ab}(g) \frac{\partial}{\partial g_{ab}} \right) \sum_{\rho, J} \exp[-\mathcal{H}(\rho, \mathbf{J})] = 0 \tag{58}$$

where $\beta_{ab}(g) \equiv \epsilon \partial / \partial \epsilon g_{ab}$ is the β function associated with (58).

Recall that $\mathcal{H}(\rho, \mathbf{J})$ is explicitly provided by (in the given approximation)

$$\mathcal{H}(\rho, \mathbf{J}) = \langle \alpha p(\rho) + \alpha^i J_i \rangle_\epsilon \sum_h \mathrm{vol}(B(x_h, \epsilon)) , \tag{59}$$

where the average $\langle \alpha p(\rho) + \alpha^i J_i \rangle_\epsilon$ is a functional of the three-metric g_{ab}, here thought of as the running coupling constant.

Thus, in order that Eq. (58) is satisfied, it is *sufficient* that

$$\left(-\epsilon \frac{\partial}{\partial \epsilon} + \beta_{ab}(g) \frac{\partial}{\partial g_{ab}} \right) \langle \alpha p(\rho) + \alpha^i J_i \rangle_\epsilon = 0 . \tag{60}$$

[Note that $\sum_h \mathrm{vol}(B(x_h, \epsilon))$ remains invariant under (55).]

The renormalization group Eq. (60) states that increasing the cutoff length (i.e., the radius of the averaging balls) from ϵ to $e^\gamma \epsilon$, while deforming the metric g_{ab} by flowing along the β function $\beta_{ab}(g)$ for a parameter time γ, has no net effect on the long distance properties of the considered system (see Fig. 10).

According to the above remarks, and Eq. (24), it can be verified that the β function yielding for Eq. (60), defined by

$$\epsilon \frac{\partial}{\partial \epsilon} g_{ab} = \beta_{ab}(g) , \tag{61}$$

is exactly provided by Eq. (55), namely,

$$\beta_{ab}(g) = \frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} \langle R(\eta) \rangle g_{ab}(\eta) - 2R_{ab}(\eta) , \tag{62}$$

where the parameter η is the logarithmic change of the cutoff length ϵ.

Since the manifold Σ is compact, Eq. (55) has to be interpreted as a renormalization group equation in a finite geometry, and thus the relevant phenomena are here related to finite size scaling (see, e.g., [21]). A continuous theory, describing the (universal) properties of the cosmological sources and of the corresponding geometry, may arise when the correlation length associated with the distribution of cosmological matter is of the order of the size of the underlying manifold.

The metric flow Eq. (55) is known as the Ricci-Hamilton flow [22], studied in connection with the quasiparabolic flows on manifolds; quite independently it has

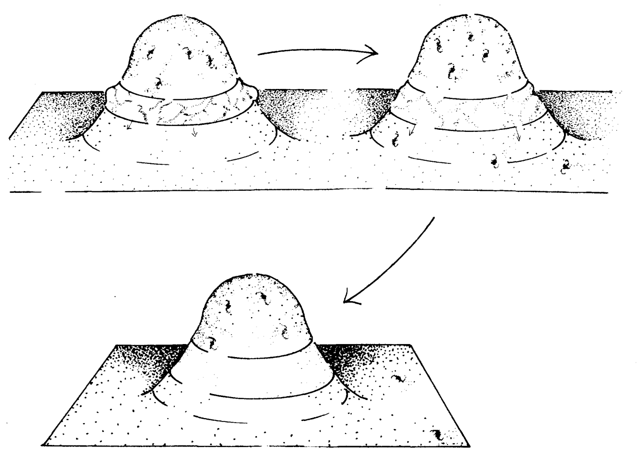

FIG. 10. If we enlarge the ball from ϵ to $e^{\eta}\epsilon$, while deforming the metric according to the Ricci-Hamilton flow Eq. (55) (for a *parameter time* $\eta = \gamma$), then the average value remains *as constant as it can be* since Eq. (55) smooths the curvature fluctuations in the annulus $B(x, e^{\gamma}\epsilon)\backslash B(x, \epsilon)$.

been discussed in investigating the renormalization group flow for general σ models (see, e.g., [23] and references quoted therein). The Ricci-Hamilton flow is always solvable [22] for sufficiently small η and has a number of useful properties (apart from being volume preserving, which is simply a consequence of the normalization chosen), namely, any symmetries of $g_{ab}(\eta_0)$ are preserved along the $g_{ab}(\eta)$ flow for all $\eta > \eta_0$, and the limiting metric (if attained) $\bar{g}_{ab} = \lim_{\eta\to\infty} g_{ab}(\eta)$ has constant sectional curvature. Thus Eq. (55), with the initial condition $g_{ab}(0) = g_{ab}$, defines (when globally solvable) a smooth family of deformations of the initial three-manifold, deforming it into a three-space of constant curvature.

B. Fixed points and basins of attraction

The point of the above discussion is that in order to arrive at a *fixed point* of the RG Eq. (17), the geometry has to be deformed according to the Ricci-Hamilton flow Eq. (55). In this setting of the problem, the Ricci-Hamilton equations appear naturally and in fact the approach proposed enables us to attain a physical meaning to them within the coarse-graining picture. This element was lacking in [3] where Hamilton's theorem appeared rather *ad hoc*. On the other hand, our approach demonstrates that the smoothing issue is deeply connected with the geometry and exhibits how this relationship works.

First some general remarks. A fixed point is a point in the coupling constants space that satisfies

$$g^*_{ab} = \mathcal{R}(g^*_{ab}) \ ; \tag{63}$$

i.e., it is mapped onto itself by RG transformation.

Under RG transformation length scales are reduced by a factor $m+1$. Namely, for the block variables, the correlation length measured in units appropriate for them, $L_m(\epsilon_m)$, is smaller than the correlation length L_0 of the initial system measured in units of ϵ_0. The actual physical value of the correlation length L is of course unchanged by the process of blocking, thus $L = L_m(\epsilon_m) = L_0\epsilon_0$, so

$$L_m = \frac{L_0}{m+1} \ . \tag{64}$$

Since $L_m < L_0$, the system with Hamiltonian $\mathcal{H}^{(m)}$ must be further from criticality than the original system. Thus, we conclude that the system is at a new effective "temperature" $g^{(m)}_{ab}$. At a fixed point Eq. (63), L^* can be only zero or infinity since then $L^* \to L^*/(m+1)$.

As is standard in RG analysis, we will refer to a fixed point with $L = \infty$ as a *critical fixed point*; when $L = 0$ we will call it *trivial.* Each fixed point has its domain or *basin* of *attraction*, namely, the points in the coupling constants space in such a basin necessarily flow towards and end up at the fixed point, after an infinite number of iterations of RG transformation.

Let us, for the purpose of clarity, employ for a moment the ferromagnetic analogy. In this case, for a system exhibiting a phase transition, there are two attractive fixed points. One is the high-temperature fixed point which attracts each point with $T > T_{\rm crit}$ in the coupling constants space, and it corresponds to the effective Hamiltonian for the system as $T \to \infty$. In this phase the variables assume random values and are uncorrelated. Upon a sensible blocking of such a system the probability distribution of the block variables remains unchanged.

The second fixed point is the low-temperature fixed point which is the effective Hamiltonian for the system when $T \to 0$. This corresponds to a system in a complete spin alignment and the block variables are ordered then. Every point corresponding to $T < T_{\rm crit}$ is eventually attracted to this fixed point.

Across the Hamiltonian space there should exist then a surface, the so-called *critical surface*, which separates the effective Hamiltonians flowing to the high-temperature fixed point from those flowing to the low-temperature fixed point. Notice that the word *surface* here has rather a heuristic meaning. Indeed, the *set* of metrics, separating those flowing towards a "low-temperature" fixed point from those flowing towards a "high-temperature" fixed point, has quite a complex structure whose understanding is deeply connected with some, yet unsolved, conjectures in three-dimensional (3D) manifold topology (see below for more details).

If we now choose to start with a point on the critical surface, then upon RG transformation it will stay within the critical surface. There is a possibility that as the number of iterations of RG goes to infinity, the Hamiltonian will tend to a finite limit $\mathcal{H}^*$. This point is the critical fixed point and within the critical surface it is attractive (this is roughly speaking the basic mechanism for universality); along the direction out of the critical surface it is repulsive. This fixed point is related to the singular critical behavior of the system due to the fact that all points in its basin of attraction have infinite correlation length. The simplest case is when the fixed points are isolated points, but it is also possible to have lines or surfaces of fixed points.

With these preliminary remarks along the way, let us discuss how some of the above general characteristics of the renormalization group flows find their proper counterpart in the particular situation we are analyzing.

In the previous paragraphs we showed that it was possible to replace the discrete operation of increasing the scale size of the observational averaging region by a "transformation," smoothly deforming the background metric g_{ab}, which turned out to be the Ricci-Hamilton flow. In this setting, following the example above, we would like to adopt the fundamental hypothesis linking RG to the critical phenomena, namely, the existence of a "critical" metric (on the critical surface) $g_{ab}^{\rm crit}$, and of a "fixed point" metric g_{ab}^{*}, such that

$$\lim_{m\to\infty} \mathcal{R}^{(m)}(g_{ab}^{\rm crit}) = g_{ab}^{*} \,. \tag{65}$$

In Eq. (65) g_{ab}^{*} is a mathematical object invariant under RG and we assume that $g_{ab}^{\rm crit}$ represents the physics (in a sense to be clarified further) of curvature fluctuations of a manifold at its critical point ("critical manifold"). Since the Ricci-Hamilton flow can be interpreted as a dynamical system on a set of closed Riemannian manifolds, we can adopt the following interpretation of Eq. (65). Suppose, that we look at our manifold through a "microscope" and are able to discern the curvature fluctuations down to a size ϵ_m. Then $\mathcal{R}^{(m)}$ represents the operation of decreasing the magnification factor, by m, say; i.e., the sample seen appears to shrink by this factor. We have to assume that the system is sufficiently large so that the edges of the sample will not appear in the view. The hypothesis Eq. (65) states that if we decrease the magnification by a sufficiently large amount, we shall not see any change if we decrease it even further.

We are going to describe to what extent such hypothesis holds. We already said that the Ricci-Hamilton flow Eq. (55), while always solvable for sufficiently small η [22], may not provide a nonsingular solution as η increases. Hamilton noticed that there are patterns in the kind of singularities that may develop. Typically, the curvature blows up, but in a very regular way (e.g., for $S^1 \times S^2$ with the standard symmetric metric). This has led him to a research program which, roughly speaking, amounts to saying that any three-manifold can be decomposed into pieces on which the Ricci-Hamilton flow is global, and thereby each of these pieces can be smoothly deformed into a locally homogeneous three-manifold. Singularities may develop in the regions connecting the smoothable pieces, but such singularities should be a finite number of types and all of a rather symmetric nature (namely, if they are blown up, they should be associated with symmetric manifolds as, e.g., $S^2 \times S^2$ [24,25]).

It may be said that Hamilton's program is an analytic approach to prove Thurston's conjecture, which claims that any closed three-manifold can be cut into pieces, such that each of them admits a locally homogeneous geometry. The rationale, underlying this *analytical* program towards Thurston's geometrization conjecture, lies in the above nice structural properties of the Ricci-Hamilton flow. Several steps are involved in this program. Let us briefly recall them since, even if some of them are yet unproven, they shed light on the assumption Eq. (65) and on Eq. (55) when interpreted as a renormalization group flow.

The first step is the assignment of an *arbitrary* metric g on the three-manifold Σ. In the renormalization group approach, this corresponds to picking up an inhomogeneous and anisotropic geometry for the physical space (here equivalent to the *high-temperature phase*). Such a choice may not conform to the actual *quasihomogeneous* three-geometry of the physical space as is experienced now. However this quasihomogeneity, in our opinion, may be related to the possibility that the actual universe is near criticality, a circumstance that we want to discuss rather than assume from the outset.

The second step is to deform this metric g via the Ricci-Hamilton flow, Eq. (55). In general, this flow develops local singularities which should be related to the manifold decomposition in Thurston's conjecture. Away from each of the local singularities it is conjectured (but not yet proved) that the Ricci-Hamilton flow approaches that of a locally homogeneous geometry in each disconnected piece.

This picture may be consistent with our blocking procedure yielding for Eq. (55) as the renormalization group flow. According to the analysis, carried out in Sec. III A a highly inhomogeneous and anisotropic geometry (Σ, g) can be characterized by minimal geodesic ball coverings $\{B(x_i, \epsilon_0)\}$, whose balls are to a good approximation largely uncorrelated when seen on a suitable scale. This means that the values of the scalar curvatures, $R(i)$, evaluated at the centers of the balls of the covering, are random variables. Upon enlarging the balls and rescaling, certain regions of the manifold might be such that correlations develop among the corresponding $R(i)$, whereas in other regions, no matter how we enlarge the balls and rescale, the $R(i)$ will remain independently distributed random variables. The former regions, exhibiting a persistence length, are those that should approach a locally homogeneous geometry under blocking and rescaling [i.e., under the flow Eq. (55)]. The latter regions should rather develop singularities under Eq. (55), since no matter how much we block and rescale, the curvature will maintain its *white noise* character.

The third step is to study the behavior of Eq. (55) for the locally homogeneous geometries, for this accounts for the structure of the critical set of Eq. (55). This has been accomplished [25], and one finds that, depending on the initial locally homogeneous geometry, the Ricci-Hamilton flow either (i) converges to a constant curvature metric, (ii) asymptotically approaches (as $\eta \to \infty$) a flat degenerate geometry, of either two or one dimensions (pancake or cigar degeneracy), with the curvature decaying at the rate $1/\eta$, or (iii) hits a curvature singularity in finite time, with this singularity being that of the Ricci-Hamilton flow for the standard metric on $S^2 \times S^1$. Note that constant curvature geometries always occur whenever the manifold can support them (in dimension three, constant curvature manifolds and Einstein are synonymous). It is also quite interesting to note that the Ricci-Hamilton flow of homogeneous metrics usually (with a few exceptions) tends to approach or converge to the maximally symmetric homogeneous metric in the class considered (see [25] for details).

Interpreting Eq. (55) as the renormalization group flow, it follows that locally homogeneous geometries evolving under Eq. (55) towards an isolated constant curvature manifold are sinks describing a stable phase of the corresponding cosmological model. For instance, a FLRW model (with closed spatial sections) characterizes such a phase. Nonisolated constant curvature manifolds (e.g., flat three-tori) provide less trivial examples of limiting behavior of Eq. (55) (see, e.g., [26,24]). The locally homogeneous manifolds nonadmitting Einstein manifolds [e.g., there are no left-invariant Einstein metrics on the group SL(2,$\mathbb{R}$)] provide even more interesting behavior. In this case, a metric renormalized under the action of Eq. (55) develops degeneracies and one gets, in our setting, an *effective* cosmological model with spatial sections of lower dimensionality.

All these limit points of Eq. (55), either fixed or not, have their own basins of attraction. Rigorously speaking, they are the sets of three-metrics flowing under Eq. (55) to the respective limit points, briefly discussed above. There are nine such basins of attraction, corresponding to the nine classes of homogeneous geometries that can be used to model (by passing to the universal cover) the local inhomogeneous geometries on closed three-manifolds. By labeling these classes according to the minimal isometry group of the geometries considered, we distinguish the following basins (here we follow closely the exposition in [26]; note also that the use of nine classes of locally homogeneous geometries rather than the standard eight classes used by Thurston is dictated by the fact that in discussing the Ricci-Hamilton flow one needs to consider metric not of maximal symmetry):

(i) The $\mathbb{R}^3$ basin. It contains all three-metrics flowing towards the homogeneous flat $\mathbb{R}^3$ metrics. This basin is eventually attracted by flat space (flat tori, when reverting to the original manifold rather than to its universal cover).

(ii) The SU(2) basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous SU(2) metrics. This class admits Einstein metrics, in particular the round metrics on the three-sphere. This basin is exponentially attracted to the round three-sphere (modulo identifications). It is the basin of attraction yielding for closed FLRW cosmological models.

(iii) The SL(2,$\mathbb{R}$) basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous SL(2,$\mathbb{R}$) metrics. This class does not admit Einstein metrics. This basin goes degenerate, yielding for a *pancake* degeneracy whereby a two-dimensional geometry survives: Two of the components of the metric increase without bound while the other shrinks to zero.

(iv) The Heisenberg basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous Nil-metrics. Again, this class does not contain any Einstein metrics. This basin too undergoes a pancake degeneracy.

(v) The $E(1,1)$ basin, where $E(1,1)$ is the group of isometries of the plane with flat Lorentz metric. It contains all three-metrics flowing towards the three-parameter family of homogeneous Solv metrics. Also this basin fails to contain Einstein metrics. This basin eventually exhibits a *cigar* degeneracy: The curvature dies away, and while one diameter expands without bound, the other two diameters shrink to zero.

(vi) The $E(2)$ basin, where $E(2)$ is the group of isometries of the Euclidean plane. It contains all three-metrics flowing towards the three-parameters family of homogeneous Solv-metrics containing the flat geometry. This basin is eventually attracted by flat metrics.

(vii) The $H(3)$ basin, where $H(3)$ is the group of isometries of hyperbolic three-space. It contains all three-metrics flowing towards the one-parameter family of homogeneous metrics constant multiples of the standard hyperbolic metric. This basin is attracted to hyperbolic space.

(viii) The SO(3) $\times$ $\mathbb{R}^1$ basin. It contains all three-metrics flowing towards the two-parameter family of homogeneous metrics obtained by rescaling the standard product metric on $S^2 \times \mathbb{R}^1$. It does not contain Einstein metrics. This is a singular basin, it is attracted towards a curvature singularity: the round two-sphere shrinks while the scale on $\mathbb{R}^1$ (or if you prefer, the S^1 factor in the original manifold), expands.

(ix) The $H^2 \times \mathbb{R}^1$ basin, where $H(2)$ is the group of isometries of the hyperbolic plane. It contains all three-metrics flowing towards the two-parameter family of homogeneous metrics obtained by rescaling the product metric on the product manifold, $\mathbb{R}^1 \times H^2$. Again, this basin does not contain Einstein manifolds, and it is attracted towards a pancake degeneracy.

The basins of attraction just described, and in particular those yielding for fixed points (Einstein manifolds), are relatively uninteresting in connection with the renormalization group interpretation of Eq. (55). Such fixed points, e.g., the round three-sphere, or the flat three-tori, are all totally attractive. As already recalled they can be thought of as distinct stable phases yielding for distinct cosmological models, characterized by the nature of the metric (and of its infinitesimal deformations) at the fixed point. For instance, the SU(2)-basin characterizes FLRW models with closed spatial sections, and the related homogeneous anisotropic model (see Sec. VI for more details).

C. Critical fixed points: an example

The existence of critical fixed points [critical in the sense of Eq. (65)], characterizing the (universal) properties of (continuous) phase transitions between two different cosmological regimes, cannot be immediately read off from the above analysis. It is rather the consequence of the (conjectured) existence of the decomposition of a manifold into pieces that is associated, according to Hamilton, with the local singularities of the flow Eq. (55) (see the second step of Hamilton-Thurston geometrization program). The origin of this connection between critical behavior of Eq. (55), in the sense of the renormalization group, and Hamilton's program can be seen by considering the following detailed example.

Let us assume that topologically Σ is a three-sphere, $\Sigma \simeq S^3$. We shall consider on Σ a metric g_1 obtained by gluing through a smooth connected sum two copies of a round three-sphere

$$\Sigma \simeq S^3_{(1)} \sharp S^3_{(2)} \ , \tag{66}$$

and endowing each $S^3_{(i)}, i = 1,2$, factor with a round metric of volume $v_1 = 1$, and the joining tube

$$S^2 \times ([0,1] \subset \mathbb{R}^1) \ , \tag{67}$$

with the standard product metric of volume const $e^{-3\tau}$ (τ is a suitable parameter, see below), for $\tau \gg 1$.

To explicitly construct this latter metric we can proceed as follows. Let y_1 and y_2, respectively, denote two chosen points in both factor copies, $S^3_{(1)}$ and $S^3_{(2)}$. Let $h_{(i)} : \mathbb{R}^3 \to S^3_{(i)}, i = 1,2$, be two imbeddings given by the exponential mappings

$$\exp_{y1} : T_{y1}S^3_{(1)} \simeq \mathbb{R}^3 \to S^3_{(1)} \ ,$$

$$\exp_{y2} : T_{y2}S^3_{(2)} \simeq \mathbb{R}^3 \to S^3_{(2)} \ . \tag{68}$$

We assume that $h_{(1)}$ preserves the orientation while $h_{(2)}$ reverses it.

Let $\alpha : (0,\infty) \to (0,\infty)$ denote an orientation reversing diffeomorphism, and define $\alpha_3 : \mathbb{R}^3/\{0\} \to \mathbb{R}^3/\{0\}$ by the map $\alpha_3(v) = \alpha(|v|)v/|v|$ for every vector $v \in \mathbb{R}^3/\{0\}$. For every point $x_1 = h_1(v)$ in the geodesic ball $B(y_1, r)/\{0\} \subset S^3_{(1)}$, with $0 < r \le \pi/2$ (with $\pi/2$ the injectivity radius of the unit three-sphere $S^3_{(i)}$), we identify $h_1(v)$ with $h_2(\alpha_3(v)) \in B(y_2, r)/\{0\} \subset S^3_{(2)}$.

The space obtained in this way,

$$S^3_{(1)} \sharp S^3_{(2)} = (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)}\alpha_3 h^{-1}_{(1)}} (S^3_{(2)} - \{y_2\}) \ , \tag{69}$$

is a particular realization of the connected sum [27] of two copies of unit three-spheres.

In order to give to the neck, joining the two $S^3_{(i)}$, a cylindrical shape we blow up [28] the metrics of the three-spheres in the neighborhoods of the points y_1 and y_2. Consider, for simplicity, only the $S^3_{(1)}$ factor since the argument goes in an analogous way also for the $S^3_{(2)}$ factor (see Fig. 11).

Exploiting the exponential mapping, the round metric of $S^3_{(1)}$ (actually *any* sufficiently smooth metric) can be written in a neighborhood of y_1 in geodesic polar coordinates as

$$g(x) = dr^2 + r^2 h_{ij} d\theta^i d\theta^j + O(r^4) \ , \tag{70}$$

where $r = \mathrm{dist}(x, y_1)$ is the distance between y_1 and the point considered, $h_{ij} d\theta^i d\theta^j$ is the metric on the two-dimensional unit spheres S^2, and as usual, the higher order correction terms involve the curvature.

Now, if we blow up this metric by rescaling it through r^2 we get, up to curvature corrections, a distorted cylindrical metric

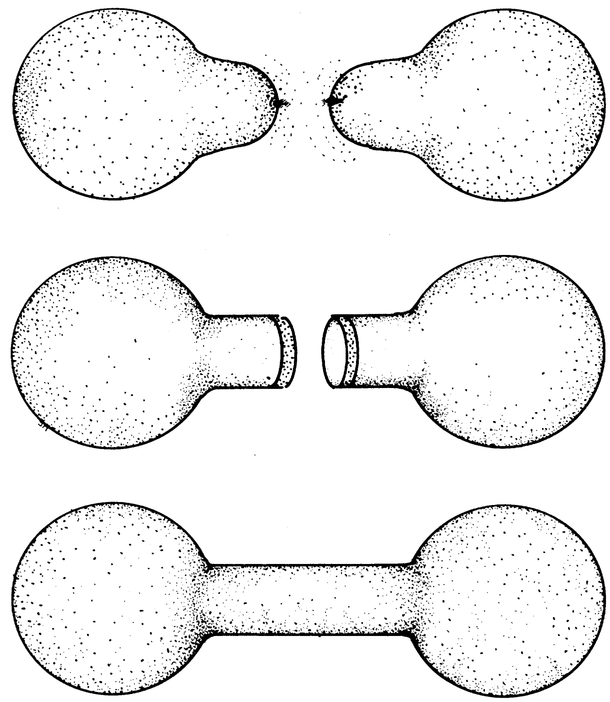

FIG. 11. If we blow up the metric of two three-spheres $S^3_{(1)}$ and $S^3_{(2)}$ (here represented as two-spheres), around the points y_1 and y_2 and then join the resulting manifolds, we obtain the connected sum $S^3_{(1)} \sharp S^3_{(2)}$. According to this procedure, the neck $S^2 \times \mathbb{R}^1$ inherits a cylindrical metric up to exponentially small correction terms.

$$\tilde{g}(x) = \frac{g(x)}{r^2} = \frac{dr^2}{r^2} + h_{ij} d\theta^i d\theta^j + O(r^2) \ . \tag{71}$$

In order to eliminate the axial distortion due to dr^2/r^2, we substitute for the radial variable r, introducing a new coordinate τ defined as

$$r = \exp[-\tau] \ . \tag{72}$$

When expressed in terms of τ we get that the blown up metric $\tilde{g}(x)$ reduces to

$$\tilde{g}(x) = d\tau^2 + h_{ij} d\theta^i d\theta^j + O(e^{-2\tau}) \ . \tag{73}$$

Thus, the blown up metric approaches the cylindrical metric exponentially fast as $\tau \to \infty$. In order to introduce smoothly such a metric on the neck of $S^3_{(1)} \sharp S^3_{(2)}$ we can proceed as follows.

Choose smooth functions $\delta_{(i)}(r), i = 1,2$, satisfying

$$\begin{aligned} &\delta_{(i)}(r_i) = \frac{1}{r_i^2}, \quad 0 < r_i \le \frac{\pi}{4} \ , \\ &\delta_{(i)}(r_i) = C^\infty, \quad \text{decreasing}, \quad \frac{\pi}{4} \le r_i \le \frac{\pi}{2} \ , \\ &\delta_{(i)}(r_i) = 1, \quad r_i \ge \frac{\pi}{2} \ , \end{aligned} \tag{74}$$

where $r_i = \mathrm{dist}(x, y_{(i)})$. Also, let us introduce the inclusion maps

$$\chi_{(1)} : S^3_{(1)} - \{y_1\} \hookrightarrow (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)}\alpha_3 h^{-1}_{(1)}} (S^3_{(2)} - \{y_2\}) ,$$
$$\chi_{(2)} : S^3_{(2)} - \{y_2\} \hookrightarrow (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)}\alpha_3 h^{-1}_{(1)}} (S^3_{(2)} - \{y_2\}) , \quad (75)$$
$$\chi_{(3)} : h_1(T_{y1} S^3_{(1)} - \{0\}) \cup h_2(T_{y2} S^3_{(2)} - \{0\}) \hookrightarrow (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)}\alpha_3 h^{-1}_{(1)}} (S^3_{(2)} - \{y_2\}) ,$$

($\chi_{(1)}$ and $\chi_{(2)}$ denote the inclusions of the spheres, minus the points $y_{(i)}$, into the connected sum; $\chi_{(3)}$ is the inclusion of the neck).

If ξ_α is a partition of unity associated with the covering corresponding to the above inclusions $\chi_{(\alpha)}$, with $\alpha = 1, 2, 3$, we can then define the following metric $\tilde{g}$

$$\tilde{g}(x) = \sum_\alpha \xi_\alpha \chi^*_{(\alpha)}[g(x)\delta_{(i)}(\mathrm{dist}(x, y_{(i)}))] . \quad (76)$$

For x not in the geodesic balls (of radius $\pi/2$), centered on y_1 and y_2, this is the standard round metric on the unit three-sphere; for x in the geodesic balls of radius $\pi/4$, centered on y_1 and y_2, this is, up to curvature corrections, the cylindrical metric introduced above. For x in the annuli between such balls $\tilde{g}$ is a smooth interpolating metric joining the spheres to the cylindrical neck.

By expressing Eq. (76) in terms of the variable τ, we get the metric on $S^3_{(1)} \sharp S^3_{(2)}$ which is the round metric on each $S^3_{(i)}/B(y_i, e^{-\tau})$ factor, and these factors are connected, for τ large enough (i.e., nearby y_i) by a thin flat cylinder.

The Ricci-Hamilton evolution of $\tilde{g}$ can be explicitly constructed as follows. According to [24] let us write the metric on $S^2 \times \mathbb{R}^1$ as

$$\tilde{g}|_{\mathrm{neck\ of} S^3_{(1)} \sharp S^3_{(2)}} = D g_{\mathbb{R}^1} + E g_{S^2} , \quad (77)$$

where $g_{\mathbb{R}^1}$ is the metric on $\mathbb{R}^1$, g_{S^2} is the round metric on the unit two-sphere, and D and E are constants. The Ricci-Hamilton flow Eq. (55) preserves the structure of this metric and reduces to the coupled system of ordinary differential equations

$$\frac{d}{d\eta} E = -\frac{2}{3} ,$$

$$\frac{d}{d\eta} D = \frac{4}{3}\left(\frac{D}{E}\right) , \quad (78)$$

which immediately integrate to

$$E = E_0 - \frac{2}{3}\eta ,$$

$$D = \frac{D_0 E_0^2}{[E_0 - (2/3)\eta]^2} , \quad (79)$$

where $E_0^2 = E^2(\eta = 0)$ is the initial radius of the round two-sphere, whereas $D_0 = D(\eta = 0)$ is the scale on $\mathbb{R}^1$.

Given this solution of Eq. (55), we can construct the Ricci-Hamilton evolution of $\tilde{g}$ by looking for a solution of Eq. (55) in the form

$$\tilde{g}(x; \eta) = \sum_\alpha \xi_\alpha \chi^*_{(\alpha)}(\eta)[g(x)\delta_{(i)}(\mathrm{dist}_\eta(x, y_{(i)}))] , \quad (80)$$

where the inclusion maps $\chi_{(\alpha)}$ depend now on the deformation parameter η. We assume that the metric $g(x)$ (the round metric on the spheres $S^3_{(i)}$ and the standard product metric on the neck $\simeq S^2 \times \mathbb{R}^1$), being locally homogeneous, is preserved by Eq. (55) since the Ricci-Hamilton flow preserves isometries. As the *plumbing* between the spheres and the neck shrinks as η increases (as the S^2 factor in the neck), the inclusions $\chi_{(\alpha)}$ are necessarily η dependent. The dynamics of $\chi_{(\alpha)}$ can be obtained as follows (see Fig. 12).

The Ricci-Hamilton flow Eq. (55) for $\tilde{g}(x; \eta)$ is given by

$$\frac{\partial \tilde{g}_{ab}(\eta)}{\partial \eta} = \frac{2}{3}\langle \tilde{R}(\eta)\rangle \tilde{g}_{ab}(\eta) - 2\tilde{R}_{ab}(\eta) ,$$
$$\tilde{g}_{ab}(\eta = 0) = \tilde{g}_{ab} ,$$

and in terms of the pulled-back metric it is

$$\frac{\partial}{\partial \eta}[\chi^*_{(\alpha)}(\eta) g(x)\delta_{(i)}]_{ab} = \frac{2}{3}\langle \tilde{R}(\eta)\rangle [\chi^*_{(\alpha)}(\eta) g(x)\delta_{(i)}]_{ab}$$
$$-2\tilde{R}(\chi^*_{(\alpha)}(\eta) g(x)\delta_{(i)})_{ab} . \quad (81)$$

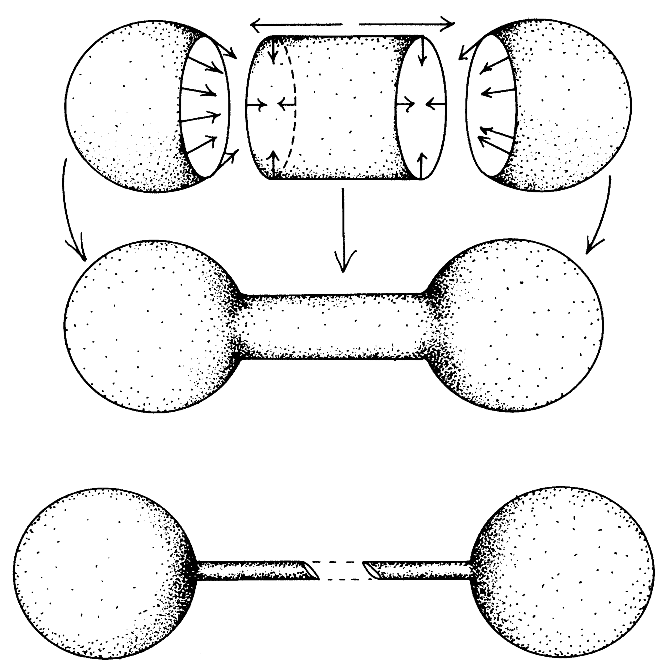

FIG. 12. The Ricci-Hamilton flow of $S^3_{(1)} \sharp S^3_{(2)}$ can be expressed in terms of the η evolution of the inclusion maps $\chi_{(\alpha)}, \alpha = 1, 2, 3$, and of the known evolution of the locally homogeneous metrics on $S^3_{(i)} \backslash B(y_i)$ and $S^2 \times \mathbb{R}^1$.

The left-hand side of Eq. (81) can be evaluated according to a suggestion first exploited by DeTurck, viz.,

$$\frac{\partial}{\partial\eta}[\chi^*_{(\alpha)}(\eta)g\delta_{(i)}]_{ab}(x) = \chi^*_{(\alpha)}(\eta)\left[\frac{\partial}{\partial\eta}[g\delta_{(i)}]_{ab}[\chi(\eta,x)]\right] + \chi^*_{(\alpha)}(\eta)[L_{w(\eta)}[g\cdot\delta_{(i)}]_{ab}[\chi(\eta,x)]] \ , \tag{82}$$

where the Lie derivative $L_{w(\eta)}[g\delta_{(i)}]_{ab}(\chi(\eta,x))$ is along the vector field $w(\eta;\alpha)$ which generates the η evolution of the inclusions $\chi_{(\alpha)}$, viz.,

$$\frac{\partial\chi_{(\alpha)}(\eta)}{\partial\eta} = w(\eta;\chi_{(\alpha)}(\eta)) \ , \quad \chi_{(\alpha)}(\eta=0) = \chi_{(\alpha)} \ . \tag{83}$$

Let us denote by $\langle R\rangle_{(\alpha)}$ the average of the scalar curvature $R(\chi_{(\alpha)}(x))$ over the images of the inclusions $\chi_{(\alpha)}$ (viz., for $\alpha = 1,2$, $\langle R\rangle_{(\alpha)}$ is the average over $S^3_{(i)}$, while for $\alpha = 3$, the average is over the neck). In terms of these averages, the Ricci-Hamilton flow Eq.(81) can be written as

$$\begin{aligned}\frac{\partial}{\partial\eta}[\delta_{(i)}g_{ab}(\chi_{(\alpha)}(x))] = {}& \tfrac{2}{3}\langle R(\eta)\rangle_{(\alpha)}\delta_{(i)}g_{ab}(\chi_{(\alpha)}(x)) - 2R_{ab}(\chi_{(\alpha)}(x)) \\ & + \tfrac{2}{3}\delta_{(i)}g_{ab}(\chi_{(\alpha)}(x))[\langle R(\eta)\rangle - \langle R(\eta)\rangle_{(\alpha)}] - L_{w(\eta)}[g\delta_{(i)}]_{ab}(\chi(\eta,x)) \ .\end{aligned} \tag{84}$$

As can be checked, the Ricci-Hamilton flow preserves the local homogeneous geometries over the spheres $S^3_{(i)}$, and on the neck, if and only if the vector field $w(\eta;\alpha)$ satisfies

$$\tfrac{2}{3}\delta_{(i)}g_{ab}(\chi_{(\alpha)}(x))[\langle R(\eta)\rangle - \langle R(\eta)\rangle_{(\alpha)}] = L_{w(\eta)}[g\delta_{(i)}]_{ab}(\chi(\eta,x)) \ . \tag{85}$$

For $\alpha = 1,2$, i.e., on $S^3_{(i)} - B(y_i,r(\eta))$, Eq. (85) can be interpreted in terms of the geometry of the two-sphere S^2_η, boundary of $S^3_{(i)} - B(y_i,r(\eta))$. Consider, in $S^3_{(i)} - B(y_i,r(\eta))$, a tubular neighborhood of $\partial(S^3_{(i)} - B(y_i,r(\eta)))$ foliated by a one-parameter family of two-spheres S^2_η with outer normals $\hat{w}_a = w_a/|w|$. Let $\bar{\nabla}$ be the Riemannian connection with respect to $g\delta_{(i)}$. Let us respectively denote by $h_ac \equiv g_ac - \hat{w}_a\hat{w}_c$ and by $\sigma_{cd} = h^a_c h^b_d \bar{\nabla}_a\hat{w}_b$ the first and the second fundamental form of the embedding of S^2_η in ${}^3_{(i)} - B(y_i,r(\eta))$. Then (85) can be rewritten as (say, for $\alpha = 1$, the argument is completely symmetrical for $\alpha = 2$)

$$\tfrac{1}{3}\delta_{(i)}h_{cd}[\langle R(\eta)\rangle - \langle R(\eta)\rangle_{(1)}] = |w(\eta)|\sigma_{cd}(\eta) \ . \tag{86}$$

By taking the trace with respect to h_{cd}, we get

$$\tfrac{2}{3}[\langle R\rangle - \langle R\rangle_{(1)}] = |w|\sigma^c_c \ . \tag{87}$$

Since for $\alpha = 1,2$ $\langle R\rangle - \langle R\rangle_{(\alpha)}$ is proportional to the average scalar curvature of the neck, the term on the left side of Eq. (87) blows up as $(E_0 - \frac{2}{3}\eta)^{-1}$ as η increases. On the other hand, by the Gauss-Codazzi relations for the embedding of S^2_η into $S^3_{(i)} - B(y_i,r(\eta))$ we get

$$R(S^2_\eta) = \tfrac{1}{3}R(S^3_{(i)}) + \tfrac{1}{2}(\sigma^c_c)^2 \ , \tag{88}$$

which implies that the blowing up of σ^c_c as $\eta \to \frac{3}{2}E_0$ is associated to the blowing up of the curvature of the η-dependent two-sphere boundary of $S^3_{(i)} - B(y_i,r(\eta))$. Thus, Eq. (87) implies that as η increases, the surface area of $S^2(\eta)$ shrinks to zero, as $(E_0 - \frac{2}{3}\eta)$, by moving along the outer normal direction of $S^2(\eta) \subset (S^3_{(i)} - B(y_i,r(\eta)))$.

For $\alpha = 3$, i.e., for the neck, Eq. (87) yields that as the $S^2(\eta)$ factor in the neck shrinks, the scale length of the $\mathbb{R}^1$ factor grows, so that the volume $\mathrm{vol}(S^2(\eta)\times\mathbb{R}^1)$ remains constant during the Ricci-Hamilton evolution (since $\langle R\rangle - \langle R\rangle_{(\alpha=3)}$ is, up to small correction terms, coming from the collars joining the spheres to the neck, the average curvature over the spheres $S^3_{(i)}, i = 1,2$).

Notice that by introducing, as above, a new variable

$$\mathrm{dist}_\eta(x,y_{(i)}) = \exp[-\tau(\eta)] \ , \tag{89}$$

with

$$\tau(\eta) = \frac{\tau}{[1-\frac{2}{3}\eta]} \ , \tag{90}$$

and by rescaling the unit two-sphere metric h_{ij} according to

$$h_{ij}(\eta) = [1-\tfrac{2}{3}\eta]h_{ij} \ , \tag{91}$$

the above analysis of the Ricci-Hamilton flow Eq. (81) provides on the neck the metric

$$\tilde{g}(x;\eta) = d\tau^2(\eta) + h_{ij}(\eta)d\theta^i d\theta^j + O(e^{-2\tau(\eta)}) \ , \tag{92}$$

and on $S^3_{(i)}$ the standard round metric.

Notice also that Eq. (92) goes cylindrical exponentially fast in η. As η increase $0 \le \eta < \frac{3}{2}$, the neck becomes longer and longer while getting thinner and thinner. In the limit, $\eta \to \frac{3}{2}$, we get (with a slight abuse of notation)

$$(S^3_{(1)}\sharp S^3_{(2)},\tilde{g}) \to S^3_{(1)}\coprod S^3_{(2)} \ , \tag{93}$$

where the three-spheres $S^3_{(i)}$ carry the round metric of volume 1, while the smooth joining regions shrink exponentially fast around the points $y_{(i)}, i = 1, 2$ (see Fig. 13).

This pinching phenomenon, as described by (93), can be naturally extended to the connected sum of more than two copies of three-spheres, and the resulting solution of the Ricci-Hamilton flow

$$(S^3_{(1)} \sharp S^3_{(2)} \sharp \cdots \sharp S^3_{(n)}, \tilde{g}) \to S^3_{(1)} \coprod S^3_{(2)} \coprod \cdots \coprod S^3_{(n)} , \tag{94}$$

can be thought of as describing a high-temperature fixed point of the renormalization group flow, yielding for a random, uncorrelated, three-geometry. This high-temperature fixed point attracts initial data sets (on three-manifolds which are topologically S^3, in the particular example constructed above), which are associated with three-geometries which are sufficiently inhomogeneous. In this phase the three-geometry tends to randomly pinch-off uncorrelated S^3 baby universes.

The previous example can be easily modified as to provide also unstable solutions to The Ricci-Hamilton flow (these solutions characterize the critical fixed points). The strategy for constructing such solutions is to consider the Ricci flow for initial three-manifolds constructed by gluing through a smooth connected sum two distinct copies of a round three-sphere

$$\Sigma \simeq (S^3_{(1)} \sharp S^3_{(2)}, g_{\text{cap}}) \tag{95}$$

by endowing the $S^3_{(1)}$ factor with a round metric of volume $v_1 = 1 - c$, and the $S^3_{(2)}$ factor (the small cap), with a round metric of volume $v_1 = c$ where c is a constant with $\frac{1}{2} < c < 1$, and by eliminating the connecting neck. Explicitly, this can be done by marking a point y_1 on the three-sphere $S^3_{(1)}$, and remove a ball $B(y_1, r)$ of sufficiently small radius $r < 1$. The two-sphere S^2 boundary of $S^3_{(1)} - B(y_1, r)$ has diameter πr. Glue on this S^2 the three-spherical cap obtained by the three-sphere $S^3_{(2)}$ whose equatorial two-sphere is the given S^2. The connecting region, topologically $S^2 \times \mathbb{R}^1$, in this $S^3_{(1)} \sharp S^3_{(2)}$ is smoothed out so as to provide a C^∞ transition metric between the large $S^3_{(1)} - B(y_1, r)$ and the small three-spherical cap. The Ricci-Hamilton flow with such a manifold as initial datum can be cast again in the form (84) where now the index α refers ($\alpha = 1$) to the large sphere $S^3_{(1)}$, and to ($\alpha = 2$) the small spherical cap $S^3_{(2)}$. The Ricci-Hamilton evolution of this particular connected sum can be still geometrically described by Eq. (87). We have three distinct possibilities (we wish to stress that actually a more careful analysis is needed to discuss this point, but for the sake of simplicity we defer such a treatment to a paper in preparation). The first possibility follows from the fact that now the Ricci-Hamilton flow preserves the local geometries over the spherical caps $S^3_{(1)}$ and $S^3_{(2)}$ only up to a rescaling. Under the only constraint that the total volume of $S^3_{(1)} \sharp S^3_{(2)}$ is preserved, by slightly perturbing the metric in the joining collar $S^2 \times \mathbb{R}^1$, we may have that the $S^3_{(2)}$ factor spreads over $S^3_{(1)}$; this implies that $\langle R \rangle - \langle R \rangle_{(\alpha)} \to 0$ as η increases. Correspondingly $\sigma^c_c \to 0$, viz., the joining two-sphere becomes the equatorial sphere in $S^3_{(1)}$, and as a final outcome we get the round three-metric on $S^3_{(1)}$. The perturbation of the metric of the collar may be such that we enhance its cylindrical shape, in this case the $S^3_{(2)}$ cap shrinks, and σ^c_c blows up. The final outcome is that of $S^3_{(1)}$ on which the $S^3_{(2)}$ cap degenerates in a longer and longer and thinner and thinner tube, possibly connecting two three-spheres whose total volume is the one of the original manifold. Finally, we do not perturb the joining neck, and let the Ricci-Hamilton flow fix the respective original round metrics on $S^3_{(1)}$ and $S^3_{(2)}$. The trace of the second fundamental form σ^c_c of the joining two-sphere then stays constant during the evolution. What happens is that the small spherical cap $S^3_{(2)}$ is constant in size but possibly moves around on the larger $S^3_{(1)}$, as η increases. Such an $(S^3_{(1)} \sharp S^3_{(2)}, g_{\text{cap}})$ is a critical fixed point for the Ricci-Hamilton flow, a fact that has important consequences for our renormalization group approach to relativistic cosmology.

D. Critical surfaces and topological crossover

Let g_0 be a metric on Σ with positive Ricci curvature,

$$\text{Ric}(g_0) > 0 \tag{96}$$

and with volume vol $(\Sigma, g_0) = 2$.

By means of g_0 and the metric $\tilde{g}$ introduced in the preceding section, we can construct on Σ a smooth one-parameter ($0 \leq t \leq 1$) family of metrics g_t, with $g_{t=0} =$

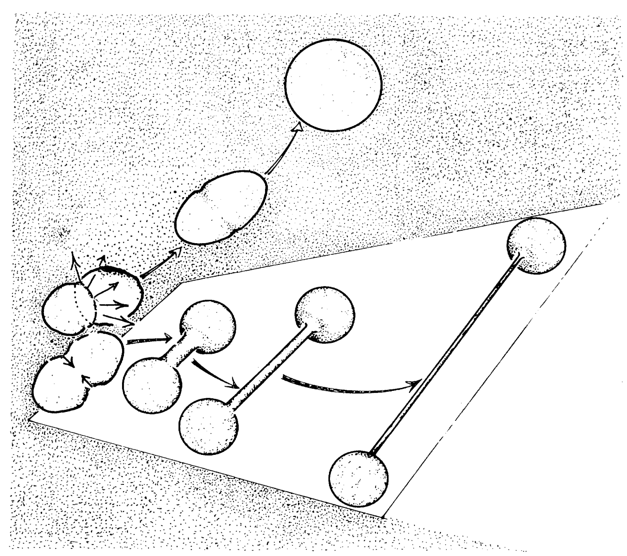

FIG. 13. Different deformations of the initial three-manifold Σ may have quite different fates. If the waist of Σ is increased enough by the deformation (i.e., if it is rounded), the Ricci-Hamilton flow will evolve Σ toward the round three-sphere. If the waist is shrunk enough by the initial deformation, the Ricci-Hamilton flow will evolve Σ toward $S^3_{(1)} \coprod S^3_{(2)}$.

g_0 and $g_{t=1} = \tilde{g}$ by setting

$$g_t \equiv (1-t)g_0 + t\tilde{g} \ , \quad 0 \le t \le 1 \ . \tag{97}$$

According to Hamilton's theorem [22], there is a right-open neighborhood of $t = 0$ such that all metrics g_t in this neighborhood are in SU(2) basins and are attracted, under the action of the Ricci-Hamilton flow Eq. (55), towards the round metrics on S^3 with volumes v_t (since the volume of g_t changes with t, we have a family of fixed points, namely, round three-spheres parameterized by the corresponding volumes v_t which are kept constant under the Ricci-Hamilton flow; thus $v_{t=0} = 2$).

On the other hand, according to the remarks above, we have an open neighborhood of $t = 1$ such that all metrics g_t in this neighborhood go *singular* under the Ricci-Hamilton flow. Indeed, for $t = 1$, the Ricci-Hamilton flow of $\tilde{g}$ fixes the round S^3 factors while the joining tube, $S^2 \times \mathbb{R}^1$, is driven towards a curvature singularity. By continuity, this behavior extends to a left-open neighborhood of $t = 1$, and a three-sphere in this neighborhood splits apart into two round spheres.

It follows that there is a neighborhood of the $\tilde{g}$ metric such that some of the metrics in this neighborhood are driven towards attractive SU(2) basins, while others are driven towards a *singular* $(S^3_{(1)}\sharp S^3_{(2)})$ basin.

According to the discussion of the previous paragraphs, the set of metrics separating these two distinct behaviors is provided by the metrics driven towards the metrics of the form $(S^3_{(1)}\sharp S^3_{(2)}, g_{\rm cap})$ defining the critical fixed points. Metrics flowing towards any of the small spherical cap metrics $(S^3_{(1)}\sharp S^3_{(2)}, g_{\rm cap})$ define a *critical surface.*

Note, however, that in this particular case, the critical point is not necessarily related to a phase transition. As mentioned before, we are here in presence of significant finite-size effects which are concerned with a dimensional crossover and they show up, as usual, as an *effective reduction of dimensionality* [21]. Indeed, geometrically speaking, the critical fixed point $(S^3_{(1)}\sharp S^3_{(2)}, g_{\rm cap})$, and the corresponding *critical surface*, separates two stable phases under the renormalization generated by Eq. (55). One is given by the manifolds eventually evolving towards the round three-sphere of volume 2. The other is generated by those manifolds which pinch off and eventually evolve towards two round three-spheres, each of volume 1. The *pinching off* through thinner and thinner necking is necessary for such a topological crossover.

From a physical point of view, and as we shall see in Sec. VI, the fixed point described above separates two possible different closed FLRW regimes. One with closed spatial sections which, at some particular instant, are a three-sphere of volume 2, while in the other regime we have *two* distinct closed FLRW universes having spatial sections (at a given instant) of volume 1. We may have also many different necks corresponding to a regime whereby the spatial section Σ yields for many closed FLRW universes.

It is clear that the above explicit construction of a critical fixed point for Eq. (55) can in principle be generalized to more general situations. The strategy is to take two or more trivial fixed points, such as those associated with the SU(2) and $H(3)$ basins, and connect them through the degenerate basins [such as $\mathrm{SO}(3)\times\mathbb{R}^1$, as in the above example, or through the $H(2)\times\mathbb{R}^1$ basin, etc.]. In this connection notice that the connected sum mechanism yielding for the $(S^3_{(1)}\sharp S^3_{(2)})$ critical fixed point, can be generalized to an operation of joining two (or more) manifolds (corresponding to stable attractors) along tubular neighborhoods of surfaces (rather than points, as in the case for the standard connected sum).

A particularly interesting connecting geometry would be $H(2)\times\mathbb{R}^1$ (by compactifying the hyperbolic plane in a closed surface). In this latter case the scale of the hyperbolic geometry goes to infinity under the Ricci-Hamilton flow (*pancake degeneracy* [25]). Finite size effects are again at work, but this time the effective reduction of dimensionality is more interesting than the one in the $(S^3_{(1)}\sharp S^3_{(2)})$ case. Indeed two out of three dimensions are infinite, and there will be a crossover to a critical behavior with critical exponents characteristic of a two-dimensional system.

One can consider the analysis presented above as a physically nontrivial application of Hamilton-Thurston's geometrization program. In some of its aspects it is rather conjectural and speculative, but in our opinion, it is quite intriguing that motivations coming from geometry and a physical problem, like the one addressed here, namely, the construction of cosmological models out of a local gravitational theory, go hand in hand in such a way.

V. LINEARIZED RG FLOW

The relative slopes of $\langle f\rangle_{\epsilon_m}(g_1)$ and $\langle f\rangle_{\epsilon_m}(g_2)$, with f given by Eq. (38), as $m\to\infty$, and for g_1 in a neighborhood of g_2, are of some relevance to our discussion. In a standard RG analysis such relative slopes are related to critical exponents. Given the blocking procedure for $\langle f\rangle_{\epsilon_m}(g_1)$ as $m\to\infty$, yielding for a g_1 "renormalized" according to the Ricci-Hamilton flow Eq. (55), one can sensibly ask what happens if g_1 is slightly perturbed, namely, if we replace it by

$$g_{ab} \to g_{ab} + \delta K_{ab} \ , \tag{98}$$

where K_{ab} is a symmetric bilinear form (a choice of the symbol is quite intentional, since later the above consideration will be applied to the second fundamental form). It can be shown that if g_1 is scaled, according to the Ricci-Hamilton flow Eq. (55), then K_{ab} gets renormalized according to the linearized Ricci-Hamilton flow, namely (η in the brackets suppressed)

$$\frac{\partial}{\partial\eta}K_{ab} = \tfrac{2}{3}\langle R\rangle K_{ab} + \tfrac{2}{3}g_{ab}[\tfrac{1}{2}\langle Rg^{ab}K_{ab}\rangle - \tfrac{1}{2}\langle R\rangle\langle g^{ab}K_{ab}\rangle - \langle R^{ab}K_{ab}\rangle] - \Delta_L K_{ab} + 2[\mathrm{div}^*(\mathrm{div}(K - \tfrac{1}{2}(\mathrm{Tr}K)g))]_{ab} \ , \tag{99}$$

with the initial data $K_{ab}(\eta = 0) = K_{ab}$, where $K \in S^2\Sigma$ is a given symmetric bilinear form, Δ_L is the Lichnerowicz–de Rham Laplacian on bilinear forms

$$\Delta_L K_{ab} \equiv -\nabla^s\nabla_s K_{ab} + R_{as}K^s_b + R_{bs}K^s_a - 2R_{asbt}K^{st} , \tag{100}$$

and the operations Δ_L, div*, div, and Tr are considered with respect to the flow of metric $(g, \eta) \to g(\eta)$, solution of Eq. (55). The div (here, minus the usual divergence) is the divergence operator on $S^2\Sigma$, div* is the L^2 adjoint of div, acting from the space of vector fields on Σ to $S^2\Sigma$ (it can be identified with $\frac{1}{2}$ [Lie derivative] of the metric tensor along a vector field).

Note that $K(\eta)$ solution of the linear (weakly) parabolic initial value problem Eq. (99) always exists and is unique [22], and represents an infinitesimal deformation of metrics connecting the two neighboring flows of metrics, $g(\eta)$ and $g'(\eta)$ [obtained as solutions of problem of Eq. (55) with initial data $g(\eta = 0) = g$ and $g'(\eta = 0) = g(\eta = 0) + \epsilon K(\eta = 0) + O(\epsilon^2)$, respectively]. For what concerns the structure of this solution, one can verify that corresponding to the "trivial" initial datum $K(\eta = 0) = L_X g$(where $X : \Sigma \to T\Sigma$ is a smooth vector field on Σ), the solution of Eq. (99) is provided by

$$K_{ab}(\eta) = L_X g_{ab}(\eta) . \tag{101}$$

This property expresses the Diff(Σ) equivariance of the Ricci-Hamilton flow. (Notice that X is η independent.)

The above fact follows by noticing that along the trajectories of the flow $(\eta, g) \to g(\eta)$, solution of Eq. (55), we have

$$\begin{aligned}\frac{\partial}{\partial\eta} L_X g_{ab}(\eta) &= L_X \left(\frac{\partial}{\partial\eta} g_{ab}(\eta)\right) \\ &= \frac{2}{3}\langle R(\eta)\rangle_\eta L_X g_{ab}(\eta) - 2L_X R_{ab}(\eta) .\end{aligned} \tag{102}$$

But the Diff(Σ) equivariance of the Ricci tensor, i.e., the fact that $\mathrm{Ric}(\varphi^* g) = \varphi^*\mathrm{Ric}(g)$ for any smooth diffeomorphism $\varphi : \Sigma \to \Sigma$, implies that

$$L_X R_{ab} = D\,\mathrm{Ric}(g) L_X g_{ab} , \tag{103}$$

where $D\,\mathrm{Ric}(g)K$ is the formal linearization of Ric(g), around g, in the direction K:

$$\begin{aligned}D\,\mathrm{Ric}(g)K &\equiv \frac{d}{dt}[\mathrm{Ric}(g + tK)]_{t=0} \\ &= \tfrac{1}{2}\Delta_L K - \mathrm{div}^*[\mathrm{div}(K - \tfrac{1}{2}(\mathrm{Tr}K)g)] .\end{aligned} \tag{104}$$

Upon introducing Eq. (103) in Eq. (102) we get

$$\begin{aligned}\frac{\partial}{\partial\eta} L_X g_{ab}(\eta) = &\tfrac{2}{3}\langle R(\eta)\rangle_\eta L_X g_{ab}(\eta) \\ &-2D\,\mathrm{Ric}(g(\eta)) L_X g_{ab}(\eta) .\end{aligned} \tag{105}$$

One can check that the right-hand side of the above expression coincides with the right-hand side of Eq. (99), when the latter is evaluated for $K_{ab}(\eta) = L_X g_{ab}(\eta)$. Hence $L_X g_{ab}(\eta)$ solves the partial differential Eq. (99) and, since for $\eta = 0, K_{ab} = L_X g_{ab}$, the uniqueness of any solution of the initial value problem Eq. (99) implies that $K_{ab}(\eta) = L_X g_{ab}(\eta)$, whenever $K_{ab}(\eta = 0) = L_X g_{ab}$, as stated.

Moreover, if $K(\eta)$ is a solution of Eq. (99), with initial datum $k(\eta = 0) = K$, then the space average of $\mathrm{Tr}K(\eta)$ over $(\Sigma, g(\eta))$ is preserved along the flow $(\eta, g) \to g(\eta)$, namely,

$$\langle \mathrm{Tr}K(\eta)\rangle_\eta = \langle \mathrm{Tr}K\rangle_0, \quad 0 \le \eta < \infty . \tag{106}$$

This property of the solutions of Eq. (99) is an immediate consequence of the volume-preserving character of the Ricci-Hamilton flow.

Finally, another relevant property of the initial value problem Eq. (99) can be stated as follows. If $(\eta, K_{ab}) \to K_{ab}(\eta)$ is the flow solution of Eq. (99), with initial datum $K_{ab}(\eta = 0) = K_{ab}$, then it can always be written as [23]

$$K_{ab}(\eta) = \hat{K}_{ab}(\eta) + L_{v(\eta)} g_{ab}(\eta) , \tag{107}$$

where the bilinear form $\hat{K}_{ab}(\eta)$ and the η-dependent vector field $v(\eta)$, respectively, are the solutions of the initial value problems

$$\begin{aligned}\frac{\partial}{\partial\eta}\hat{K}_{ab} = &\frac{2}{3}\langle R\rangle \hat{K}_{ab} + \frac{2}{3} g_{ab}\left(\frac{1}{2}\langle R g^{ab}\hat{K}_{ab}\rangle \right.\\ &\left. -\frac{1}{2}\langle R\rangle\langle g^{ab}\hat{K}_{ab}\rangle - \langle R^{ab}\hat{K}_{ab}\rangle\right) - \Delta_L \hat{K}_{ab} , \\ &\hat{K}_{ab}(\eta = 0) = K_{ab} ,\end{aligned} \tag{108}$$

and

$$\frac{\partial}{\partial\eta} v_a(\eta) = -\nabla^c(\hat{K}_{ca} - \tfrac{1}{2}\hat{K}^{rs} g_{rs} g_{ca}), \quad v(\eta = 0) = 0 . \tag{109}$$

To summarize, as $\eta \to \infty, K_{ab}(\eta)$ may either approach a Lie derivative term, such as $L_{v(\eta)} g_{ab}(\eta)$, or a nontrivial deformation $\hat{K}_{ab}(\eta)$ [23]. The nontrivial deformation is present only if the corresponding Ricci-Hamilton flow for $g_{ab}(\eta)$ approach an Einstein metric on Σ which is not isolated. In such a case (e.g., flat tori), there is a finite dimensional set of such Einstein metrics, and the nontrivial $\hat{K}_{ab}$ simply are the infinitesimal deformations connecting one Einstein metric $\bar{g}_1$ in Σ and the infinitesimally nonequivalent one. Also in this case the Lie derivative term may be present. What it represents is a reparametrization of the metric $\bar{g}_1$ [under the action of the infinitesimal group of diffeomorphisms generated by v; see Eq. (107)] ("gauge artifact"). This latter Lie derivative term is the only surviving term when $\bar{g}$ is isolated (like, e.g., in the case of the round three-sphere. As is known, the round metric $\bar{g}$ on the three-sphere S^3 is isolated, in the sense that there are not volume-preserving infinitesimal deformations of $\bar{G}$ mapping it to another inequivalent constant curvature metric $\bar{g}'$) (see Fig. 14).

The flow Eq. (108), Eq. (109), tells us how to renormalize the second fundamental form in such a way that the blocking prescription $\langle f\rangle_{\epsilon_m} \to \langle f\rangle_{\epsilon_{m+1}}$ works both for the initial metric g as well as for the *perturbed* metric $g + \delta K_{ab}$.

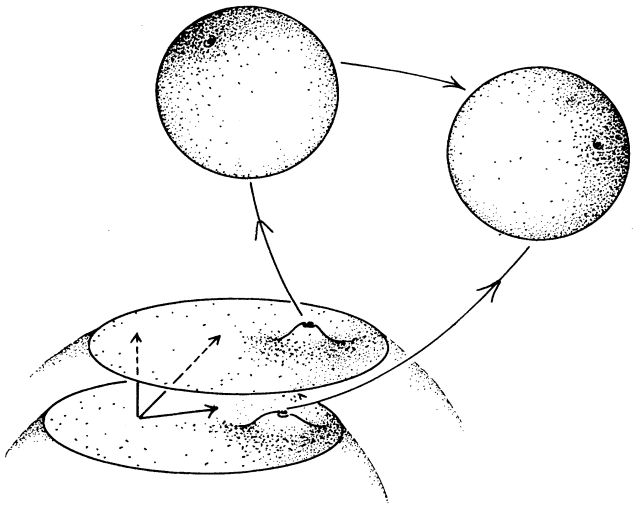

FIG. 14. Two neighboring three-manifolds (Σ, g) and $(\Sigma, g+\delta K)$, with the same volume, evolving under the Ricci-Hamilton flow toward isometric round three-spheres $(S^3, g_{\rm can})$ and $(S^3, g^*_{\rm can})$. In general, $g_{\rm can}$ and $g^*_{\rm can}$ differ by an infinitesimal diffeomorphism ϕ generated by a vector field v, i.e., $g^*_{\rm can} = g_{\rm can} + \delta(L_v g_{\rm can} - \frac{2}{3} g_{\rm can}\nabla_i v^i)$. By exploiting the freedom in choosing the shift vector field $\vec{\alpha}$, we can get rid of ϕ.

A. Scaling and critical exponents

From the characterization of critical fixed points g^*_{ab} for Eq. (55) (see Sec. IV A) we can get information on the critical exponent characterizing critical behavior of the metrics nearby g^*_{ab}. This is the content of this section. In particular, we shall discuss the critical exponents related to the high temperature fixed point $(S^3_{(1)}\sharp S^3_{(2)}, \tilde{g})$. Even if this point is not a thermodynamically interesting critical point, it exhibits many of the general features of the more interesting type of singularities.

In order to characterize these critical exponents we can use the linearized Ricci-Hamilton flow associated with the one-parameter family of metrics $(S^3_{(1)}\sharp S^3_{(2)}, \tilde{g})$. In the following we shall, however, proceed more directly and examine the properties of the two-point correlation function associated with $(S^3_{(1)}\sharp S^3_{(2)}, \tilde{g})$ and the probability law of relevance to our analysis.

Let $u_i \in S^3_{(i)}, i = 1, 2$, be the two points in $(S^3_{(i)}\sharp S^3_{(2)}, \mathcal{G}_{\rm crit})$ around which the round metrics of $S^3_{(i)}$ have been blown up. Let f denote a non-negative scalar field on $(S^3_{(1)}\sharp S^3_{(2)})$, distributed according to the probability law formally defined by

$$dP \equiv \frac{\exp[-H(f)]\prod_x df(x)}{\int \exp[-H(f)]\prod_x df(x)} \ , \qquad (110)$$

where the functional integration is over the space of fields $f : (S^3_{(1)}\sharp S^3_{(2)}) \to \mathbb{R}$, equipped with the L^2 inner product

$$(f|f') \equiv \int_{(S^3_{(1)}\sharp S^3_{(2)})} f(x)f'(x)d\mu_g(x) \ , \qquad (111)$$

and where

$$H(f) = \int_{(S^3_{(1)}\sharp S^3_{(2)})} f(x)d\mu_g(x) \ . \qquad (112)$$

(We could consider f as related to the matter field, as in $f = \alpha\rho + \alpha^i J_i$, but the following analysis is quite independent from a particular meaning of f.) Notice also that the measure $d\mu_g(x)$ in the above formulas is the Riemannian measure on $(S^3_{(1)}\sharp S^3_{(2)})$ associated with the metric $\tilde{g}(x)$ defined by Eq. (76).

Let us now concentrate on the behavior of Eq. (110) when the neck of $(S^3_{(1)}\sharp S^3_{(2)})$ gets thinner and longer under the action of Eq. (55). It can then be checked that along the Ricci-Hamilton flow Eq. (79) associated with $\tilde{g}|_{\rm neck}$ both the L^2 inner product Eq. (111) and the Gibbs factor $\exp[-H(f)]$ corresponding to Eq. (112), are invariant. Thus it follows that the probability measure Eq. (110) is invariant under Eq. (55), and the correlation function defined by

$$E_{dP}[f(y_1)f(y_2)] \ , \qquad (113)$$

where E_{dP} denotes the expectation with respect to dP, and is well defined over the fixed points $\mathcal{G}_{\rm crit}$.

In Sec. IV A we have interpreted the Ricci-Hamilton flow Eq. (55) as the RG flow in a finite geometry, characterized by the length scale L, which is large compared to the microscopic scale (in particular, it is much larger than the radius of the geodesic ball coverings used to discretize the theory). The correlation function depends on such a dimensional parameter. If

$$L(y_1, y_2) \equiv \int_{y_1}^{y_2} \sqrt{D}\, dz \qquad (114)$$

denotes the distance between the points y_1 and y_2 along the cylindrical neck of $(S^3_{(1)}\sharp S^3_{(2)})$, and if this distance is large (as compared with the radius of geodesic ball coverings), then the *correlation length* ξ associated with the two-point (connected) correlation function can be read off from

$$\xi \simeq_{L(y_1,y_2)\gg 1} \frac{-L(y_1, y_2)}{\ln E_{dP}(f(y_1)f(y_2))_{\rm conn}} \ . \qquad (115)$$

Since the correlation function remains invariant under the Ricci-Hamilton deformation of the cylindrical neck, we get that along Eq. (79), $\xi/L(y_1, y_2)$ remains constant which implies that on $(S^3_{(1)}\sharp S^3_{(2)}, \tilde{g})$ the correlation length ξ behaves as

$$\xi \simeq \sqrt{D_0}E_0[E_0 - \tfrac{2}{3}\eta]^{-1} \ . \qquad (116)$$

According to standard usage, we can define the *critical exponent* ν associated with the correlation length of a finite size system (with typical size L) by the condition

$$\frac{\partial}{\partial p}\xi(L, p)|_{p=p_c} \simeq L^{1+1/\nu} \ , \qquad (117)$$

where p is a parameter driving the system to criticality, and p_c is its corresponding critical value. In our case, it is natural to set $p = \eta$, with $p_c = \eta_c = \frac{3}{2}$. This immediately yields for the critical exponent ν the value

$$\nu = 1 \ . \tag{118}$$

A similar computation for the critical exponent associated with the correlation length can be carried out when the connected geometry is $H^2 \times \mathbb{R}^1$. This takes place when two Riemannian manifolds M_1 and M_2, which are supposed to evolve nicely under the Ricci-Hamilton flow, are connected through a tubular neighborhood of a surface S_h of genus h, viz., $\sigma = M_1 \sharp s_h M_2$.

In this case, the metric on the neck can be written as [25]

$$g_{\text{neck}} = D g_{\mathbb{R}^1} + E g_{H^2} \ , \tag{119}$$

where g_H is the metric on the hyperbolic plane. The Ricci-Hamilton flow equations take a form similar to the $S^2 \times \mathbb{R}^1$ case, up to a minus (important) sign, namely,

$$\frac{d}{d\eta} E = \frac{2}{3} \ ,$$

$$\frac{d}{d\eta} D = -\frac{4}{3} \left(\frac{D}{E} \right) \ , \tag{120}$$

which upon integration provide

$$E = E_0 + \frac{2}{3} \eta \ ,$$

$$D = \frac{D_0 E_0^2}{[E_0 + (2/3)\eta]^2} \ , \tag{121}$$

where $E_0^2 = E^2(\eta = 0)$ is the initial scale of the hyperbolic geometry, whereas $D_0 = D(\eta = 0)$ is the scale on $\mathbb{R}^1$.

The scale E of the hyperbolic geometry increases linearly with η, while the scale factor D on the line $\mathbb{R}^1$ shrinks. It can be checked [25] that corresponding to this scale dynamics, the curvature decays according to $\|\text{Ric}\| = \sqrt{2}/(E_0 + \frac{2}{3}\eta)$, and we get in the limit $\eta \to \infty$ a *pancake* degeneracy.

The relevant correlation function is now $E_{dP}(f(y_1) f(y_2))_{\text{conn}}$, with y_1 and y_2 fixed points in the H^2 factor, (i.e., on the surface S_h). Again, owing to the symmetries of the geometry involved, it immediately follows that the ratio between the correlation length ξ and the distance $L(y_1, y_2)$ must remain constant under the Ricci-Hamilton flow. This implies that the correlation length behaves as

$$\xi \simeq \sqrt{E_0 + \tfrac{2}{3}\eta} \ , \tag{122}$$

to which corresponds again a critical exponent $\nu = 1$ [in this case one cannot apply Eq. (117) since the system is actually going to an infinite size].

A striking feature of these topological crossover phenomena, associated with the renormalization of the cosmological matter distribution, is that their pattern resembles the linear sheetlike (or sponge-like) structure in the distribution of galaxies on large scales (e.g., [29]). It is evident that if $(\Sigma, g, K, \rho, \mathbf{J})$, the initial data set for the real universe (see the next section), is close to *criticality*, in the sense discussed in the preceding section, then the corresponding averaged model exhibits a tendency to topological crossover in various regions (the ones where the inhomogeneities are larger). Filamentlike and sheetlike structures would emerge, and the overall situation would be the one where such structures appear together with regions of high homogeneity and isotropy, in some sort of hierarchy. This situation is akin to that of a ferromagnetic nearby its critical temperature, whereby we have islands of spins up and down in some sort of nested pattern. Even if there is a tendency to homogeneity at very large scales, the picture just sketched, of the "hierarchy of structures," might be qualitatively valid in a good part of the universe; indeed recent observational data seem to suggest the existence of still larger and larger structures (e.g., [30]). Notice that this picture bears some resemblance to the "cascade of fluctuations" in critical phenomena [31]. Droplet fluctuations nucleated at the lattice scale in the critical state can grow to the size of the correlation length where the details of the lattice structure become lost and the scale invariant distributions of the large "droplets" are universal.

A possible (though not yet clarified) connection of this whole picture with the self-organized criticality (SOC) [32] can be envisaged, whereby the problem of structure formation in terms of growth phenomena could be tackled in the framework of avalanche activity used in SOC (cf. [33]). It would be particularly interesting to estimate the scales in the Universe, where it is necessarily critical and trapped into self-organized (critical) states (a similar suggestion was recently posed in [34]).

VI. EFFECTIVE COSMOLOGICAL MODELS

The results of the previous sections have interesting consequences when applied in a cosmological setting.

Let us assume that at the scale over which general relativity is experimentally verified, a cosmological model of our universe is provided by evolving a set of consistent initial data $(\Sigma, g, K, \rho, \mathbf{J})$, according to the evolutive part of Einstein's equations.

The data $(\Sigma, g, K, \rho, \mathbf{J})$ describing the interaction between the actual distribution of sources and the inhomogeneous geometry of the physical space, (Σ, g, K), are required to satisfy the Hamiltonian and the divergence constraints, respectively:

$$8\pi G \rho = R(g) - K^{ab} K_{ab} + k^2 \ , \tag{123}$$

$$\nabla^i K_{ih} - \nabla_h k = 16\pi G J_h \ , \tag{124}$$

where $k \equiv K^a{}_a$ (see also Sec. IV, where they were written in terms of the three-metric g_{ab} and its associated conjugated momentum π_{ab}).

According to the blocking and renormalization procedure, discussed at length in this paper, we can implement a coarse-graining transformation on this actual data set by suitably renormalizing the metric and the second fundamental form, while retaining the functional form and the validity of the constraints.

The metric is renormalized according to the Ricci-Hamilton flow Eq. (55):

$$\frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3}\langle R(\eta)\rangle g_{ab}(\eta) - 2R_{ab}(\eta)$$

$$g_{ab}(\eta = 0) = g_{ab} \ .$$

In order to renormalize the second fundamental form, let us rewrite it in terms of the *deformation tensor* H_{ab} defined by

$$H_{ab} \equiv K_{ab} + L_\alpha g_{ab} \ , \tag{125}$$

where $\vec{\alpha}$ is the shift vector field providing the three-velocity of chosen instantaneous observers on Σ, with respect to which we are implementing the renormalization procedure. We renormalize K_{ab} by rescaling this deformation tensor H_{ab} according to Eq. (99), viz.,

$$\frac{\partial}{\partial \eta}H_{ab} = \frac{2}{3}\langle R\rangle H_{ab} + \frac{2}{3}g_{ab}\left(\frac{1}{2}\langle Rg^{ab}H_{ab}\rangle - \frac{1}{2}\langle R\rangle\langle g^{ab}H_{ab}\rangle - \langle R^{ab}H_{ab}\rangle\right)$$
$$-\Delta_L H_{ab} + 2\left[\mathrm{div}^*\left(\mathrm{div}\left(H - \frac{1}{2}(\mathrm{Tr}H)g\right)\right)\right]_{ab} \ , \tag{126}$$

with the initial data $H_{ab}(\eta = 0) = (K_{ab} + L_\alpha g_{ab})_{\eta=0}$. Notice that we renormalize the deformation tensor H_{ab}, rather than K_{ab} directly, because in this way we can get rid of the possible Diff-induced shear which may develop in $\lim_{\eta\to\infty} K_{ab}(\eta)$. Indeed, according to Eq. (107), as $\eta \to \infty$ the solution of this initial value problem, $K_{ab}(\eta)$, approaches a nontrivial deformation $\lim_{\eta\to\infty} \hat{K}_{ab}(\eta)$ plus a Lie derivative term, $\lim_{\eta\to\infty} L_{v(\eta)}g_{ab}(\eta)$. The former is present only if the Ricci-deformed metric $\bar{g}_{ab} = \lim_{\eta\to\infty} g_{ab}(\eta)$ is not isolated.

Since at this stage we are mainly interested in FLRW space-times, let us assume that $\bar{g}_{ab}$ is isolated, while in order to take care of the Diff-induced shear, $\lim_{\eta\to\infty} Lv(\eta)g_{ab}(\eta)$, we can choose the shift vector field α^i in such a way that $\lim_{\eta\to\infty} L_{v(\eta)}g_{ab}(\eta)$ is compensated by $L_\alpha g_{ab}$. Since $L_\alpha g_{ab}$ is a trivial datum for Eq. (99), it is thus sufficient to choose

$$\alpha^i = \lim_{\eta\to\infty} v(\eta) \ . \tag{127}$$

Notice that α^i is the three-velocity vector of the chosen instantaneous observers on Σ, thus Eq. (127) provides a map identifying corresponding points between the initial *actual* manifold $(\Sigma, g, K, \rho, \mathbf{J})$ and its renormalized counterpart $\lim_{\eta\to\infty}(\Sigma, g, K, \rho, \mathbf{J})(\eta)$.

In this section, and mainly for actual computational purposes, we assume that the original inhomogeneous initial data set is such that the Ricci-Hamilton flow is global. As already mentioned, we are interested in connecting an inhomogeneous cosmological space-time to its corresponding FLRW model. This is the case, in particular, if we assume that the original manifold (Σ, g) has a positive Ricci tensor (this case is obviously quite similar to the analysis in [3]; there are, however, important differences that we are going to emphasize). Or more generally, this is the case if we assume that the original manifold (Σ, g) is in the SU(2) basin of attraction or *nearby* the critical point $\sharp_i S^3_{(i)}$, with $i = 1, 2, \dots$, yielding for a manifold (Σ, g), nucleating under the Ricci-Hamilton flow Eq. (81) (extended to many connected sums), to disjoint three-spheres $S^3_{(i)}$.

Given this setting, we see that due to the properties of the Ricci-Hamilton flow we have $\lim_{\eta\to\infty} K_{ab}(\eta) = \frac{1}{3}\bar{k}\bar{g}_{ab}$. The given K_{ab} is deformed by gradual elimination of its shear $K_{ab} - \frac{1}{3}kg_{ab}$ and the original (position-dependent) rate of volume expansion k is being replaced with its corresponding average value.

Since the constraints, Eq. (123) and Eq. (124), are required to hold at each step of the renormalization procedure, we get

$$8\pi G(\eta)\rho(\eta) = R(g(\eta)) - K^{ab}(\eta)K_{ab}(\eta) + k^2(\eta) \ , \tag{128}$$

$$\nabla(\eta)^i K_{ih}(\eta) - \nabla(\eta)_h k = 16\pi G(\eta)J_h(\eta) \ , \tag{129}$$

where we have explicitly introduced a possible η dependence into the gravitational coupling G.

Let us now explore the consequences of Eqs. (128) and (129). From the stated hypothesis on the Ricci-Hamilton flow it follows immediately that, $K_{ab}(\eta) \to \frac{1}{3}\bar{k}\bar{g}_{ab}$ as $\eta \to \infty$, thus Eq. (129) implies that

$$\lim_{\eta\to\infty} J_h(\eta) = 0 \ . \tag{130}$$

In order to analyze Eq. (128), we will make use of a property of the Ricci-Hamilton flow, namely, that the flow $K(\eta)$, solution of Eq. (99), is such that $\partial/\partial\eta\langle k(\eta)\rangle_\eta = 0$, i.e., the space average of the trace of the second fundamental form remains constant during the deformation. This allows us to write

$$\langle k\rangle_0^2 = \bar{k}^2 \ , \tag{131}$$

since in the limit, the volume expansion is simply a constant, and where $\langle\cdots\rangle_0$ denotes the full average of the enclosed quantity with respect to the initial metric.

Equation (131) provides the Hubble constant on the FLRW time slice associated with the smoothed data corresponding $(\Sigma, g, K, \rho, \mathbf{J})$.

More explicitly, let us write the FLRW metric in the standard form (units $c = 1$)

$$ds^2 = -dt^2 + S^2(t)d\sigma^2 \ , \tag{132}$$

where $d\sigma^2$ is the metric of a three-space of constant curvature and it is time independent. As we are interested in the three-sphere case, the metric $d\sigma^2$ can be written as

$$d\sigma^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{133}$$

Since the volume $\mathrm{vol}(\Sigma, g)$ of the original inhomogeneous manifold is preserved by the Ricci-Hamilton flow, we can relate the factor $S^2(t)$, providing the inverse (sectional) curvature of the FLRW slice $t = t_0$, to $\mathrm{vol}(\Sigma, g)$,

$$S^2(t_0) = \left(\frac{\mathrm{vol}(\Sigma, g)}{2\pi^2}\right)^{2/3} \ . \tag{134}$$

Notice in particular that the scalar curvature, towards which $R(g(\eta))$ evolves under the Ricci-Hamilton flow, is given by

$$\bar{R} \equiv \lim_{\eta\to\infty} (g(\eta)) = \frac{3}{S^2} = 3\left(\frac{\mathrm{vol}(\Sigma, g)}{2\pi^2}\right)^{-2/3} \ . \tag{135}$$

Having said this, Eq. (128) becomes, in the limit, after extracting a trace-free part of K_{ab},

$$8\pi\bar{G}\bar{\rho} = \bar{R} + \frac{2}{3}\bar{k}^2 \ , \tag{136}$$

since no residual shear survives, and where we have introduced the renormalized gravitational coupling

$$\bar{G} \equiv \lim_{\eta\to\infty} G(\eta) \ . \tag{137}$$

On the other hand, Eq. (128) gives

$$\frac{2}{3}k^2(\eta) = 8\pi G\rho(\eta) - R(g(\eta)) + \tilde{K}^{ab}\tilde{K}_{ab}(\eta) \ , \tag{138}$$

where the shear $\tilde{K}_{ab} \equiv K_{ab} - \frac{1}{3}kg_{ab}$ has been explicitly introduced.

We can rewrite the last equation upon taking the average with respect to the initial metric, as

$$\frac{2}{3}\langle k^2\rangle_0 = 8\pi G\langle\rho\rangle_0 - \langle R\rangle_0 + \langle\tilde{K}^{ab}\tilde{K}_{ab}\rangle_0 \ . \tag{139}$$

By exploiting Eqs. (131) and (139) we can get the Hubble constant of the FLRW model we want to use to describe the time evolution of the renormalized initial data set $\lim_{\eta\to\infty}(\Sigma, g, K, \rho, \mathbf{J})(\eta)$.

A. The renormalized Hubble constant

Taking into account Eq. (131), and noting that the Hubble constant on the FLRW slice corresponding, via the Ricci-Hamilton flow, to (Σ, g, K) is

$$H_0^2 = \left(\frac{1}{S}\frac{dS}{dt}\right)^2 = \frac{2}{9}\bar{k}^2 \ , \tag{140}$$

we get

$$H_0^2 = \frac{8\pi G}{3}\langle\rho\rangle_0 - \frac{1}{3}\langle R\rangle_0 + \frac{1}{3}\langle\tilde{K}^{ab}\tilde{K}_{ab}\rangle_0 - \frac{2}{9}(\langle k^2\rangle_0 - \langle k\rangle_0^2) \ . \tag{141}$$

This is a rather tautological rewriting of Eq. (140), justified by the fact that in this form H_0^2 clearly shows a contribution from the spatial average of the shear. In particular, if originally (i.e., for $\eta = 0$) k is spatially constant, Eq. (141) reduces to

$$H_0^2 = \frac{8\pi G}{3}\langle\rho\rangle_0 - \frac{1}{3}\langle R\rangle_0 + \frac{1}{3}\langle\tilde{H}^{ab}\tilde{H}_{ab}\rangle_0 + \frac{1}{3}\langle\tilde{L}_\alpha g_{ab}\tilde{L}_\alpha g_{cd}g^{ac}g^{bd}\rangle_0 \ , \tag{142}$$

where we have explicitly used the decomposition $K_{ab} = H_{ab} - L_\alpha g_{ab}$ [see Eq. (125)], and

$$\tilde{L}_\alpha g_{ab} \equiv \nabla_a\alpha_b + \nabla_b\alpha_a - \tfrac{2}{3}g_{ab}\nabla_c\alpha^c \tag{143}$$

is the conformal Lie derivative of the metric g_{ab} along the vector field α.

The above expressions for H_0 correspond, through Eq. (136), to the matter density distribution

$$8\pi\bar{G}\bar{\rho} = 8\pi G\langle\rho\rangle_0 + \bar{R} - \langle R\rangle_0 + \langle\tilde{K}^{ab}\tilde{K}_{ab}\rangle_0 + \tfrac{2}{3}(\langle k^2\rangle_0 - \langle k\rangle_0^2) \ . \tag{144}$$

We have to consider this expression for$\bar{G}\bar{\rho}$ as the *renormalized* effective sources entering into the Friedmann equation, if we want to describe the real locally inhomogeneous Universe through a corresponding idealized FLRW model. As expected, the renormalized matter density shows contributions of geometric origin, either coming from the shear anisotropies or from the local fluctuations in curvature and volume expansion.

At this stage, a few comments are in order concerning, in particular, the effective Hubble constant H_0 Eq. (141). First, we wish to emphasize that this is the theoretical expression for the Hubble constant if one wishes to model the real locally anisotropic and inhomogeneous Universe with a corresponding FLRW one. Expression Eq. (141) clearly shows that apart from the expected contributions to the expansion rate coming from matter and curvature, there are two further contributions:

(i) A negative contribution coming from the local fluctuations in the expansion rate. Apparently, this is not a very significant term, since we may choose, as a convenient initial data set describing the real locally inhomogeneous Universe, a data set supported on a three-manifold Σ of constant *extrinsic time* (i.e., k spatially constant on Σ). Actually, the optimal choice of the data set with respect to which smooth-out the lumpiness in matter and geometry is a deep and interesting question on which we comment later on, in Sec. VI C.

(ii) A positive contribution coming from the shear term $\frac{1}{3}\langle\tilde{H}^{ab}\tilde{H}_{ab}\rangle_0$, and from the observer-dependent term $\frac{1}{3}\langle\tilde{L}_\alpha g_{ab}\tilde{L}_\alpha g_{cd}g^{ac}g^{bd}\rangle_0$ which depends on the three-velocity $\vec{\alpha}$ of the observers with respect to the renormalization is carried out.

The latter terms are the most important nonstandard contributions to H_0^2, and have their origin both in the presence of the gravitational radiation

$$\frac{1}{3}\langle \tilde{H}_{\perp}^{ab}\tilde{H}_{\perp ab}\rangle_0 \ , \tag{145}$$

where $\tilde{H}_{\perp}$ is the divergence-free part of the deformation tensor $\tilde{H}_{ab}$, and in the anisotropies generated by the motion of matter

$$\tfrac{1}{3}\langle \tilde{H}_{\parallel}^{ab}\tilde{H}_{\parallel ab}\rangle_0 \ , \tag{146}$$

where $\tilde{H}_{\parallel}$ is the longitudinal part of the deformation tensor, obtained as a solution to the equation

$$\nabla^i(\tilde{K}_{\parallel})_{ih} - \tfrac{2}{3}\nabla_h k = 16\pi G J_h + \nabla^i(\tilde{L}_\alpha g_{ih}) \ . \tag{147}$$

Recall that the shift vector field $\vec{\alpha}$ is connected to the current density $\vec{J}$ by the requirement that it is *chosen* in such a way as to eliminate the longitudinal shear in $\lim_{\eta\to\infty} K_{ab}(\eta)$. Also, notice that

$$\langle \tilde{H}^{ab}\tilde{H}_{ab}\rangle_0 = \langle \tilde{H}_{\perp}^{ab}\tilde{H}_{\perp ab}\rangle_0 + \langle \tilde{H}_{\parallel}^{ab}\tilde{H}_{\parallel ab}\rangle_0 \ . \tag{148}$$

In practice, the deformation energy $\langle \tilde{H}^{ab}\tilde{H}_{ab}\rangle_0$ yields a contribution to the Hubble constant H_0 which can be roughly estimated by exploiting the anisotropy measurements in the cosmic microwave background (CMB) radiation, as long as the frame used in averaging (i.e., the lapse α and the shift α^i) is, on the average, comoving with the cosmological fluid. We also require that the original locally inhomogeneous manifold (Σ, g, K) does not differ too much for a standard FLRW t =const slice. In such a case one can apply the analysis of [35] to conclude that, at the present epoch, the ratio between the deformation shear $\tilde{H}_{ab}$ and expansion is of the order

$$\left(\frac{|\tilde{H}_{ab}|}{H}\right)_0 < 4\epsilon \ , \tag{149}$$

where $\epsilon \equiv \max(\epsilon_1, \epsilon_2, \epsilon_3)$ denotes the upper limit of currently observed anisotropy in the CMB radiation temperature variation, and where $\epsilon_1, \epsilon_2, \epsilon_3$, respectively, denote the dipole, quadrupole, and octopole temperature anisotropies. On choosing $\epsilon \simeq 10^{-4}$, as indicated by the recent CMB radiation anisotropy measurements, one gets that the shear deformation is at most about 10^{-3} of the expansion [35].

Thus, as long as we assume that the original locally inhomogeneous manifold (Σ, g, K) does not differ too much from a standard FLRW $t =$ const slice, the contribution to H_0 from $\langle \tilde{H}^{ab}\tilde{H}_{ab}\rangle_0$ is certainly quite small, and we can reliably write

$$H_0^2 \simeq \frac{8\pi G}{3}\langle \rho\rangle_0 - \frac{1}{3}\langle R\rangle_0 + \frac{1}{3}\langle \tilde{L}_\alpha g_{ab}\tilde{L}_\alpha g_{cd} g^{ac}g^{bd}\rangle_0 \ . \tag{150}$$

The last term involving the shift vector $\vec{\alpha}$ is a velocity effect term which is by no means small, at least *a priori*. Thus the renormalized Hubble constant needed, to describe by a model FLRW a locally inhomogeneous and anisotropic universe, is *not* provided by a naive average even if the actual universe, to be smoothed out in FLRW modeling, is not too far from homogeneity and isotropy. Clearly, we are expecting that in such a case the actual contribution of the observer-velocity term is rather small, but this is a question which is difficult to handle. Technically speaking, we would need estimates on the size of the Diff-induced shear generated upon smoothing by the Ricci-Hamilton flow. This is an issue under current investigation. We wish also to stress that in this velocity term there is hidden a nontrivial scale dependence. Indeed, assuming that we wish to model the actual universe by a FLRW one only up to a certain scale, then the shift vector $\vec{\alpha}$ needed to cure the Diff-induced shear (which is tantamount to saying the proper selection of a frame with respect to which we are carrying out the partial smoothing) depends on such a scale, and the larger the scale the larger the contribution.

There is another intriguing possible explanation for a larger than expected Hubble constant. Indeed, if we take seriously the possibility that the real Universe may be close to the critical phase, as argued in the previous sections, then the contribution from the shear is not just a conceptual value. For example the original data set (Σ, g, K) may be near the critical surface associated with the $\sharp_i S^3_{(i)}$ critical point. In this case we may generate, upon smoothing, a whole family of disconnected T =const FLRW slices, each one with its own Hubble constant $H_0(i)$. Explicitly, let us assume that the initial data set (Σ, g, K) is attracted upon averaging towards a critical $(S^3_{(1)}\sharp S^3_{(2)}\sharp\cdots\sharp S^3_{(n)})$, this being the case if the original metric g exhibits regions of large inhomogeneities. Assuming, for simplicity, that the rate of volume expansion is spatially constant, we get that each $S^3_{(i)}$ factor, $i = 1, \ldots, n$, inherits a Hubble constant provided by

$$H_0^2(i) = \frac{8\pi G}{3}\langle \rho_0\rangle_0 - \frac{1}{3}\langle R\rangle_0 + \frac{1}{3}\langle \tilde{K}^{ab}\tilde{K}_{ab}\rangle_0 \ . \tag{151}$$

These Hubble constants can be quite dominated by the large anisotropies $\langle \tilde{K}^{ab}K_{ab}\rangle_0$ of the original manifold. Indeed, the previous estimate on the smallness of the shear term is, strictly speaking, valid only in the observable domain defined by our past light cone from last scattering to the present day. Thus, if the $S^3_{(i)}$ factors resulting from the critical behavior of (Σ, g, K) are not smaller than the spatial sections intercepted by the interior of this light cone, we would not notice the contribution from the local shear (since we would have been looking at a rather homogeneous and isotropic island), the large contribution would come from the regions of large inhomogeneities and anisotropies which, under the Ricci-Hamilton renormalization, undergo the topological crossover. Thus, a value of the Hubble constant may quite well depend on a possible large shear outside our observable domain.

The behavior just described also shows that the RG procedure developed preserves the size of the regions where spatial homogeneity and isotropy is observed, even if large inhomogeneity may be globally present. This tells us what are the averaging scales on which a FLRW model is a good approximation. More precisely, in order to determine the averaging scale associated with a FLRW model we want to use, to describe (Σ, g, K), we need to understand more in detail the (conjectured) Hamilton-Thurston decomposition of (Σ, g, K). The specific example discussed in Sec. IV C may serve as an indication.

B. The scale dependence of the matter distribution

In the above analysis, we also introduced a renormalized gravitational coupling. In a sense this is superfluous since the three-dimensional metric g of Σ is acting as the running coupling constant, and we can always reabsorb G into a definition of g. Nevertheless, the use of the renormalized coupling $\bar{G}$ may be helpful if one wishes to use the standard average of matter $\langle\rho\rangle_0$ in the Friedmann equation, rather than the effective matter distribution $\bar{\rho}$. The explicit expression for $\bar{G}$ can be easily obtained by setting $\bar{\rho} = \langle\rho\rangle_0$ in Eq. (144):

$$8\pi\bar{G} = 8\pi G + \frac{\bar{R} - \langle R\rangle_0 + \langle\tilde{K}^{ab}\tilde{K}_{ab}\rangle_0 - \frac{3}{3}(\langle k^2\rangle_0 - \bar{k}^2)}{\langle\rho\rangle_0} . \tag{152}$$

Notice however, that it is $G(\eta)\rho(\eta)$ which is inferred from measurements for different scales, and thus the use of $\bar{G}$ is not particularly remarkable.

In this connection, it is more important to discuss the dependence of $G(\eta)\rho(\eta)$as the local scale is varied, namely as η increases (recall that η is the logarithmic change of the cutoff length associated with the geodesic ball coverings). For simplicity, we do this only for the case in which no shear is present ($K_{ab} = 0$), and the rate of volume expansion is spatially constant (k =const). Under such hypothesis, we get for the scale dependence of the average $\langle G\rho\rangle$

$$\frac{\partial}{\partial\eta}[8\pi\langle G(\eta)\rho(\eta)\rangle] = \frac{\partial}{\partial\eta}\langle R(g(\eta))\rangle = 2\langle\tilde{R}^{ik}\tilde{R}_{ik}\rangle + \frac{1}{3}(\langle R\rangle^2 - \langle R^2\rangle) , \tag{153}$$

where $\tilde{R}_{ik} = R_{ik} - \frac{1}{3}g_{ik}R$ is the trace-free part of the Ricci tensor. From this expression we see that, not only shear anisotropies but, also metric anisotropies favor an increasing in $G(\eta)\rho(\eta)$ with the scale. To give an explicit example, let us consider as an initial metric to be smoothed out a locally homogeneous and anisotropic SU(2) metric g. Following the notation and the analysis in the paper of Isenberg and Jackson [25], we can write such a metric and its Ricci-Hamilton evolution, in terms of a left-invariant one-form basis $\{\theta^a\}$, $a = 1, 2, 3$, on SU(2) as

$$g = A(\eta)(\theta^1)^2 + B(\eta)(\theta^2)^2 + C(\eta)(\theta^3)^2 , \tag{154}$$

where A, B, C are scale (η)-dependent variables. With respect to this parametrization, the scalar curvature is given by

$$R(\eta) = \tfrac{1}{2}\{[A^2 - (B-C)^2] + [B^2 - (A-C)^2] + [C^2 - (A-B)^2]\} . \tag{155}$$

While the squared trace-free part of the Ricci tensor is given by

$$\begin{aligned}\|\tilde{\mathrm{Ric}}\|^2 = {} & \tfrac{1}{6}\{[A^2 - (B-C)^2]^2 + [B^2 - (A-C)^2]^2 + [C^2 - (A-B)^2]^2\} \\ & - \tfrac{1}{6}\{[A^2 - (B-C)^2][B^2 - (A-C)^2] + [A^2 - (B-C)^2][C^2 - (A-B)^2]\} \\ & + \tfrac{1}{6}[B^2 - (A-C)^2][C^2 - (A-B)^2] .\end{aligned} \tag{156}$$

The Ricci-Hamilton flow for this metric g exponentially converges to the fixed point $A = B = C = 1$, with the normalization $ABC = 1$, and with $A(\eta) \geq B(\eta) \geq C(\eta)$ for all η [25]. From the above expression for $R(\eta)$ it follows that $R(\eta)$ monotonically increases from its initial value $R(\eta = 0)$ and exponentially approaches

$$\lim_{\eta\to\infty} R(\eta) = \tfrac{3}{2} . \tag{157}$$

This increase is generated by the exponential damping of the anisotropic part of the Ricci tensor Eq. (156) which is smoothed by the Ricci-Hamilton flow and, roughly speaking, is redistributed uniformly in the form of scalar curvature.

Notice that if the anisotropy is large, the actual increase from $R(\eta = 0)$ to $\lim_{\eta\to\infty} R(\eta) = \bar{R}$ may be quite significant. For instance, if for $\eta = 0$, we have $A(\eta = 0) = 1$, $B(\eta = 0) = 2$, $C(\eta = 0) = 1/2$, we get $R(\eta = 0) = 7/8$, while $\bar{R} = 3/2$. The renormalized scalar curvature will have increased by nearly as much as 70% of its original value.

Thus, if we smooth-out the initial data set ($\Sigma \simeq S^3, g, K, \rho, \mathbf{J}$), with g the above SU(2) metric, $K_{ab} = \frac{1}{3}g_{ab}k$, with k =const, $\mathbf{J} = 0$, and with the matter density ρ such that the Hamiltonian constraint holds, we get that $G(\eta)\rho(\eta)$ monotonically increases as $\eta \to \infty$, exponentially approaching a fixed value.

Under the above simplifying assumptions concerning the shear and the rate of volume expansion, we get

$$8\pi\bar{G}\bar{\rho} = 8\pi G\langle\rho\rangle_0 + \bar{R} - \langle R\rangle_0 \tag{158}$$

and, as remarked above, the correction term $\bar{R} - \langle R\rangle_0$ may be quite large, of the same order of magnitude as the naive average $8\pi G\langle\rho\rangle_0$. More generally, we can easily get an expression for the variation of $G(\eta)\langle\rho(\eta)\rangle$ as η is increased from a given scale, say from η_0 to a larger scale $\eta = \eta_0 + \delta\eta$.

From Eq. (153) we immediately get

$$8\pi G(\eta)\langle\rho(\eta)\rangle = 8\pi G(\eta_0)\langle\rho(\eta_0)\rangle + 2\langle\tilde{R}^{ik}\tilde{R}_{ik}\rangle_{(\eta_0)}\delta\eta + \tfrac{1}{3}(\langle R\rangle^2 - \langle R^2\rangle)_{(\eta_0)}\delta\eta + O(\delta\eta^2) \ . \tag{159}$$

With our simplifying assumptions, that there is no shear and that the rate of volume expansion is spatially constant, we can exploit the Hamiltonian constraint and rewrite Eq. (159) as

$$G(\eta)\langle\rho(\eta)\rangle = G(\eta_0)\langle\rho(\eta_0)\rangle + \frac{1}{4\pi}\langle\tilde{R}^{ik}\tilde{R}_{ik}\rangle_{(\eta_0)}\delta\eta - \frac{8}{3}\pi G^2(\eta_0)\langle\rho(\eta_0)\rangle^2\left(\frac{\langle\rho^2(\eta_0)\rangle - \langle\rho(\eta_0)\rangle^2}{\langle\rho(\eta_0)\rangle^2}\right)\delta\eta + O(\delta\eta^2) \ . \tag{160}$$

Thus, in leading order, a large density contrast $\langle\rho^2(\eta_0)\rangle - \langle\rho(\eta_0)\rangle^2/\langle\rho(\eta_0)\rangle^2$, at a given observational scale η_0, tends to reduce the value of $G(\eta)\langle\rho(\eta)\rangle$ as the scale of observation is increased. This reduction effect should dominate on local scales where large inhomogeneities are present. On sufficiently large scales η_0, cosmological data suggest that the density contrast tends to decrease, and in such a case, as the scale of observation is further increased, Eq. (160) shows that $G(\eta)\langle\rho(\eta)\rangle$ now tends to become larger, with deviations from the original value driven by anisotropies in the curvature.

Thus, the behavior of the product $G(\eta)\langle\rho(\eta)\rangle$ under a variation of the observational scale is related to a competition between anisotropy and density contrast, and it may be of significance in the correct interpretation of recent cosmological data. In particular, a sufficiently large spatial anisotropy (even maintaining local homogeneity) may increase $G(\eta)\langle\rho(\eta)\rangle$ in a significant way as η is increased.

C. The choice of the initial data set

The general picture arising from the above analysis is that we pick up an appropriate initial data set which, when evolved, gives rise to the *real* space-time. The description of this data set and of the resulting space-time is too detailed for being of relevance to cosmology. Intuitively, one would like to eliminate somehow all the unwanted (coupled) fluctuations of matter and space-time geometry on small scales, and thus extract the effective dynamics capturing the global dynamics of the original space-time. The possibility of actually implementing such an approach is strongly limited by the fact that we do not know *a priori* the structure of the space-time we are dealing with. But we may alternatively decide to handle the unwanted fluctuations at the level of data sets, since the time evolution of the initial data set for the Einstein equations is actually determined by the very constraints which that data set has to satisfy. As we have seen above, this can be done quite effectively in the framework of the renormalization group which is naturally well suited to this purpose. However, and here we come to the point we wish to make clear, *different initial data sets giving rise to the same inhomogeneous and anisotropic space-time, may yield smoothed data set giving rise to different FLRW space-times* (see Fig. 15). In other words, the renormalization procedure and the dynamics do not commute. The dynamics gives rise to the crossover between different FLRW space-times or more generally, between different renormalized models of the same original irregular space-time.

This situation is in fact not so paradoxical as it may seem. From a thermodynamical point of view, we have seen that one of the members of the initial data set, namely the three-metric g, plays the role of a temperature. Thus, by varying the metric, one can move through the possible *pure phases* of the thermodynamical system considered. In this sense, a real, locally inhomogeneous universe is to be considered as akin to a generic complex thermodynamical system. The possible locally homogeneous cosmological models, arising from it by suitable choices of initial data set to smooth out, correspond to its distinct pure phases. The resulting dynamics yields, in an analogy with common statistical system, a dynamical crossover between different pure phases. This makes accessible in cosmology too, the whole subject of critical phenomena with a plethora of interesting consequences. Critical phenomena are always manifested macroscopically, as phase transitions are collective phenomena in their nature. This aspect may turn out to be of importance for the study of structure formation and clustering in the universe.

These remarks relate familiar statistical mechanics behavior to the most delicate aspect of coarse-graining in cosmology, namely, its covariance properties [36]. It is clear that if we adopt the standard position of demanding full (space-time) covariance of the coarse-graining procedure adopted, then a renormalization group approach as ours is on a rather shaky ground. However, from a physical point of view, coarse graining in cosmology naturally makes a spatial three-metric (rather than the space-time metric), act as a running coupling constant. This indicates that the averaging issue is deeply connected to the optimal choice of the slicing (the *time coordinate*), with respect to which the coarse-graining must be implemented. For instance, in our approach, different choices of the slicing may drive the system to quite distinct fixed points under the renormalization procedure. Thus our viewpoint is that as long as we address the averaging issue in cosmology we cannot use full space-time covariance at a naive face value, and progress on this problem is strictly related to a proper selection of a physical frame of reference with respect to which the averaging is carried out.

Thus, the issue that now needs consideration concerns the existence of natural initial data sets, in the real lumpy universe, with respect to which the above renormalization group smoothing can be implemented. As remarked previously, the formalism is particularly well suited to a

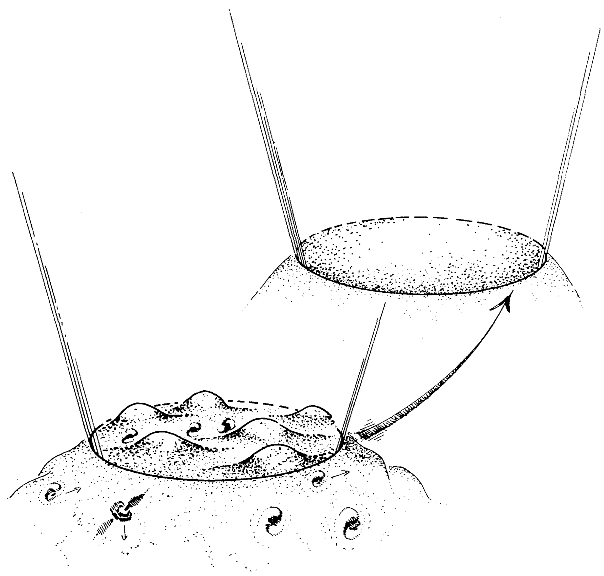

FIG. 15. The time evolution of an inhomogeneous and anisotropic initial data set and its Ricci-Hamilton renormalized counterpart. Note that $(\bar{M}^4, \bar{g}^{(4)})$ depends in a *sensible way* on the particular initial data set $(\Sigma, g, K, \rho, \mathbf{J})$ which is renormalized according to the Ricci-Hamilton flow. Different initial data sets, corresponding to the same inhomogeneous and anisotropic space-time $(M^4, g^{(4)})$, may generate *distinct* model space-times $(\bar{M}^4, \bar{g}^{(4)})$. This seemingly paradoxical situation has a natural explanation in terms of the RG approach.

data set supported on slices of constant extrinsic time (viz., spatially constant rate of volume expansion, k). This is so simply because the Ricci-Hamilton flow, characterizing the scale dependence in the fluctuations of geometry, is most conveniently normalized to preserve volume. However, depending on the particular geometry we wish to smooth, different normalizations can be envisaged too [25], and the relevance of constant extrinsic time initial data set can be simply traced back to its importance in the standard analysis of the initial value problem in relativistic cosmology.

A suitable slice of a frame comoving with matter is another natural choice that almost immediately comes about. However, in general, we cannot choose surfaces orthogonal to the cosmic fluid flow lines, since if rotation is present, they do not exist. A more proper choice would be to select the surface of constant matter density as the surfaces of constant time (see [37] and references quoted therein). It has been argued recently that this choice is rather optimal. In such a slicing [37], and in the observable domain, the space-time metric takes the form

$$g^{(4)} = -A^2(x_\alpha) dt \otimes dt + S^2(t, x^h) f_{ij}(t, x^h) dx^i \otimes dx^j \ , \qquad (161)$$

where $\alpha = 1,2,3,4,\ i,j = 1,2,3$. From an obervative point of view, the function $A^2(x^\alpha)$ is nearly constant since the acceleration of the corresponding timelike congruence is small; $S^2(t, x^h)$ is nearly independent of time since the expansion of the congruence is almost spatially constant, and finally also $f_{ij}(t, x^h)$ is nearly independent of time, because the shear is nearly zero. These remarks suggest that the slices of constant time characterized in this way are, at least in our observative domain, nearly spaces of constant curvature, so that they are the most suitable for implementing the above smoothing procedure.

In this connection note that for an initial data set associated with such a surface of constant matter density, some of the formulas of the previous sections are simplified. We refer in particular to the expression Eq. (160) providing the scale dependence of $G(\eta)\langle\rho(\eta)\rangle$, which now reduces to

$$G(\eta)\langle\rho(\eta)\rangle = G(\eta_0)\langle\rho(\eta_0)\rangle + \frac{1}{4\pi}\langle \tilde{R}^{ik}\tilde{R}_{ik}\rangle_{(\eta_0)}\delta\eta + O(\delta\eta^2) \ . \qquad (162)$$

Thus, for an initial data set associated with a surface of constant matter density, the product $G(\eta)\rho(\eta)$ increases with the averaging scale as we average out the local anisotropies in the geometry. And, as remarked in the preceding section, $\bar{G}\bar{\rho} - G\rho$ can be quite large, of the same order of magnitude of $G\rho$, if $f(x^h, t)$ is sufficiently spatially anisotropic.

Finally, one may wish to choose initial data set corresponding to a frame minimizing the anisotropies in the CMB radiation. This is characterized by that unique four-velocity which eliminates the CMB dipole. This corresponds to a family of observers moving with the background radiation. This choice has an advantage that it can be accurately determined by local observations, but it is manifestly not very suitable to use in the Ricci-Hamilton formalism, as developed here.

VII. CONCLUDING REMARKS

According to the contents of this paper the key idea on which our whole analysis rests is that of RG, namely, the involved physics is that of the running (scale dependence, be it energy or momentum scales) of the couplings and the relevant quantities, accordingly. In each case it is the presence of "fluctuations" (of any kind) that requires a scale-dependent redefinition ("dressing") of the physical parameters which can, in turn, modify them, as well as the very structure of the theory, in a nontrivial way. Applications of the RG to particle physics have usually been in the ultraviolet limit (e.g., in QED, QCD, GUT) whereas in condensed matter physics they have been in the infrared limit, in the study of critical phenomena and phase transitions.

We have taken this infrared direction is cosmology. The application of the concept of running of the physical quantities, motivated by RG, appears to be a new important feature in a cosmological setting, providing (at least) partial explanation of some controversies of standard cosmology, which we are going to discuss below. Let us first point out that generally running, also, of cosmological quantities is as such motivated by the asymptotically free higher derivative quantum gravity, according to which the gravitational constant is asymptotically free [38]. Taking into account this fact, of running G, one can explore its consequences in the standard FLRW cosmology (cf. [39,40]).

A word of caution is in order—since we do not have an ultimate theory of quantum gravity (QG), such approaches to cosmology are not on a rigorous basis from a theoretical point of view and should rather be taken as phenomenological. In principle, once we have a valid QG theory will one be able to directly derive a RG equations for various cosmological quantities. Although we have taken here a more standard view in which a split exists between the background (associated with infrared effects) and renormalization of fluctuations, there may well be scales where such a split is not sensible at all. However, a RG capable of interpolating between the qualitatively different degrees of freedom in a parameter space of gravity, remains to be developed which may after all be possible as well after QG theory is within our reach.

One of the major issues in modern cosmology is concerned with the value of the Hubble constant and the apparent conflict between the observed age of the Universe and the predicted one, in the standard FLRW model, based on the recent measurements of the Hubble constant. Namely, the recent measurements of the Hubble constant using the Virgo cluster (distance $\simeq$ 15 Mpc) strongly support that the Hubble constant H_0 has somewhat larger value[6] $h = 0.87 \pm 0.07$ [41]. At the same time, other distance indicators yield a systematically smaller value of H_0, e.g., around 0.55 ± 0.08 using NGC 5253 at a distance of 4 Mpc [42], while an analysis of the gravitational lensing of quasistellar object (QSO) 0.957+561 indicates $h = 0.50 \pm 0.17$ [43]. When one now calculates the age of the Universe, using the larger value of H_0, in the FLRW model one runs into a serious problem, as the predicted age turns out to be too small to accommodate the measured ages ($\sim$ 14–18 Gyr) of the globular clusters in our galaxy [44,45].

Moreover, typical inferred values of the density parameter $\Omega_0 = \rho_0/\rho_0^{\rm crit}$ [ρ_0 is the present value of the total energy density of the Universe and $\rho_0^{\rm crit}$ its present critical energy density, defined as $\rho_0^{\rm crit} = 3H_0^2/(8\pi G)$ where G is Newton's gravitational constant] increase correspondingly with the increasing size of various structures (e.g., [46]). These measurements can at most account for a fraction of Ω_0 which, according to the inflationary paradigm, should be equal to 1. This in turn is one of the reasons for postulating the existence of nonbaryonic dark matter (DM) which is also required to explain the structure formalism.

Various people have since then looked at possible theoretical alternatives to these DM scenarios, such as, e.g., introducing a cosmological constant in the Einstein equations or *ad hoc* modifications of the usual theory of gravity. The important point in this respect may as well be the one addressed in this paper. It is usually taken for granted that, on large scales, the Universe is described by the FLRW solution. There is no alternative really since we do not know any solution of Einstein's equations capable of describing a clumpy universe. Nevertheless, even in the absence of explicit fine-grained models, we would like to know how in principle, and when, one could extract a background model from an inhomogeneous one, such that (i) they both obey, "approximately," the Einstein equations despite the averaging or smoothing involved, and (ii) observational determinations of cosmological parameters ($H_0, \Omega_0, \ldots$) correspond in a sensible way to that mathematical averaging procedure. Thus an issue of importance for cosmology [47], is the question on what scale is the FLRW model supposed to describe the universe. Likewise, what averaging scale are we referring to when we give the value of Ω_0, whose definition necessarily refers to an idealized, i.e., smoothed background model?

There has been recently an increased effort in this direction with some interesting results, as, e.g., that the coarse-graining effects could be non-negligible in the context of affecting the age of the universe. For example, [48] considered a model, of locally open (underdense) universe embedded in the spatially flat universe, in which the expansion rate in our local universe is larger than the global average. A similar model was considered in [49], where a local void in the global FLRW model was studied and the inhomogeneity described by the Lemître-Tolman-Bondi solution. The results indicate that if we happened to live in such a void, but insisted on interpreting the observations by the FLRW model, the Hubble constant measurements could give results depending on the separation of the source and the observer, providing a possible explanation for the wide range of their reported values and capable of resolving the age-of-Universe problem.

On the other hand [39] studied the QG effects at cosmological scales (the phenomenon in question is exactly that of quantum coherence, known to happen on macroscopic scales of the order of cm), assuming asymptotic freedom of the gravitational constant and incorporating running G, according to the appropriate RG equations, into the FLRW model. Such G takes the value of Newton's constant G_N at short distances but then slowly rises as distance increases. However, as mentioned in [39], the RG equations used there might not be applicable in the infrared regime studied, but these concerns were put aside, having in mind the absence of any other available β function for QG in the infrared regime.

One can also approach the averaging problem, modifying the FLRW metric (or equivalently cosmological principle) and the Einstein equations, by an introduction of a generalized scale factor which depends both on t and the scale r [50]. This introduces the scale dependence of $(G\rho)$ and running of other cosmological quantities such as H_0, Ω_0, and the age of the universe t_0, as functions of distance scales.

Let us note that this picture of running of cosmological quantities comes about naturally in our approach which is physically motivated by RG, namely, due to increasing of the gravitational constant with scale (and possibly increasing amount of DM), as discussed in the preceding section, Ω_0 has effectively increased too. Moreover, since the scale factor is governed by the scale dependent

[6] h is h_0 measured in units of 100 km/sec Mpc.

$(G\rho)$, it now seems to depend on the scale r as well, i.e., it increases at any fixed time with the increasing distance. Consequently, the value of H_0 is not the same everywhere in the observable Universe and depends on a scale where it is measured. Moreover, since $H_0 \sim 1/t_0$ the Universe becomes older when its age is estimated on a smaller scale. This is not to be taken as implying that the age depends on where one calculates (every observer using the same scale r at some time t_1 will obtain the same age). The key point to emphasize is that, having in mind the RG arguments and interpretation, a direct comparison of cosmological quantities makes sense only when they are measured (or calculated) with respect to the same scale, since the same quantity can take different values at different scales.

Notice that independently, quantum cosmology also advocates, although in a different context of bubble universes, a possibility that we may live in a Universe in which the value of Hubble constant and the measured density are different in different places and in our local neighborhood Ω_0 may well be less than 1 [51]. This makes our proposal even more promising.

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