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Non-Abelian soliton operators

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We construct soliton operators for the Wess-Zumino-Witten (WZW) model with a chiral $O(N)\times O(N)$ invariance, and use them to derive non-Abelian bosonization properties directly. The soliton operators express the fermion fields as nonlocal functions of the boson currents and provide a direct map from the boson to the fermion fields. With them we determine the fermion equivalent Lagrangian for level $k = N$ together with the fermion bilinears equivalent to the boson fields. We generalize this construction to arbitrary values of the coupling constant and find the equivalent fermion model, which has current-current interactions.

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I. INTRODUCTION

Bosonization is the means by which a fermion field theory can be expressed entirely in terms of boson fields, the best known example being the massive Thirring model equivalence with the sine-Gordon model [1,2] in two space-time dimensions. Bosonization offers many insights into models of interacting fermions; for example, by using semiclassical approximations in the boson formulation one can gain nonperturbative information about the fermions. Bosonization is also useful in the study of quantum critical phenomena, having been used [3] in the description of quantum antiferromagnetic chains.

In two-dimensional field theory the concepts of bosonization can be stated precisely: the boson currents of one model can be identified with the fermion currents of another model and explicit mappings between the fundamental boson and fermion fields established. The dynamics of the boson field then implies certain dynamics for the fermion field and vice versa. Mandelstam [2] derived the essential features of bosonization for the massive Thirring model by constructing a soliton operator (so named because it could be viewed as creating as sine-Gordon soliton) which expressed the fermion field directly as a nonlocal function of the boson fields. With this operator Mandelstam was able to establish the correspondence between the fermion and boson currents, identify the mass and potential terms of the two models, and derive the fermion field equations from those of the boson field. Renormalization of the fermion theory follows automatically from the renormalization of the boson theory together with suitable regularization of the soliton operator. Subsequently, Witten [4] described a boson-fermion equivalence between the boson Wess-Zumino-Witten (WZW) model, with a chiral $O(N) \times O(N)$ invariance and including a Wess-Zumino term, and a free massless fermion theory; this equivalence is a non-Abelian generalization of the special case $N = 2$ of the Abelian sine-Gordon and massive Thirring model equivalence in the zero mass limit, with the coupling constant $\beta^2 = 4\pi$.

Non-Abelian bosonization has been formulated in a path integral setting by several authors; see, for example, Redlich and Schnitzer [5], and more recently [6,7] where a path integral approach is used to determine "smooth bosonization" in which mixed fermionic and bosonic descriptions are possible. Our investigation, however, is more in the spirit of Mandelstam, in which the equivalence is determined by a direct mapping between the quantum fields. We generalize Mandelstam's soliton operators to the non-Abelian case by constructing operators, as nonlocal functions of the boson currents, which satisfy the same commutation relations with respect to the currents as do the fermion fields and which after suitable renormalization we identify as fermion fields. Central to our construction is an operator factorization of the boson matrix g (described in Sec. III) which is consistent with the classical solution of the boson field equations and from which the soliton operators can be obtained directly using properties of the current algebra. As a consequence of this construction, we are able to examine non-Abelian bosonization for the case when the level k of the current algebra takes the value $k = N$, when the quantum WZW model exhibits a high degree of symmetry which is manifest in the fermion formulation (Sec. V). We also derive the equivalence between the fermion and boson currents and between the boson matrix and fermion bilinears. Although we use properties of the classical fields, this derivation applies to the quantum fields once the soliton operator is regularized and suitably normalized. The regularization of divergences, discussed in Sec. VII, arising from the multiplication of fermion operators at the same point is also necessary in order to determine the precise anticommutation properties of the fermion fields, and we indicate how this regularization may be performed without deriving precise details.

Because the essential properties of the soliton operators follow solely from the current (Kac-Moody) algebra, the formulas are not restricted to the WZW model

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and its fermion equivalent but can be extended to include, for example, massive boson models. Furthermore, bosonization using soliton operators is not restricted to the case in which the coupling constant λ takes only the values $\lambda^2 = 4\pi/k$, for which the WZW model is conformally invariant. We generalize the current algebra to arbitrary λ and construct soliton operators satisfying the same equal-time commutation relations with the currents as do the fermion fields. The equivalent fermion model has current-current interactions which comprise a non-Abelian generalization of the Thirring model and which we determine from the properties of the soliton operators, as demonstrated in Sec. VI. Finally, in Sec. VII, we consider the regularization and renormalization of the soliton operators which is necessary in order to establish the equivalence between the two quantum field theories. We indicate several ways in which these operators and the quantum equations they satisfy might be regularized and renormalized, although full details of the renormalization required for the four-point fermion couplings remain to be determined.

The boson Lagrangian may be written

$$L = \frac{1}{4\lambda^2}\int \mathrm{Tr}(\partial_\mu g \partial^\mu g^{-1}) d^2x + k\Gamma \ , \tag{1}$$

where $g \in \mathrm{O}(N)$, $\lambda^2 = 4\pi/k$, where k, the central charge or the level of the current algebra, is an integer which we take to be positive, and Γ is the Wess-Zumino term. Witten showed that this boson model is equivalent to a model of $2N$ free Majorana fermion fields $\psi_i^\pm$ with the left and right currents identified according to

$$J_{ij}^- = (\partial_- g g^{-1})_{ij} = i\lambda^2 \psi_i^- \psi_j^- \ ,$$
$$J_{ij}^+ = (g^{-1}\partial_+ g)_{ij} = -i\lambda^2 \psi_i^+ \psi_j^+ \ , \tag{2}$$

where $\partial_\pm = \partial/\partial t \pm \partial/\partial x$. These currents satisfy $\partial_+ J_{ij}^- = 0 = \partial_- J_{ij}^+$, and the fermion field equations are $\partial_- \psi_i^+ = 0 = \partial_+ \psi_i^-$. The equal-time current commutation relations are

$$\begin{aligned}[J_{ij}^\pm(x), J_{kl}^\pm(y)] = \pm i\lambda^2 \delta(x-y)[\delta_{ik} J_{jl}^\pm(x) + \delta_{jl} J_{ik}^\pm(x) \\ -\delta_{jk} J_{il}^\pm(x) - \delta_{il} J_{jk}^\pm(x)] \\ \pm 2i\lambda^2 \delta'(x-y)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \ ,\end{aligned} \tag{3}$$

and $[J_{ij}^+(x), J_{kl}^-(y)] = 0$. In addition to the current equivalence, Witten also identified the boson fields with fermion bilinears according to

$$M g_{ij} = -i\psi_i^- \psi_j^+ \ , \tag{4}$$

where M is a mass renormalization. This formula can be investigated in detail using soliton operators, including the case $N = 2$ when a different formula applies [8]. It is also of interest to investigate the consequences of the classical boson field equations $\partial_+(\partial_- g g^{-1}) = 0 = \partial_-(g^{-1}\partial_+ g)$, which imply that g factorizes in the form

$$g(x) = A(x^-)B(x^+) \ , \tag{5}$$

where A and B are elements of $\mathrm{O}(N)$. By contrast, since ψ_i^- depends only on x^- and ψ_i^+ depends only on x^+ (because the fermion field equations are $\partial_- \psi_i^+ = 0 = \partial_+ \psi_i^-$), we see that in (4) each element g_{ij} has been factorized, whereas classically Eq. (5) shows that the matrix g must be factorized. By requiring consistency between (4) and (5), we will find soliton operators for the case $k = N$ which allow us to generalize (4) in a way which still reduces to the known formula for $N = 2$.

In order to construct the appropriate non-Abelian soliton operators, we return briefly to the Abelian case $N = 2$ where we can use the Mandelstam soliton operators to determine the correct form of the bilinear equivalence (4) and then progress to the construction of non-Abelian soliton operators.

II. ABELIAN BOSONIZATION

Let us parametrize $g \in \mathrm{SO}(2)$ according to

$$g = \begin{pmatrix} \cos\beta\phi & \sin\beta\phi \\ -\sin\beta\phi & \cos\beta\phi \end{pmatrix} \ , \tag{6}$$

where $\beta^2 = 4\pi$ (in Coleman's [1] notation), giving, for the currents,

$$J_{12}^\pm = \beta\left(\pi \pm \frac{\partial\phi}{\partial x}\right) \ .$$

Define

$$A^\pm(x) = \pm\frac{\beta}{2}\phi(x) - \frac{2\pi}{\beta}\int_{-\infty}^{x} \pi(\xi) d\xi \ , \tag{7}$$

which, upon discarding a boundary term and using $\beta^2 = 4\pi$, can also be written

$$A^\pm(x) = -\tfrac{1}{2}\int_{-\infty}^{x} J_{12}^\pm(\xi) d\xi \ . \tag{8}$$

Then, following Mandelstam, we can form the soliton operators $\exp(iA^\pm)$, which we identify with Dirac fermions, or equivalently we can form Majorana fermions according to

$$\begin{aligned}\psi_1^+ = :\cos A^+: \ , \quad \psi_2^+ = -:\sin A^+: \ , \\ \psi_1^- = :\sin A^-: \ , \quad \psi_2^- =: \cos A^-: \ ,\end{aligned} \tag{9}$$

where we have indicated that normal ordering must be performed and it is understood that suitable regularization is also to be carried out (for a discussion, see Refs. [2,9]). By using the soliton operators we can now verify directly [8] that, for $N = 2$, Eq. (4) is replaced by

$$M g_{ij} = i(\varepsilon_{ik}\psi_k^- \psi_j^+ - \psi_i^- \varepsilon_{jk}\psi_k^+) \ . \tag{10}$$

In matrix form this can be written

$$M g = i\begin{pmatrix} \psi_2^- & \psi_1^- \\ -\psi_1^- & \psi_2^- \end{pmatrix}\begin{pmatrix} \psi_1^+ & \psi_2^+ \\ -\psi_2^+ & \psi_1^+ \end{pmatrix} \ , \tag{11}$$

which is consistent with the dynamics as expressed in (5) because the boson matrix is in a manifestly factorized form. Equation (11) can be rewritten in terms of the soliton operators (9), and we find that g can be written in the suggestive form $g = g^- g^+$, where

$$g^{\pm} = \exp\left(\pm\tfrac{1}{2}L_{12}\int_{-\infty}^{x} J_{12}^{\pm}(\xi)d\xi\right) ,$$
$$L_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{12}$$

where we have used $J_{12}^{+} - J_{12}^{-} = 2\beta\partial\phi/\partial x$. We see that $g = g^- g^+$ is an operator factorization of g which is independent of, but consistent with, the boson field dynamics. Furthermore, from the factors of g we can form the soliton operators themselves, for by choosing ψ^- to be a column of g^- and ψ^+ to be a row of g^+ we obtain correctly the fermion fields, as (11) shows.

III. NON-ABELIAN BOSONIZATION

The factorization of g for $N = 2$ relies only on the properties of the quantum boson fields and the current algebra derived from these fields, but can be regarded as the starting point of the boson-fermion equivalence because it determines the soliton operators and hence the fermion fields. Now let us generalize this construction to the non-Abelian case. By definition, the soliton operators satisfy the same equal-time commutation relations with respect to the currents as do the fermion fields: namely,

$$[J_{ij}^{\pm}(x), \psi_k^{\pm}(y)] = \pm i\lambda^2\delta(x-y)(\delta_{ik}\psi_j^{\pm} - \delta_{jk}\psi_i^{\pm}) ,$$
$$[J_{ij}^{+}(x), \psi_k^{-}(y)] = [J_{ij}^{-}(x), \psi_k^{+}(y)] = 0 . \tag{13}$$

These equations can also be regarded as the definition of the fermion fields, given the fermion currents $J_{ij}^{\pm}$. This method of introducing fermion fields follows the approach of Dell'Antonio, Frishman, and Zwanziger [10] and also Johnson [11], in which the direct introduction of the singular fermion anticommutation relations is avoided. Instead, one begins with the current commutation relations and then postulates the commutation relations of the fermion fields with the currents, allowing for renormalized arbitrary constants which are determined from the quantum field equations. The renormalized fermion anticommutation relations can then be derived in operator form by means of the expansion of operator products on the light cone.

We will solve the commutation relations (13) by regarding J_{ij}^{+} as given boson currents in order to obtain the soliton operators. Let us define antisymmetric matrices L_{ij}, which span the Lie algebra of $O(N)$, by

$$(L_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj} \tag{14}$$

$(i, j, k, l = 1, \ldots, N)$ and define

$$J^{\pm} = \tfrac{1}{2}\sum_{i,j} L_{ij}J_{ij}^{\pm} .$$

The commutation relations of the currents and the boson fields are [4]

$$[J_{ij}^{-}(x), g(y)] = -i\lambda^2\delta(x-y)L_{ij}g ,$$
$$[J_{ij}^{+}(x), g(y)] = -i\lambda^2\delta(x-y)gL_{ij} . \tag{15}$$

Let us suppose that g can be factorized in the form $g = g^- g^+$, where $g^{\pm} \in O(N)$ and g^- depends (nonlocally) on J_{ij}^{-} and g^+ depends on J_{ij}^{+}. This factorization accords with the $N = 2$ case shown in (12) and is consistent with the classical boson dynamics given in (5) for the WZW model. Using $[J_{ij}^{-}, J_{kl}^{+}] = 0$, we then determine from (15) that

$$[J_{ij}^{-}(x), g^-(y)] = -i\lambda^2\delta(x-y)L_{ij}g^-(y) , \tag{16a}$$
$$[J_{ij}^{+}(x), g^+(y)] = -i\lambda^2\delta(x-y)g^+(y)L_{ij} . \tag{16b}$$

Now let us choose ψ^- to be any column of g^- and ψ^+ any row of g^+; then, the components $\psi_i^{\pm}$ satisfy precisely the relations (13). In other words, by factorizing g appropriately we obtain immediately the soliton operators, from which the boson-fermion equivalence can be established.

Rather than seek an operator factorization of g directly, let us proceed as in the Abelian case by solving Eqs. (16) for g^- and g^+. The dynamics of the free boson field is of assistance in obtaining a trial solution. Using $\partial_+ g^- = 0$ and $J^- = \partial_- g^-(g^-)^{-1}$, which follows from the assumption $g = g^- g^+$, we find that, for any fixed time,

$$\frac{\partial g^-(x)}{\partial x} = -\tfrac{1}{2}J^-(x)g^-(x) . \tag{17}$$

The solution, for given J^-, can be expressed as a path-ordered exponential:

$$g^-(x) = P\left[\exp\left(-\tfrac{1}{2}\int_{-\infty}^{x} J^-(\xi)d\xi\right)\right] , \tag{18}$$

where P denotes path ordering; i.e., factors are ordered such that operators $J^-(\xi)$ with larger arguments ξ stand to the left. Explicitly,

$$g^-(x) = \sum_{n=0}^{\infty}(-\tfrac{1}{2})^n \int \left(\prod_{i=1}^{n}\theta(\xi_{i-1}-\xi_i)J^-(\xi_i)\right) d^n\xi , \tag{19}$$

where $\xi_0 \equiv x$. We adopt (18) therefore as the definition of g^- and must now prove that g^- actually satisfies the commutation relation (16a); we do this by finding $B(x)$ such that

$$g^-(x)A = B(x)g^-(x) , \tag{20}$$

where $A = J_{ij}^{-}(y)$ (suppressing the dependence of A, B, on i, j, y). First, we differentiate (20) using (17), to get

$$\frac{dB(x)}{dx} = \tfrac{1}{2}[B(x), J^-(x)]$$

and upon integrating find

$$B(x) = A + \tfrac{1}{2}\int_{-\infty}^{x}[B(\xi), J^-(\xi)]d\xi \ .$$

This equation can be solved iteratively; however, it is sufficient to verify by direct substitution that the solution is $B(x) = i\lambda^2\delta(x-y)L_{ij} + J^-_{ij}(y)$ by using the current commutation relations (3). Hence (16a) is satisfied. Similarly, the expression for g^+ which satisfies (16b) is

$$g^+(x) = P^{-1}\left[\exp\left(\tfrac{1}{2}\int_{-\infty}^{x} J^+(\xi)d\xi\right)\right] \ , \qquad (21)$$

where P^{-1} indicates path ordering in the sense opposite to P.

Hence we have solved Eqs. (16) for g^-, g^+ as functions of J^-, J^+, and we refer to $g^\pm$ as soliton operators by analogy with the Abelian case, where these operators have a soliton interpretation. The soliton operators are not unique, because g^-h^- and h^+g^+ also satisfy (16a) and (16b), respectively, where h^-, h^+ are constant orthogonal matrices, and we also retain the freedom to set overall normalization factors.

Path-ordered exponentials of currents appear in several contexts in non-Abelian models, although not previously as soliton operators. For example, Bardakci, Crescimanno, and Hotes [12] have realized parafermion fields in non-Abelian coset models as path-ordered exponentials of free currents (with finite base points, whereas our base point is taken at infinity) which may be quantized to obtain quantum parafermion operators. The finite base point is associated with a Wilson line which is attached to each parafermion field, and gauge invariance of the WZW currents ensures that physical quantities are independent of the base point. Alekseev, Faddeev, and Semenov-Tian-Shansky [13] have investigated properties of the monodromy matrix M, defined as a path-ordered exponential of the currents, using a lattice regularization and have shown that properties of the monodromy are determined by the quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$. This observation, that quantum groups can be used to determine properties of the regularized quantum field theory including path-ordered exponentials of the currents, is applicable to quantum soliton operators and this is discussed briefly in Sec. VII.

IV. PROPERTIES OF SOLITON OPERATORS

Now let us develop properties of the soliton operators and, in particular, derive details of the boson-fermion correspondence. From the classical fields $g^\pm$ we define quantum operators $\psi^\pm$ according to

$$\psi^-_{ia}(x) =: g^-_{ia}(x):\, , \quad \psi^+_{ai}(x) =: g^+_{ai}(x):\, , \qquad (22)$$

which we wish to identify with Majorana fermion fields. We note first that we have in fact constructed N^2 such fields, which is consistent with a fermion theory with a central charge of N. The operators (22) are normal ordered with respect to a Fock space representation of the quantum currents (where normal ordering is performed by writing the currents as a sum of components with positive and negative frequencies) and are also regularized and normalized as necessary. The regularization will include a suitable cutoff for the integral in the exponential shown in (18), as well as regularized step functions θ in (19). Indeed, suitable regularization is necessary in order to determine whether the fields $\psi^\pm$ defined by (22) anticommute for different arguments x, y. In the form shown, the elements of $g^-(x)$ in fact commute with those of $g^+(y)$, because the left and right currents commute, and it can also be shown that the elements of g^- commute among themselves for $x \neq y$ and similarly for g^+. This is true even for $N = 2$; i.e., when $A^\pm$ is defined as in (8), the Majorana fields (9) commute for arguments $x \neq y$. On the other hand, when $A^\pm$ is given by (7) the correct anticommuting properties are satisfied, as shown by Mandelstam. Since (7) and (8) differ only by a boundary term, which vanishes when the operators are regularized, we conclude that anticommutation properties of $\psi^\pm$ can be determined only when the operators (22) are correctly regularized.

Without considering details of regularization, which are discussed further in Sec. VII, we can see that fermion anticommutation relations for $\psi^\pm$ are not possible for odd values of N. Each element of an orthogonal matrix is equal to its minor in the matrix and so, considering g^-, for example, the first column can be expressed in terms of the remaining columns. Therefore we may write

$$\psi^-_{i1} \sim \sum \varepsilon_{ii_2\cdots i_N}\psi^-_{i_2 2}\ldots\psi^-_{i_N N} \ , \qquad (23)$$

where "$\sim$" means equality up to multiplicative normalization and where the right-hand side is evaluated by point splitting and then renormal ordered to give the left-hand side. Equation (23) shows that for odd N the components of ψ^- cannot consistently anticommute for different arguments x, y. For even N, however, let us assume that in a regularized form $\psi^\pm$ will satisfy fermion anticommutation relations, and hence we refer to the components of $\psi^\pm$ as fermion operators. Since normal ordering amounts to a multiplicative renormalization, the quantum fermion operators satisfy the relations (13).

Now we consider the product g^-g^+, which is an orthogonal matrix with commuting elements, which we can therefore identify with the boson matrix g; the expression $g = g^-g^+$ factorizes g in a way similar to the Abelian case. From (22) we find that

$$g_{ij} \sim \sum_{a=1}^{N}\psi^-_{ia}\psi^+_{aj} \ , \qquad (24)$$

which is the generalization to level $k = N$ of Eq. (4). By substituting for dependent columns or rows of ψ^- and ψ^+, as shown in (23), we obtain the precise non-Abelian analogue of (10). Using also $g^- = g(g^+)^{-1}$, we can write

$$\psi_{ia}^- \sim \sum_j g_{ij}\psi_{aj}^+ \ , \tag{25}$$

which shows that the boson matrix transforms the right Majorana components into the left components.

From the soliton operators we can determine the fermion currents equivalent to the boson currents $J^\pm$. By using (17) we obtain the expansion

$$g^-(x) = g^-(y) - \tfrac{1}{2}(x-y)J^-(y)g^-(y) + O((x-y)^2)$$

and therefore

$$\lim_{y\to x}\sum_{a=1}^N \psi_{ia}^-(x)\psi_{ja}^-(y) \sim \lim_{y\to x}[g^-(x)g^-(y)^{-1}]_{ij} \sim J_{ij}^-(x) \ , \tag{26}$$

where we subtracted a c-number constant in the last step to obtain J_{ij}^-. Hence, for $k=N$, Eq. (2) generalizes in the natural way to read

$$\begin{aligned} J_{ij}^- &= (\partial_- g g^{-1})_{ij} \sim \sum_{a=1}^N \psi_{ia}^-\psi_{ja}^- \ , \\ J_{ij}^+ &= (g^{-1}\partial_+ g)_{ij} \sim \sum_{a=1}^N \psi_{ai}^+\psi_{aj}^+ \ . \end{aligned} \tag{27}$$

V. EQUIVALENT FERMION MODEL

The field equations satisfied by the fermion fields follow immediately from those of the boson fields, i.e., $\partial_- J^+ = 0 = \partial_+ J^-$ implies $\partial_-\psi_{ai}^+ = 0 = \partial_+\psi_{ia}^-$, and so the fermion Lagrangian which corresponds to the boson theory is given by

$$L = i\int \sum_{i,a}(\psi_{ia}^-\partial_+\psi_{ia}^- + \psi_{ai}^+\partial_-\psi_{ai}^+)d^2x \ , \tag{28}$$

where $\psi^\pm$ are constrained to be orthogonal matrices, as follows from the orthogonality of $g^\pm$ and the identification in (22).

The equivalence of the WZW model for general k with a constrained fermion theory is well known; Antoniadis and Bachas [14] found that the boson model is equivalent to a gauged theory of Majorana fermions or, alternatively, to a theory of free fermions obeying para-Fermi statistics. This approach has been developed by Chang, Kumar, Mohapatra [15] and also Bardakci, Crescimanno, and Rabinovici [16] and Redlich and Schnitzer [5] using path integrals. Let us demonstrate the equivalence of the constrained fermion Lagrangian (28) with that obtained by gauging a free fermionic theory and show that the orthogonality of the fermion fields in (28) can be expressed as a set of constraints on certain currents. Consider therefore the free fermion Lagrangian density

$$\mathcal{L} = \sum_{i,j,a}\bar\psi_i^a\gamma^\mu[\delta_{ij}\partial_\mu + (A_\mu)_{ij}]\psi_i^a \ , \tag{29}$$

where A_μ is a gauge field taking values in $\mathrm{O}(N)$ and ψ is a set of Dirac fermions which can be written in Majorana form according to

$$\psi_i^a = \begin{pmatrix}\psi_{ai}^+ \\ \psi_{ia}^-\end{pmatrix} \ . \tag{30}$$

The gauge field A_μ is not dynamical, but acts as a Lagrange multiplier to project out the currents of the algebra, which implies that the constraints can be written

$$\sum_{a,k,l}\bar\psi_k^a\gamma^\mu(L_{ij})_{kl}\psi_l^a = 0 \ , \tag{31}$$

where L_{ij} is defined in (14). By choosing $\mu=0,1$ separately, we find that the constraints take the form

$$\sum_a[\psi_{ai}^+,\psi_{aj}^+] = 0 = \sum_a[\psi_{ia}^-,\psi_{ja}^-] \ , \tag{32}$$

which become, for $i\neq j$, $\sum_a\psi_{ai}^+\psi_{aj}^+ = 0 = \sum_a\psi_{ia}^-\psi_{ja}^-$. Up to an overall normalization we find, therefore, that the constraints (31) are equivalent to specifying that the Majorana fields $\psi^\pm$ should form orthogonal matrices, and so the fermion model given by (28) is equivalent to (29) with the gauged free fermions. It should be pointed out that the bilinear constraints (32) must be understood in terms of point splitting and do not contradict the fact that bilinear fermion fields generate the currents, as shown in (27). More precisely, we expand $\sum_a\psi_{ai}^+(x)\psi_{aj}^+(x+\varepsilon)$ in powers of ε, and the lowest-order contribution will vanish unless $i=j$ and the terms of next highest order will be proportional to the currents J_{ij}^+.

The Lagrangian (28) displays a symmetry not present in the boson Lagrangian (1), namely, invariance with respect to $\mathrm{O}(N)$ transformations on the index a, corresponding to right and left transformations on g^-, g^+, respectively, by elements of $\mathrm{O}(N)$. This invariance, however, is present in the quantized boson theory, for it corresponds to the transformation $g = g^-g^+ \to g' = g^-h^-h^+g^+ \in \mathrm{O}(N)$, where $h^\pm$ are constant orthogonal matrices. Since this transformation relies on the operator factorization $g=g^-g^+$, this symmetry appears only in the quantum theory, whereas the $\mathrm{O}(N)\times\mathrm{O}(N)$ chiral invariance is manifest classically. This hidden symmetry was observed by Antoniadis and Bachas [14] for a general level k and is an $\mathrm{O}(k)\times\mathrm{O}(k)$ invariance of the fermion theory.

Because the soliton operators are defined entirely in terms of the boson currents, their properties are not restricted to the WZW model, but depend only on properties of the boson currents. If we add a mass term Trg to the Lagrangian (1), then the current commutation relations (3) are unchanged and so the soliton operators g^-g^+ given by (18) and (21) still satisfy the same commutation relations (16). In this case the classical solution of the boson field equations is no longer of the form (5), and J^-, J^+ depend on both x^-, x^+; however, the operator factorization $g=g^-g^+$ and other algebraic properties of the soliton operators are still valid because the current algebra is unchanged. This includes the current equivalence (27) and the boson matrix and fermion bilinear equivalence (24), but now the fermion field equations are determined directly from the soliton operators by using the nonlinear boson field equations. More directly, we can regard the mass term $\mathrm{Tr}g$ as a perturbation within the WZW model and (24) then shows that

$$\mathrm{Tr}g \sim \sum_{i,a=1}^{N} \psi_{ia}^{-}\psi_{ai}^{+} , \tag{33}$$

and so, by adding the right-hand side to (28), we obtain the equivalent massive fermion theory. The quantization of the massive WZW model has been discussed by Gepner [17].

VI. ARBITRARY COUPLING CONSTANT

Let us now generalize the bosonization results by removing the restriction that the coupling constant λ should take only the values $\lambda^2 = 4\pi/k$, and so we consider the Lagrangian (1) for arbitrary λ. The WZW model no longer displays conformal symmetry, but we can still find a fermion equivalent model. In the Abelian case this equivalent model is the massive Thirring model, and so we expect that for the non-Abelian case the fermion Lagrangian (28) will need to be modified by the addition of current-current interactions. The generalization proceeds by finding the current algebra for arbitrary values of the coupling constant λ and factorizing the boson matrix g into a product $g^L g^R$ of left and right components which can be identified as fermion operators.

The Lagrangian (1) for general λ is chirally invariant and so has two conserved currents, the left and right currents corresponding to the chiral group $\mathrm{O}(N)\times\mathrm{O}(N)$. These currents are

$$\begin{aligned} J^\mu &= g^{-1}\partial^\mu g - \frac{k\lambda^2}{4\pi}\varepsilon^{\mu\nu} g^{-1}\partial_\nu g , \\ K^\mu &= \partial^\mu g g^{-1} + \frac{k\lambda^2}{4\pi}\varepsilon^{\mu\nu}\partial_\nu g g^{-1} , \end{aligned} \tag{34}$$

each of which is conserved, as follows directly from the field equations

$$\partial_\mu(g^{-1}\partial^\mu g) = \frac{k\lambda^2}{4\pi}\varepsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) . \tag{35}$$

For $\lambda^2 = 4\pi/k$, the current densities J^0, K^0 reduce to J^+, J^-, respectively, as defined in (2). The equal-time commutation relations satisfied by J^0 and K^0 are given [18], in matrix form, by

$$\begin{aligned} [J_{ij}^0(x), J^0(y)] &= -i\lambda^2\delta(x-y)[J^0(x), L_{ij}] \\ &\quad + \frac{2ik\lambda^4}{4\pi}\delta'(x-y)L_{ij} , \\ [K_{ij}^0(x), K^0(y)] &= i\lambda^2\delta(x-y)[K^0(x), L_{ij}] \\ &\quad - \frac{2ik\lambda^4}{4\pi}\delta'(x-y)L_{ij}, \\ [J_{ij}^0(x), K_{kl}^0(y)] &= 0 \end{aligned} \tag{36}$$

(we have verified these relations explicitly for $N = 3$). The two commuting algebras each generate an invariant chiral algebra, with Schwinger terms contributing a quantized central charge k. Again, these commutation relations reduce to (3) when $\lambda^2 = 4\pi/k$. The commutators of the currents with the fundamental fields $g_{ij}(x)$ are

$$\begin{aligned} [J_{ij}^0(x), g(y)] &= -i\lambda^2\delta(x-y)g(y)L_{ij} , \\ [K_{ij}^0(x), g(y)] &= -i\lambda^2\delta(x-y)L_{ij}g(y) . \end{aligned} \tag{37}$$

We can now write down operators in terms of path-ordered exponentials which satisfy the same equal-time commutation relations as do the currents with the fermion operators (designated as left and right fermion fields ψ^L, ψ^R). These commutators, which we can regard as defining the fermion operators (again following Dell'Antonio, Frishman, and Zwanziger [10]), are

$$\begin{aligned} &[J_{ij}^0(x), \psi_k^R(y)] = i\lambda^2\delta(x-y)(\delta_{ik}\psi_j^R - \delta_{jk}\psi_i^R) , \\ &[K_{ij}^0(x), \psi_k^L(y)] = -i\lambda^2\delta(x-y)(\delta_{ik}\psi_j^L - \delta_{jk}\psi_i^L) , \\ &[J_{ij}^0(x), \psi_k^L(y)] = [K_{ij}^0(x), \psi_k^R(y)] = 0 . \end{aligned} \tag{38}$$

Define now, in analogy with (18), the operators (for fixed time)

$$\begin{aligned} g^L(x) &= P\left[\exp\left(-\frac{4\pi}{2k\lambda^2}\int_{-\infty}^{x} K^0(\xi)d\xi\right)\right] , \\ g^R(x) &= P^{-1}\left[\exp\left(\frac{4\pi}{2k\lambda^2}\int_{-\infty}^{x} J^0(\xi)d\xi\right)\right] . \end{aligned} \tag{39}$$

The commutation relations satisfied by these operators are

$$\begin{aligned} [K_{ij}^0(x), g^L(y)] &= -i\lambda^2\delta(x-y)L_{ij}g^L(y) , \\ [J_{ij}^0(x), g^R(y)] &= -i\lambda^2\delta(x-y)g^R(x)L_{ij} , \\ [J_{ij}^0(x), g^L(y)] &= 0 , \\ [K_{ij}^0(x), g^R(y)] &= 0 . \end{aligned} \tag{40}$$

These relations are proved in a way similar to (16) for the operators (18) and (21), by using the properties

$$\begin{aligned} \frac{\partial g^L}{\partial x} &= -\frac{4\pi}{2k\lambda^2}K^0 g^L , \\ \frac{\partial g^R}{\partial x} &= \frac{4\pi}{2k\lambda^2}g^R J^0 . \end{aligned} \tag{41}$$

Equations (40) can be viewed as integrated forms of the commutators (36) since, by differentiating (40) with respect to y (at fixed time), we regain (36). Now, by identifying fermion operators according to the normal-ordered operators

$$\psi_{ia}^L =: (g^L)_{ia} : , \quad \psi_{ai}^R =: (g^R)_{ai} : , \tag{42}$$

we find that precisely the relations (38) are satisfied. Again, we obtain N^2 fermion fields for each of the left and right components, and hence we have obtained the desired generalization of the soliton operators to the case of general λ. As before, since normal ordering is multiplicative, the quantum fermion operators (42) satisfy the commutation relations (38).

Several properties follow immediately, again in the same way as for the particular case $\lambda^2 = 4\pi/k$. First, we can identify the combination $g^L g^R$ with the matrix of boson fields g, corresponding to (24), i.e.,

$$g_{ij} \sim \sum_{a=1}^{N} \psi_{ia}^L \psi_{aj}^R , \tag{43}$$

and we may also identify the currents as fermion bilinear combinations:

$$\begin{aligned} K_{ij}^0 &\sim \sum_{a=1}^{N} \psi_{ia}^L \psi_{ja}^L , \\ J_{ij}^0 &\sim \sum_{a=1}^{N} \psi_{ai}^R \psi_{aj}^R . \end{aligned} \tag{44}$$

Here we used the expansion (for J^0)

$$\begin{aligned} g^R(x) = g^R(y) &+ \frac{4\pi}{2k\lambda^2}(x-y)g^R(y)J^0(y) \\ &+ O((x-y)^2) , \end{aligned} \tag{45}$$

and then the identification in (44) follows in the same way as for (26). The spatial components of the currents can also be identified with fermion bilinears by expressing J^1 and K^1 as linear combinations of $g^{-1}K^0 g, J^0$ and $g^{-1}J^0 g, K^0$, respectively. From expansion (45) we find

$$\begin{aligned} g^R(x)g^R(y)^{-1} = 1 &+ \frac{4\pi}{2k\lambda^2}(x-y)g^R(y)J^0(y)g^R(y)^{-1} \\ &+ O((x-y)^2) \end{aligned} \tag{46}$$

and similarly for $g^{-1}K^0 g$, giving

$$\begin{aligned} (g^{-1}K^0 g)_{ab} &= [(g^L)^{-1}K^0 g^L]_{ab} \sim \sum_{i=1}^{N} \psi_{ia}^L \psi_{ib}^L , \\ (gJ^0 g^{-1})_{ab} &= [g^R J^0 (g^R)^{-1}]_{ab} \sim \sum_{i=1}^{N} \psi_{ai}^R \psi_{bi}^R , \end{aligned} \tag{47}$$

where we used the decomposition $g = g^L g^R$ and the fact that g^R commutes with K^0 and similarly for J^0.

Next, we wish to establish the form of the equivalent fermion model directly using the soliton operators. For $\lambda^2 = 4\pi/k$ this is straightforward since we can obtain the equations satisfied by the fermion fields immediately, i.e., $\partial_- \psi_{ai}^+ = 0 = \partial_+ \psi_{ia}^-$, from which the Lagrangian (28) follows. In general, however, the fermion fields depend on both x^-, x^+ and we now use the Hamiltonian formalism to calculate the time and spatial rates of change of the fermion fields. We can then in principle deduce the form of the fermion Lagrangian from the corresponding field equations. With this approach, however, we encounter several highly singular commutators which we seek to avoid by regarding the corresponding terms in the Hamiltonian as perturbations. Hence the procedure we adopt is to identify part of the boson Hamiltonian as the "free" part H_0, from which the time rate of change of the soliton operators can be calculated without encountering singular commutators. We regard the remaining part of the Hamiltonian H_I, which comprises current-current interactions and could also include a mass term, as a perturbative interaction which is multiplied by a parameter that can later be set to unity.

We identify H_I directly with corresponding terms in the fermion Hamiltonian by using the equivalences given in Eqs. (43)–(47). This is analogous to the treatment of the massive Thirring model (Coleman [1]) in which the mass term is introduced in the Lagrangian as an operator within the massless Thirring model. We point out that one cannot directly identify all terms of the boson Lagrangian with those of the fermion Lagrangian, since the equivalence of the two theories occurs only at the extrema of the respective actions, when the quantum fields satisfy the quantum Euler-Lagrange equations. In particular, we cannot directly identify the two kinetic terms of the boson and fermion models.

The Hamiltonian and momentum density operators for the WZW model are given by

$$\begin{aligned} \mathcal{H} &= -\frac{1}{4\lambda^2}\mathrm{Tr}[(g^{-1}g_x)^2 + (g^{-1}g_t)^2] , \\ \mathcal{P} &= -\frac{1}{2\lambda^2}\mathrm{Tr}[g^{-1}g_x g^{-1}g_t] , \end{aligned} \tag{48}$$

and can be written in terms of the charge densities of the currents:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_I \\ &= -\frac{1}{16\lambda^2}\mathrm{Tr}\left\{\left[1 + \left(\frac{4\pi}{k\lambda^2}\right)^2\right](K_0^2 + J_0^2) \right. \\ &\qquad \left. + 2\left[1 - \left(\frac{4\pi}{k\lambda^2}\right)^2\right](g^{-1}K_0 g J_0)\right\} , \\ \mathcal{P} &= -\frac{4\pi}{8k\lambda^4}\mathrm{Tr}[J_0^2 - K_0^2] , \end{aligned} \tag{49}$$

where the "free" part of the Hamiltonian density is given by

$$\mathcal{H}_0 = -\frac{1}{16\lambda^2}\left[1 + \left(\frac{4\pi}{k\lambda^2}\right)^2\right]\mathrm{Tr}[K_0^2 + J_0^2] , \tag{50}$$

and the remaining term is denoted $\mathcal{H}_I$.

The spatial derivative ψ' of the fermion operator ψ is given by $\psi' = i[P, \psi]$, where P is the total momentum, and so, according to the equivalence (42), we calculate

$$\begin{aligned} &[\mathrm{Tr}\{J_0^2(x) - K_0^2(x)\}, g^R(y)] \\ &\qquad = i\lambda^2 \delta(x-y) \sum_{p,q} \{g^R(y)L_{pq}, J_{pq}^0(y)\} , \end{aligned} \tag{51}$$

from which follows

$$\frac{\partial \psi^R_{ai}}{\partial x} = \frac{\pi}{k\lambda^2} \sum_p \{\psi^R_{ap}, J^0_{pi}\} \ . \quad (52)$$

Evidently, the right-hand side corresponds to a current-current interaction in the fermion Lagrangian and is interpreted as before with point splitting between the fermion and current terms. Next, we calculate the commutator $i[\mathcal{H}_0, \psi]$ in order to obtain the time derivative of the fermion field, with the further contribution to the field equations from $\mathcal{H}_I$ to be considered separately. The calculation is similar to that leading to (52), and we obtain

$$\frac{\partial \psi^R_{ai}}{\partial t} = \frac{1}{8}\left[1 + \left(\frac{4\pi}{k\lambda^2}\right)^2\right] \sum_p \{\psi^R_{ap}, J^0_{pi}\} \ . \quad (53)$$

By combining (52) and (53) we find

$$\partial_- \psi^R_{ai} = \frac{1}{8}\left(1 - \frac{4\pi}{k\lambda^2}\right)^2 \sum_p \{\psi^R_{ap}, J^0_{pi}\} \ , \quad (54)$$

which corresponds to a fermion Lagrangian with a current-current coupling of strength κ given by

$$\kappa = \frac{1}{32}\left(1 - \frac{4\pi}{k\lambda^2}\right)^2 \ , \quad (55)$$

which, as in the case of the massive Thirring model equivalence with the sine-Gordon model, corresponds to a strong fermion coupling (large κ) for a weak boson interaction (small λ); in addition, the case $\kappa = 0$ of free massless fermions corresponds to the conformally invariant boson theory for which $\lambda^2 = 4\pi/k$.

The interaction term $\mathcal{H}_I$ in the Hamiltonian (49) can be identified directly with a corresponding fermion term in the following way. By writing

$$\mathrm{Tr}[g^{-1}K^0 g J^0] = \mathrm{Tr}\{[(g^L)^{-1}K^0 g^L][g^R J^0 (g^R)^{-1}]\} \quad (56)$$

and identifying the fermion fields as in (47), we find

$$\mathrm{Tr}[g^{-1}K^0 g J^0] \sim \psi^L_{ia}\psi^L_{ib}\psi^R_{bj}\psi^R_{aj} \ , \quad (57)$$

where repeated indices are summed from $1,\ldots,N$. (We have discarded additive renormalization constants, and there will also be a multiplicative renormalization.) If we now include the four-point interaction (57) in the fermion Lagrangian, along with the terms equivalent to $\mathrm{Tr}[J_0^2 + K_0^2]$ and also a mass term, we obtain

$$\begin{aligned}\mathcal{L} = {}& i\psi^L_{ia}(\delta_{ij}\partial_+ + A^+_{ij})\psi^L_{ia} + i\psi^R_{ai}(\delta_{ij}\partial_- + A^-_{ij})\psi^R_{ai} \\ &+\kappa\psi^L_{ia}\psi^L_{ja}\psi^L_{ib}\psi^L_{jb} + \kappa\psi^R_{ia}\psi^R_{ja}\psi^R_{ib}\psi^R_{jb} \\ &+\kappa'\psi^L_{ia}\psi^L_{ib}\psi^R_{aj}\psi^R_{bj} + m_0\psi^L_{ia}\psi^R_{ai} \ , \end{aligned} \quad (58)$$

where repeated indices are summed from $1,\ldots,N$. The coefficient κ' is related to κ in a way that is dependent on the precise renormalization procedure. We have included nondynamical gauge fields $A^\pm$ in order to enforce the constraints that the matrices ψ^R and ψ^L each be orthogonal, as follows from (42) from the orthogonality of g^R and g^L.

Quantization of the constrained fermionic system defined by (58), consisting of N^2 interacting left and right Majorana fermions, may be carried out by means of the path integral formalism or by canonical quantization. The problems posed by the constraints in the latter approach have been discussed by Chang, Kumar, and Mohapatra [15], where it is noted that classically the vanishing currents in (31) comprise a set of first class constraints. However, in quantum theory the algebra generated by these currents includes a Schwinger term which converts the constraints from first class to second class. Quantization in the presence of such Schwinger terms has also been discussed by Faddeev [19]; the Poisson brackets of the constraints must be modified by the addition of a two-cocyle, at least for the model of fermion-gauge boson interactions considered by Faddeev, which may be determined by suitable operator ordering and point splitting in order to define the multiplication of singular operators. One can expect that a similar procedure must be undertaken for the model (58) and that the constraints must be treated as second class. Faddeev [19] has also developed a path integral approach to quantization in the presence of second-class constraints, by adding auxiliary fields, and for constrained fermionic systems this has also been discussed by Chang, Kumar, and Mohapatra [15]. The precise meaning of the path integral, in particular the form of the trace anomaly, depends on the method of regularization.

The Lagrangian (58), without the mass term, again displays symmetries present only in the quantum WZW model, specifically the $\mathrm{O}(N)\times\mathrm{O}(N)$ invariance arising from $\mathrm{O}(N)$ transformations on the indices a, b and which corresponds to right and left orthogonal transformations on g^L and g^R, respectively. The method of bosonization which we have developed here via soliton operators provides a very direct way of demonstrating this quantum symmetry. Indeed, we have shown that soliton operators provide a direct means of determining the main features of the boson-fermion equivalence for WZW models with mass terms and arbitrary couplings and non-Abelian generalizations of massive Thirring models.

VII. REGULARIZATION AND RENORMALIZATION

Although the definition (39) of soliton operators as path-ordered exponentials of the boson currents is given in classical terms, as are several properties such as the commutation relations (40), nevertheless these relations extend to the corresponding quantum fields when the currents become quantum operators. This follows once the soliton operators are normal ordered because the renormalization is multiplicative. Let us consider therefore how soliton operators may be regularized and hence how the infinities arising from singular commutators may be renormalized. We regard the current densities K^0, J^0 in the definition (39) as quantum fields and regularize the spatial integral over ξ by introducing a cutoff in the form of a factor $\exp(\varepsilon\xi)$ in the integrand. In order to effectively regularize the δ functions in the commutators (40) (and hence also in the current commutation relations), we insert a set of $\frac{1}{2}n(n-1)$ regularized step functions $\theta_{ij}(\xi)$ into the integrand, where θ_{ij} interpolates between 0 and 1. A possible definition is

$$\theta_{ij} = \theta\left(\frac{\xi}{\varepsilon_{ij}}\right) .$$

where $\theta(x)$ is a smooth function which interpolates between 0 and 1. For small ε_{ij} the function θ_{ij} approximates the step function $\theta(x)$ and the derivative θ'_{ij} is a regularized δ function. Hence we write the regularized soliton operator as

$$\psi^R(x) = N(\varepsilon) : P^{-1}\left[\exp\left(\frac{4\pi}{2k\lambda^2}\int_{-\infty}^{\infty}\exp(\varepsilon\xi)\tfrac{1}{2}\sum_{ij}\theta_{ij}(\xi)L_{ij}J^0_{ij}(\xi)d\xi\right)\right] : ,$$

where $N(\varepsilon)$ is an overall normalization that depends on the regulators $\varepsilon, \varepsilon_{ij}$. This soliton operator satisfies a regularized form of the commutation relations (38).

We avoid singularities arising from the multiplication of operator-valued distributions at the same point in the standard way by point splitting. For example, we replace products of the form $\psi^L(x)\psi^R(x)$ by

$$\lim_{\varepsilon\to 0}\varepsilon^\sigma\psi^L(x)\psi^R(x+\varepsilon) , \quad (59)$$

where σ is calculated so as to produce a finite result as $\varepsilon \to 0$. Point splitting for soliton operators has the same effect as multiplicative renormalization, as follows from the semigroup property of path-ordered exponentials:

$$g^R(x+\varepsilon) = g^R(x)P^{-1}\left[\exp\left(\frac{4\pi}{2k\lambda^2}\int_x^{x+\varepsilon}J^0(\xi)d\xi\right)\right] .$$

The exponent σ in (59) is calculated by renormal ordering the product $\psi^L(x)\psi^R(x)$ of soliton operators.

Another way of regularizing the soliton operator is an adaption of the method of Alekseev, Faddeev, and Semenov-Tian-Shansky [13], in which one introduces a direct cutoff in the integration and defines

$$\psi^R(x_0,x) = N(x_0) : P^{-1}\left[\exp\left(\frac{4\pi}{2k\lambda^2}\int_{x_0}^{x}J^0(\xi)d\xi\right)\right] : ,$$

so that now the fermion fields ψ depend on a cutoff x_0 (also called a base point in [12]) which can be different for each fermion field. We assume that for large negative values of x_0 physical quantities such as expectation values are independent of x_0. We may further regularize the operators $\psi^R(x_0,x)$ by placing these fields on a lattice of n points, and so we define

$$L_i = P^{-1}\left[\exp\left(\frac{4\pi}{2k\lambda^2}\int_{x_i}^{x_{i+1}}J^0(\xi)d\xi\right)\right] ,$$

for $i = 0,\ldots,n-1$, and then write $\psi^R(x_0,x)$ as an ordered product of fields on the lattice points $x_0 < x_1 < \cdots < x_{n-1} < x_n = x$ by using the semigroup property of path-ordered exponentials:

$$\psi^R(x_0,x) = N(x_0,\varepsilon) : L_0L_1\cdots L_{n-1} : ,$$

where $N(x_0,\varepsilon)$ is a normalization depending on x_0 and the lattice spacing $\varepsilon = x_{i+1} - x_i$. The fields L_i may each be normal ordered, and it is found [13] that $L_i = e^a : L_i :$, where the exponent a has been calculated for the case $\lambda^2 = 4\pi/k$ in [13] using an operator expansion and properties of the R matrix for the quantum group. [In [13] the fields L_i are lattice variables comprising the monodromy of the current and have properties determined by the quantum group $\mathcal{U}_q(\mathfrak{sl}(2))$, which suggests that quantum groups can play a similar role for the soliton operators in lattice form.] The calculation of the fermion-boson current equivalence as shown in (44) and (45) may be carried out in regularized form:

$$\frac{1}{\varepsilon}\sum_{a=1}^{N}\psi^R_{ai}(x_0,x)\psi^R_{aj}(x_0,x+\varepsilon)$$

$$= \frac{A_{ij}}{\varepsilon} + \frac{4\pi}{2k\lambda^2}N^2(x_0,\varepsilon)J^0_{ij}(x) + O(\varepsilon) ,$$

where the term A/ε arising from renormal ordering can be discarded by means of an additive renormalization and $N^2(x_0,\varepsilon)$ comprises a multiplicative renormalization constant. Again, we assume that physical quantities are independent of x_0 for large negative x_0.

We conclude by noting that in principle these methods can be applied to verify other bosonization formulas in regularized form, including the equivalence of singular current-current terms such as in (56), but that at present the precise expression of these equivalences in regularized form, particularly for the case of arbitrary coupling λ, remains to be determined.

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