

Renormalization-group-inspired approaches and estimates of the tenth-order corrections to the muon anomaly in QED

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We present the estimates of the five-loop QED corrections to the muon anomaly using the scheme-invariant approaches and compare them with other similar results existing in the literature.

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I. INTRODUCTION

Direct analytical or numerical calculations of the higher-order terms to the physical quantities in concrete renormalization schemes provide important information about the behavior of the corresponding perturbative approximations. However, there are also some other approaches to treat the problem of the extraction of certain information from the truncated perturbative series. These approaches are the principle of minimal sensitivity (PMS) [1] and the effective charges (ECH) prescription [2], which is equivalent *a posteriori* to scheme-invariant perturbation theory [3]. Of course, it is better to use these approaches directly in the concrete orders of the perturbation theory, as was done in QCD in Refs. [4–8]. However, if one adopts the point of view that these methods really pretend to the role of “optimal” procedures in the sense that they might provide a better convergence of the corresponding approximations in the nonasymptotic regime, it is possible to try to go one step further and apply the procedure of reexpansion of the “optimized” expressions in the coupling constant of an initial scheme. One can consider the residual $(N+1)$ th order term as the estimate of the $(N+1)$ th order correction in the initial scheme [1].

The reexpansion procedure was already applied for the analysis of the perturbative predictions for $(g-2)_\mu$ in QED [1, 9] (for related considerations see Ref. [10]) and for estimates of the QCD corrections to definite physical quantities. In these works, the quantities under study are the Drell-Yan cross section at the $O(\alpha_s^2)$ level [11], $R(s) = \sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, $R_\tau = \Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})/\Gamma(\tau \rightarrow \nu_\tau \bar{\nu}_e e)$, nonpolarized and polarized Bjorken sum rules at the $O(\alpha_s^4)$ and even $O(\alpha_s^5)$ levels [12, 13], and the singlet contribution to the Ellis-Jaffe sum rule at the $O(\alpha_s^3)$ order [14].

It is clear that the reexpansion formalism, which is similar to the procedure used in Ref. [15] to predict the

renormalization-group- (RG-) controllable $\ln(m_\mu/m_e)$ terms from the expression for $(g-2)_\mu$ through the effective coupling constant $\bar{\alpha}(m_\mu/m_e)$, correctly reproduces the RG-controllable terms [1, 16]. One can also hope that it can give an impression about the possible values of the constant terms as well. This hope is based on the observation of the existence of a satisfactory agreement of the results of application of the reexpansion procedure in QED [9] and QCD [12, 13] with results of the explicit calculations. It should be stressed that contrary to the RG considerations of Ref. [15], the “optimization methods” deal with the full RG invariance of the quantities under consideration, which produce additional equations, relevant to the freedom of the choice of higher-order coefficients of the β function. The solution of these equations gives the possibility to define the sets of scheme invariants [1] which are the cornerstones of the “optimization” methods.

However, in definite cases the procedure of reexpansion of the “optimized” results can run against some barrier, which was overlooked in the process of some previous applications [1, 9, 10]. In the case of the analysis of the perturbative series for $(g-2)_\mu$ this problem grows from the noncareful treatment of the light-by-light scattering graphs with the electron loop coupled to the external photon line.

In Sec. II of this work we describe the basis of the formalism used by us. The exact expressions for the terms in the reexpansion formulas are derived. It is demonstrated that the estimates obtained using the reexpansion of the ECH expressions are identical to the results of calculations of $(N+1)$ th order corrections in the special scheme, where all lower-order coefficients of the physical quantities and the β function are defined in a certain fixed scheme [in the case of QED the on-shell (OS) scheme is usually used] and the $(N+1)$ th order coefficient of the β function coincides with the $(N+1)$ th order scheme-invariant coefficient of the ECH β function β_{eff} .

In Sec. III, using the information about the four-loop coefficient of the QED β function in the OS scheme [17] we generalize the considerations of Refs. [1, 9, 10] to the five-loop level. We follow the proposals of Ref. [18] and consider the light-by-light scattering graphs mentioned

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above separately in our RG-inspired analysis. We show that this empirical improvement leads to more satisfactory and thus more reliable estimates of the five-loop contributions to $(g-2)_\mu$ than in the case of the nonseparation of the light-by-light scattering contributions.

II. DESCRIPTION OF THE FORMALISM

Consider first the order $O(a^N)$ approximation of a renormalization-group-invariant quantity

$$D_N = d_0 a \left(1 + \sum_{i=1}^{N-1} d_i a^i \right), \quad (1)$$

with $a = \alpha/\pi$ being the solution of the corresponding renormalization group equation for the β function which is defined as

$$\mu^2 \frac{\partial a}{\partial \mu^2} = \beta(a) = \beta_0 a^2 \left(1 + \sum_{i=1}^{N-1} c_i a^i \right). \quad (2)$$

The coefficients $d_i, i \geq 1$ and $c_i, i \geq 2$ are scheme dependent. In order to calculate them in practice it is necessary to specify the scheme of subtractions of the ultraviolet divergences. In QED the OS scheme is commonly used. However, this scheme is not the unique prescription for fixing the renormalization scheme ambiguities, which affect the values of these coefficients. In both phenomenological and theoretical studies other methods are also widely applied.

The PMS [1] and ECH [2] prescriptions stand out from various methods of treating scheme-dependence ambiguities. Indeed, they are based on the conceptions of the scheme-invariant quantities, which are defined as the combinations of the scheme-dependent coefficients in Eqs. (1) and (2). Both these methods pretend to the role of “optimal” prescriptions, in the sense that they might provide better convergence of the corresponding approximations in the nonasymptotic regime, and thus allow an estimation of the uncertainties of the perturbative series in the definite order of perturbation theory. Therefore, applying these “optimal” methods one can try to estimate the effects of the order $O(a^{N+1})$ corrections starting from the approximations $D_N^{\text{opt}}(a_{\text{opt}})$ calculated in a certain “optimal” approach [1, 9, 16].

Let us follow the considerations of Ref. [1] and reexpand $D_N^{\text{opt}}(a_{\text{opt}})$ in terms of the coupling constant a of the particular scheme

$$D_N^{\text{opt}}(a_{\text{opt}}) = D_N(a) + \delta D_N^{\text{opt}} a^{N+1}, \quad (3)$$

where

$$\delta D_N^{\text{opt}} = \Omega_N(d_i, c_i) - \Omega_N(d_i^{\text{opt}}, c_i^{\text{opt}}), \quad (4)$$

are the numbers which simulate the coefficients of the order $O(a^{N+1})$ corrections to the physical quantity, calculated in the particular initial scheme. The coefficients Ω_N can be obtained from the following system of equations:

$$\frac{\partial}{\partial \tau} (D_N + \Omega_N a^{N+1}) = O(a^{N+2}),$$

$$\frac{\partial}{\partial c_i} (D_N + \Omega_N a^{N+1}) = O(a^{N+2}), \quad i \geq 2, \quad (5)$$

where the parameter $\tau = \beta_0 \ln(\mu^2/\Lambda^2)$ represents freedom in the choice of the renormalization point μ . The conventional scale parameter Λ will not explicitly appear in all our final formulas. The system of these equations can be solved following the lines of Ref. [1]. Let us stress again that the difference between the “optimization” equations and the RG approach of Ref. [15] lies in the fact that the latter one deals with the first equation from the system of Eq. (5) only. The quantities Ω_l can be related to the scheme invariants ρ_l in the following way:

$$\rho_l = d_l + \frac{1}{l-1} c_l - \Omega_l(d_1, \dots, d_{l-1}; c_1, \dots, c_{l-1}). \quad (6)$$

Note that the general expressions of the scheme invariants ρ_l and of the correction terms Ω_l can be defined in different ways. Various definitions differ by scheme-independent constant terms. We are choosing these correlated constant terms by imposing the condition that the expressions for the scheme invariants ρ_l are connected with the coefficients c_i^{ECH} of the ECH β function:

$$\beta_{\text{eff}}(a_{\text{ECH}}) = \beta_0 a_{\text{ECH}}^2 \left(1 + c_1 a_{\text{ECH}} + \sum_{i \geq 2} c_i^{\text{ECH}} a_{\text{ECH}}^i \right) \quad (7)$$

as

$$\rho_l = \frac{c_l^{\text{ECH}}}{l-1}, \quad (8)$$

where

$$D(a_{\text{ECH}}) = d_0 a_{\text{ECH}}(a). \quad (9)$$

Concrete expressions for the invariants ρ_l and thus for the correction terms Ω_l can be derived from the equation

$$\beta_{\text{eff}}(a_{\text{ECH}}) = \frac{\partial a_{\text{ECH}}}{\partial a} \beta(a). \quad (10)$$

We present here the final expressions, which are already known [1],

$$\Omega_2 = d_0 d_1 (c_1 + d_1), \quad (11)$$

$$\Omega_3 = d_0 d_1 (c_2 - \frac{1}{2} c_1 d_1 - 2d_1^2 + 3d_2), \quad (12)$$

and the new term which we evaluated:

$$\Omega_4 = \frac{d_0}{3} (3c_3 d_1 + c_2 d_2 - 4c_2 d_1^2 + 2c_1 d_1 d_2 - c_1 d_3 + 14d_1^4 - 28d_1^2 d_2 + 5d_2^2 + 12d_1 d_3). \quad (13)$$

These terms reproduce the RG-controllable logarithmic contributions. In the case of the five-loop level one can reobtain the QED results presented in Ref. [19]. We discuss this point in more detail in the next section.

It should be stressed that in the ECH approach $d_i^{\text{ECH}} \equiv 0$ for all $i \geq 1$. Therefore one gets the following expressions for the higher-order corrections in Eq. (3):

$$\delta D_2^{\text{ECH}} = \Omega_2(d_1, c_1), \quad (14)$$

$$\delta D_3^{\text{ECH}} = \Omega_3(d_1, d_2, c_1, c_2), \quad (15)$$

$$\delta D_4^{\text{ECH}} = \Omega_4(d_1, d_2, d_3, c_1, c_2, c_3). \quad (16)$$

One can understand from Eqs. (6), (8) that the expressions for Ω_N and for the corrections δD_N^{ECH} in Eqs. (14)–(17) are the *exact* numbers which are related to the special scheme. This scheme is identical to the initial scheme at the lower-order levels and is defined by the condition $c_N = c_N^{\text{ECH}}$ at the $(N+1)$ order, where c_N^{ECH} is considered as an unknown number. This means that the correction coefficients δD_N are related to the initial scheme only partly. However, it was shown in Refs. [12, 13] that in certain cases the numerical values of these coefficients are in satisfactory agreement with the results of the explicit calculations. *A posteriori* we consider this fact as an argument in favor of the possibility of the application of the reexpansion procedure in the cases discussed by us.

In order to find similar corrections to Eq. (3) in the

N th order of perturbation theory starting from the PMS approach [1], it is necessary to use the relations obtained in Ref. [20] between the coefficients d_i^{PMS} and c_i^{PMS} ($i \geq 1$) in the expression for the order $O(a_{\text{PMS}}^N)$ approximation $D_N^{\text{PMS}}(a_{\text{PMS}})$ of the physical quantity under consideration:

$$d_i^{\text{PMS}} = \frac{1}{i+1} \left(\frac{N-2i-1}{N-1} \right) c_i^{\text{PMS}} + O(a_{\text{PMS}}), \quad (17)$$

where $c_1^{\text{PMS}} = c_1$. Using now Eq. (17) it is possible to find the following additional correction terms in Eq. (4) which result from the application of the PMS approach:

$$\begin{aligned} \Omega_2(d_i^{\text{PMS}}, c_i^{\text{PMS}}) &= -\frac{d_0 c_1^2}{4}, \\ \Omega_3(d_i^{\text{PMS}}, c_i^{\text{PMS}}) &= 0. \end{aligned} \quad (18)$$

The expression for $\Omega_4(d_i^{\text{PMS}}, c_i^{\text{PMS}})$ derived by us is more complicated:

$$\Omega_4(d_i^{\text{PMS}}, c_i^{\text{PMS}}) = \frac{d_0}{3} \left[\frac{1}{4} c_1 c_3^{\text{PMS}} - \frac{4}{81} (c_2^{\text{PMS}})^2 - \frac{5}{81} c_1^2 c_2^{\text{PMS}} + \frac{7}{648} c_1^4 \right], \quad (19)$$

where

$$\begin{aligned} c_2^{\text{PMS}} &= \frac{9}{8} \left(c_2^{\text{ECH}} + \frac{7}{36} c_1^2 \right) + O(a_{\text{PMS}}) \\ &= \frac{9}{8} \left(d_2 + c_2 - d_1^2 - c_1 d_1 + \frac{7}{36} c_1^2 \right) + O(a_{\text{PMS}}) \end{aligned} \quad (20)$$

and

$$c_3^{\text{PMS}} = 4 \left(d_3 + \frac{1}{2} c_3 - c_2 d_1 - 3 d_1 d_2 + 2 d_1^3 \right) + \frac{1}{2} c_1 \left(d_2 + c_2 + 3 d_1^2 - c_1 d_1 + \frac{1}{108} c_1^2 \right) + O(a_{\text{PMS}}). \quad (21)$$

The expressions for Eqs. (19)–(21) are the pure numbers, which do not depend on the choice of the initial scheme. We will show in the next section that in the case of the consideration of perturbative series for $(g-2)_\mu$ the numerical values of $\Omega_4(d_i^{\text{PMS}}, c_i^{\text{PMS}})$ are small and thus the *a posteriori* approximate equivalence of the ECH and PMS approaches, which follows from the small value of $\Omega_2(d_i^{\text{PMS}}, c_i^{\text{PMS}})$ and from the condition $\Omega_3(d_i^{\text{PMS}}, c_i^{\text{PMS}}) = 0$, is preserved for the quantity under consideration at this level also.

In certain considerations we will need to use a generalization of the expression for Ω_2 to the case when the initial perturbative series is starting from corrections of order $O(a^p)$ with $p > 1$:

$$D_N^{(p)} = d_0 a^p \left(1 + \sum_{i \geq 1} d_i a^i \right). \quad (22)$$

In this case the expression for the correction terms reads

$$\Omega_2^{(p)} = \frac{p+1}{2p} d_0 d_1^2 + d_0 d_1 c_1. \quad (23)$$

The corresponding correction related to the PMS-improved expression was originally obtained in Ref. [1].

It differs from Eq. (23) by the additional small contribution

$$-\Omega_2^{(p)}(d_i^{\text{PMS}}, c_i^{\text{PMS}}) = \frac{p}{2(p+1)} d_0 c_1^2. \quad (24)$$

III. APPLICATIONS TO $(g-2)_\mu$

It is well known that the expressions for anomalous magnetic moments of the electron $a_e = (g-2)_e/2$ and muon $a_\mu = (g-2)_\mu/2$ are known at four-loop order from the results of calculations of Ref. [21] and Refs. [22, 23] respectively. The three-loop correction to a_e is now known with more accuracy than previously [24]. Combining the currently available information about the coefficients of the perturbative series for a_e and a_μ we have the expressions

$$\begin{aligned} a_e &= 0.5a - 0.329\,478\,9\dots a^2 + 1.176\,19(21)a^3 \\ &\quad - 1.434(138)a^4, \end{aligned} \quad (25)$$

$$\begin{aligned} a_\mu - a_e &= 1.094\,335\,83(7)a^2 + 22.869\,265(4)a^3 \\ &\quad + 127.55(41)a^4, \end{aligned} \quad (26)$$

where the expansion parameter $a = \alpha/\pi$ is related to the

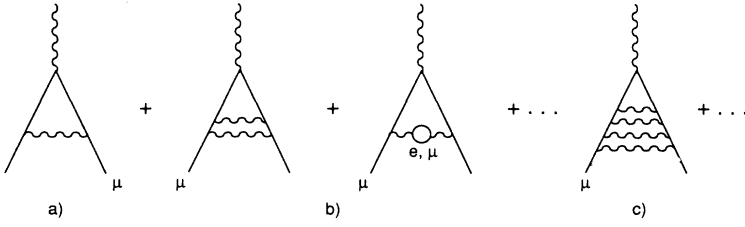


FIG. 1. The diagrams which are included in the set (I).

fine structure constant α and the last term in Eq. (26) is the result of the most recent calculations of Ref. [23] stimulated by the work of Ref. [17]. Combining Eq. (25) with Eq. (26) we arrive at the following approximate expression for a_μ :

$$a_\mu = 0.5a + 0.76585a^2 + 24a^3 + 126a^4 + O(a^5). \quad (27)$$

The order $O(a^5)$ correction to a_μ is only partly known [22]. Our aim will be to try to estimate the existing uncertainty due to the totally noncalculated order $O(a^5)$ contribution to Eq. (27) using the reexpansion procedure outlined in the previous section.

It is known that in the OS scheme the coefficients of the corresponding perturbative series depend on the large $\ln(m_\mu/m_e)$ contributions starting from the two-loop level. The parts of these effects are governed by the RG method [15, 25] (for a recent application of the RG method to a_μ , see Refs. [26, 19]). However, there are also certain $\ln(m_\mu/m_e)$ contributions, which are not governed by the RG method. They are associated with the light-by-light-scattering electron loop insertions coupled to the external photon line. These contributions appear first in the three-loop graphs, which were subsequently calculated numerically in the works of Refs. [27, 22] and recently evaluated analytically in the work of Ref. [28].

In view of the different origin of the lower $\ln(m_\mu/m_e)$ contributions we divide all diagrams into two classes. The first class contains all diagrams with an external muon vertex and dressed internal photon lines (see Fig. 1). As well as in Ref. [22] we will not include the diagrams with electron loops to which four internal photon lines are attached. However, we will include four-loop diagrams typical to a_e but with substitution of the external electron vertex to the muon one. The second class of diagrams includes diagrams with an electron light-by-light scattering subgraph, to which three and four internal photon lines are attached (see Fig. 2). Let us stress that all $\ln(m_\mu/m_e)$ terms of the diagrams contributing to the first class are totally controlled by the RG method, while in class (II) only part of these contributions is governed by the RG technique.

In accordance with our classification we represent the

expression for a_μ in the form

$$a_\mu = a_\mu^{(I)} + a_\mu^{(II)}. \quad (28)$$

The concrete contributions to Eq. (28) read

$$a_\mu^{(I)} = d_0^{(I)} a (1 + d_1^{(I)} a + d_2^{(I)} a^2 + d_3^{(I)} a^3 + \dots), \quad (29)$$

$$a_\mu^{(II)} = d_0^{(II)} a^3 (1 + d_1^{(II)} a + d_2^{(II)} a^2 + \dots). \quad (30)$$

Note that the coefficients d_i ($i \geq 1$) contain the RG-controllable $\ln(x) = \ln(m_\mu/m_e)$ terms. Indeed, the corresponding contributions to a_μ are governed by the RG equation

$$\left(m^2 \frac{\partial}{\partial m^2} + \beta(a) \frac{\partial}{\partial a} \right) a_\mu^{(I,II)} = 0, \quad (31)$$

where $\beta(a)$ is the QED β function in the OS scheme, which is defined as

$$m^2 \frac{\partial a}{\partial m^2} = \beta(a) = \sum_{i \geq 0} \beta_i a^{i+2}. \quad (32)$$

From our point of view, the separation of all diagrams into the two classes mentioned above is respected by the property of the RG invariance. At least we do not know any arguments why the sums of the diagrams which belong to the class (I) and to the class (II) should not obey the RG equations separately.

The coefficients of the QED β function in the OS scheme are known at the four-loop level [17]. They have the form

$$\begin{aligned} \beta_0 &= \frac{1}{3}, \\ \beta_1 &= \frac{1}{4}, \\ \beta_2 &= -\frac{121}{288} = -0.42, \\ \beta_3 &= \left(\frac{5561}{5184} - \frac{23}{9} \zeta(2) + \frac{8}{3} \zeta(2) \ln(2) - \frac{7}{8} \zeta(3) \right) \frac{1}{2} \\ &= -0.571. \end{aligned} \quad (33)$$

Thus, the related coefficients $c_i = \beta_i/\beta_0$ ($i \geq 1$) read

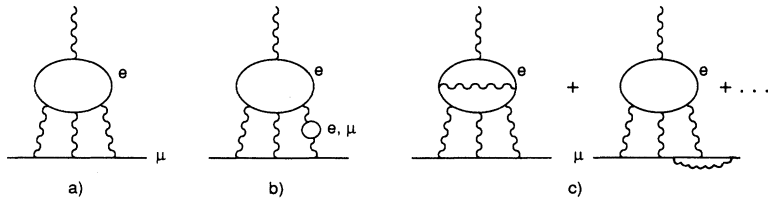


FIG. 2. The diagrams which are included in the set (II).

$c_1 = 3/4$, $c_2 = -1.26$, $c_3 = -1.713$. Let us write down the asymptotic expansions of the coefficients of the contributions a_μ as

$$\begin{aligned} d_0^{(I)} &= B_1, \\ d_0^{(I)} d_1^{(I)} &= B_2 + C_2 \ln(x), \\ d_0^{(I)} d_2^{(I)} &= B_3 + C_3 \ln(x) + D_3 \ln^2(x), \\ d_0^{(I)} d_3^{(I)} &= B_4 + C_4 \ln(x) + D_4 \ln^2(x) + E_4 \ln^3(x), \\ d_0^{(I)} d_4^{(I)} &= B_5 + C_5 \ln(x) + D_5 \ln^2(x) + E_5 \ln^3(x) \\ &\quad + F_5 \ln^4(x) \end{aligned} \quad (34)$$

and

$$\begin{aligned} d_0^{(II)} &= \bar{B}_1, \\ d_0^{(II)} d_1^{(II)} &= \bar{B}_2 + \bar{C}_2 \ln(x), \\ d_0^{(II)} d_2^{(II)} &= \bar{B}_3 + \bar{C}_3 \ln(x) + \bar{D}_3 \ln^2(x). \end{aligned} \quad (35)$$

The coefficients C_i , D_i , E_i , F_i and \bar{C}_i , \bar{D}_i can be related to the coefficients of the β function using either the RG considerations of Refs. [15, 25, 19] or the explicit expressions for the coefficients Ω_i and $\Omega_i^{(p)}$ in the corresponding reexpansion formulas [see Eqs. (11)–(13) and Eq. (23)]. The results of the corresponding analysis have the form

$$C_2 = 2\beta_0 B_1, \quad (36)$$

$$\begin{aligned} C_3 &= 4\beta_0 B_2 + 2\beta_1 B_1, \\ D_3 &= 4\beta_0^2 B_1, \end{aligned} \quad (37)$$

$$\begin{aligned} C_4 &= 6\beta_0 B_3 + 4\beta_1 B_2 + 2\beta_2 B_1, \\ D_4 &= 12\beta_0^2 B_2 + 10\beta_0 \beta_1 B_1, \\ E_4 &= 8\beta_0^3 B_1, \end{aligned} \quad (38)$$

$$\begin{aligned} C_5 &= 8\beta_0 B_4 + 6\beta_1 B_3 + 4\beta_2 B_2 + 2\beta_3 B_1, \\ D_5 &= 24\beta_0^2 B_3 + 28\beta_0 \beta_1 B_2 + 6\beta_1^2 B_1 + 12\beta_0 \beta_2 B_1, \\ E_5 &= 32\beta_0^3 B_2 + \frac{104}{3} \beta_0^2 \beta_1 B_1, \\ F_5 &= 16\beta_0^4 B_1, \end{aligned} \quad (39)$$

$$\begin{aligned} \bar{C}_2 &= 6\beta_0 \bar{B}_1, \\ \bar{C}_3 &= 8\beta_0 \bar{B}_2 + 6\beta_1 \bar{B}_1, \\ \bar{D}_3 &= 24\beta_0^2 \bar{B}_1. \end{aligned} \quad (40)$$

Note that in the case of the diagrams of set (II) the corresponding coefficients \bar{B}_1 , \bar{B}_2 , and \bar{B}_3 contain the contributions of the $\ln(x)$ terms that are noncontrollable by the RG method.

Let us first discuss the applications of the procedure of Sec. II to the diagrams of set (I). In this case the correction terms Ω_2 – Ω_4 reproduce all $\ln(x)$ contributions presented in Eqs. (34). Moreover, one can get from the reexpansion procedure the exact values of the constant terms B_i ($i \geq 3$) which do not depend on the $\ln(x)$ terms. In the case of the application of the ECH-improved variant of the OS scheme these constant terms are defined by the conditions

$$\begin{aligned} B_i &= \Omega_{i-1}(d_0^{\text{OS}}, d_1^{\text{OS}}, \dots, d_{i-2}^{\text{OS}}, c_1, \dots, c_{i-2}^{\text{OS}}) \\ &= \Omega_{i-1}(B_1^{\text{OS}}, \dots, B_{i-1}, c_1, \dots, c_{i-2}^{\text{OS}}). \end{aligned} \quad (41)$$

Similar terms which arise from the PMS-improved expressions can be obtained after taking into account the additional small scheme-independent contributions derived in Sec. II.

Concrete values of the coefficients $B_1, B_2^{\text{OS}}, B_3^{\text{OS}}$ are known from a comparison of the results of the RG-inspired analysis with the results of the analytical and numerical calculations [22]. The coefficient $B_1 = 0.5$ is of course well known. The asymptotic expression of the coefficient B_2 , derived in the limit $m_e/m_\mu \rightarrow 0$, can be found in Ref. [22]: $B_2^{\text{OS}} = -\frac{25}{36} + a_e^{(4)} = -1.022923$. The value of the coefficient $B_3^{\text{OS}} = 2.741$ was obtained in Ref. [22] after subtracting the contributions of the light-by-light scattering graphs of the set (II) and of the RG-controllable contribution of Eq. (34) from the expression for the three-loop correction to a_μ .

In order to determine the value of the coefficient B_4^{OS} we used the expression

$$B_4^{\text{OS}} = a_\mu^{(8)} - A_\mu^{(8)}(\gamma\gamma) - C_4 \ln(x) - D_4 \ln^2(x) - E_4 \ln^3(x), \quad (42)$$

where C_4 , D_4 , and E_4 are determined by Eqs. (38) and the value of $A_\mu^{(8)}(\gamma\gamma) \approx -116.7$ is the sum of the eighth-order contributions of the diagrams with electron light-by-light scattering subgraphs [22]. The numerical value of the coefficient B_4^{OS} is thus $B_4^{\text{OS}} = -7.74$.

In order to study the predictive abilities of the reexpansion procedure described in Sec. II we present in Table I the numerical results of our estimates of the coefficients B_i ($i \geq 3$) and compare them with the exact results for B_3^{OS} and B_4^{OS} presented above.

One can see that the reexpansion procedure used by us reproduces well enough the values of the coefficients B_3 and B_4 (it gives the correct sign and predicts the order of magnitude of these coefficients). Therefore, we hope that the estimate of the five-loop constant term B_5 is also rather realistic. Notice also the sign-alternating character of the results of the estimates presented in Table I. This feature has something in common with the expectation that the RG-improved QED series for the Euclidean physical quantities should have sign-alternating behavior [29].

Taking now into account the numerical value of the RG-controllable terms in Eqs. (34), (39) we arrive at the following estimate of the five-loop contributions of the diagrams of set (I) into a_μ :

$$\begin{aligned} a_\mu^{(10)}(\text{I}) &= B_5^{\text{OS}} + 8.55 \\ &= 50.1 \quad (\text{ECH}) \\ &= 50.2 \quad (\text{PMS}). \end{aligned} \quad (43)$$

TABLE I. Estimated values of the coefficients B_i for the diagrams of set (I).

Order	B_i^{OS}	$B_i(\text{ECH})$	$B_i(\text{PMS})$
$i = 1$	0.5	—	—
$i = 2$	-1.022923	—	—
$i = 3$	2.741	1.326	1.396
$i = 4$	-7.74	-5.48	-5.48
$i = 5$	—	41.6	41.7

This estimate is almost nonsensitive to the concrete realization of the method of optimization. Notice also the effect of the reduction of the value of the RG-controllable five-loop contributions presented in Refs. [19, 17],

$$a_\mu^{(10)}(\text{I}) = B_5^{\text{OS}} + 83, \quad (44)$$

which was obtained from Eqs. (39) using the value of the constant term $B_4^{\text{OS}} = -2.503(55)$ given in Ref. [22]. The difference between this value and the value $B_4^{\text{OS}} = -7.74$, which follows from Eq. (42), is explained by two facts. First, it is necessary to use the corrected expressions obtained in Refs. [17, 23] of the certain four-loop graphs contributing to a_μ . Second, it is necessary to add to the value of B_4^{OS} of Ref. [22] the negative constant terms due to the four-loop graphs typical to a_e but with a substitution of the electron vertex and internal electron loops for the muon ones. This contribution was missed in Ref. [22] while deriving the expression for B_4^{OS} . As is known from Ref. [30] the addition of the negative contributions of the five-loop graphs containing vacuum polarization insertions on fourth-order vertex graphs to the positive contributions of the five-loop diagrams with vacuum polarization insertions in the second-order graph, as calculated in Ref. [19], leads to strong cancellations. Comparison of the RG-controllable contributions of Eq. (43) with the ones of Eq. (44) indicates the same pattern. The origin of this cancellation is the same, namely, the inclusion in the considerations of the diagrams with vacuum polarization insertions to the eighth-order vertex graph. Our final estimate of Eq. (43) should be also compared with the conservative estimate $a_\mu^{(10)} = \pm 140$ of the RG-controllable contributions, given in Ref. [22].

Let us now discuss the applications of the outlined procedure for the estimates of the five-loop contributions of the diagrams with the light-by-light scattering subgraphs of Fig. 2. The most precise value of the coefficient $d_0^{(\text{II})} = \bar{B}_1 = 20.94792\dots$ is known from the results of the analytical calculations of Ref. [28]. The numerical result for the sum of the corresponding four-loop graphs reads [22]

$$d_0^{(\text{II})} d_1^{(\text{II})} = 116.7. \quad (45)$$

Using now Eqs. (23), (24) we arrive at the following numerical estimate of the sum of the corresponding five-loop graphs:

$$\begin{aligned} a_\mu^{(10)}(\text{II}) &= 520.8 \quad (\text{ECH}) \\ &= 525.2 \quad (\text{PMS}), \end{aligned} \quad (46)$$

which includes the contribution of both RG-controllable and non-RG-controllable $\ln(x)$ terms.

Our estimate of Eq. (46) should be compared with the result

$$a_\mu^{(10)} = 570 + 176(35), \quad (47)$$

where the first contribution comes from the exact calculations made in Ref. [22] of the diagrams belonging to the set of the light-by-light-scattering diagrams with 2 one-loop electron vacuum polarization insertions into the internal photon lines (see Fig. 3). The second contribution to Eq. (47) comes from the estimates of Ref. [31]

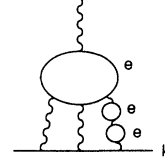


FIG. 3. The example of the five-loop diagram, calculated in Ref. [22].

of the tenth-order diagrams depicted in Fig. 4 which were not calculated in Ref. [22]. These diagrams are formed by the insertion of the two-loop electron loop into the internal photon line of the lower light-by-light-type diagram. In order to understand the uncertainties of this estimate of Ref. [31] better it is useful to write down a RG relation analogous to Eqs. (35) for this set of diagrams separately. Notice that this contribution should be proportional to the two-loop coefficient of the QED β function (which is determined by the graphs inserted into the internal photon line). Using this observation we arrive at the relation

$$a_\mu^{(10)}(\text{Fig. 4}) = \bar{B}_3(\text{Fig. 4}) + 6\bar{B}_1\beta_1 \ln(x). \quad (48)$$

The main contribution to the estimate of Ref. [31] comes from the $\ln(x)$ term. Indeed, it has the numerical value $6\bar{B}_1\beta_1 \ln(x) = 167.47$. This expression should be compared with the estimate $a_\mu^{(10)}(\text{Fig. 4}) = 176 \pm 35$ given in Ref. [31]. One can see that this estimate is relevant to the RG-controllable contribution only. However, from the reexpansion procedure we conclude that the contributions noncontrollable by the RG methods might be non-negligible [see Eq. (43)] and might affect the final numerical value of the diagrams belonging to this set. In order to study this guess in detail it is of interest to calculate the diagrams of Fig. 4 explicitly. This calculational project is rather realistic [32].

It is also interesting to understand deeper the uncertainties due to other diagrams which are included neither in the “optimized” estimates of Eqs. (43), (46) nor in the estimates of Eq. (47). These diagrams, depicted in Fig. 5, form a new class of diagrams, which cannot be touched by the RG-inspired analysis. Indeed, one can hardly expect that any resummation procedures dealing with light-by-light-type graphs with three internal photon lines will be able to give the estimate of the light-by-light-type graph with five internal photon lines. The expressions for the $\ln(x)$ -terms the non-controllable by the RG-method for this type of graphs can be read from the considerations of Refs. [33]. The result was used in Ref. [31] where the following estimate of the diagrams of

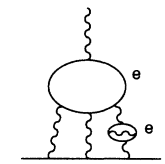


FIG. 4. The five-loop diagram, estimated in Ref. [31].

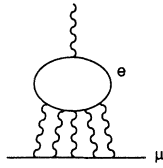


FIG. 5. The five-loop diagram, estimated in Ref. [31], which cannot be estimated by the approach considered in this work.

Fig. 5 was presented:

$$a_{\mu}^{(10)}(\text{Fig. 5}) = 185 \pm 85. \quad (49)$$

Combining our estimates of Eqs. (43) and (46) with the ones of Eq. (49) we get the final result of applications of the reexpansion procedure supplemented by the estimates of the diagrams of new structure which are not touched on by this method:

$$a_{\mu}^{(10)} \approx 750. \quad (50)$$

Let us stress again that the new ingredient of our analysis, which distinguishes it from previous applications of the reexpansion procedures in QED [1, 9, 10], is the separation of the considered initial diagrams into two classes, one of which consists of the diagrams relevant to the effects of “new physics,” discussed in more detail in Refs. [34, 33]. This procedure finds its support in the theoretical considerations of Ref. [18].

Moreover, we checked that in spite of the good agreement of the application of the reexpansion procedure to the nonseparated sixth-order expressions for a_{μ} with results of eighth-order calculations [9], the straightforward application of Eq. (13) to the nonseparated eighth-order approximation of Eq. (27) results in the uncomfortably large tenth-order estimate $a_{\mu}^{(10)} \approx 2160$. It is possible to understand that the reason for the success of the appli-

cation of the reexpansion procedure to the nonseparated sixth-order approximation is connected to the fact that the use of Eq. (12) (and more definitely its last term) gives for the eighth-order light-by-light-type term the estimate $a_{\mu}^{(8)}(\gamma\gamma) = 3d_0d_1d_2(\gamma\gamma) = 6a_{\mu}^{(4)}a_{\mu}^{(6)}(\gamma\gamma)$, which is known to be in good agreement with the results of direct numerical calculations [22]. However, at the next level of perturbation theory the expression for the correction term Ω_4 of Eq. (13) has a more complicated structure and thus the resulting nonseparated estimates turn out to be uncomfortably large.

Another interesting question is connected to the problem of the comparison of our estimates with the results of the recent applications of the Padé resummation technique to the perturbative series for a_{μ} [35] and $a_{\mu} - a_e$ [35, 36]. It should be stressed that in their analysis the authors of Refs. [35, 36] did not consider the light-by-light scattering graphs separately. Note also that the coefficients of the corresponding Padé approximants depend on the $\ln(x)$ terms. In spite of the fact that our results for a_{μ} are in qualitative agreement with the results of the applications of the Padé resummation method, namely, $a_{\mu}^{(10)} \approx 656$ [35] and $a_{\mu}^{(10)} = 705(275)$ [36], it is interesting to try to understand the predictive abilities of the Padé resummation methods better. Clearly, this problem is connected to the necessity of a more detailed understanding of the relations of the Padé results to the ones obtained using the RG-inspired analysis. Note that the Padé resummation methods are not able to analyze the problem of reproducing the structure of the RG-controllable $\ln(x)$ terms.

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