Equivalence of the Maxwell-Chern-Simons theory and a self-dual model

R. Banerjee,* H. J. Rothe, and K. D. Rothe
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany
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We study the connection between the Green functions of the Maxwell-Chern-Simons theory and a self-dual model by starting from the phase-space path-integral representation of the Deser-Jackiw master Lagrangian. Their equivalence is established modulo time-ordering ambiguities.

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In a recent interesting paper [1] the bosonization of the massive Thirring model in $2+1$ dimensions was discussed by relating it in the large mass limit to the Maxwell-Chern-Simons (MCS) theory [2]. As an intermediary step, use has been made of the equivalence [3] of this theory to that of a self-dual (SD) model discussed in [4]. This analysis has been carried out on the level of the configuration space path-integral expressions of the partition functions. Because of the constraint structure associated with the various Lagrangians involved in the argument, a complete investigation of the problem must start from a proper phase-space path-integral formulation. This is done in the present paper. Starting from the master Lagrangian of Deser and Jackiw [3], we follow the general line of reasoning of Ref. [1] and establish the equivalence, modulo time-ordering ambiguities, of the SD model, and the MCS theory on the level of Green's functions.

Consider the symmetrized form of the master Lagrangian given in [3]

$$\mathcal{L} = \tfrac{1}{2}f^\mu f_\mu - \tfrac{1}{2}\epsilon^{\mu\nu\lambda}f_\mu\partial_\nu A_\lambda - \tfrac{1}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu f_\lambda + \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{1}$$

The primary constraints [5] are given by

$$\Omega_0 = \pi_0 \approx 0, \quad \Omega_i = \pi_i + \tfrac{1}{2}\epsilon_{ij}f^j - \frac{m}{2}\epsilon_{ij}A^j \approx 0,$$
$$\Omega_0^{(f)} = \pi_0^{(f)} \approx 0, \quad \Omega_i^{(f)} = \pi_i^{(f)} + \tfrac{1}{2}\epsilon_{ij}A^j \approx 0, \tag{2}$$

where $\pi_\mu(\pi_\mu^{(f)})$ are the momenta canonically conjugate to $A^\mu(f^\mu)$. The canonical Hamiltonian is given by

$$H_c = \int d^2x[-\tfrac{1}{2}f^\mu f_\mu + A_0\epsilon^{ij}(\partial_i f_j - m\partial_i A_j) + \epsilon^{ij}f_0\partial_i A_j]. \tag{3}$$

The persistence of the first-class constraints Ω_0 and $\Omega_0^{(f)}$ in time leads, respectively, to the secondary constraints

$$\Omega_3 = \epsilon^{ij}\partial_i f_j - m\epsilon^{ij}\partial_i A_j \approx 0,$$
$$\Omega_3^{(f)} = f_0 - \epsilon^{ij}\partial_i A_j \approx 0. \tag{4}$$

Although, apart from π_0, all other constraints appear to be second class, there actually exists a linear combination of the constraints that is first class. This constraint is given by

$$\Omega = \vec{\nabla}\cdot\vec{\Omega} + \Omega_3 \approx 0, \tag{5}$$

which can be checked to be the generator of the gauge transformations $A^i \to A^i + \partial^i\lambda$, $f^i \to f^i$. There are no further constraints. Hence, we have two first-class constraints, Ω_0 and Ω, and six second-class constraints $\Omega_0^{(f)}$, $\Omega_i^{(f)}$, $\Omega_3^{(f)}$, and Ω_i $(i = 1,2)$. Since the equivalence to be demonstrated refers to the observables of the SD and MCS models, we are free to choose the Coulomb gauge for our discussion. The phase-space partition function [6] in this gauge is then given by

$$Z = \int Df^\mu D\pi_\mu^{(f)} DA^\mu D\pi_\mu \delta(A_0)\delta(\vec{\nabla}\cdot\vec{A}) \times\delta(\Omega_0)\delta(\Omega)\delta(\Omega_1)\delta(\Omega_2) \times\prod_{\alpha=0}^{3}\delta(\Omega_\alpha^{(f)})\exp\left(i\int d^3x(\pi_\mu\dot{A}^\mu + \pi_\mu^{(f)}\dot{f}^\mu - \mathcal{H}_c)\right). \tag{6}$$

The Faddeev-Popov determinants associated with the constraints and the gauge-fixing are all trivial and, hence, do not appear in the functional integral. The momentum integrations in (6) can easily be performed, and one obtains

$$Z = \int Df^\mu DA^\mu \delta(\Omega_3^{(f)})\delta(\vec{\nabla}\cdot\vec{A})\exp\left(i\int d^3x\,\mathcal{L}\right). \tag{7}$$

To arrive at (7) we have expressed $\delta(\Omega_3)$ as a Fourier integral and have redefined the A^0 field in order to obtain a manifestly Lorentz-covariant action. We next couple the gauge-invariant fields f^μ and $F^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ to external sources in order to establish the equivalence of the MCS and SD models on the level of Green's functions. From (7) we are led to consider the generating functional

*On leave of absence from S.N. Bose National Centre for Basic Sciences, DB-17, Sec 1, Salt Lake, Calcutta-700064, India.

$$Z[J,j] = \int Df^\mu DA^\mu \delta(\Omega_3^{(f)})\delta(\vec{\nabla}\cdot\vec{A}) \exp\left(i\int d^3x[\mathcal{L} + J_\mu F^\mu + j_\mu f^\mu]\right). \tag{8}$$

The f_μ integration is easily done to yield

$$Z[J,j] = \int DA^\mu \delta(\vec{\nabla}\cdot\vec{A}) \exp\left(i\int d^3x[\mathcal{L}_{\rm MCS} + F_\mu(J^\mu + j^\mu) + \tfrac{1}{2}\vec{j}^2]\right), \tag{9}$$

where

$$\mathcal{L}_{\rm MCS} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{10}$$

is the familiar MCS Lagrangian [2]. For vanishing sources this is the partition function of the MCS theory in the Coulomb gauge.

Alternatively, one may perform the A_μ integration. To this end, we first integrate over the A_0 field, leading to

$$Z[J,j] = \int \mathcal{D}f^\mu \mathcal{D}A^i \delta(\vec{\nabla}\cdot\vec{A})\delta(\Omega_3^{(f)})\delta(mf_0 - \epsilon^{ij}\partial_i f_j + \epsilon_{ij}\partial^i J^j)\exp\left(i\int d^3x[\mathcal{L}' + f_\mu j^\mu + F_0 J^0 - \epsilon_{ij}J^i\partial_0 A^j]\right), \tag{11}$$

where

$$\mathcal{L}' = \tfrac{1}{2}f_\mu f^\mu - \frac{m}{2}\epsilon^{ij}A_i\partial_0 A_j - \epsilon^{ij}(f_0\partial_i A_j - f_i\partial_0 A_j). \tag{12}$$

The Gaussian A^i integration may be performed by expanding the A^i fields about the classical solution of the constraint equation in the Coulomb gauge:

$$A_i^{\rm cl}(\vec{x},t) = \epsilon_{ij}\partial^j \int d^2x' D(\vec{x}-\vec{x}')f_0(\vec{x}',t), \tag{13}$$

where $\vec{\nabla}^2 D(\vec{x}-\vec{x}') = \delta(\vec{x}-\vec{x}')$. One then finds that

$$Z = \int \mathcal{D}f_\mu \delta(mf_0 - \epsilon^{ij}\partial_i f_j + \epsilon_{ij}\partial_i J^j) \times \exp\left\{i\int\left[\mathcal{L}_{\rm SD} + f_\mu\left(J^\mu + \frac{1}{m}\epsilon^{\mu\nu\lambda}\partial_\nu J_\lambda\right) - \frac{1}{2m}\epsilon^{\mu\nu\lambda}J_\mu\partial_\nu j_\lambda\right]\right\}, \tag{14}$$

where

$$\mathcal{L}_{\rm SD} = \tfrac{1}{2}f_\mu f^\mu - \frac{1}{2m}\epsilon^{\mu\lambda\nu}f_\mu\partial_\lambda f_\nu \tag{15}$$

is the self-dual Lagrangian of [4]. We note that the source J^i appears in the argument of the δ function. A more convenient form for the computation of Green's functions is obtained by performing the integration over f_0. Then it can be verified that the resulting path-integral expression can also be written in the form

$$Z[j,J] = \int \mathcal{D}f_\mu \delta(mf_0 - \epsilon^{ij}\partial_i f_j)\exp\left(i\int \mathcal{L}_{\rm SD}\right) \times \exp\left\{i\int\left[\tilde{f}_\mu J^\mu + j^\mu f_\mu - \frac{1}{2m}\epsilon^{\mu\nu\lambda}J_\mu\partial_\nu J_\lambda - \frac{1}{2m^2}(\epsilon^{ij}\partial_i J_j)^2 - \frac{1}{m}j^0\epsilon^{ij}\partial_i J_j\right]\right\}, \tag{16}$$

where

$$\tilde{f}_\mu = \frac{1}{m}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda \tag{17}$$

is the dual of f_μ.

In the absence of sources, expressions (14) or (16) reduce to the partition function associated with the SD Lagrangian derived from the phase-space path-integral representation. Recalling the alternative representation (9), we infer from here the equivalence of the partition functions corresponding to the MCS and SD models.

We next consider this equivalence on the level of Green's functions. Because of the Gaussian character of the models, it is sufficient to consider the respective two-point functions. Functionally differentiating the partition functions (9) and its equivalent (16) with respect to the sources j^μ and j^ν, we obtain

$$\langle F_\mu(x)F_\nu(y)\rangle_{\rm MCS} - i\delta_{\mu i}\delta_{\nu i}\delta(x-y) = \langle f_\mu(x)f_\nu(y)\rangle_{\rm SD}. \tag{18}$$

Alternatively, by functionally differentiating (9) and (16) with respect to the sources J^μ and J^ν, and making use of (18), one finds that

$$\langle f_\mu(x)f_\nu(y)\rangle_{\rm SD} = \langle \tilde{f}_\mu(x)\tilde{f}_\nu(y)\rangle_{\rm SD} + S_{\mu\nu}(x-y), \tag{19}$$

where

$$S_{\mu\nu}(x-y) = -\frac{i}{m}\epsilon_{\mu\nu\lambda}\partial^\lambda\delta(x-y) -\frac{i}{m^2}\epsilon_{0\mu\lambda}\epsilon_{0\nu\rho}\partial^\lambda\partial^\rho\delta(x-y) -i\delta_{\mu i}\delta_{\nu i}\delta(x-y). \quad (20)$$

Finally, by differentiating (9) and (16) with respect to j^μ and J^ν, and again making use of (18), one is led to the relation

$$\langle f_\mu(x)f_\nu(y)\rangle_{\rm SD} = \langle f_\mu(x)\tilde{f}_\nu(y)\rangle_{\rm SD} + S'_{\mu\nu}(x-y), \quad (21)$$

where

$$S'_{\mu\nu}(x-y) = -i\delta_{\mu i}\delta_{\nu i}\delta(x-y) + \frac{i}{m}\delta_{\mu 0}\delta_{\nu j}\epsilon_{jk}\partial_k\delta(x-y). \quad (22)$$

The contributions $S_{\mu\nu}$ and $S'_{\mu\nu}$ appearing in (19) and (21), which contain Schwinger-like terms, can be recognized as arising from a time-ordering ambiguity in the $\tilde{f}_\mu$ fields. This can be verified by expressing these fields in terms of the f_μ's, which are the actual integration variables in the path integral, and making use of the commutators of the f_μ fields given in [3]. From Eqs. (18)–(22) we conclude that modulo a contact term and time-ordering ambiguities the following identifications hold:

$$F^\mu \leftrightarrow f^\mu \leftrightarrow \tilde{f}^\mu.$$

It is instructive to compare the above results connecting the correlation functions involving f_μ and $\tilde{f}_\mu$ with the corresponding relations obtained by not including [1,3] the constraint appearing in the functional measure of (16). Thus, consider the generating functional,

$$Z[j,\tilde{j}] = \int Df_\mu \exp\left(i\int d^3x[\mathcal{L}_{\rm SD} + f_\mu j^\mu + \tilde{f}_\mu\tilde{j}^\mu]\right). \quad (23)$$

Performing the integration one is led to the action

$$I_{\rm SD}[j,\tilde{j}] = -\tfrac{1}{2}\int d^3x\,d^3y\left[j_\mu\Gamma^{\mu\nu}j_\nu + \tilde{j}_\mu\Gamma^{\mu\nu}\tilde{j}_\nu + 2j_\mu\Gamma^{\mu\nu}\tilde{j}_\nu - 2j_\mu\tilde{j}^\mu - \tilde{j}_\mu\tilde{j}^\mu - \frac{1}{m}\epsilon_{\mu\alpha\beta}(\partial^\alpha\tilde{j}^\beta)\tilde{j}^\mu\right], \quad (24)$$

where

$$\Gamma^{\mu\nu} = \frac{1}{\Box + m^2}(m^2 g^{\mu\nu} + \partial^\mu\partial^\nu + m\epsilon^{\mu\alpha\nu}\partial_\alpha)\delta(x-y). \quad (25)$$

For $\tilde{j}^\mu = 0$ this expression reduces to Eq. (18) of Ref. [3]. From (24) we are immediately led to the following relations, which are the analogues of (19) and (21):

$$\langle f_\mu(x)f_\nu(y)\rangle = \langle f_\mu(x)\tilde{f}_\nu(y)\rangle + ig_{\mu\nu}\delta(x-y) = \langle \tilde{f}_\mu(x)\tilde{f}_\nu(y)\rangle + ig_{\mu\nu}\delta(x-y) - \frac{i}{m}\epsilon_{\mu\nu\alpha}\partial^\alpha\delta(x-y). \quad (26)$$

Note that again the self-duality relation $f_\mu = \tilde{f}_\mu$ is realized modulo nonpropagating contact terms. But now these terms can no longer be interpreted as time-ordering ambiguities, as was the case before.

As a final comment we point out that, while the Hamiltonian of the SD model is mapped onto the MCS model by the above identifications [3], this is not the case for the respective Lagrangians. The present phase-space analysis has shown, however, in what sense the mapping is realized on the level of Green's functions.

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