Classical moduli $O(\alpha')$ hair

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We extend existing treatments of black hole solutions in string gravity to include moduli fields. We compute the external moduli and dilaton hair, as well as their associated axions, to $O(\alpha')$ in the framework of the loop-corrected superstring effective action for a Kerr-Newman black hole background.

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Superstring [1] theory is our best existing candidate for a consistent quantum theory of gravity which also has the prospect of unification with all other interactions. Einstein's theory, which has been very successful as a classical theory of gravitation, is incorporated in this more general framework. However, superstrings involve a characteristic length of the order of the Planck scale and are expected to lead to drastic modifications of the Einstein action at short distances. These modifications arise either as a result of the contribution of the infinite tower of massive string modes, appearing as α' corrections, or as a result of quantum loop effects. An effective low energy Lagrangian [2] that incorporates the above, involving only the massless string modes, can be derived from string theory using a perturbative approach in both the string tension α' and the string coupling. The relevant massless fields, apart from the graviton and other gauge fields, are the dilatons, which play the role of the field-dependent string couplings that parametrize the string loop expansion and the moduli fields that describe the size and the shape of the internal compactification manifold.

In Einstein gravity, minimally coupled to other fields, the most general black hole solution is described by the Kerr-Newman family of rotating charged black hole solutions [3]. In agreement with the "no-hair" theorem [4] at the classical level, the only external fields present are those required by gauge invariance. A qualitative new feature present in the superstring effective action is the appearance of external field strength hair for the axion and dilaton fields [5–11]. The tree level effective action has been calculated up to several orders in the α' expansion. It turns out that there is no dependence on the moduli fields at the tree level. The one-loop corrections to gravitational and gauge couplings have been calculated in the context of orbifold compactifications of the heterotic superstring [12]. It has been shown that there are no moduli-dependent corrections to the Einstein term, while there are nontrivial $\mathcal{R}^2$ contributions appearing in a Gauss-Bonnet combination multiplied by a moduli-dependent coefficient function. This term is subject to a nonrenormalization theorem which implies that all higher-loop moduli-dependent $\mathcal{R}^2$ contributions vanish. It is interesting to note the existence of singularity-free [13] solutions of the field equations in a Friedmann-Robertson-Walker background depending crucially on the presence of the Gauss-Bonnet (GB) term.

In the present article we extend existing treatments [5–11] of black hole solutions in string gravity to include moduli fields. Our action is the low energy effective action derived in the context of orbifold compactifications of the heterotic superstring to one loop and α' order. We compute to α' order the moduli and dilaton hair together with the corresponding two-axion hair. The result, although expected from previous existing investigations without the moduli fields, serves to establish even better the qualitatively new features of string gravity in contrast with Einstein gravity characterized by the "no-hair" theorem. In our action we have not introduced any potential for the above fields, although it is likely that in the full quantum string theory such a potential and (small) mass are generated through nonperturbative effects. Nevertheless, if the black hole size or the distance from the black hole is small compared to their inverse mass, the solutions found are valid to a good approximation.

Let us consider the universal part of the effective action of any four-dimensional heterotic superstring model which describes the dynamics of the graviton, gauge fields, the dilaton S, and, for simplicity, the common modulus fields T. At the tree level and up to first order in α', it takes the form

$$S_{\rm eff}^{(0)} = \int d^4x\sqrt{-g}\left(\tfrac{1}{2}R + \frac{|DS|^2}{(S+\bar{S})^2} + 3\frac{|DT|^2}{(T+\bar{T})^2} + \frac{\alpha'}{8}({\rm Re}S)(\mathcal{R}^2_{\rm GB} - F^{\mu\nu}F_{\mu\nu}) + \frac{\alpha'}{8}({\rm Im}S)(\mathcal{R}\tilde{\mathcal{R}} - F\tilde{F})\right), \qquad (1)$$

where

$$\mathcal{R}^2_{\rm GB} \equiv R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \qquad (2)$$

and

$$\mathcal{R}\tilde{\mathcal{R}} \equiv \eta^{\mu\nu\rho\sigma} R^{\kappa\lambda}{}_{\mu\nu} R_{\rho\sigma\kappa\lambda}, \tag{3}$$

$$F\tilde{F} \equiv \eta^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{4}$$

In what follows we shall consider only the case of an Abelian gauge field. Note that[1] $\eta^{\mu\nu\rho\sigma} \equiv \epsilon^{\mu\nu\rho\sigma}(-g)^{-1/2}$. We have chosen units such that $k \equiv \sqrt{8\pi G_N} \equiv 1$.

The one-loop corrections give a modulus dependence to the quadratic gravitational and gauge terms that are of the form

$$S^{(1)}_{\text{eff}} = \int d^4x \sqrt{-g}[\alpha' \Delta(T,\bar{T}) \mathcal{R}^2_{\text{GB}} + \alpha' \Theta(T,\bar{T}) \mathcal{R}\tilde{\mathcal{R}} + \alpha' \hat{\Delta}(T,\bar{T}) F^{\mu\nu} F_{\mu\nu} + \alpha' \hat{\Theta}(T,\bar{T}) F\tilde{F}]. \tag{5}$$

The functions $\Delta(T,\bar{T})$, $\Theta(T,\bar{T})$, $\hat{\Delta}(T,\bar{T})$, and $\hat{\Theta}(T,\bar{T})$ have been derived in Ref. [12] and depend multiplicatively through a coefficient on the supermultiplet content of the string model. Introducing the notation

$$S \equiv (e^{\phi} + ia)/g^2, \quad T \equiv e^{\sigma} + ib, \tag{6}$$

and referring to ϕ as the dilaton, σ as the modulus, a and b as the axions, and g^2 as the string coupling, we can write the effective one-loop, $O(\alpha')$ Lagrangian as

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2} R + \tfrac{1}{4}(\partial_\mu \phi)^2 + \tfrac{1}{4} e^{-2\phi}(\partial_\mu a)^2 + \tfrac{3}{4}(\partial_\mu \sigma)^2 + \tfrac{3}{4} e^{-2\sigma}(\partial_\mu b)^2 + \alpha' \left(\frac{e^{\phi}}{8g^2} + \Delta\right) \mathcal{R}^2_{\text{GB}} + \alpha' \left(\frac{a}{8g^2} + \Theta\right) \mathcal{R}\tilde{\mathcal{R}}$$
$$+ \alpha' \left(-\frac{e^{\phi}}{8g^2} + \hat{\Delta}\right) F^{\mu\nu} F_{\mu\nu} + \alpha' \left(-\frac{a}{8g^2} + \hat{\Theta}\right) F\tilde{F}. \tag{7}$$

The equations of motion resulting from (6) are four equations for the scalar and pseudoscalar fields,

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \partial^\mu \phi] = -e^{-2\phi}(\partial_\mu a)^2 + \frac{\alpha'}{4g^2} e^{\phi} (\mathcal{R}^2_{\text{GB}} - F^{\mu\nu} F_{\mu\nu}), \tag{8}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} e^{-2\phi} \partial^\mu a] = \frac{\alpha'}{4g^2} (\mathcal{R}\tilde{\mathcal{R}} - F\tilde{F}), \tag{9}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \partial^\mu \sigma] = -e^{-2\sigma}(\partial_\mu b)^2 + \frac{2\alpha'}{3}\left(\frac{\delta\Delta}{\delta\sigma}\right) \mathcal{R}^2_{\text{GB}} + \frac{2\alpha'}{3}\left(\frac{\delta\Theta}{\delta\sigma}\right) \mathcal{R}\tilde{\mathcal{R}} + \frac{2\alpha'}{3}\left(\frac{\delta\hat{\Delta}}{\delta\sigma}\right) F^{\mu\nu} F_{\mu\nu} + \frac{2\alpha'}{3}\left(\frac{\delta\hat{\Theta}}{\delta\sigma}\right) F\tilde{F}, \tag{10}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} e^{-2\sigma} \partial^\mu b] = \frac{2\alpha'}{3}\left(\frac{\delta\Delta}{\delta b}\right) \mathcal{R}^2_{\text{GB}} + \frac{2\alpha'}{3}\left(\frac{\delta\Theta}{\delta b}\right) \mathcal{R}\tilde{\mathcal{R}} + \frac{2\alpha'}{3}\left(\frac{\delta\hat{\Delta}}{\delta b}\right) F^{\mu\nu} F_{\mu\nu} + \frac{2\alpha'}{3}\left(\frac{\delta\hat{\Theta}}{\delta b}\right) F\tilde{F}, \tag{11}$$

the gravitational equation[2]

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \alpha'(g_{\mu\rho} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\rho}) \eta^{\kappa\lambda\alpha\beta} D_\gamma (\tilde{R}^{\gamma\rho}{}_{\alpha\beta} D_\kappa f_1) - 8\alpha' D_\rho (\tilde{R}_\mu{}^\lambda{}_\nu{}^\rho D_\lambda f_2) + 4\alpha' f_3 (F_\mu{}^\sigma F_{\nu\sigma} - \tfrac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma})$$

$$= -\tfrac{1}{2}(\partial_\mu \phi)(\partial_\nu \phi) + \tfrac{1}{4} g_{\mu\nu} (\partial_\rho \phi)^2 - \frac{e^{-2\phi}}{2} (\partial_\mu a)(\partial_\nu a) + \tfrac{1}{4} g_{\mu\nu} e^{-2\phi} (\partial_\rho a)^2$$
$$- \tfrac{3}{2}(\partial_\mu \sigma)(\partial_\nu \sigma) + \tfrac{3}{4} g_{\mu\nu} (\partial_\rho \sigma)^2 - \tfrac{3}{2} e^{-2\sigma} (\partial_\mu b)(\partial_\nu b) + \tfrac{3}{4} e^{-2\sigma} g_{\mu\nu} (\partial_\rho b)^2 \tag{12}$$

and the equation for the gauge field

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} f_3 F^{\mu\nu}] + \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} f_4 \tilde{F}^{\mu\nu}] = 0. \tag{13}$$

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[1] $\epsilon^{0ijk} = -\epsilon_{ijk}$.

[2] $\tilde{R}^{\mu\nu}{}_{\kappa\lambda} = \eta^{\mu\nu\rho\sigma} R_{\rho\sigma\kappa\lambda}$.

We have introduced the functions

$$f_1 \equiv \frac{e^{\phi}}{8g^2} + \Delta, \quad f_2 \equiv \frac{a}{8g^2} + \Theta, \quad f_3 \equiv -\frac{e^{\phi}}{8g^2} + \hat{\Delta}, \quad f_4 \equiv -\frac{a}{8g^2} + \hat{\Theta}. \tag{14}$$

At this point we introduce the Kerr-Newman metric, the most general black hole solution of the standard Einstein equation minimally coupled to an Abelian gauge field. It is

$$ds^2 = \left(\frac{\rho^2 - 2Mr + q^2}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Lambda} dr^2 - \rho^2 d\theta^2 + \frac{2A\sin^2\theta(2Mr - q^2)}{\rho^2} dt\, d\varphi - \frac{\sin^2\theta}{\rho^2}\Sigma^2 d\varphi^2, \tag{15}$$

where $\rho^2 \equiv r^2 + A^2\cos^2\theta$, $\Lambda \equiv r^2 + A^2 - 2Mr + q^2$, and $\Sigma^2 \equiv (r^2 + A^2)^2 - \Lambda A^2 \sin^2\theta$. A stands for the angular momentum per unit mass and $q^2 = q_e^2 + q_m^2$ for the total charge of the black hole. It will be shown very shortly that the Kerr-Newman metric satisfies our gravitational equation (12) to $O(\alpha')$.

Since, as we declared in the Introduction, we plan to determine solutions to $O(\alpha')$, let us first obtain the zeroth order solutions for the scalar and pseudoscalar fields. Introducing a rescaled axion field $\partial_\mu \tilde{a} \equiv e^{-2\phi}\partial_\mu a$, we can write the dilatonic-axion equation of motion for a Kerr-Newman background in the form

$$\frac{\partial}{\partial r}\left[(r^2 - 2Mr + A^2 + q^2)\frac{\partial \tilde{a}}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\tilde{a}}{\partial\theta}\right] = 0. \tag{16}$$

It has a general solution of the form

$$\tilde{a} = \sum_{l=0}^{\infty} P_l(\cos\theta)[A_l Q_l(z) + B_l P_l(z)], \tag{17}$$

where $z \equiv (r - M)/\sqrt{M^2 - A^2 - q^2}$. Imposing the black hole boundary condition[3] $r \to r_H$ or $z \to 1$ forces us to require $A_l = 0$, $\forall l$. On the other hand, requiring finiteness at $r \to \infty$ or $z \to \infty$ forces us to set $B_l = 0$, $\forall l \geq 1$. Thus only the constant solution $\tilde{a} = B_0$ is possible. Using that, the dilaton equation reduces, to zeroth order, to the form $D^2\phi = 0$, which for the same reasons as in the case of the axion $\tilde{a}$ leads us to the conclusion that to this order the dilaton is a constant. Following the same procedure for the modulus and its associated axion, we also arrive at constant zeroth order values.

The gravitational equation to $O(\alpha')$ takes the form of the minimal Einstein-Yang Mills equation

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu} = -4\alpha'(f_3)_{\phi_0,\sigma_0,b_0}(F_\mu{}^\sigma F_{\nu\sigma} - \tfrac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}). \tag{18}$$

At this point we can introduce for the gauge field the ansatz

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \frac{Q_e r}{\rho^2}[dt - A\sin^2\theta\, d\varphi] + \frac{Q_m\cos\theta}{\rho^2}[A dt - (r^2 + A^2)d\varphi]. \tag{19}$$

This ansatz describes a dyon since it possesses both an electric Q_e and a magnetic Q_m charge. Note that for vanishing q^2 the gravitational equation can be satisfied with the standard Kerr metric and there is no correction of $O(\alpha')$. In the opposite case, there must be a $O(\alpha')$ correction from the gauge sector in the metric. Thus the charge q^2 should come out to be $O(\alpha')$. We can easily derive the relation

$$q^2 = \alpha' Q^2\left[\frac{e^{\phi_0}}{4g^2} - 2[\hat{\Delta}(T,\bar{T})]_{\sigma_0,b_0}\right], \tag{20}$$

where $Q^2 = Q_e^2 + Q_m^2$. Because of the fact that the source terms are already of $O(\alpha')$, any $O(\alpha')$ correction to the metric will not affect the solution for the fields. Also, in the limit $\alpha' \to 0$, with Q^2 fixed, $q^2 \to 0$, and we can use the Kerr metric [which follows from Eq. (15) if we set $q^2 = 0$] for our computations. For this metric we can calculate

$$\mathcal{R}\tilde{\mathcal{R}} = \frac{192M^2Ar\cos\theta(3r^2 - A^2\cos^2\theta)(r^2 - 3A^2\cos^2\theta)}{(r^2 + A^2\cos^2\theta)^6}, \tag{21}$$

$$\mathcal{R}^2_{\rm GB} = \frac{48M^2(r^2 - A^2\cos^2\theta)[(r^2 + A^2\cos^2\theta)^2 - 16r^2A^2\cos^2\theta]}{(r^2 + A^2\cos^2\theta)^6}. \tag{22}$$

Similarly, in terms of the $\mathcal{A}$ expression and the Kerr metric, we can calculate

$$F^{\mu\nu}F_{\mu\nu} = -\frac{2(Q_e^2 - Q_m^2)[(r^2 - A^2\cos^2\theta)^2 - 4A^2r^2\cos^2\theta]}{(r^2 + A^2\cos^2\theta)^4} - \frac{16Q_eQ_mAr\cos\theta(r^2 - A^2\cos^2\theta)}{(r^2 + A^2\cos^2\theta)^4}, \tag{23}$$

[3] The horizon of a Kerr-Newman black hole is $r_H = M + \sqrt{M^2 - A^2 - q^2}$.

$$F\tilde{F} = -\frac{16(Q_e^2 - Q_m^2)Ar\cos\theta(r^2 - A^2\cos^2\theta)}{(r^2 + A^2\cos^2\theta)^4} + \frac{8Q_eQ_m[(r^2 - A^2\cos^2\theta)^2 - 4A^2r^2\cos^2\theta]}{(r^2 + A^2\cos^2\theta)^4}. \tag{24}$$

Following the same procedure as with the Kerr-Newman metric, we can see that the zeroth order solutions for the Kerr metric are still constants. The solution for the dilaton, modulus, and axion fields will be derived from the $O(\alpha')$ equations (8)–(11) setting the zeroth order solutions for the fields in the right-hand side. As a result, the kinetic terms $(\partial_\mu a)^2$ and $(\partial_\mu b)^2$ vanish, the derivatives of Δ, Θ, $\hat{\Delta}$, and $\hat{\Theta}$ are taken at the point $(\sigma = \sigma_0, b = b_0)$, and the quadratic gravitational and gauge terms are given from expressions (21)–(24).

In order to proceed and obtain the $O(\alpha')$ solutions, we need the static axisymmetric Green's function defined by the equation

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu G(x-y)] = \frac{\delta^{(3)}(x-y)}{\sqrt{-g}}, \tag{25}$$

which for the Kerr metric becomes

$$\frac{\partial}{\partial r}\left[(r^2 + A^2 - 2Mr)\frac{\partial G}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial G}{\partial\theta}\right] = -\delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\varphi - \varphi_0) \tag{26}$$

for a point source located at r_0, θ_0, φ_0. Demanding finiteness at $r = r_H$ and at infinity, we obtain

$$G(r,\theta,\varphi; r_0,\theta_0,\varphi_0) = \sum_{l=0}^{\infty} R_l(r, r_0) P_l(\cos\gamma), \tag{27}$$

with

$$\cos\gamma \equiv \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0) \tag{28}$$

and

$$R_l(r, r_0) = -\frac{2l+1}{4\pi\sqrt{M^2 - A^2}}\left[P_l\left(\frac{r_0 - M}{\sqrt{M^2 - A^2}}\right)Q_l\left(\frac{r - M}{\sqrt{M^2 - A^2}}\right)\theta(r - r_0)\right.$$
$$\left. + P_l\left(\frac{r - M}{\sqrt{M^2 - A^2}}\right)Q_l\left(\frac{r_0 - M}{\sqrt{M^2 - A^2}}\right)\theta(r_0 - r)\right]. \tag{29}$$

Using the Green's function, we can write the external dilaton solution as

$$\phi(r,\theta,\varphi) = \int_{r_H}^{\infty} dr_0 \int_{-1}^{1} d\cos\theta_0 \int_0^{2\pi} d\varphi_0 (r_0^2 + A^2\cos^2\theta_0) G(r,\theta,\varphi; r_0,\theta_0,\varphi_0)\mathcal{J}(r_0,\theta_0,\varphi_0), \tag{30}$$

where the source $\mathcal{J}$ is the right-hand side of Eq. (8). Similar expressions hold for the rest of the scalar and pseudoscalar fields[4] $\sigma, \tilde{a}, \tilde{b}$ in terms of the corresponding source terms. It is straightforward but tedious to obtain the $O(\alpha')$ expressions for these fields. The $O(\alpha')$ fields are

$$\phi(r,\theta) = \phi_0 - \frac{\alpha' e^{\phi_0}}{g^2}\left[\frac{1}{A^2}\ln\left(\frac{r - M + \sqrt{M^2 - A^2}}{\sqrt{A^2 + r^2}}\right) + \frac{2A^2 - M^2}{2A^3 M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right.$$
$$\left. + \frac{Mr}{(A^2 + r^2)^2} + \frac{2A^2 + Mr}{2A^2(A^2 + r^2)} + \frac{Q_e^2 - Q_m^2}{4AM}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right]P_0(\cos\theta)$$
$$+ \frac{3\alpha' Q_e Q_m e^{\phi_0}}{4AMg^2}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)P_1(\cos\theta) + \cdots, \tag{31}$$

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[4] $\partial_\mu\tilde{b} = e^{-2\sigma}\partial_\mu b$.

$$\tilde{a}(r,\theta) = \tilde{a}_0 + \frac{\alpha' Q_e Q_m}{AMg^2}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)P_0(\cos\theta) + \left(\frac{3\alpha'(Q_e^2 - Q_m^2)}{2AMg^2}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)\right.$$
$$-\frac{6\alpha' A}{g^2(M^2-A^2)}\left\{(r-M)\left[\frac{A^2-M^2}{A^4}\ln\left(\frac{r-M+\sqrt{M^2-A^2}}{\sqrt{A^2+r^2}}\right)\right.\right.$$
$$\left.+\frac{2A^2+Mr-M^2}{2A^2(A^2+r^2)} + \frac{A^2-M^2}{A^3M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right] + \frac{A^2\sqrt{M^2-A^2}}{Mr_H^3}$$
$$\left.\left.+\frac{3M-4r_H}{2r_H^2} + \frac{M(M^2+r^2)}{(A^2+r^2)^2}\right\}\right)P_1(\cos\theta) + \cdots, \tag{32}$$

$$\sigma(r,\theta) = \sigma_0 + \left\{-\frac{8\alpha'}{3}\left(\frac{\partial\Delta}{\partial\sigma}\right)_{\sigma_0,b_0}\left[\frac{1}{A^2}\ln\left(\frac{r-M+\sqrt{M^2-A^2}}{\sqrt{A^2+r^2}}\right)\right.\right.$$
$$\left.+\frac{2A^2-M^2}{2A^3M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right) + \frac{2A^2+Mr}{2A^2(A^2+r^2)} + \frac{Mr}{(A^2+r^2)^2}\right]$$
$$\left.+\left(\frac{\partial\hat{\Delta}}{\partial\sigma}\right)_{\sigma_0,b_0}\frac{2\alpha'(Q_e^2-Q_m^2)}{3AM}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right) - \left(\frac{\partial\hat{\Theta}}{\partial\sigma}\right)_{\sigma_0,b_0}\frac{8\alpha' Q_e Q_m}{3AM}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right\}P_0(\cos\theta)$$
$$-\left(\left(\frac{\partial\Theta}{\partial\sigma}\right)_{\sigma_0,b_0}\frac{48\alpha' A}{3(M^2-A^2)}\left\{(r-M)\left[\frac{A^2-M^2}{A^4}\ln\left(\frac{r-M+\sqrt{M^2-A^2}}{\sqrt{A^2+r^2}}\right)\right.\right.\right.$$
$$\left.+\frac{2A^2+Mr-M^2}{2A^2(A^2+r^2)} + \frac{A^2-M^2}{A^3M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right] + \frac{A^2\sqrt{M^2-A^2}}{Mr_H^3}$$
$$\left.+\frac{3M-4r_H}{2r_H^2} + \frac{M(M^2+r^2)}{(A^2+r^2)^2}\right\} + \left(\frac{\partial\hat{\Delta}}{\partial\sigma}\right)_{\sigma_0,b_0}\frac{2\alpha' Q_e Q_m}{AM}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)$$
$$\left.+\left(\frac{\partial\hat{\Theta}}{\partial\sigma}\right)_{\sigma_0,b_0}\frac{4\alpha'(Q_e^2-Q_m^2)}{3AM}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)\right)P_1(\cos\theta) + \cdots, \tag{33}$$

$$\tilde{b}(r,\theta) = \tilde{b}_0 + \left\{-\frac{8\alpha'}{3}\left(\frac{\partial\Delta}{\partial b}\right)_{\sigma_0,b_0}\left[\frac{1}{A^2}\ln\left(\frac{r-M+\sqrt{M^2-A^2}}{\sqrt{A^2+r^2}}\right)\right.\right.$$
$$\left.+\frac{2A^2-M^2}{2A^3M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right) + \frac{2A^2+Mr}{2A^2(A^2+r^2)} + \frac{Mr}{(A^2+r^2)^2}\right]$$
$$\left.+\left(\frac{\partial\hat{\Delta}}{\partial b}\right)_{\sigma_0,b_0}\frac{2\alpha'(Q_e^2-Q_m^2)}{3AM}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right) - \left(\frac{\partial\hat{\Theta}}{\partial b}\right)_{\sigma_0,b_0}\frac{8\alpha' Q_e Q_m}{3AM}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right\}P_0(\cos\theta)$$
$$-\left(\left(\frac{\partial\Theta}{\partial b}\right)_{\sigma_0,b_0}\frac{48\alpha' A}{3(M^2-A^2)}\left\{(r-M)\left[\frac{A^2-M^2}{A^4}\ln\left(\frac{r-M+\sqrt{M^2-A^2}}{\sqrt{A^2+r^2}}\right)\right.\right.\right.$$
$$\left.+\frac{2A^2+Mr-M^2}{2A^2(A^2+r^2)} + \frac{A^2-M^2}{A^3M}\left(\frac{\pi}{2} - \arctan\frac{r}{A}\right)\right] + \frac{A^2\sqrt{M^2-A^2}}{Mr_H^3}$$
$$\left.+\frac{3M-4r_H}{2r_H^2} + \frac{M(M^2+r^2)}{(A^2+r^2)^2}\right\} + \left(\frac{\partial\hat{\Delta}}{\partial b}\right)_{\sigma_0,b_0}\frac{2\alpha' Q_e Q_m}{AM}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)$$
$$\left.+\left(\frac{\partial\hat{\Theta}}{\partial b}\right)_{\sigma_0,b_0}\frac{4\alpha'(Q_e^2-Q_m^2)}{3AM}\left(\frac{r}{A}\arctan\frac{A}{r} - 1\right)\right)P_1(\cos\theta) + \cdots\ . \tag{34}$$

The leading modulus and $\tilde{b}$-hair behavior is that of a monopole term analogous to the dilaton and $\tilde{a}$ hair. This is evident from the slow rotation limit of the dilaton solution

$$\phi(r,\theta)=\phi_0+\frac{\alpha' e^{\phi_0}}{4g^2}\left\{\left[-\frac{2}{Mr}\left(1+\frac{M}{r}+\frac{4M^2}{3r^2}\right)+\frac{A^2}{2M^3r}\left(\frac{1}{2}+\frac{M}{r}+\frac{12M^2}{3r^2}+\frac{6M^3}{r^3}+\frac{64M^4}{5r^4}\right)\right.\right.$$
$$\left.\left.-\frac{(Q_e^2-Q_m^2)}{Mr}+\cdots\right]P_0(\cos\theta)-\frac{AQ_eQ_m}{Mr^2}P_1(\cos\theta)+\cdots\right\} \tag{35}$$

and the $\tilde{a}$-hair solution

$$\tilde{a}(r,\theta)=\tilde{a}_0+\frac{\alpha'}{g^2}\left\{\left[\frac{Q_eQ_m}{Mr}+\cdots\right]P_0(\cos\theta)\right.$$
$$\left.+\left[-\frac{(Q_e^2-Q_m^2)A^2}{2Mr^2}-\frac{5A}{4Mr^2}\left(1+\frac{2M}{r}+\frac{18M^2}{5r^2}\right)+\cdots\right]P_1(\cos\theta)+\cdots\right\}. \tag{36}$$

Note that the coefficient functions [12] Δ, Θ, $\hat{\Delta}$, and $\hat{\Theta}$ could be such that they have an extremum at the self-dual point $\sigma_0=b_0=0$. Perturbing around the self-dual solution leads to vanishing modulus hair to $O(\alpha')$. The infinite continuum of nonzero σ_0, b_0 values allows for the nonvanishing modulus and $\tilde{b}$-axion hair given by (33) and (34).

Although the existence of nontrivial dilaton, moduli, and axion fields outside a Kerr-Newman black hole seems to violate the letter of the "no-hair theorem," it does not violate the spirit since the solution is uniquely characterized by mass, charge, and angular momentum. In the terminology introduced by Coleman, Preskill, and Wilczek [14], the external moduli and dilaton hair are examples of "secondary" hair.

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