Casimir effect for soft boundaries

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In quantum field theory with confining "hard" (e.g., Dirichlet) boundaries, the latter are represented in the Schrödinger equation defining spatial quantum modes by infinite step-function potentials. One can instead introduce confining "soft" boundaries, represented in the mode equation by some smoothly increasing potential function. Here the global Casimir energy is calculated for a scalar field confined by harmonic-oscillator (HO) potentials in one, two, and three dimensions. Combinations of HO and Dirichlet boundaries are also considered. Some results differ in sign from comparable hard-wall ones.

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I. INTRODUCTION

Boundaries in quantum field theory (QFT) are traditionally "hard," i.e., smooth, static, and have precise spatial location and shape. Hard often means impenetrable (as for a Dirichlet boundary) but not always. Penetrability is decided by the boundary condition imposed at the boundary. From Casimir's original paper [1] to the present time, investigators of the boundary-induced Casimir effect have assumed hard boundaries, perhaps without exception (see the reviews [2]). Being specific objects one can imagine these boundaries experiencing well-defined macroscopic vacuum forces such as those revealed by Casimir's investigation. However, there is a deeper and more complete way to understand what actually occurs. The boundary-induced Casimir effect is the distortion, extending out into space away from each boundary, of the quantum vacuum or virtual-particle background belonging to the quantum field constrained by these boundaries. In maintaining this distortion boundaries experience macroscopic back forces. The latter are of course the direct experimental signature of the Casimir effect. However, the rather beautiful (and in general position-dependent) phenomenon of vacuum distortion is the essence of the Casimir effect. This phenomenon continues to occur when boundaries other than the traditional hard ones are introduced.

Hard boundaries in QFT are unquestionably an idealization. No boundary made of matter can be perfectly smooth and static under arbitrary magnification. Yet most (if not literally all) work on the Casimir effect makes this assumption. Quite possibly the accumulated experience in Casimir theory depends strongly on the boundary-smoothness assumption, and in ways which may not be predictable. The only way to find out is to relax this assumption and see what the consequences are. We begin doing this here, leaving walls static but making them "soft" by means of a semiclassical device familiar in soliton theory.

The spatial modes $\Phi_m(\vec{x})$ of a scalar quantum field $\hat{\Phi}$ can be subjected to a confining spatial potential $V(\vec{x})$ in the Schrödinger-like equation

$$[-\Delta + V(\vec{x})]\Phi_m(\vec{x}) = \omega_m^2 \Phi_m(\vec{x})$$

defining them. The potential $V(\vec{x})$ grows without limit in one or more spatial directions, suppressing all modes (hence all zero-point fluctuations) at large distance in these directions. Hard Dirichlet walls are a limiting case in which $V(\vec{x}) \equiv 0$ ($\equiv \infty$) inside (outside) the wall. Less drastic confining potentials soften the wall. Any potential $V(\vec{x})$ in the mode equation is perhaps best viewed as representing some distribution of matter which interacts with the quantum field. In spatial regions where $V(\vec{x})$ is large, only the high modes are not suppressed. Where $V(\vec{x})$ is small, all modes of the field fluctuate much as in free space. Obviously any $V(\vec{x})$ will distort the vacuum of $\hat{\Phi}$. This distorted vacuum must be regarded as belonging both to $\hat{\Phi}$ and to the distribution of matter modeled by $V(\vec{x})$. Casimir effects unquestionably are generated and conventional methods of analysis can be brought to bear (always assuming one can handle the Schrödinger problem). Local and global Casimir shifts in the vacuum energy can be computed. However, there are no specific walls on which vacuum forces can act. The Casimir forces involved act on the distribution of matter represented by $V(\vec{x})$.

Soliton quantization became familiar 20 years ago (see, e.g., Refs. [3]). Solitons, or localized classical solutions $\Phi_c(\vec{x})$ of some physical nonlinear equation, become in QFT a soft localized "spatial property" represented by a potential in a Schrödinger equation whose solutions are the modes of the quantum field $\hat{\Phi}$ quantized about $\Phi_c(\vec{x})$ rather than about $\Phi(\vec{x}) = 0$. The difference between soliton quantization and what we shall be doing here is obvious. For solitons one has a localized $V(\vec{x})$ and the modes are scattering modes (plus perhaps a finite set of bound modes). Given the Schrödinger energy spectrum $\{\omega_n^2\}$ and the modes $\Phi_n(\vec{x})$ one can calculate local

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quantities such as $\langle T_{00}(\vec{x})\rangle$ within and around the soliton. Traditionally, attention has centered on global quantities, however, especially the mass of the quantized soliton:

$$M_{\text{quantum}} = M_{\text{cl}} + \tfrac{1}{2}\sum_n \omega_n \,,$$

where the mode sum representing quantum effects diverges and needs regularization. This quantum correction to the classical soliton mass M_{cl} is really a kind of Casimir energy, although a scattering rather than bound spectrum is involved. Just as one "quantizes about a soliton," one could characterize a typical Casimir problem as "quantization within a confining potential." This characterization applies to hard and soft walls. Let us proceed now to describe the calculations presented in this paper. It seems important to contrast the soft-wall mathematics with standard hard-wall language, so we begin with the latter.

The spatial modes of a scalar quantum field $\hat{\Phi}$ confined in the x_1 direction between infinite parallel planar Dirichlet boundaries at $x_1 = 0, L$ are (with δ-function normalization)

$$\Phi_{mq_{2,3}}(\vec{x}) = \begin{cases} \sqrt{\frac{2}{L}}\sin(k_{1m}x_1)\frac{1}{2\pi}e^{i(q_2x_2+q_3x_3)}, & 0 \le x_1 \le L, \\ 0, \ x_1 < 0 \ \text{and} \ x_1 > L, \end{cases}$$
$$k_{1m} = \frac{m\pi}{L}, \quad m = 1,2,3,\ldots\,, \tag{1.1}$$

where $k_1(q_{2,3})$ represent momentum in the x_1 $(x_{2,3})$ directions. The sine factor in (1.1) is of course the set of solutions of a one-dimensional Schrödinger equation with the Dirichlet walls represented by an infinite square-well potential.

To have tractable soft-wall modeling we must choose a confining potential whose Schrödinger problem can be solved, analytically or numerically. As an obvious first choice we constrain the quantum field by a harmonic oscillator (HO) potential in the x_1 direction:

$$\left[-\frac{d^2}{dx_1^2} + \tfrac{1}{4}\alpha^4 x_1^2\right]\Phi_n(x_1) = k_{1n}^2\Phi_n(x_1)\,. \tag{1.2}$$

The familiar orthonormal solutions are

$$\Phi_n(x_1) = \left[\frac{\alpha}{n!\sqrt{2\pi}}\right]^{1/2}\exp(-\tfrac{1}{4}\alpha^2x_1^2)H_n(\alpha x_1)\,, \tag{1.3}$$
$$k_{1n} = \alpha\sqrt{n+\tfrac{1}{2}}, \quad n = 0,1,2\ldots\,,$$

where the functions $H_n(x)$ are Hermite polynomials, solutions of

$$y'' - xy' + ny = 0\,.$$

When the factor $\sqrt{2/L}\sin(m\pi x_1/L)$ in (1.1) is replaced by $\Phi_n(x_1)$, the hard Dirichlet walls at $x_1 = 0, L$ have been replaced by soft walls modeled by the HO potential in (1.2) and arranged symmetrically on either side of $x_1 = 0$. In classical language, particles are able to travel freely in the $x_{2,3}$ directions, but experience the force of a spring in the x_1 direction. In quantum language the modes have plane-wave form in the $x_{2,3}$ directions, while the exponential damping of $\Phi_n(x_1)$ with increasing $|x_1|$ makes it very unlikely that real or virtual particles can penetrate far into the classically forbidden region. This is a much softened version of the abrupt reflection occurring at a hard Dirichlet wall. The parameter α in Eqs. (1.2) and (1.3) has dimension 1/length, and perhaps it makes sense to assign a "characteristic position" to each soft HO wall, these positions being the characteristic length $L = 1/\alpha$ on either side of $x_1 = 0$. As α decreases the soft HO walls move outward, receding to $\pm\infty$ as $\alpha \to 0$ and the confining potential is removed.

Because one knows as much about the modes (1.3) as one does about the modes (1.1) for parallel hard walls, it is possible to compute everything for the soft-boundary system one computes for hard walls. Clearly many other absolutely confining potentials could in principle be used in addition to $V(x) = cx^2$ to model soft boundaries, and two directions are open for future research. One can choose confining potentials whose Schrödinger problem is solvable analytically, or solvable numerically. Quite different potentials are involved and both approaches seem important. We intend to pursue both paths in subsequent work. Here we proceed analytically, calculating for the HO potential in one, two, and three spatial directions the global Casimir energy of a confined massless scalar field. We feel confident that other soft confining potentials would yield results qualitatively similar to those presented here. The essential step is the one away from hard boundaries, and for this the harmonic-oscillator potential appears quite adequate. Local quantum variables such as energy density are at least as interesting to compute as global quantities, and this also can be done using the modes (1.3). Quite fundamental questions arise at the local level. For example, what happens to the well-known local divergences (see, e.g., Ref. [4]) associated with hard walls? What are the effects of finite temperature? How is vacuum regularization affected by boundary softening? These questions will be addressed elsewhere.

In the limit $\alpha \to 0$ which removes the confining HO potential one expects the set of one-dimensional (1D) discrete modes (1.3) to be replaced by the continuum of plane-wave modes $(2\pi)^{-1/2}e^{ik_1x_1}$ with $-\infty < k_1 < \infty$.

The following asymptotic formulas for the Hermite polynomials (see, e.g., Ref. [5]) show this is indeed the case.

(i) n even, $n \to \infty$:

$$\Phi_n(x_1) = \left[\frac{\alpha}{n!\sqrt{2\pi}}\right]^{1/2} (-1)^{n/2}(n-1)!\,2^{n/2} \times[\cos(k_{1n}x_1) + O(n^{-1/4})]\,; \tag{1.4a}$$

(ii) n odd, $n \to \infty$:

$$\Phi_n(x_1) = \left[\frac{\alpha}{n!\sqrt{2\pi}}\right]^{1/2} (-1)^{(n-1)/2}n(n-2)!2^{n/2} \times[\sin(k_{1n}x_1) + O(n^{-1/4})]\,. \tag{1.4b}$$

Here the appearance of the exact "momentum" $k_{1n} = \alpha\sqrt{n+1/2}$ in the arguments of cosine and sine is very important. For arbitrary α one can interpret Eqs. (1.4a) and (1.4b) as telling us the very high or short wavelength modes care little about global features of the potential function $V(x_1)$. Over short distances $V(x_1)$ is practically constant and the modes are essentially plane waves $\exp(ik_{1n}x_1)$ after reconstruction from $\cos(k_{1n}x_1)$ and $\sin(k_{1n}x_1)$ as one would expect. In the limit $\alpha \to 0$ the size of the region over which $V(x_1)$ is practically flat increases without limit. The asymptotic formulas (1.4a) and (1.4b) show that plane waves fill up all of space as $\alpha \to 0$, and that the transition from a discrete spectrum to a continuous one is smooth.

One would expect the modes of any other soft confining potential $V(x_1)$ to behave in a similar fashion when $V(x_1)$ is made to flatten and vanish. Suppose that $x_1 = 0$ is the minimum of $V(x_1)$ and $V(0) = 0$. Then near $x_1 = 0$ the *exact* mode equation looks like

$$\left[-\frac{d^2}{dx_1^2} + O(x_1^p)\right]\Phi_n(x_1) = k_{1n}^2\Phi_n(x_1), \quad p > 0\,,$$

where k_{1n}^2 continues to be the *exact* Schrödinger spectrum. Ignoring the small potential term $O(x_1^p)$ one has just the mode equation for free space whose real solutions are $\cos(k_{1n}x_1)$ and $\sin(k_{1n}x_1)$, with the exact discrete momentum k_{1n} in place of the continuous momentum one would have if space really were free. This approximation to the exact modes should be good whenever the kinetic term $-d^2/dx_1^2$ in the Schrödinger operator is much larger than the potential term $V(x_1)$, as is the case for very high modes. Thus one can really *predict* Eqs. (1.4a) and (1.4b). Moreover, in a large class of Schrödinger problems, essentially the same asymptotic form (expressed in terms of the exact momentum spectrum for that problem) can be expected for the high modes.

A distinguishing feature of HO modes is the discrete momentum spectrum $k_n = \alpha\sqrt{n+1/2}$ growing like $\sqrt{n}$. The spectrum for hard walls $k_n = cn$ grows linearly with n [see, e.g., (1.1)]. Global Casimir energy calculations for a rectangular hard cavity require the evaluation of $\det(-\Delta) = \prod\lambda_{\vec{k}}$ where $\lambda_{\vec{k}} = (c_1n_1)^2 + (c_2n_2)^2 + (c_3n_3)^2$. The ζ functions encountered are of the relatively familiar Epstein-type with (quadratic summand)$^{-s}$. For a rectangular cavity with soft walls one must evaluate $\det[-\Delta + \frac{1}{4}(\alpha_1^4x_1^2 + \alpha_2^4x_2^2 + \alpha_3^4x_3^2)] = \prod\lambda_{\vec{k}}$ where $\lambda_{\vec{k}} = \alpha_1^2(n_1+\frac{1}{2}) + \alpha_2^2(n_2+\frac{1}{2}) + \alpha_3^2(n_3+\frac{1}{2})$. The ζ functions encountered have (linear summand)$^{-s}$ and are unfamiliar, but fortunately not difficult to work with [6]. All calculations of global Casimir energy are done by the ζ-function method, in Euclidean four-dimensional spacetime.

We evaluate the global Casimir energy for a scalar field confined between soft opposing walls, confined within a soft waveguide, and confined within a soft-walled cavity. In all three cases the Casimir energy is negative. For comparison and completeness we briefly rederive the corresponding results for rectangular Dirichlet walls (these can also be found in the literature). The hard-wall Casimir energies are negative, positive, and negative in the same order. Thus there seems to be a significant difference between the hard- and soft-wall Casimir effects for long waveguides, while for 1D and 3D confinement there is qualitative similarity.

A nontrivial variant of our calculation will also be given. One can in Eq. (1.3) convert one of the HO walls into a Dirichlet wall at $x_1 = 0$ by discarding all of the (n = even) modes, leaving the odd modes which all vanish at $x_1 = 0$. The odd modes become sine functions [Eq. (1.4b)] as they should when $\alpha \to 0$ and the soft wall is removed, leaving an isolated Dirichlet boundary. In two and three confining dimensions the sign of the Casimir energy is affected by combining soft and hard walls.

II. ONE-DIMENSIONAL CONFINEMENT

Parallel HO walls

The spatial modes for parallel harmonic-oscillator walls are as in Eq. (1.1) with $\Phi_m(x_1)$ given by Eq. (1.3) The global spacetime (or four-dimensional) ζ function for the system is (see the Appendix)

$$Z(s) = \frac{\mu^{2s}}{(2\pi)^3}\int d^3q\sum_{n=0}^{\infty}[\alpha^2(n+\tfrac{1}{2}) + q^2]^{-s} = \frac{\pi^{3/2}\Gamma(s-\frac{3}{2})}{(2\pi)^3\Gamma(s)}\left(\frac{\mu}{\alpha}\right)^{2s}\alpha^3\zeta\left(s-\tfrac{3}{2},\tfrac{1}{2}\right)\,, \tag{2.1}$$

where $\zeta(s,a)$ is the Hurwitz ζ function

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}, \quad \mathrm{Re}\, s > 1\,. \tag{2.2}$$

Numerical evaluation of $\zeta(s,a)$ will be done using MATHEMATICA [7]. In Eq. (2.1) μ is a mass parameter needed for analytic continuation in the complex variable s. The dimension [mass]^3 of $Z(s)$ arising from integration over the three continuous energy-momentum components $q = (q_0, q_2, q_3)$ is necessarily expressed in terms of the one available dimensional parameter α. A global vacuum energy density with dimension [mass]^3 can be obtained by the usual ζ-function method (see the Appendix):

$$\epsilon_{\text{eff}} \equiv -Z'(0) = -\frac{\alpha^3}{6\pi}\zeta(-\tfrac{3}{2},\tfrac{1}{2}), \qquad (2.3)$$

where

$$\zeta(-\tfrac{3}{2},\tfrac{1}{2}) = 0.016\,475\,.$$

This negative Casimir energy has an evident interpretation as the vacuum energy contained within a tube of unit cross section (and in principle infinite length, although quantum fluctuations rapidly die out as $|x_1|$ increases) parallel to the x_1 axis. The energy density ϵ_{eff} becomes increasingly negative as the stiffness parameter α of the HO potential increases, reminiscent of an attractive force between parallel walls. Here there are no physical walls. However, if one regards $V(x_1)$ as representing a distribution of matter within which the quantum field exists, then Eq. (2.3) can be interpreted as a tendency for the matter in the half spaces $x_1 < 0$ and $x_1 > 0$ to be attracted inward due to the distortion of $\hat{\Phi}$. Obviously one must investigate this locally to understand it in more detail.

Parallel HO and Dirichlet walls

As already mentioned one of the HO walls can be made into a hard Dirichlet wall at $x_1 = 0$ by discarding the even values of n in Eq. (2.1) above. The other HO wall is not affected. The Casimir energy is still negative:

$$\epsilon_{\text{eff}} = -\frac{\alpha^3\sqrt{2}}{3\pi}\zeta\left(-\tfrac{3}{2},\tfrac{3}{4}\right) \qquad (2.4)$$

with

$$\zeta(-\tfrac{3}{2},\tfrac{3}{4}) = 0.020\,93\,.$$

In some sense the Dirichlet wall and the half space $x_1 > 0$ (if $x_1 < 0$ lies beyond the hard wall) attract one another.

Parallel Dirichlet walls

Replacing $\alpha^2(n+\frac{1}{2})$ by $(m\pi/L)^2$ with $m = 1, 2, 3, \ldots$ in Eq. (2.1) converts this ζ function into the one for parallel Dirichlet boundaries:

$$Z(s) = \frac{\pi^{3/2}\Gamma(s-\frac{3}{2})}{(2L)^3\Gamma(s)}\left(\frac{\mu L}{\pi}\right)^{2s}\zeta(2s-3)$$
$$= \frac{(\mu L)^{2s}\Gamma(2-s)}{8\pi^2 L^3\Gamma(s)}\zeta(4-2s), \qquad (2.5)$$

where $\zeta(s) = \zeta(s,1)$ is the Riemann ζ function (see, e.g., [5]) and the final equality is reached using the reflection formula or functional equation for $\zeta(s)$. The global Casimir energy corresponding to (2.3) is (see also [8])

$$\epsilon_{\text{eff}} = -Z'(0) = -\frac{1}{8\pi^2 L^3}\zeta(4) = -\frac{\pi^2}{720L^3}\,. \qquad (2.6)$$

Here ϵ_{eff} represents the vacuum energy within a tube of unit cross section and length L extending from one wall to the other. The negative result (2.6) is traditionally understood to represent a global attractive force between the planar boundaries, just as the original Casimir effect [1] arising from the electromagnetic vacuum between uncharged metal plates is an attraction between these plates.

III. TWO-DIMENSIONAL CONFINEMENT

HO cylinder

For spatial potential $V(\vec{x}) = (\alpha_1^4 x_1^2 + \alpha_2^4 x_2^2)/4$ the 4D ζ function is

$$Z(s) = \frac{\mu^{2s}}{(2\pi)^2}\int d^2q \sum_{n_{1,2}=0}^{\infty}[\alpha_1^2(n_1+\tfrac{1}{2}) + \alpha_2^2(n_2+\tfrac{1}{2}) + q^2]^{-s}$$
$$= \frac{\mu^{2s}}{4\pi(s-1)}\sum_{n_{1,2}=0}^{\infty}[\alpha_1^2(n_1+\tfrac{1}{2}) + \alpha_2^2(n_2+\tfrac{1}{2})]^{-s+1}\,. \qquad (3.1)$$

An explicit evaluation of this linear ζ function for arbitrary $\alpha_{1,2}$ can be found in Ref. [6]. We do not wish to go into as much detail here.

For the cylindrically symmetric potential $\alpha_{1,2} = \alpha$ explicit calculation is simple:

$$Z(s) = \frac{1}{4\pi(s-1)}\left(\frac{\mu}{\alpha}\right)^{2s}\alpha^2\sum_{n_{1,2}=0}^{\infty}(n_1+n_2+1)^{-s+1}$$
$$= \frac{1}{4\pi(s-1)}\left(\frac{\mu}{\alpha}\right)^{2s}\alpha^2\zeta(s-2)\,. \qquad (3.2)$$

Here we have used [6]

$$L_2(s,a) = \sum_{n_{1,2}=0}^{\infty}(n_1+n_2+a)^{-s}$$
$$= a^{-s} + 2[\zeta(s,a) - a^{-s}] + \sum_{m_{1,2}=1}^{\infty}(m_1+m_2+a)^{-s}$$
$$= -a^{-s} + 2\zeta(s,a) + \sum_{m=1}^{\infty}(m-1)(m+a)^{-s}$$
$$= \zeta(s-1,a) + (1-a)\zeta(s,a)\,. \qquad (3.3)$$

To reach the final equality one only has to notice that $m-1 = m+a-a-1$ and extend the sum to include $m=0$. To reach the second to last equality one needs

$$\sum_{m_{1,2}=1}^{\infty} f(m_1+m_2) = \sum_{m=1}^{\infty}(m-1)f(m)\,,$$

where $m-1$ is the number of partitions of $m = m_1 + m_2$ into two positive integers. The dimension [mass]^2 of $Z(s)$ comes from the double momentum integral in Eq. (3.1) over the momentum components $q_{0,3}$ parallel to the unbounded spacetime directions $x_{0,3}$. Thus the Casimir energy for $\alpha_{1,2} = \alpha$,

$$\epsilon_{\rm eff} = -Z'(0) = \frac{\alpha^2}{4\pi}\zeta'(-2) = -\frac{\alpha^2}{16\pi^3}\zeta(3)\,, \quad \zeta(3) = 1.202\,06\,, \tag{3.4}$$

which is negative, has the interpretation of vacuum energy per unit length within the HO waveguide under discussion. Quantum fluctuations try to contract the HO waveguide.

To investigate the stability of the result just found let us choose an asymmetric potential $\alpha_1^2 = 2\alpha_2^2 = 2\alpha^2$ for which calculation remains simple. $Z(s)$ is given by Eq. (3.2) with $\zeta(s-2)$ replaced by $f_2(s-1, \frac{3}{2})$ where

$$\begin{aligned} f_2(s,a) &= \sum_{n_{1,2}=0}^{\infty} (2n_1 + n_2 + a)^{-s} \\ &= 2^{-s}\left[L_2\left(s, \frac{a}{2}\right) + L_2\left(s, \frac{a+1}{2}\right)\right] \\ &= \tfrac{1}{2}\zeta(s-1,a) + \tfrac{1}{2}(1-a)\zeta(s,a) \\ &\quad + 2^{-s-1}\zeta\left(s, \frac{a}{2}\right). \end{aligned} \tag{3.5}$$

Here Eq. (3.3) is used and also the identity $2^s\zeta(s,a) = \zeta(s,a/2) + \zeta(s,(a+1)/2)$. One can verify that $f_2(-1, \frac{3}{2}) = 0$. [For this use $\zeta(-n,a) = -(n+1)^{-1} B_{n+1}(a)$ where the $B_n(a)$ are Bernoulli polynomials [5].] Thus

$$\begin{aligned} \epsilon_{\rm eff} &= -Z'(0) = \frac{\alpha^2}{4\pi} f_2'\left(-1, \tfrac{3}{2}\right) \\ &= \alpha^2[-0.004\,892] \end{aligned} \tag{3.6}$$

and the Casimir energy of the asymmetric cylinder continues to be negative.

Half cylinder

Now imagine cutting the HO cylinder lengthwise through its center with a Dirichlet plane positioned at $x_1 = 0$, and discarding the $(x_1 < 0)$ half of the original cylinder. The appropriate ζ function for the remaining half cylinder is obtained from Eq. (3.1) by deleting all even values of n_1. $Z(s)$ for $\alpha_{1,2} = \alpha$ is given by Eq. (3.2) with $\zeta(s-2)$ replaced by

$$f_2(s-1,2) = \tfrac{1}{2}\zeta(s-2) + [2^{-s} - \tfrac{1}{2}]\zeta(s-1)\,. \tag{3.7}$$

Because $f_2(-1,2) = -\frac{1}{24}$ does not vanish the Casimir energy depends on $\ln(\mu/\alpha)$:

$$\begin{aligned} \epsilon_{\rm eff} &= -Z'(0) \\ &= \frac{\alpha^2}{4\pi}\left\{f_2'(-1,2) - \frac{1}{24}\left[1 + \ln\left(\frac{\mu}{\alpha}\right)^2\right]\right\}, \end{aligned} \tag{3.8}$$

where $f_2'(-1,2) = -0.040\,172\,5$. Here the sign of $\epsilon_{\rm eff}$ depends on μ/α and there appears to be a sign reversal at a critical value of this ratio:

$$\ln\left(\frac{\alpha}{\mu}\right) = \tfrac{1}{2} - 12 f_2'(-1,2) = 0.982\,07\,. \tag{3.9}$$

This possibility will be discussed in Sec. V, and rejected.

Quarter cylinder

In addition to the Dirichlet plane at $x_1 = 0$ let us position a second plane at $x_2 = 0$, retaining only the part $x_{1,2} \geq 0$ of the original HO cylinder. Even values of both $n_{1,2}$ are deleted from Eq. (3.1). In Eq. (3.2), $\zeta(s-2)$ is replaced by $2^{1-s}L_2(s-1, \frac{3}{2})$ where from Eq. (3.3) one can verify

$$\begin{aligned} L_2(s-1, \tfrac{3}{2}) &= (2^{s-2} - 1)\zeta(s-2) \\ &\quad - \tfrac{1}{2}(2^{s-1} - 1)\zeta(s-1)\,. \end{aligned} \tag{3.10}$$

Because $L_2(-1, \frac{3}{2}) = -\frac{1}{48}$ does not vanish the Casimir energy again depends on $\ln(\mu/\alpha)$:

$$\begin{aligned} \epsilon_{\rm eff} &= -Z'(0) \\ &= \frac{\alpha^2}{2\pi}\left[L_2'\left(-1, \tfrac{3}{2}\right) + \frac{1}{48}\ln\left(\frac{2\alpha^2}{\mu^2}\right) - \frac{1}{48}\right], \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} L_2'\left(-1, \tfrac{3}{2}\right) &= -\tfrac{3}{4}\zeta'(-2) + \tfrac{1}{4}\zeta'(-1) + \tfrac{1}{48}\ln 2 \\ &= \frac{3}{16\pi^2}\zeta(3) + \tfrac{1}{48}\ln 2 + \tfrac{1}{4}\zeta'(-1) \\ &= -0.040\,784\,. \end{aligned} \tag{3.12}$$

Rectangular Dirichlet waveguide

Replacing $\alpha_a^2(n_a + \frac{1}{2})$ by $(m_a\pi/L_a)^2$ in Eq. (3.1) leads to

$$Z(s) = \frac{\pi}{4(s-1)}\left(\frac{\mu}{\pi}\right)^{2s} D_2(s-1|L_{1,2})\,, \tag{3.13}$$

where

$$D_N(s|L_a) \equiv \sum_{m_a=1}^{\infty}\left[\left(\frac{m_1}{L_1}\right)^2 + \cdots + \left(\frac{m_N}{L_N}\right)^2\right]^{-s}. \tag{3.14}$$

This ζ function can be written [9] in terms of the Epstein ζ function [10]:

$$Z_N(s|L_a) \equiv \sum_{\substack{m_a = -\infty \\ \text{exclude } (0,\ldots,0)}}^{\infty} \left[\left(\frac{m_1}{L_1}\right)^2 + \cdots + \left(\frac{m_N}{L_N}\right)^2\right]^{-s}, \quad \text{Re}\, s > \frac{N}{2}\,. \tag{3.15}$$

One easily verifies

$$4D_2(s|L_{1,2}) = Z_2(s|L_{1,2}) - 2(L_1^{2s} + L_2^{2s})\zeta(2s)\,. \quad (3.16)$$

Important properties of $Z_N(s|L_a)$ are $Z_N(0|L_a) = 0$, $Z_N(-p|L_a) = 0$ for $p = 1, 2, \ldots$, and the reflection formula (extremely useful for analytic continuation into $\mathrm{Re}\, s < N/2$) [10]

$$\Gamma(s) Z_N(s|L_a) = (L_1 L_2 \cdots L_N)\pi^{2s-N/2}\Gamma\left(\frac{N}{2} - s\right) \times Z_N\left(\frac{N}{2} - s \,\middle|\, \frac{1}{L_a}\right)\,. \quad (3.17)$$

From Eq. (3.17),

$$Z_N'(-p|L_a) = (-1)^p (L_1 \cdots L_N) p! \frac{\Gamma(N/2 + p)}{\pi^{2p+N/2}} \times Z_N\left(\frac{N}{2} + p \,\middle|\, \frac{1}{L_a}\right), \quad p = 1, 2, \ldots\,.$$

We obtain, for the vacuum energy per unit length within the Dirichlet waveguide [9,11],

$$\begin{aligned}\epsilon_{\text{eff}} &= -Z'(0) = \frac{\pi}{4} D_2'(-1|L_{1,2}) \\ &= \frac{\pi}{16} Z_2'(-1|L_{1,2}) - [L_1^{-2} + L_2^{-2}]\zeta'(-2)\frac{\pi}{4} \\ &= -\frac{L_1 L_2}{16\pi^2} Z_2\left(2 \,\middle|\, \frac{1}{L_{1,2}}\right) + \frac{1}{16\pi}\zeta(3)\left[\frac{1}{L_1^2} + \frac{1}{L_2^2}\right]\,. \end{aligned} \quad (3.18)$$

In the limit $L_2 \to \infty$, $\epsilon_{\text{eff}}/L_2$ smoothly becomes the parallel-plane result Eq. (2.6). For the equilateral case $L_{1,2} = L$,

$$\epsilon_{\text{eff}} = \frac{1}{8\pi L^2}\left[\zeta(3) - \frac{\pi}{3}\beta(2)\right] = \frac{1}{L^2}[0.009\,66]\,, \quad (3.19)$$

where $Z_2(s|1) = 4\zeta(s)\beta(s)$ (see [12]) with

$$\beta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} \quad (3.20)$$

and $\beta(2) \approx 0.915$ is Catalan's constant. Equation (3.19) agrees with the numerical result in [9]. The energy density ϵ_{eff} is positive [the contrary of Eqs. (3.4) and (3.6)]. Vacuum fluctuations tend to expand the Dirichlet waveguide when L_1 and L_2 do not differ greatly. However, for $L_2 \gg L_1$ as just mentioned the attractive parallel-plane result is recovered. Thus for the rectangular Dirichlet waveguide the sign of the Casimir energy depends on the ratio L_1/L_2.

IV. THREE-DIMENSIONAL CONFINEMENT

HO cavity

The global 4D ζ function for spatial potential $V(\vec{x}) = (\alpha_1^4 x_1^2 + \alpha_2^4 x_2^2 + \alpha_3^4 x_3^2)/4$ is

$$Z(s) = \mu^{2s}\frac{\sqrt{\pi}\Gamma(s - \frac{1}{2})}{2\pi\Gamma(s)} \times \sum_{n_{1,2,3}=0}^{\infty}\left[\sum_{i=1}^{3}\alpha_i^2(n_i + \tfrac{1}{2})\right]^{-s+1/2}\,. \quad (4.1)$$

This linear ζ function can be evaluated for arbitrary $\alpha_{1,2,3}$. In the spherically symmetric case $\alpha_{1,2,3} = \alpha$,

$$Z(s) = \frac{\sqrt{\pi}\Gamma(s - \frac{1}{2})}{2\pi\Gamma(s)}\left(\frac{\mu}{\alpha}\right)^{2s}\alpha L_3(s - \tfrac{1}{2}, \tfrac{3}{2})\,, \quad (4.2)$$

where

$$\begin{aligned}L_3(s,a) &= \sum_{n_{1,2,3}=0}^{\infty}(n_1 + n_2 + n_3 + a)^{-s} \\ &= \tfrac{1}{2}\zeta(s-2,a) + (\tfrac{3}{2} - a)\zeta(s-1,a) \\ &\quad + \tfrac{1}{2}(a-1)(a-2)\zeta(s,a)\,, \end{aligned} \quad (4.3)$$

which can be derived much like Eq. (3.3) as is explained in [6]. The vacuum energy $E_{\text{cav}} = -Z'(0)$ of the cavity is

$$\begin{aligned}E_{\text{cav}} &= \alpha[\tfrac{1}{2}\zeta(-\tfrac{5}{2}, \tfrac{3}{2}) - \tfrac{1}{8}\zeta(-\tfrac{1}{2}, \tfrac{3}{2})] \\ &= \alpha[-0.011\,117] \end{aligned} \quad (4.4)$$

and is negative. Vacuum fluctuations tend to contract the symmetric HO cavity. To investigate stability consider the cylindrically symmetric potential $\alpha_1^2 = 2\alpha_{2,3}^2 = 2\alpha^2$. Then $Z(s)$ is given by Eq. (4.2) with L_3 replaced by $f_3(s - \frac{1}{2}, 2)$ where

$$\begin{aligned}f_3(s,a) &= \sum_{n_{1,2,3}=0}^{\infty}(2n_1 + n_2 + n_3 + a)^{-s} \\ &= 2^{-s}\left[L_3\left(s, \frac{a}{2}\right) + 2L_3\left(s, \frac{a+1}{2}\right) + L_3\left(s, \frac{a}{2} + 1\right)\right]\,. \end{aligned} \quad (4.5)$$

The cavity energy $E_{\text{cav}} = -Z'(0) = \alpha f_3(-\frac{1}{2}, 2) = \alpha[-0.0194]$ continues to be negative.

Half cavity

Position a Dirichlet plane at $x_1 = 0$ and discard the half cavity in $x_1 < 0$. The ζ function is obtained by deleting even n_1 in Eq. (4.1). $Z(s)$ for $\alpha_{1,2,3} = \alpha$ is now given by Eq. (4.2) with $f_3(s - \frac{1}{2}, \frac{5}{2})$ in place of L_3. The cavity energy $E_{\text{cav}} = -Z'(0) = \alpha f_3(-\frac{1}{2}, \frac{5}{2}) = \alpha[0.014\,65]$ is now positive.

Quarter cavity

Position a second Dirichlet plane at $x_2 = 0$, discarding all of the cavity outside $x_{1,2} \geq 0$. Even values of $n_{1,2}$

are deleted from Eq. (4.1) to obtain $Z(s)$ for one quarter of the original HO cavity, bounded by the flat Dirichlet walls at $x_{1,2} = 0$. $Z(s)$ for $\alpha_{1,2,3} = \alpha$ is given by Eq. (4.2) with L_3 replaced by $g_3(s - \frac{1}{2}, \frac{7}{2})$ where

$$g_3(s,a) = \sum_{n_{1,2,3}=0}^{\infty} (2n_1 + 2n_2 + n_3 + a)^{-s}$$
$$= 2^{-s}\left[L_3\left(s, \frac{a}{2}\right) + L_3\left(s, \frac{a+1}{2}\right)\right].$$

The cavity energy $E_{\rm cav} = -Z'(0) = \alpha g_3(-\frac{1}{2}, \frac{7}{2}) = \alpha[0.0340]$ is positive.

$\frac{1}{8}$ cavity

Position a hard Dirichlet plane at $x_3 = 0$ which, together with the planes at $x_{1,2} = 0$, cuts off one octant of the original HO cavity. Even values of $n_{1,2,3}$ are discarded in Eq. (4.1) and L_3 is replaced in Eq. (4.2) by $2^{-s+1/2}L_3(s - \frac{1}{2}, \frac{9}{4})$. The cavity energy is $E_{\rm cav} = \alpha\sqrt{2}L_3(-\frac{1}{2}, \frac{9}{4}) = \alpha[0.0303]$ and is positive.

Rectangular Dirichlet cavity

For a rectangular cavity with Dirichlet walls

$$Z(s) = \left(\frac{\mu}{\pi}\right)^{2s} \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{2\Gamma(s)} D_3(s - \tfrac{1}{2}) \qquad (4.6)$$

in the notation of Eq. (3.14). One readily verifies

$$8D_3(s|L_{1,2,3}) = Z_3(s|L_{1,2,3}) - [Z_2(s|L_{1,2}) + Z_2(s|L_{1,3}) + Z_2(s|L_{2,3})] + 2\sum_{i=1}^{3} L_i^{2s}\zeta(2s). \qquad (4.7)$$

The vacuum energy of the cavity $E_{\rm cav} \equiv -Z'(0) = \pi D_3(-\frac{1}{2})$ is given by [9]

$$E_{\rm cav} = -\frac{L_1L_2L_3}{16\pi^2} Z_3\left(2\left|\frac{1}{L_{1,2,3}}\right.\right) + \frac{1}{32\pi}\left[L_1L_2Z_2\left(\frac{3}{2}\left|\frac{1}{L_{1,2}}\right.\right) + L_1L_3Z_2\left(\frac{3}{2}\left|\frac{1}{L_{1,3}}\right.\right) + L_2L_3Z_2\left(\frac{3}{2}\left|\frac{1}{L_{2,3}}\right.\right)\right] - \frac{\pi}{48}\left[\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right]. \qquad (4.8)$$

In the limit $L_3 \to \infty$, $E_{\rm cav}/L_3$ smoothly becomes the energy per unit length (3.18) of the rectangular Dirichlet waveguide.

For a cubic cavity Eq. (4.8) simplifies to

$$E_{\rm cav} = \frac{1}{L}\left[-\frac{1}{16\pi^2}Z_3(2) + \frac{3}{8\pi}\zeta\left(\frac{3}{2}\right)\beta\left(\frac{3}{2}\right) - \frac{\pi}{16}\right] = \frac{1}{L}[-0.032]. \qquad (4.9)$$

This negative result agrees numerically with [9].

V. DISCUSSION

Although there may be other interpretations for them, let us for the sake of explicitness regard a boundary in QFT as modeling some distribution of matter which interacts with the quantum field(s). As mentioned in the Introduction, boundaries have traditionally been assumed to be hard (and smooth or piecewise smooth) in Casimir theory. The word "hard" implies the boundary has a precise location in space. In this sense the standard calculations in electromagnetic Casimir theory which involve dielectrics rather than metals, and therefore represent objects partially transparent to the electromagnetic field, are hard-boundary calculations. (In scalar terms, dielectrics correspond to finite step-function potentials in the Schrödinger mode equation.) For bulk purposes the hardness assumption may be justifiable. There is little doubt, however, that this assumption is highly idealized, and cannot be microscopically realistic.

The present article is the first attempt known to the authors to relax the hardness assumption. One can imagine doing this in different ways, characterized by different potential functions $V(\vec{x})$ in the Schrödinger mode equation. This program is very broad, and one can say a great deal about it. Detailed discussions are in preparation [13]. In the present article we have chosen an extreme form of boundary softening. The hard rectangular boundaries of the Dirichlet box have been replaced by completely soft boundaries which act to confine a scalar field, but do not themselves have spatial position. These soft boundaries are represented by HO potentials in the Schrödinger mode equation. Such boundaries could model nonuniform distributions of matter which suppress the quantum field more and more strongly the farther from the center one looks.

The notion of soft confining boundaries has been studied quantitatively in this paper by means of examples. A massless scalar field $\hat{\Phi}$ was subjected to confining HO potentials in one, two, and three spatial directions. Performing global calculations which exactly parallel the global calculations in standard hard-wall Casimir theory, we obtained the vacuum energy of $\hat{\Phi}$ for one- two-, and three-dimensional soft-wall confinement. In all cases these vacuum energies are negative, and they increase (toward zero) as the confining HO potentials are weakened. In other words, it costs energy to displace the HO potentials, or the matter these potentials represent, outward to infinity. This qualitative statement appears to be independent of the confining HO geometry. On the other hand, using a simple mathematical device, we were able to insert hard Dirichlet walls in various ways through our HO boundaries, to obtain additional confining systems having both hard and soft walls. For these, the results were strongly dependent on the confining geometry.

Parallel planes

From Eq. (2.3) we find the global vacuum energy

$$\epsilon_{\rm eff}(\text{tube}) = -\alpha^3[0.000\,874] \tag{5.1}$$

within an infinite tube of unit cross section orthogonal to the parallel HO boundaries. From Eq. (2.4) the corresponding energy within a unit semi-infinite tube extending from the Dirichlet wall at $x_1 = 0$ into the region $x_1 > 0$ ($\hat{\Phi}$ is excluded from $x_1 < 0$) is

$$\epsilon_{\rm eff}(\tfrac{1}{2}\text{tube}) = -\alpha^3[0.003\,14]\,. \tag{5.2}$$

Comparing total energies in the semi-infinite tubes in $x_1 \geq 0$ we find

$$\epsilon_{\rm eff}(\tfrac{1}{2}\text{tube}) < \tfrac{1}{2}\epsilon_{\rm eff}(\text{tube})\,. \tag{5.3}$$

Predictably, the Dirichlet wall does not simply halve the vacuum energy in the infinite tube. The Dirichlet wall adds its own distortion of $\hat{\Phi}$ to the distortion caused by the HO potential. Its insertion is a complicated and highly local process, best studied at the local level.

For an *isolated* Dirichlet wall at $x_1 = 0$ the distortion of $\hat{\Phi}$ as measured by the canonical vacuum stress tensor is

$$\langle \hat{T}_{00}(x)\rangle = -1/(4\pi x_1^2)^2\,. \tag{5.4}$$

This energy density is intrinsic to the Dirichlet wall, even though it extends outward into space from the wall. [Just think of displacing the wall. The entire function (5.4), and, more importantly, the distortion of $\hat{\Phi}$ this function represents, gets rigidly displaced along with the wall.] The energy density (5.4) diverges at $x_1 = 0$, a consequence of the assumed perfect smoothness of the hard boundary. This surface divergence can be removed by a little surface roughening and is, in any case, unimportant for our present discussion because it does not participate in the global Casimir effect. Only finite parts of $\langle \hat{T}_{00}(x)\rangle$ some distance away from $x_1 = 0$ participate in the global Casimir effect. It is not simple to compute the exact $\langle \hat{T}_{\mu\nu}(x)\rangle$ for the parallel HO dirichlet wall system. This is being done and will be reported elsewhere. Here we only make a few qualitative remarks.

The HO potential suppresses at least the lower modes of $\hat{\Phi}$, and this suppression increases without limit as $x_1 \to \infty$. For large x_1 the hard wall at $x_1 = 0$ has little effect. However, for small x_1 its suppression of $\hat{\Phi}$ is strong. Right where the HO potential has little effect, the Dirichlet wall has a strong effect. Together, the two boundaries suppress $\hat{\Phi}$ much more strongly than does the HO potential alone. The result is Eq. (5.3)—a substantial diminution of the total vacuum energy in the half tube $x_1 > 0$.

In the limit $\alpha \to 0$, Eq. (5.1) describes the restoration of free field in all directions, while Eq. (5.2) describes the restoration of free field in $x_1 > 0$, with a now isolated Dirichlet wall at $x_1 = 0$. These are quite different systems, yet the final global energy looks the same: $\epsilon_{\rm eff} \to 0-$. The system with the isolated Dirichlet wall has vanishing global Casimir energy because all of the energy (5.4) of the distorted field $\hat{\Phi}$ is inseparable from this wall. The mathematics we are using correctly assigns it to the wall, not to $\hat{\Phi}$.

Waveguide

Equations (3.4), (3.8), and (3.11) give us the global vacuum energies for the cylindrical HO waveguide, and for half- and quarter-cylinder waveguides formed by the insertion of one or two perpendicular Dirichlet walls through the cylinder axis:

$$\epsilon_{\rm eff}(\text{cylinder}) = -\alpha^2[0.002\,42]\,, \tag{5.5}$$

$$\epsilon_{\rm eff}(\tfrac{1}{2}\,\text{cylinder}) = -\alpha^2\left[0.006\,51 + \frac{1}{48\pi}\ln\frac{\mu}{\alpha}\right]\,, \tag{5.6}$$

$$\epsilon_{\rm eff}(\tfrac{1}{4}\,\text{cylinder}) = -\alpha^2\left[0.007\,51 + \frac{1}{48\pi}\ln\frac{\mu}{\alpha}\right]\,. \tag{5.7}$$

Disregarding for the moment the $\ln(\mu/\alpha,)$ terms, each of the energies (5.6) and (5.7) is negative, predicting as for the cylindrical HO waveguide an attractive Casimir effect. The $\ln(\mu/\alpha)$ terms enhance this effect if $\mu/\alpha > 1$. We remind the reader that, in ζ function theory, μ is an ultraviolet regularization parameter linked with vacuum renormalization. One usually thinks of μ as being arbitrary. However, if μ truly were arbitrary, then whenever $\ln\mu$ contributed to a Casimir energy [because the spacetime ζ function $Z(0) \neq 0$] the sign (not to mention the magnitude) of the Casimir energy and effect would be undefined and, therefore, meaningless. We do not believe the actual situation can be this bad. Perhaps a physical interpretation can be found for μ in Eqs. (5.6) and (5.7). Remember, we are dealing here with a vacuum energy problem and not, say, with a scattering process in which an external energy may determine the renormalization scale.

Let us change to the UV regularization length $l = 1/\mu$. If $\alpha l > 1$ or $\ln(\mu/\alpha) = -\ln(\alpha l) < 0$, the strength of the Casimir effect in Eqs. (5.6) and (5.7) is eroded. At some critical value $l_c > 1/\alpha$ of l these Casimir energies would change sign. This gives us reason to think that $l > 1/\alpha$ is somehow excluded on physical grounds. In this problem there is only one available length $L = 1/\alpha$. L sets the broad scale of nonuniformity of the background matter with which $\hat{\Phi}$ interacts. We suggest that l might be regarded as a *smaller* length scale characterizing possible lumpiness or irregularity in this background. This finer structure is only crudely taken into account because, in the mode equation, the HO potential is ignorant of it.

This idea attempts to distinguish a physical meaning for μ. A different question is as follows: Why does μ appear at all? In standard Casimir theory the presence of $\ln\mu$ terms is linked with the confining geometry. This appears to be the case here as well.

Cavity

Finally, we note again that a sign change occurs in the Casimir energy of a symmetric cavity ($\alpha_{1,2,3} = \alpha$) when hard walls are inserted through its center:

$$E_{\rm cav}(\text{sphere}) = -\alpha[0.011\,12]\,, \qquad (5.8)$$

$$E_{\rm cav}(\tfrac{1}{2}\,\text{sphere}) = \alpha[0.014\,65]\,, \qquad (5.9)$$

$$E_{\rm cav}(\tfrac{1}{4}\,\text{sphere}) = \alpha[0.0340]\,, \qquad (5.10)$$

$$E_{\rm cav}(\tfrac{1}{8}\,\text{sphere}) = \alpha[0.0303]\,. \qquad (5.11)$$

Quantum fluctuations within the spherical soft cavity try to contract it—i.e., to draw inward the matter represented by the HO potential. Contraction of the cavity means $\alpha \to \infty$, with $E_{\rm cav} \to -\infty$. If instead this matter recedes to infinity, then $E_{\rm cav} \to 0-$ and the free field is regained (but with spherically distributed matter at infinity).

Somehow, when one or more Dirichlet planes are inserted, the situation reverses. Consider first the hemispherical cavity obtained by inserting an infinite Dirichlet plane at $x_1 = 0$ which confines $\hat{\Phi}$ to $x_1 > 0$. Quantum fluctuations now try to push the hemispherically symmetric distribution of matter outward from the center. This is so unlike the attractive parallel boundary system that one may wonder how both can be true. We emphasize these two systems *are* quite different. Here, as α is tuned up and down, the edge of the hemispherelike distribution of background matter sweeps inward and outward directly along the plane $x_1 = 0$. Behind this edge the suppression of $\hat{\Phi}$ is quite strong, right up to the plane. In the parallel-boundary system there is no such edge; all the way out to infinity the soft-wall material is no closer to the Dirichlet wall than it is in the center. The distortion of $\hat{\Phi}$ in these two systems is therefore very different. Similar remarks can be made about the $\frac{1}{4}$ and $\frac{1}{8}$ sphere configurations.

We believe that the sign change in $E_{\rm cav}$ can be understood. Local calculations will decide this. We remind the reader that the sign changes going from Eqs. (2.6) and (3.19) to (4.9) for hard walls are not predictable by any known (and compelling) argument. One must compute these signs.

The global calculations reported in this article are obvious first steps in our program of softening the traditional hard boundaries of Casimir theory. The HO potentials used are prototypical of the type of boundary we have called "soft." There is another category between hard and soft, which we call "semihard" [13]. A semihard boundary has a potential $V(\vec{x})$ which grows smoothly from $V = 0$ far from the boundary, to $V = \infty$ on the boundary surface ∂M. This potential $V(\vec{x})$ gives ∂M a surface "texture." The latter is smoothly eliminated when $V(\vec{x})$ smoothly becomes the infinite step-function potential of a Dirichlet boundary at ∂M. The spectra (and therefore the mathematics) associated with semihard boundaries are basically similar to the spectra associated with hard boundaries. Semihardening only finitely distorts a hard boundary problem. Softening is more extreme. When one smoothly eliminates a soft boundary, free space is the result. Therefore very different spectra are produced by soft boundaries. Both the semihard and soft categories of nontraditional boundary are interesting. Local calculations are needed to measure the distortion of the quantum field characteristic of each type of boundary. Both semihard and soft boundaries appear to be new topics in QFT, with interesting physical applications.

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APPENDIX

For notational convenience we summarize here the ζ-function method [14] for evaluating Casimir energies. It begins with the global spacetime ζ function $Z(s) \equiv \sum_m (\mu^2/\lambda_m)^s$, $\text{Re}s > 2$, where λ_m is the spectrum of the Schrödinger operator on one's 4D Euclidean spacetime of interest; s is a complex variable; finally μ is an arbitrary mass parameter needed to have a dimensionless quantity μ^2/λ_m raised to a complex power. The defining series for $Z(s)$ converges absolutely in $\text{Re}s > 2$. Continuation into $\text{Re}s < 2$ reveals that $Z(s)$ is meromorphic, having its rightmost pole at $s = 2$ on the real axis and other possible poles to the left. With this interpretation, $Z(s)$ assigns a unique finite value to its divergent defining series everywhere in $\text{Re}s < 2$ [excepting the poles of $Z(s)$].

Now consider Euclidean spacetime $E^1 \times \mathcal{M}$ where E^1 and $\mathcal{M}$ represent the imaginary time line and 3D space, respectively. $Z(s)$ is defined by

$$Z(s) = \frac{\mu^{2s}}{2\pi}\int_{-\infty}^{\infty} dk_0 \sum_n (k_0^2 + \omega_n^2)^{-s}$$

$$= \frac{\mu^{2s}}{\sqrt{4\pi}\,\Gamma(s)}\Gamma(s - \tfrac{1}{2}) \sum_n \omega_n^{1-2s}\,. \qquad (\text{A1})$$

$(-\Delta + V)\Phi_n = \omega_n^2 \Phi_n$ is the spatial eigenmode problem.

Differentiating the second equality in (A1) yields a formal equation (with $\mu = 1$)

$$-Z'(0) = \sum_n \omega_n . \qquad \text{(A2)}$$

This quantity has the evident interpretation of a (*finite*) global vacuum energy or energy density, whose dimension depends on how many momentum components are continuous. Integration over each of these provides $Z(s)$ with one dimensional power of [mass], affecting its interpretation as mentioned in the text.

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