

Static black hole solutions without rotational symmetry

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We construct static black hole solutions that have no rotational symmetry. These arise in theories (including the standard electroweak model) that contain charged vector mesons with mass $m \neq 0$. In such theories, a magnetically charged Reissner-Nordström black hole with a horizon radius less than a critical value of order m^{-1} is classically unstable against the development of a nonzero vector meson field just outside the horizon, indicating the existence of static black hole solutions with vector meson hair. For the case of unit magnetic charge, spherically symmetric solutions of this type have previously been studied. For other values of the magnetic charge, general arguments show that any new solution with hair cannot be spherically symmetric. In this paper we develop and apply a perturbative scheme (which may have applicability in other contexts) for constructing such solutions in the case where the Reissner-Nordström solution is just barely unstable. For a few low values of the magnetic charge the black holes retain a rotational symmetry about a single axis, but this axial symmetry disappears for higher charges. While the vector meson fields vanish exponentially at distances greater than $O(m^{-1})$, the magnetic field and the metric have higher multipole components that decrease only as powers of the distance from the black hole.

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I. INTRODUCTION

One of the many remarkable features of black holes is the symmetry and simplicity of the time-independent black hole solutions. The static vacuum black holes are all spherically symmetric and determined by a single parameter. Adding electromagnetism gives the possibility of endowing the black hole with electric or magnetic charge, but the static solutions remain spherically symmetric, with purely Coulomb electromagnetic fields. Even if one considers solutions that are stationary, but not static, a rotational symmetry about one axis remains. This situation stands in contrast with that of electromagnetism in flat spacetime, which possesses static (even if singular) solutions corresponding to point multipole moments of arbitrarily high order. An explanation for the absence of static gravitational solutions with higher multipoles comes from the no-hair theorems [1] that sharply constrain the possible structure of black holes both in the electrovac case and for gravity coupled to a number of types of matter.

However, it has become clear in recent years that if the theory governing the matter fields has sufficient structure, it is in fact possible to have black holes with nontrivial static fields outside the horizon; i.e., black holes with hair. In particular, theories with electrically charged massive vector mesons can have two types of magnetically charged [2,3] black hole solutions.¹ One is the trivial generalization of the Reissner-Nordström solution to the coupled Einstein-Maxwell equations. The other, which exists

only if the horizon radius is sufficiently small, has nonzero massive vector fields just outside the horizon. For the case of the SU(2) gauge theory with a triplet Higgs field, which has a nonsingular magnetic monopole solution in flat spacetime, one finds a new solution with unit magnetic charge that may be viewed as a Schwarzschild-like black hole embedded in the center of a 't Hooft-Polyakov monopole [5]. Although this solution was first found directly, a signal of its existence is the fact that the Reissner-Nordström black hole develops a classical instability when its horizon radius becomes smaller than the radius of a magnetic monopole core [6].

Similar arguments [7] based on instabilities of Reissner-Nordström solutions suggest the existence of new black hole solutions with higher magnetic charges. However, in the presence of a magnetic monopole a spherically symmetric charged spin-one field is possible [8,9] only if the product of the magnetic charge of the monopole and the electric charge of the field is unity.² Hence, these new black holes can be at most axially symmetric. Whether or not they actually possess such symmetry is a question not of general principle, but of detailed dynamics. In this paper we will show that, at least for certain ranges of parameters, such black holes have no rotational symmetry at all.

²This follows from the absence of vector spherical harmonics with total angular momentum zero. The absence of such harmonics can be understood by considering the motion of a particle with electric charge e about a monopole with magnetic charge Q_M . The total angular momentum is the sum of the spin angular momentum, the orbital angular momentum, and a contribution of magnitude eQ_M directed along the line from the monopole to the charge; if $eQ_M \neq 1$, the sum of these three terms can never vanish.

¹For other types of black hole solutions in theories with massive vector mesons, see Ref. [4].

A theory with sufficient structure to yield these new black holes has matter fields described by the flat-spacetime Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^*W^{\mu\nu} - m^2W_{\mu}^*W^{\mu} \\ & -\frac{ieg}{4}F^{\mu\nu}(W_{\mu}^*W_{\nu} - W_{\nu}^*W_{\mu}) \\ & -\frac{\lambda e^2}{4}|W_{\mu}^*W_{\nu} - W_{\nu}^*W_{\mu}|^2, \end{aligned} \quad (1.1)$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad (1.2)$$

$$W_{\mu\nu} = D_{\mu}W_{\nu} - D_{\nu}W_{\mu}, \quad (1.3)$$

$$D_{\mu}W_{\nu} = (\partial_{\mu} - ieA_{\mu})W_{\nu}. \quad (1.4)$$

The fourth term in the Lagrangian is an anomalous magnetic moment term, with the constant g arbitrary. In order that the energy be bounded from below, we must require that $\lambda \geq g^2/4$ [7].

If we add to the theory a neutral scalar field ϕ with appropriate self-interactions and give the vector field a ϕ -dependent mass $m = e\phi$, then for $g = 2$ and $\lambda = 1$, Eq. (1.1) is simply the unitary gauge form of the Lagrangian for an SU(2) gauge theory spontaneously broken to U(1) by a triple Higgs field. Similarly, for $g = 2$, $\lambda = 1/\sin^2\theta_W$, and $m = e\phi/2$ we obtain the unitary gauge form of the standard electroweak Lagrangian, but with all terms involving the Z or fermions omitted. It is a straightforward matter to extend the analysis of this paper to such models.

We are seeking static black hole solutions to the theory obtained by coupling the Lagrangian of Eq. (1.1) to general relativity. In the absence of rotational symmetry, the static field equations are a set of coupled partial differential equations in three variables. An exact analytic solution of these is beyond our abilities. Instead, we use a perturbative approach. We begin by considering a Reissner-Nordström black hole with radial magnetic field

$$F_{\theta\phi} = \frac{q}{e} \sin\theta \quad (1.5)$$

corresponding to a magnetic charge $Q_m = q/e$ (where q is restricted by the Dirac quantization condition to integer or half-integer values) and vanishing W field. The metric is

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1.6)$$

with

$$B(r) = 1 - \frac{2MG}{r} + \frac{4\pi Gq^2}{e^2 r^2} = \frac{(r - r_H)(r - r_-)}{r^2}. \quad (1.7)$$

We choose the mass M to be such that the outer horizon radius r_H is less than r_{cr} , the critical value for instability. It is in this mass range that we expect there

to be a second black hole solution with nontrivial W field and with electromagnetic field strengths and metric that differ from the Reissner-Nordström form. It is often the case that the exponentially growing eigenmodes about an unstable static solution give a good indication of the nature of a nearby stable solution, particularly in the case where the original solution is just barely unstable. Guided by this intuition, we linearize the static field equations about the Reissner-Nordström solution. This leads to an eigenvalue problem that is closely related to, although not exactly the same as, that encountered in the stability analysis [10]. For r_H close to r_{cr} , there is a single negative eigenvalue, whose magnitude tends to zero as $r_H \rightarrow r_{\text{cr}}$. With $r_{\text{cr}} - r_H$ sufficiently small, this eigenvalue becomes a small parameter that can serve as the basis for a perturbative expansion.

In Sec. II, we illustrate our method with a simple toy model consisting of a scalar field coupled to a fixed, but spatially inhomogeneous, external source. In Sec. III, we set up the formalism for treating the case in which we are actually interested, that of a black hole in the theory described by the Lagrangian of Eq. (1.1). We assume that Gm^2/e^2 is small; this allows one to solve (to leading order) for the charged vector field and the perturbations of the electromagnetic field before dealing with the metric perturbations. For technical reasons, it turns out that the details of the subsequent analysis are considerably simpler if g is positive and $q \geq 1$. We exploit these simplifications in Sec. IV, where we determine the leading perturbations of the electromagnetic field in terms of those of W_{μ} . In Sec. V we examine the lowest order contributions to the charged vector field and show that there is a parameter range for which the solution is not even axially symmetric. In Sec. VI we obtain the leading corrections to the Reissner-Nordström metric. Some concluding remarks are included in Sec. VII. An appendix describes some results needed to construct Green's functions that we use.

II. A TOY MODEL

Consider a real scalar field whose dynamics is governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_{\mu}\phi)^2 - \frac{1}{2}F(\mathbf{x})\phi^2 - \frac{\lambda}{4}\phi^4, \quad (2.1)$$

where $F(\mathbf{x})$ arises from the coupling to a spatially inhomogeneous but static external source. Static solutions obey

$$\begin{aligned} 0 = & [-\nabla^2 + F(\mathbf{x})]\phi + \lambda\phi^3 \\ \equiv & \mathcal{M}\phi + \lambda\phi^3 \end{aligned} \quad (2.2)$$

and, to have finite energy, must satisfy the boundary condition that ϕ vanish at spatial infinity.

The trivial configuration $\phi(x) = 0$ is a solution for any choice of $F(\mathbf{x})$. It is easy to see that this is the only static solution if \mathcal{M} is a positive operator. However, if \mathcal{M} has negative eigenvalues, this trivial solution is unstable,

implying the existence of a new, spatially inhomogeneous, solution whose form we seek. To this end, let us assume that \mathcal{M} has only a single negative eigenvalue, with an eigenfunction ψ obeying

$$\mathcal{M}\psi = -b^2\psi \quad (2.3)$$

and normalized so that

$$\int d^3x \psi^2(\mathbf{x}) = 1. \quad (2.4)$$

We now write ϕ as the sum of a term proportional to ψ and one orthogonal to it; i.e.,

$$\phi(x) = k\psi(x) + \tilde{\phi}(x) \quad (2.5)$$

with

$$\int d^3x \psi(\mathbf{x})\tilde{\phi}(\mathbf{x}) = 0. \quad (2.6)$$

The static field equation (2.2) then implies that

$$\mathcal{M}\tilde{\phi} + \lambda(k\psi + \tilde{\phi})^3 + \Gamma\psi = 0, \quad (2.7)$$

where Γ is a Lagrange multiplier that enforces the orthogonality condition (2.6). It can be calculated by multiplying both sides of this equation by ψ and then integrating over all space to obtain

$$\Gamma = -\lambda \int d^3x \psi(k\psi + \tilde{\phi})^3. \quad (2.8)$$

Variation of the action with respect to k gives the addi-

tional equation

$$\frac{\partial I}{\partial k} = 0, \quad (2.9)$$

where

$$\begin{aligned} I &= \int d^3x \left[\frac{1}{2}k^2\psi\mathcal{M}\psi + \frac{\lambda}{4}(k\psi + \tilde{\phi})^4 \right] \\ &= -\frac{1}{2}k^2b^2 + \frac{\lambda}{4} \int d^3x (k\psi + \tilde{\phi})^4. \end{aligned} \quad (2.10)$$

Thus far we have made no approximations; together, Eqs. (2.7), (2.8), and (2.9) are completely equivalent to Eq. (2.2). We now recall that if $b = 0$ the scalar field ϕ , and therefore k and $\tilde{\phi}$, must vanish. Hence, for small b it should be possible to expand these quantities as power series in b . Furthermore, since it is the existence of the negative eigenvalue which makes a nontrivial solution possible, we may view $k\psi$ as providing the source for $\tilde{\phi}$ [through Eq. (2.7)]. We therefore expect that $\tilde{\phi}$ is of higher order in b than $k\psi$. Assuming this to be the case, Eqs. (2.9) and (2.10) give

$$k^2 = \frac{b^2}{\lambda} \left[\int d^3x \psi^4(\mathbf{x}) \right]^{-1} + O(b^3), \quad (2.11)$$

while Eq. (2.8) implies

$$\begin{aligned} \Gamma &= -\lambda k^3 \int d^3x \psi^4(\mathbf{x}) + O(b^4) \\ &= \frac{b^3}{\sqrt{\lambda}} \left[\int d^3x \psi^4(\mathbf{x}) \right]^{-1/2} + O(b^4). \end{aligned} \quad (2.12)$$

These may be substituted into Eq. (2.7) to give

$$\begin{aligned} \mathcal{M}\tilde{\phi}(\mathbf{x}) &= -\lambda k^3 \left[\psi^3(\mathbf{x}) - \psi(\mathbf{x}) \int d^3y \psi^4(\mathbf{y}) \right] + O(b^4) \\ &= \frac{b^3}{\sqrt{\lambda}} \left[\int d^3x \psi^4(\mathbf{x}) \right]^{-3/2} \left[\psi^3(\mathbf{x}) - \psi(\mathbf{x}) \int d^3y \psi^4(\mathbf{y}) \right] + O(b^4). \end{aligned} \quad (2.13)$$

This shows that $\tilde{\phi}$ is of order b^3 , and justifies the assumption above that it is of higher order than $k\psi$. Thus, to leading order the static solution is approximated by the negative eigenvalue fluctuation about the vacuum solution, multiplied by a scale factor whose magnitude is determined by the nonlinear term in the Lagrangian. By substitution of the lower order results back into the original equations, k and $\tilde{\phi}$, and hence ϕ itself, can be calculated to arbitrarily high order in b .

III. CHARGED VECTOR MESON MODEL

We now apply this method to the theory in which we are actually interested, namely that described by the Lagrangian of Eq. (1.1). The first step is to identify the un-

stable (i.e., exponentially growing in time) modes about the unperturbed Reissner-Nordström solution. This was done in Ref. [10], whose results we now briefly summarize. When the field equations are linearized, the perturbations in the gauge field and the metric decouple from those in the massive vector field. The linear perturbation problem for the former two modes is the same as in the pure Einstein-Maxwell theory, where it was shown some time ago [11] that the Reissner-Nordström solution is stable. Hence, the stability analysis reduces to a study of the linearized W field equations. It is convenient to define $\mathcal{M}_{\mu\nu}$ by

$$\begin{aligned} \mathcal{M}^{\mu\nu}W_\nu &= -\frac{1}{\sqrt{g}}\bar{D}_\alpha(\sqrt{g}W^{\alpha\mu}) + m^2W^\mu \\ &\quad -\frac{ieg}{2}\bar{F}^{\alpha\mu}W_\alpha, \end{aligned} \quad (3.1)$$

where here, and for the remainder of the paper, we adopt the convention that $\bar{g}_{\mu\nu}$, \bar{A}_μ , and $\bar{F}_{\mu\nu}$ denote the corresponding unperturbed quantities while \bar{D}_μ is the gauge covariant derivative taken with respect to the unperturbed potential; indices are raised and lowered with the unperturbed metric. The unstable modes are solutions of

$$\mathcal{M}_\mu{}^\nu W_\nu = 0 \quad (3.2)$$

whose time dependence is of the form

$$W_\nu(\mathbf{x}, t) = f_\nu(\mathbf{x})e^{\omega t} \quad (3.3)$$

with real ω . The spherical symmetry of the unperturbed solution allows one to choose the solutions of Eq. (3.2) to be eigenfunctions of both \mathbf{J}^2 and J_z , where \mathbf{J} is the total angular momentum operator. Because of the extra angular momentum of an electric charge in the field of a magnetic monopole, the corresponding eigenvalues are not the usual ones. Instead, J runs in integer steps upward from the minimum value $J_{\min} = q - 1$, unless $q = 1/2$, in which case $J_{\min} = 1/2$. For each value of J , unstable modes exist if the horizon radius r_H is less than a critical value $r_{\text{cr}}(J)$ that is of order m^{-1} , provided that g lies in an appropriate range ($g > 0$ for $J = q - 1$, $g > 2$ for $J = q$, and either $g < 0$ or $g > 2$ for $J > q$). For a given value of g , $r_{\text{cr}}(J)$ is greatest for the smallest J that can have unstable modes with that g . Thus, if \hat{J} denotes the value that maximizes r_{cr} , we have $\hat{J} = q - 1$ for $g > 0$ and $\hat{J} = q + 1$ for $g < 0$ and $q \geq 1$. If $q = 1/2$, $\hat{J} = 1/2$ if $g > 2$ and $3/2$ if $g < 0$; if $0 \leq g \leq 2$, there is no instability.

The modes that will form the basis for our new solutions are the static eigenfunctions of \mathcal{M} with negative eigenvalue; i.e., the time-independent solutions of

$$\mathcal{M}_\mu{}^\nu \psi_\nu = -\beta^2 m^2 \psi_\mu \quad (3.4)$$

with real β . (A factor of m^2 has been extracted to make β dimensionless.)

This eigenvalue equation must be supplemented by boundary conditions. At spatial infinity we merely require that ψ_μ not diverge. For negative eigenvalues (indeed for all eigenvalues less than m^2) this implies that ψ_μ in fact vanishes as $r \rightarrow \infty$. A second boundary condition is obtained at the horizon, where we require that ψ_μ be regular, in the sense that its components measured relative to a coordinate system that is nonsingular at the horizon (e.g., Kruskal-like coordinates) be regular. Because of the manner in which the singular metric factors enter Eq. (3.4), this constrains the behavior of ψ_μ near the horizon, as we will see more explicitly in the next section, and causes the spectrum of negative eigenvalues to be discrete.³ For this portion of the spectrum, we can require that the eigenfunctions satisfy the normalization condition⁴

³It might seem strange that the nature of the spectrum should be determined by the singularities of a metric at a horizon that is only a coordinate singularity. This happens because the condition we are imposing on the eigenfunctions, that they be static, is defined in terms of a coordinate t that is singular at the horizon.

⁴Since, as is easily shown, static solutions of Eq. (3.4) must have $\psi_t = 0$, $\psi_\mu \psi^\mu$ is positive.

$$\int d^3x \sqrt{\bar{g}} \psi_\mu^* \psi^\mu = 1, \quad (3.5)$$

where, both in this equation and hereafter, the spatial integration is understood to be restricted to the region outside the Reissner-Nordström outer horizon.

Because of both the nontrivial metric component $g_{tt}(r)$ and the possibility of a nonvanishing W_t in the time-dependent case, the eigenfunctions ψ_μ are not in general the same as the f_μ that appear in Eq. (3.3); the spectra of the ω and β are not even the same. However, a zero eigenvalue for the static operator does correspond to a zero frequency of the small oscillation problem and, furthermore, the static problem has negative eigenvalues if and only if the Reissner-Nordström solution is unstable. Hence, the conditions for instability enumerated above are also the conditions we need to be able to construct our new solutions.

In fact, the static eigenmodes for real β can be obtained from the $\omega = 0$ solutions of Eq. (3.2) with different values for the parameters. This can be seen by bringing the right hand side of Eq. (3.4) over to the left; the resulting equation is precisely that satisfied by $f_\mu(\mathbf{x})$ for $\omega = 0$, but with m^2 replaced by $m^2(1 + \beta^2)$. It follows that the value of r_H that leads to a given β for W mass m is equal to the critical value r_{cr} for a W mass $m\sqrt{1 + \beta^2}$. If $m \ll M_{\text{Pl}}$ (the case with which we will be primarily concerned), r_{cr} is much greater than the horizon size for an extremal Reissner-Nordström black hole and its dependence on the inner horizon r_- can be neglected. Dimensional arguments then show that r_{cr} is inversely proportional to m . It then follows that

$$\beta = \frac{\sqrt{r_{\text{cr}}^2 - r_H^2}}{r_H}. \quad (3.6)$$

Thus, by taking $r_H - r_{\text{cr}}(\hat{J}) \ll r_{\text{cr}}(\hat{J})$, we ensure that $\beta \ll 1$, thus providing the small parameter needed for our perturbative calculation.

In general \hat{J} is nonzero, so that, in contrast to the model of Sec. II, in which there was only a single negative eigenvalue mode, there is a degenerate multiplet of negative eigenvalue solutions ψ_μ^M to Eq. (3.4) that are distinguished by the eigenvalue of J_z . Proceeding as in that section, we write the W field as a linear combination of the unstable modes plus a remainder orthogonal to these modes:

$$W_\mu = V_\mu + \tilde{W}_\mu = m^{-1/2} \sum_{M=-\hat{J}}^{\hat{J}} k_M \psi_\mu^M + \tilde{W}_\mu, \quad (3.7)$$

where

$$\int d^3x \sqrt{\bar{g}} (\psi_\mu^M)^* \tilde{W}^\mu = 0. \quad (3.8)$$

It is useful to define a quantity a by

$$\sum_{M=-\hat{J}}^{\hat{J}} |k_M|^2 = a^2 \quad (3.9)$$

so that

$$V_\mu = O(a) . \tag{3.10}$$

Since, as will be displayed explicitly below, the source for the perturbations of the electromagnetic field is quadratic in V_μ and contains an explicit factor of e

$$\delta A_\mu = O(ea^2) . \tag{3.11}$$

(Note that the quantities ea and Gm^2/e^2 are truly dimensionless, whereas e is dimensionless only if one sets $\hbar = 1$, which would not be natural in this essentially clas-

sical context.) The source for the metric perturbations is the perturbation of the energy-momentum tensor. The leading contribution to this, of order a^2 , is from terms quadratic in V_μ and from terms linear in δA_μ . However, these enter the field equation multiplied by a factor of G , and so $\delta g_{\mu\nu} \equiv h_{\mu\nu}$ must be suppressed by an additional factor of roughly $Gm^2 = (m/M_{\text{Pl}})^2$. Hence,

$$\delta g_{\mu\nu} \equiv h_{\mu\nu} = O(Gm^2 a^2) . \tag{3.12}$$

Finally, the magnitude of \tilde{W} can be determined from the field equation

$$\begin{aligned} \mathcal{M}^{\nu\alpha} \tilde{W}_\alpha &= -\lambda e^2 (V^{*\mu} V^\nu - V^{*\nu} V^\mu) V_\mu + \frac{ieg}{2} V_\mu \delta F^{\mu\nu} - ie \delta A_\mu (\bar{D}^\mu V^\nu - \bar{D}^\nu V^\mu) \\ &+ \frac{ie}{\sqrt{g}} \bar{D}_\mu [\sqrt{g} (V^\mu \delta A^\nu - V^\nu \delta A^\mu)] - \sum \Gamma_M \psi_M^\nu + \dots . \end{aligned} \tag{3.13}$$

Here the ellipses represent terms which are either $O(e^4 a^5)$ or $O(Gm^2 a^3)$ or smaller, while the Γ_M are Lagrange multipliers introduced to enforce the orthogonality of \tilde{W}_μ and the ψ_M^M . One can solve for the Γ_M by multiplying both sides of this equation by ψ_M^M and then integrating over all space outside the horizon. Inserting the result back into Eq. (3.13), we see that

$$\tilde{W}_\mu = O(e^2 a^3) . \tag{3.14}$$

We will assume that $Gm^2/e^2 \ll 1$. The leading behavior of V_μ , \tilde{W}_μ , and δA_μ can then be obtained by solving the field equations in the background of the unperturbed Reissner-Nordström metric. Having done this, the leading perturbations of the metric can then be obtained. In fact, for calculating the lowest order metric perturbations, only δA_μ and V_μ are needed. For the remainder of this section, and the next two, we will concentrate on the determination of these two quantities. We will then return to the calculation of the metric perturbations in Sec. VI.

Linearization of the electromagnetic field equation about the unperturbed solution yields

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \delta F^{\mu\nu}) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} p^{\mu\nu}) + j^\nu , \tag{3.15}$$

where

$$p_{\mu\nu} = -\frac{ieg}{2} (V_\mu^* V_\nu - V_\nu^* V_\mu) + \dots \tag{3.16}$$

and

$$\begin{aligned} j^\nu &= ie [V_\mu^* (\bar{D}^\mu V^\nu - \bar{D}^\nu V^\mu) - V_\mu (\bar{D}^\mu V^{*\nu} \\ &- \bar{D}^\nu V^{*\mu})] + \dots \end{aligned} \tag{3.17}$$

with the ellipsis signifying higher order terms. Similarly, the Bianchi identity gives

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu \delta F_{\alpha\beta} = 0 . \tag{3.18}$$

In addition to these, we need the equations obtained by varying the action with respect to the k_M . In deriving these, we need only take into account terms in the action of up to order $e^2 a^4$ and so can approximate the action by

$$\begin{aligned} S_{\text{approx}} &= \int d^4 x \sqrt{g} \left[-V_\mu^* \mathcal{M}^{\mu\nu} V_\nu - \frac{\lambda e^2}{4} |V_\mu^* V_\nu - V_\nu^* V_\mu|^2 \right. \\ &- \frac{1}{4} (\bar{F}_{\mu\nu} + \delta F_{\mu\nu}) (\bar{F}^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{2} \delta F_{\mu\nu} p^{\mu\nu} \\ &\left. - \delta A_\nu j^\nu \right] . \end{aligned} \tag{3.19}$$

(Terms linear in both \tilde{W}_μ and V_ν , which would be of order $e^2 a^4$, are absent because of the orthogonality of \tilde{W}_μ and the ψ_M^M .) There are two ways of proceeding from here. One can substitute the expansion of Eq. (3.7) for V_μ and then vary the above expression with respect to k_M , thus obtaining an equation involving both the k_M and δA_ν . Alternatively, one can first use Eqs. (3.15) and (3.18) to solve for δA_μ and $\delta F_{\mu\nu}$ in terms of V_μ and then substitute the resulting expressions back into Eq. (3.19) to obtain an action which is a function of only the k_M ; we will follow this second approach. A number of simplifications are possible. First, the eigenvalue equation (3.4) and the normalization condition (3.5) can be used to integrate the term quadratic in V_μ . Next, by multiplying both sides of Eq. (3.15) by $\sqrt{g} \delta A_\nu$ and then integrating by parts, one obtains the identity

$$\int d^4 x \sqrt{g} \delta F_{\mu\nu} \delta F^{\mu\nu} = \int d^4 x \sqrt{g} [\delta F_{\mu\nu} p^{\mu\nu} - \delta A_\nu j^\nu] , \tag{3.20}$$

which can be used to eliminate the term quadratic in $\delta F_{\mu\nu}$. A similar procedure applied to the source-free

equation obeyed by the unperturbed field strength shows that

$$\int d^4x \sqrt{\bar{g}} \bar{F}_{\mu\nu} \delta F^{\mu\nu} = 0. \quad (3.21)$$

(In both cases, one can verify that the surface terms from the integration by parts vanish as long as the total magnetic charge is held fixed.) The term quadratic in $F_{\mu\nu}$ is obviously independent of the k_M and can be ignored. Finally, since all quantities are independent of time, it is sufficient to integrate over the spatial variables. We are thus led to the equation

$$0 = \frac{\partial I}{\partial k_M}, \quad (3.22)$$

where

$$I = -\beta^2 m a^2 + \int d^3x \sqrt{\bar{g}} \left[\frac{\lambda e^2}{4} |V_\mu^* V_\nu - V_\nu^* V_\mu|^2 - \frac{1}{4} \delta F_{\mu\nu} p^{\mu\nu} + \frac{1}{2} \delta A_\nu j^\nu \right] \quad (3.23)$$

and δA_ν is understood to be given in terms of the k_M . Since the integrand on the right-hand side is of order $e^2 a^4$, we see that a is proportional to β/e , indicating that our perturbative expansion is justified for r_H sufficiently close to r_{cr} .

IV. THE CASE $q \geq 1, q > 0$

We now specialize to the case $q \geq 1, g > 0$ for which, as was noted above, $\hat{J} = q - 1$. This allows us to take advantage of the special properties [9] of the $J = q - 1$ vector spherical harmonics, which lead to a number of technical simplifications in the analysis. For $J = q - 1$, and only for that value, there is but a single monopole vector spherical harmonic [9,12] for each value of $J_z = M$. Hence, if we denote this harmonic by $C_\mu^M(\theta, \phi)$, the unstable modes in this case can be written in the form

$$\psi_\mu^M = f(r) C_\mu^M(\theta, \phi), \quad (4.1)$$

where $f(r)$ does not depend on M .

The $J = q - 1$ harmonics have a number of special properties. Their radial and time components vanish,

$$C_r^M = C_t^M = 0, \quad (4.2)$$

and their two angular components are related by

$$C_\phi^M = i \sin \theta C_\theta^M. \quad (4.3)$$

In addition, their covariant curl, evaluated in the background Dirac vector potential, vanishes,

$$\bar{D}_\mu C_\nu^M - \bar{D}_\nu C_\mu^M = 0, \quad (4.4)$$

as does their covariant divergence

$$\frac{1}{\sqrt{\bar{g}}} \bar{D}_\mu (\sqrt{\bar{g}} C^{M\mu}) = 0. \quad (4.5)$$

A convenient choice of normalization condition is

$$\int d\phi d\theta \sin \theta [C_\mu^M(\theta, \phi)]^* C^{M\mu}(\theta, \phi) = \frac{1}{r^2}. \quad (4.6)$$

To obtain an explicit expression for the C_μ^M , we must choose a gauge. If the electromagnetic vector potential has a single nonvanishing component

$$A_\phi = \frac{q}{e} (1 - \cos \theta), \quad (4.7)$$

then

$$C_\theta^M = a_{qM} e^{i\phi} (1 + \cos \theta)^{q-1} \left[\frac{\sin \theta}{1 + \cos \theta} e^{i\phi} \right]^{q+M-1} \quad (4.8)$$

where

$$a_{qM} = \frac{1}{2^q \sqrt{2\pi}} \left[\frac{(2q-1)!}{(q+M-1)!(q-M-1)!} \right]^{1/2}. \quad (4.9)$$

With the aid of these properties, the eigenvalue equation (3.4) reduces to

$$-\frac{d}{dr} \left(B \frac{df}{dr} \right) + \left(m^2 - \frac{qg}{2r^2} \right) f = -\beta^2 m^2 f \quad (4.10)$$

where $f(r)$ can be chosen to be real. Equations (3.5) and (4.6) fix the normalization of f to be

$$\int_{r_H}^{\infty} dr |f(r)|^2 = 1. \quad (4.11)$$

Given r_{cr} , and hence β , Eq. (4.10) can be integrated numerically to obtain $f(r)$. This function is monotonic, has no zeros, and vanishes exponentially with r as $r \rightarrow \infty$. Near the horizon it behaves as

$$f(r) = A [1 - b(r - r_H)] + O[(r - r_H)^2], \quad (4.12)$$

where

$$b = \left[\frac{qg}{r_H^2} - m^2 (1 + \beta^2) \right] [B'(r_H)]^{-1} > 0. \quad (4.13)$$

Proceeding with the construction of the solution, we write

$$V_\mu = m^{-1/2} f(r) \Phi_\mu(\theta, \phi) \quad (4.14)$$

where

$$\Phi_\mu(\theta, \phi) = \sum_{M=-(q-1)}^{q-1} k_M C_\mu^M(\theta, \phi). \quad (4.15)$$

Equations (4.2) and (4.3) imply that $\Phi_r = \Phi_t = 0$ and fix the ratio of Φ_θ and Φ_ϕ . Note that Φ_μ has exactly $2(q-1)$ zeros as θ and ϕ range over the unit sphere. To show this, we use the explicit expression (4.8) for the vector harmonics and write

$$\begin{aligned} \Phi_\theta(\theta, \phi) &= e^{i\phi}(1 + \cos \theta)^{q-1} \\ &\times \sum_{M=-(q-1)}^{q-1} a_{qM} k_M z^{q+M-1}, \end{aligned} \quad (4.16)$$

where

$$z = \frac{\sin \theta}{1 + \cos \theta} e^{i\phi}. \quad (4.17)$$

The entire complex z plane maps onto the unit sphere, with $|z| = \infty$ corresponding to the south pole, $\theta = \pi$. Let \bar{M} be the largest value of M for which k_M is nonzero. The sum in Eq. (4.16) is then a polynomial of order $\bar{M} + q - 1$ in z , and thus has $\bar{M} + q - 1$ zeros at finite z . In addition, the prefactor multiplying the sum combines with the $M = \bar{M}$ term to give a zero of order $q - 1 - \bar{M}$ at $\theta = \pi$. Adding these together, we obtain the promised result.

The properties of the C_μ^M also lead to the useful identity

$$\Phi_\mu^* \Phi_\nu - \Phi_\nu^* \Phi_\mu = ir^2 \epsilon_{\mu\nu} \Phi_\alpha^* \Phi^\alpha, \quad (4.18)$$

where $\epsilon_{\mu\nu}$ is an antisymmetric tensor whose only nonzero components are

$$\epsilon_{\theta\phi} = -\epsilon_{\phi\theta} = \sin \theta. \quad (4.19)$$

We will encounter the quantity $\Phi_\mu^* \Phi^\mu$ in the source terms for the perturbations of both the electromagnetic field and of the metric. Since we will solve these equations by separation of variables, it is useful to define the expansion

$$r^2 \Phi_\mu^*(\theta, \phi) \Phi^\mu(\theta, \phi) = a^2 \sum_{jm} \sigma_{jm} Y_{jm}(\theta, \phi), \quad (4.20)$$

where

$$\sigma_{jm} = \frac{r^2}{a^2} \int d\phi d\theta \sin \theta Y_{jm}^*(\theta, \phi) \Phi_\mu^*(\theta, \phi) \Phi^\mu(\theta, \phi), \quad (4.21)$$

and a is as defined in Eq. (3.9). Ordinary, rather than monopole, spherical harmonics enter here because we are dealing with a neutral quantity. Hence, j runs over integer values although, since Φ_μ is a linear combination of monopole harmonics with angular momentum $\hat{J} = q - 1$, the σ_{jm} vanish for all $j > 2(q - 1)$. Note that, as a consequence of the normalization condition (4.6),

$$\sigma_{00} = \frac{1}{\sqrt{4\pi}}. \quad (4.22)$$

The properties of the $J = q - 1$ harmonics also simplify the electromagnetic field equations. All components of j^ν vanish, while

$$p_{\mu\nu} = \frac{eg}{2m} \epsilon_{\mu\nu} r^2 f^2 \Phi_\alpha^* \Phi^\alpha. \quad (4.23)$$

The various components of Eq. (3.15) can be written as

$$\partial_\mu(\sqrt{g} \delta F^{\mu t}) = 0, \quad (4.24)$$

$$\partial_\mu(\sqrt{g} \delta F^{\mu r}) = 0, \quad (4.25)$$

$$\frac{1}{\sqrt{g}} \partial_\mu(\sqrt{g} \delta F^{\mu\alpha}) = -\frac{eg}{2m} \epsilon^{ab} r^2 f^2 \partial_b(\Phi_\alpha^* \Phi^\alpha), \quad (4.26)$$

where we have adopted the convention that Roman indices from the beginning of the alphabet take only the values θ or ϕ . (In obtaining the last of these, we have used the fact that $\sqrt{g} \epsilon^{ab}$ is a function only of r .)

Because we are seeking time-independent solutions, the equations for the electric and magnetic fields decouple. For the former the source term vanishes, and so the equations are the same as those encountered in studying perturbations of the pure Reissner-Nordström solution, where the only allowed static perturbation of the electric field is a radial field corresponding to a variation of the black hole's electric charge. Since we are assuming vanishing electric charge, this perturbation must be excluded, and so $\delta F^{t\mu} = 0$.

The equations for the magnetic field can be solved by separation of variables. We first expand $\delta F_{\mu\nu}$ in terms of vector spherical harmonics:

$$\begin{aligned} \delta F_{\theta\phi} &= \sum_{jm} F_1^{jm}(r) \sin \theta Y_{jm}(\theta, \phi) \\ \delta F_{r\alpha} &= \sum_{jm} [F_2^{jm}(r) \epsilon_a^b \partial_b Y_{jm}(\theta, \phi) \\ &\quad + F_3^{jm}(r) \partial_a Y_{jm}(\theta, \phi)], \end{aligned} \quad (4.27)$$

where it is understood that $F_2^{00} = F_3^{00} = 0$. Substituting this expansion into Eq. (4.25) and using the identity

$$\frac{1}{\sqrt{g}} \partial_\mu(\sqrt{g} \bar{g}^{\mu\nu} \partial_\nu Y_{jm}) = -\frac{j(j+1)}{r^2} Y_{jm}, \quad (4.28)$$

we find that

$$F_3^{jm} = 0. \quad (4.29)$$

Next, the $\mu = t$ component of the Bianchi identity, Eq. (3.18), leads to

$$\frac{dF_1^{jm}}{dr} = \frac{j(j+1)}{r^2} F_2^{jm}. \quad (4.30)$$

For $j \geq 1$ this can be used to eliminate F_2^{jm} , while for $j = 0$ it implies that F_1^{00} is a constant. Since a constant F_1^{00} corresponds to a change in the magnetic charge, we set $F_1^{00} = 0$.

Finally, Eq. (4.26), together with Eqs. (4.20) and (4.30), yields

$$\frac{d}{dr} \left(B \frac{dF_1^{jm}}{dr} \right) - \frac{j(j+1)}{r^2} F_1^{jm} = -\frac{ega^2}{2mr^2} j(j+1) f^2 \sigma_{jm}. \quad (4.31)$$

It will be convenient to write

$$F_1^{jm}(r) = \frac{ega^2}{2} \sigma_{jm} \mathcal{F}_j(r), \quad j > 0, \quad (4.32)$$

where \mathcal{F}_j obeys

$$\frac{d}{dr} \left(B \frac{d\mathcal{F}_j}{dr} \right) - \frac{j(j+1)}{r^2} \mathcal{F}_j = -\frac{j(j+1)}{mr^2} f^2. \quad (4.33)$$

By multiplying this equation by \mathcal{F}_j and then integrating over r from r_H to ∞ , one can show that for $f = 0$ and $j > 0$ the only regular solution is the trivial one $\mathcal{F}_j(r) = 0$. Hence, in the presence of the source one can solve, at least formally, for \mathcal{F}_j by inverting the operator on the left-hand side of Eq. (4.33). To construct the appropriate Green's function we need the two solutions $g_j^-(r)$ and $g_j^+(r)$ of the homogeneous equation that are regular at $r = r_H$ and $r = \infty$, respectively. Using the fact that $B(r)$ tends to unity at large r , one immediately finds that these two solutions behave asymptotically as r^{j+1} and r^{-j} . (Explicit forms for these solutions are given in the Appendix.) If they are normalized so that $r^{-(j+1)}g_j^-(r)$ and $r^jg_j^+(r)$ both tend to unity as $r \rightarrow \infty$, then

$$\mathcal{F}_j(r) = -\frac{j(j+1)}{m} \int_{r_H}^{\infty} dr' G_j^{(F)}(r, r') \left[\frac{f(r')}{r'} \right]^2, \quad (4.34)$$

where

$$G_j^{(F)}(r, r') = -\frac{1}{2j+1} [\theta(r-r')g_j^+(r)g_j^-(r') + \theta(r'-r)g_j^-(r)g_j^+(r')]. \quad (4.35)$$

Note that neither $g_j^+(r)$ nor $g_j^-(r)$ can have any zeros for $r > r_H$ (i.e., in the region where $B(r) > 0$). This fact, together with Eqs. (4.34) and (4.35), implies that $\mathcal{F}_j(r)$ is positive everywhere outside the horizon.

Because $f(r)$ falls exponentially for $r \gg m^{-1}$, the contribution from the first term in the Green's function dominates at large distance and so

$$\mathcal{F}_j \sim \frac{S_j}{r^j}, \quad r \rightarrow \infty \quad (4.36)$$

where

$$S_j = \frac{j(j+1)}{2j+1} \int_{r_H}^{\infty} dr \frac{g_j^-(r)f(r)^2}{mr^2}. \quad (4.37)$$

As a check that this is indeed the proper behavior, note that this implies that the 2^j -pole components of $\delta F_{\theta\phi}$ fall as $1/r^j$, while the unperturbed monopole component of $F_{\theta\phi}$ is independent of r . Since a is of order β/e , the magnetic field perturbations that we have found correspond to magnetic 2^j poles with components equal to the σ_{jm} times quantities of order $\beta^2 r_H^j / e \sim \beta^2 / em^j$.

V. DETERMINATION OF THE k_M AND THE SYMMETRY OF THE SOLUTION

We can now use our results for V_μ and $\delta F_{\mu\nu}$ to determine the k_M . The overall scale of these, measured by

the quantity a that was defined in Eq. (3.9), determines the magnitude of the departure of our solution from the Reissner-Nordström black hole. The relative sizes of the various k_M determine the angular dependence, i.e., the shape, of the solution; for studying these it is convenient to define

$$n_M = \frac{k_M}{a} \quad (5.1)$$

that satisfy

$$\sum_M |n_M|^2 = 1. \quad (5.2)$$

Substitution of the results of the preceding section into Eq. (3.23) gives the quantity

$$I = -\beta^2 ma^2 + \frac{\lambda e^2 ma^4}{2} p \sum_{jm} |\sigma_{jm}|^2 - \frac{e^2 g^2 ma^4}{8} \sum_j q_j \sum_m |\sigma_{jm}|^2 \quad (5.3)$$

whose minimum determines the k_M . In this expression p and q_j denote the positive integrals

$$p = \int dr \frac{f(r)^4}{m^3 r^2} \quad (5.4)$$

and

$$q_j = \int dr \frac{f(r)^2}{m^2 r^2} \mathcal{F}_j(r), \quad (5.5)$$

which are all of order unity.

Equations (4.15) and (4.20) show that the σ_{jm} are homogeneous polynomials of degree 2 in the k_M . This suggests that we rewrite Eq. (5.3) as

$$I = -\beta^2 ma^2 + \frac{e^2 g^2 ma^4}{8} I_1(n_M), \quad (5.6)$$

where

$$I_1(n_M) = \frac{4\lambda}{g^2} p \sum_{j=0}^{2(q-1)} \Psi_j - \sum_{j=1}^{2(q-1)} q_j \Psi_j \quad (5.7)$$

and the rotational scalars

$$\Psi_j = \sum_{m=-j}^j |\sigma_{jm}|^2 \quad (5.8)$$

are homogeneous polynomials of degree 4 in the n_M . Minimization of I requires

$$a = \frac{2\beta}{eg} [I_1(n_M)]^{-1/2}. \quad (5.9)$$

The n_M are determined, up to an ambiguity corresponding to the rotational and global gauge symmetries of the theory, by minimizing I_1 . We begin by considering individually several low values of q .

A. Case (i): $q = 1, \mathcal{J} = 0$

The solution with unit magnetic charge is spherically symmetric (indeed, it is the only case for which spherical symmetry is possible). There is only a single n_M , of unit magnitude, whose phase has no physical significance.

B. Case (ii): $q = 3/2, \mathcal{J} = 1/2$

There are two n_M that form a complex $SU(2) \times U(1)$ doublet, where the former factor refers to spatial rotations and the latter to global phase rotations of the charged fields. It is always possible to find a symmetry transformation that brings such a doublet into the standard form $n_{1/2} = 1, n_{-1/2} = 0$. The solution is axially symmetric in the sense that it is left invariant by a combination of a rotation about the z axis and a global gauge transformation. In particular, all gauge-invariant quantities are manifestly axially symmetric. One finds that

$$r^2 \Phi_\mu^* \Phi^\mu = \frac{a^2}{4\pi} (1 - \cos \theta), \quad (5.10)$$

so that the nonzero σ_{jm} are

$$\sigma_{00} = \frac{1}{\sqrt{4\pi}}, \quad \sigma_{10} = -\frac{1}{\sqrt{12\pi}}. \quad (5.11)$$

The solution has a net magnetic dipole moment that can be attributed to the asymmetric distribution of the magnetic dipole density of the charged vector field.

C. Case (iii): $q = 2, \mathcal{J} = 1$

This case is somewhat less trivial. The three complex n_M are equivalent to a pair of real vectors \mathbf{v} and \mathbf{w} obeying $\mathbf{v} + \mathbf{w}^2 = 1$, with the correspondence being given by

$$n_{\pm 1} = \frac{1}{\sqrt{2}} [\mp (v_x + iw_x) - i(v_y + iw_y)], \quad (5.12)$$

$$n_0 = v_z + iw_z.$$

Using Eqs. (4.21) and (5.8) we find that

$$\begin{aligned} \Psi_0 &= \frac{1}{4\pi}, \\ \Psi_1 &= \frac{3}{4\pi} |\mathbf{v} \times \mathbf{w}|^2, \\ \Psi^2 &= \frac{1}{20\pi} (1 - 3|\mathbf{v} \times \mathbf{w}|^2), \end{aligned} \quad (5.13)$$

and hence that

$$\begin{aligned} I_1 &= \frac{1}{20\pi} \left(\frac{24\lambda}{g^2} p - q_2 \right) \\ &+ \frac{3}{20\pi} \left(\frac{8\lambda}{g^2} p - 5q_1 + q_2 \right) |\mathbf{v} \times \mathbf{w}|^2. \end{aligned} \quad (5.14)$$

The nature of the minimum depends on whether the coefficient of $|\mathbf{v} \times \mathbf{w}|^2$ is positive or negative. In the former case, I_1 is minimized when \mathbf{v} and \mathbf{w} are parallel. By

choosing their direction to be along the z axis and then applying a global phase rotation, we can bring the solution into the form

$$n_0 = 1, \quad n_{\pm 1} = 0. \quad (5.15)$$

It then follows that

$$r^2 \Phi_\mu^* \Phi^\mu = \frac{3a^2}{8\pi} \sin^2 \theta \quad (5.16)$$

and that

$$\sigma_{00} = \frac{1}{\sqrt{4\pi}}, \quad \sigma_{20} = -\frac{1}{\sqrt{20\pi}} \quad (5.17)$$

with all other σ_{jm} vanishing.

If instead the coefficient is negative, then I_1 is minimized when \mathbf{v} and \mathbf{w} are perpendicular and of equal length. Any such solution can be rotated so that $v_x = -w_y = -1/\sqrt{2}$ with all other components vanishing. This gives

$$n_1 = 1, \quad n_0 = n_{-1} = 0 \quad (5.18)$$

and

$$r^2 \Phi_\mu^* \Phi^\mu = \frac{3a^2}{16\pi} (1 - \cos \theta)^2. \quad (5.19)$$

The nonzero σ_{jm} are

$$\sigma_{00} = \frac{1}{\sqrt{4\pi}}, \quad \sigma_{10} = -\frac{\sqrt{3}}{4\sqrt{\pi}}, \quad \sigma_{20} = \frac{1}{4\sqrt{5\pi}}. \quad (5.20)$$

Both solutions are axially symmetric; the former is manifestly invariant under rotation about the z axis, while the latter is invariant if the rotation is supplemented by a global gauge transformation.

D. Case (iv): $q = 3, \mathcal{J} = 2$

For larger q , the minima of I_1 depend on the actual values of the integrals p and q_j , which we can only determine numerically. However, if $4\lambda/g^2$ is sufficiently large, the first term in Eq. (5.7) is dominant and the dependence on the q_j can be ignored to leading order. In fact, one only has to minimize

$$\begin{aligned} \Sigma &= \sum_{jm} |\sigma_{jm}|^2 \\ &= \frac{1}{a^4} \int d\phi d\theta \sin \theta r^4 (\Phi_\mu^* \Phi^\mu)^2. \end{aligned} \quad (5.21)$$

The integral in the second line is a sum of integrals of products of four vector harmonics. Using the explicit expression⁵ given in Eq. (4.8), we obtain

⁵Although these expressions for the vector harmonics are gauge dependent, the result for Σ is gauge independent.

$$\Sigma = \sum_{M_1, M_2, M_3, M_4} A_{M_1, M_2, M_3, M_4} n_{M_1} n_{M_2} n_{M_3}^* n_{M_4}^* , \quad (5.22)$$

where

$$A_{M_1, M_2, M_3, M_4} = \delta_{(M_1+M_2), (M_3+M_4)} \frac{[(2q-1)!]^2 (2q+M_1+M_2-2)! (2q-M_1-M_2-2)!}{4\pi(4q-3)! \sqrt{\prod_{j=1}^4 (q-M_j-1)! (q+M_j-1)!}} . \quad (5.23)$$

Our problem has now been reduced to the minimization of a quartic polynomial in $2\hat{J} + 1 = 5$ complex variables. Even after using the rotational and phase freedom to fix some of these, one is still left with a rather formidable task. We therefore used MATHEMATICA to search for minima, finding a solution that can be rotated into the form

$$n_0 = \frac{1}{\sqrt{2}}, \quad n_{\pm 1} = 0, \quad n_{\pm 2} = \pm \frac{1}{2} . \quad (5.24)$$

From this one finds that the nonzero σ_{jm} are

$$\begin{aligned} \sigma_{00} &= \frac{1}{\sqrt{4\pi}}, \quad \sigma_{3,\pm 2} = -\frac{\sqrt{5}}{4\sqrt{14\pi}}, \\ \sigma_{40} &= \frac{1}{24\sqrt{\pi}}, \quad \sigma_{4,\pm 4} = \frac{\sqrt{70}}{336\sqrt{\pi}} . \end{aligned} \quad (5.25)$$

This solution has no continuous rotational symmetry, although it is invariant under the group of finite rotations that leave the tetrahedron invariant. This is illustrated by Fig. 1, where we present a three-dimensional plot of $r^2 \Phi_\mu^* \Phi^\mu$ as a function of angle. Note that Φ_μ vanishes at the center of each of the faces of the deformed tetrahedron in this figure, in agreement with our previous remark that it should have $2(q-1)$ zeros.

Including the effects of the terms involving the q_j shifts the location of the minimum of I_1 . One can verify that

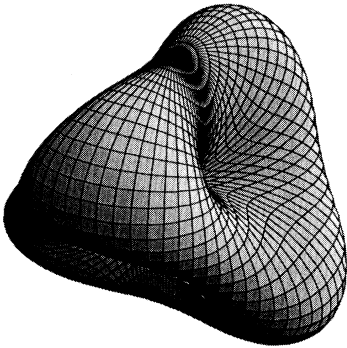


FIG. 1. A three-dimensional spherical plot of the quantity $r^2 \Phi_\mu^* \Phi^\mu$ for the $q = 3$ solution described in Eq. (5.24). If we denote the spherical coordinates of a point as (R, θ, ϕ) , then this plot shows the surface $R(\theta, \phi) = r^2 \Phi_\mu^*(\theta, \phi) \Phi^\mu(\theta, \phi)$. While θ and ϕ represent the corresponding spatial coordinates, R is unrelated to any physical spacetime coordinate. Note the tetrahedral symmetry of the surface.

(in contrast with the $q = 2$ case) the Ψ_j contain terms that are linear in the deviation of the n_M from the values given above. As a result, the full solution for the n_M changes continuously as λ/g^2 is varied.

Since the minimum found here was obtained by numerical methods, we do not have an analytic proof that it is in fact the global minimum (although we are fairly confident that it is.) However, we can demonstrate unambiguously that the global minimum is not axially symmetric. To do this, we note first that any configuration with all but one of the n_M equal to zero is invariant under rotations about the z axis (possibly supplemented by a gauge transformation). By evaluating the σ_{jm} with $m \neq 0$, it is easy to show that these are the only configurations with this symmetry. Explicit calculations for the five configurations of this form shows that they all give higher values for Σ than does the configuration of (5.24). Hence, the global minimum cannot be achieved by a configuration that is axially symmetric about the z axis; the rotational symmetry of the theory then extends this result to an arbitrary axis of rotation.

E. Larger charges

We have applied the methods used for the $q = 3$ case to higher charges also. For $q = 4$ (i.e., $\hat{J} = 3$), the lowest minimum we find for Σ has

$$n_{\pm 2} = \pm \frac{1}{\sqrt{2}} \quad (5.26)$$

with all other n_M vanishing. The nonzero σ_{jm} are

$$\begin{aligned} \sigma_{00} &= \frac{1}{\sqrt{4\pi}}, \\ \sigma_{40} &= -\frac{7}{44\sqrt{\pi}}, \quad \sigma_{4,\pm 4} = \frac{\sqrt{70}}{88\sqrt{\pi}}, \\ \sigma_{60} &= -\frac{\sqrt{13}}{572\sqrt{\pi}}, \quad \sigma_{6,\pm 4} = -\frac{7\sqrt{13}}{572\sqrt{14\pi}} . \end{aligned} \quad (5.27)$$

A three-dimensional plot of $r^2 \Phi_\mu^* \Phi^\mu$ for this solution is shown in Fig. 2. As suggested by the plot, this solution is invariant under the discrete rotational symmetries of the cube. Φ_μ has a zero on each face of this roughly cubic shape.

As we go to higher values of q , the solutions develop more small-scale structure, while at the same time appearing more symmetric when viewed on a large scale.

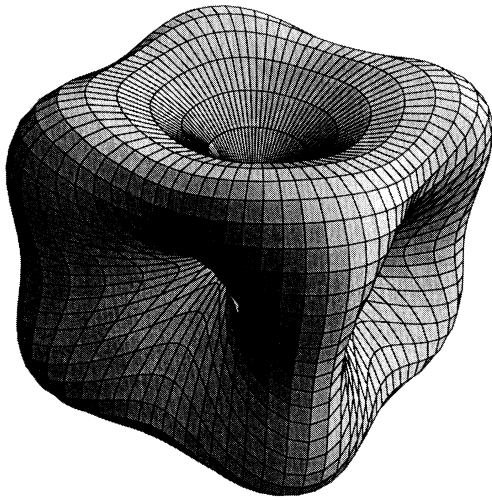


FIG. 2. A spherical plot, similar to that shown in Fig. 1, for the $q = 4$ solution of Eq. (5.26). The cubical symmetry of the solution is apparent.

(Note that a discrete polyhedral symmetry such as that exhibited by the $q = 3$ and $q = 4$ solutions is impossible for most values of q .) What we see happening is that there is a tendency for the $2(q-1)$ zeros of Φ_μ to be distributed as evenly as possible over the unit two-sphere. Lying between these zeros are maxima of $\Phi_\mu^* \Phi^\mu$. This behavior can be seen, for example, in Fig. 3, where we show a solution with $q = 12$.

VI. PERTURBATION OF THE METRIC

The deviation $h_{\mu\nu}$ of the metric from the Reissner-Nordström solution is determined to leading order by the linearized Einstein equation

$$\delta G_{\mu\nu} = -8\pi G t_{\mu\nu}. \quad (6.1)$$

Here $\delta G_{\mu\nu}$ denotes the terms in the Einstein tensor that are linear in $h_{\mu\nu}$ while $t_{\mu\nu}$ is the leading correction to the energy-momentum tensor.

In doing this calculation, we continue to restrict ourselves to the case where Gm^2/e^2 is very small; it was this assumption that allowed us to decouple the determination of $\delta F_{\mu\nu}$ from that of $h_{\mu\nu}$. For our perturbative scheme to be valid, the horizon radius of the unperturbed solution must be close to r_{cr} , and hence must be of order m^{-1} . When this is the case, the difference between the Reissner-Nordström and Schwarzschild metrics with the same value for M is never greater than order Gm^2/e^2 anywhere outside the horizon. Hence, in solving for $h_{\mu\nu}$ we can approximate the metric by the corresponding Schwarzschild metric. We will consider only the case $q \geq 1$, $g > 0$, so that we can use the results for the W field and the electromagnetic perturbations that

were obtained in Secs. IV and V.

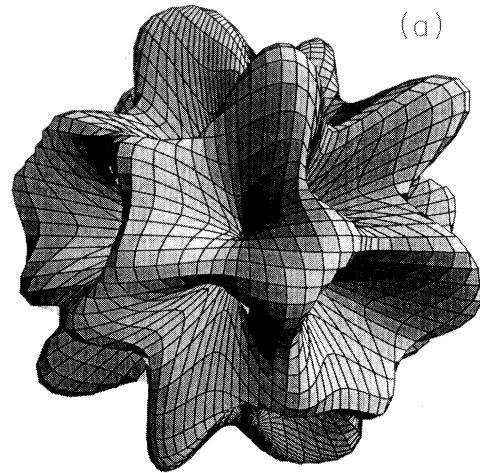
The first step is to calculate $t_{\mu\nu}$. The full energy-momentum tensor can be written as

$$T_{\mu\nu} = T_{\mu\nu}^{\text{EM}} + T_{\mu\nu}^W, \quad (6.2)$$

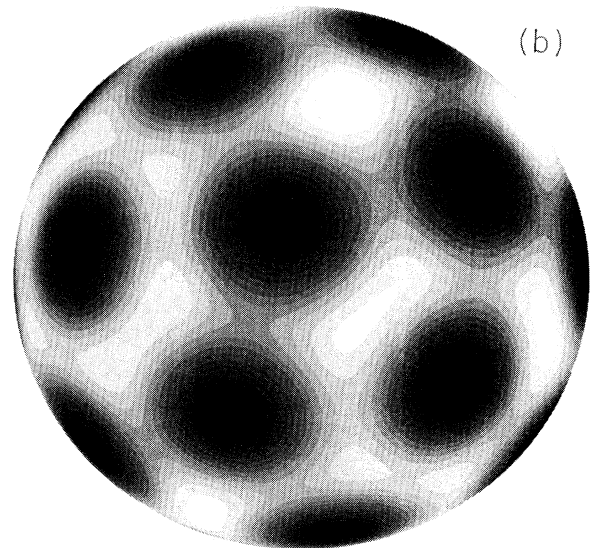
where

$$T_{\mu\nu}^{\text{EM}} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} g^{\lambda\rho} F_{\alpha\lambda} F_{\beta\rho} \quad (6.3)$$

is the purely electromagnetic part and



(a)



(b)

FIG. 3. (a) A spherical plot, similar to those in Figs. 1 and 2, for a solution with $q = 12$. There is no apparent symmetry. (b) Another presentation of the same solution. The value of the function $r^2 \Phi_\mu^* \Phi^\mu$ on the unit hemisphere ($0 \leq \phi \leq \pi$) is represented by one of 16 gray levels, with black being the minimum (zero) with white being the maximum. One can see the fairly even distribution of the zeros.

$$\begin{aligned}
T_{\mu\nu}^W &= g^{\alpha\beta} [W_{\mu\alpha}^* W_{\nu\beta} + W_{\nu\alpha}^* W_{\mu\beta}] + m^2 (W_\mu^* W_\nu + W_\nu^* W_\mu) \\
&\quad + \frac{ieg}{2} g^{\alpha\beta} [F_{\mu\alpha} (W_\nu^* W_\beta - W_\beta^* W_\nu) + F_{\nu\alpha} (W_\mu^* W_\beta - W_\beta^* W_\mu)] \\
&\quad - g_{\mu\nu} \left[\frac{1}{2} W_{\mu\nu}^* W^{\mu\nu} + m^2 W_\mu^* W^\mu + \frac{ieg}{4} F^{\mu\nu} (W_\mu^* W_\nu - W_\nu^* W_\mu) \right] + O(W^4) .
\end{aligned} \tag{6.4}$$

The $O(a^2)$ corrections to $T_{\mu\nu}^{\text{EM}}$ are the sum of a part linear in $\delta F_{\mu\nu}$ and a part linear in $h_{\mu\nu}$ that arises from the corrections to the metric in Eq. (6.3); because the latter is suppressed by an additional factor of $Gm^2 a^2$, we can ignore it here. The dominant part of $T_{\mu\nu}^W$, which is also $O(a^2)$, is obtained by substituting the unperturbed metric and field strength into Eq. (6.4). Using Eq. (1.5) for the only nonzero component of the unperturbed electromagnetic field strength, together with the results of Sec. IV, we find that the nonzero components of $t_{\mu\nu}$ can be written as

$$\begin{aligned}
t_{tt} &= B \left\{ K_{F1} + \left[B(f')^2 + \left(m^2 - \frac{qg}{2r^2} \right) f^2 \right] K_W \right\} , \\
t_{rr} &= -\frac{1}{B} \left\{ K_{F1} + \left[-B(f')^2 + \left(m^2 - \frac{qg}{2r^2} \right) f^2 \right] K_W \right\} , \\
t_{ab} &= \bar{g}_{ab} \left[K_{F1} - \frac{qg}{2r^2} f^2 K_W \right] , \\
t_{ra} &= K_{F2} ,
\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}
K_W &= \frac{1}{mr^2} (r^2 \Phi_\alpha^* \Phi^\alpha) = \frac{a^2}{mr^2} \sum_{jm} \sigma_{jm} Y_{jm}(\theta, \phi) , \\
K_{F1} &= \frac{q}{er^4} \sum_{jm} F_1^{jm} Y_{jm} = \frac{qga^2}{2r^4} \sum_{jm} \mathcal{F}_j \sigma_{jm} Y_{jm} , \\
K_{F2} &= -\frac{q}{er^4} \sum_{jm} F_2^{jm} \partial_a Y_{jm} = -\frac{qga^2}{2r^2} \sum_{jm} \frac{1}{j(j+1)} \mathcal{F}'_j \sigma_{jm} \partial_a Y_{jm} .
\end{aligned} \tag{6.6}$$

[In the last two lines we have used Eqs. (4.30) and (4.32) to relate the perturbations of the field strengths to those of W_μ ; it should be recalled that $\mathcal{F}_0 = 0$.]

The next step is to expand the components of $h_{\mu\nu}$ in terms of spherical harmonics. The space-space components of $h_{\mu\nu}$ can be decomposed into a spin-0 field and a spin-2 field, the time-time component corresponds to a spin-0 field, and the time-space components can be chosen to vanish because the solution is static. Thus, there are potentially seven functions of r entering this expansion for each value of j and m . However, examination of the parity of $t_{\mu\nu}^W$ and $t_{\mu\nu}^F$ shows that it is sufficient to consider only those terms corresponding to perturbations of parity $(-1)^j$. (In the terminology of Ref. [13], these are polar perturbations.) In general, this leaves only five modes for each value of j and m : two, with $l = j$, for the spin-0 fields, and three, with $l = j - 2, j$, and $j + 2$, for the spin-2 field. Hence, there are five radial functions, which we define by

$$\begin{aligned}
h_{tt} &= B(r) \sum_{jm} H_1^{jm}(r) Y_{jm}(\theta, \phi) , \\
h_{rr} &= \frac{1}{B(r)} \sum_{jm} H_2^{jm}(r) Y_{jm}(\theta, \phi) , \\
h_{ab} &= \sum_{jm} [\bar{g}_{ab} H_3^{jm}(r) Y_{jm}(\theta, \phi) \\
&\quad + H_4^{jm}(r) \nabla_a \nabla_b Y_{jm}(\theta, \phi)] , \\
h_{ra} &= \sum_{jm} H_5^{jm}(r) \nabla_a Y_{jm}(\theta, \phi) ,
\end{aligned} \tag{6.7}$$

where ∇_μ denotes the generally covariant derivative with respect to the unperturbed metric. For $j = 1$ there is no mode with $l = j - 2$, and so there should be only four radial functions. Indeed, the identity

$$\nabla_a \nabla_b Y_{1M}(\theta, \phi) = -\frac{\bar{g}_{ab}}{r^2} Y_{1M}(\theta, \phi) \tag{6.8}$$

shows that the H_4^{1m} is redundant and can be set equal to zero. Similarly, there should be only three radial functions for $j = 0$, and the fact that Y_{00} is a constant allows us to set $H_4^{00} = H_5^{00} = 0$.

Further simplification can be achieved by utilizing the freedom to perform coordinate transformations, which change the metric by an amount

$$\delta_G g_{\mu\nu} = \nabla_\mu e_\nu + \nabla_\nu e_\mu . \tag{6.9}$$

Writing

$$\begin{aligned}
e_t &= 0 , \\
e_r &= \sum_{jm} f_1^{jm}(r) Y_{jm}(\theta, \phi) , \\
e_a &= \sum_{jm} f_2^{jm}(r) \nabla_a Y_{jm}(\theta, \phi) ,
\end{aligned} \tag{6.10}$$

we find that

$$\begin{aligned}
\delta_G H_1^{jm} &= -B' f_1^{jm}, \\
\delta_G H_2^{jm} &= 2B f_1'^{jm} + B' f_1^{jm}, \\
\delta_G H_3^{jm} &= \frac{2B}{r} f_1^{jm}, \\
\delta_G H_4^{jm} &= 2f_2^{jm}, \\
\delta_G H_5^{jm} &= f_1^{jm} + f_2'^{jm} - \frac{2}{r} f_2^{jm}.
\end{aligned} \tag{6.11}$$

For $j \geq 2$, we choose f_2^{jm} so that $H_4^{jm} = 0$ and then f_1^{jm} so that $H_5^{jm} = 0$. For $j = 1$, we choose f_1^{1m} so that $H_1^{1m} = H_2^{1m}$ and then choose f_2^{1m} so that $H_5^{1m} = 0$. Finally, for $j = 0$ we choose f_1^{00} (the only coordinate freedom variable) to set $H_3^{00} = 0$.

We next note that t_{ab} is proportional to \bar{g}_{ab} , even though it could in principle have also contained terms proportional to $\nabla_a \nabla_b Y_{JM}(\theta, \phi)$. The absence of such terms implies that

$$\begin{aligned}
\sin^2 \theta \delta G_{\theta\theta} - \delta G_{\phi\phi} &= \frac{1}{2} \sum_{jm} (H_2^{jm} - H_1^{jm}) \\
&\quad \times (\sin^2 \theta \nabla_\theta \nabla_\theta - \nabla_\phi \nabla_\phi) Y_{jm} \\
&= 0,
\end{aligned} \tag{6.12}$$

from which it follows that

$$H_1^{jm} = H_2^{jm}, \quad j \geq 2. \tag{6.13}$$

This leaves only two independent radial functions for each value of j and m and gives us a metric of the form

$$\begin{aligned}
g_{tt} &= B(r) \left[-1 + \sum_{j=0} \sum_m H_1^{jm}(r) Y_{jm}(\theta, \phi) \right], \\
g_{rr} &= \frac{1}{B(r)} \left[1 + H_2^{00}(r) Y_{00} \right. \\
&\quad \left. + \sum_{j=1} \sum_m H_1^{jm}(r) Y_{jm}(\theta, \phi) \right], \\
g_{ab} &= \bar{g}_{ab} \left[1 + \sum_{j=1} \sum_m H_3^{jm}(r) Y_{jm}(\theta, \phi) \right].
\end{aligned} \tag{6.14}$$

All of the H_a^{jm} must vanish as $r \rightarrow \infty$. At the horizon, the tt and ab components of the metric, as well as its determinant, are nonsingular. We require the same of the perturbed metric, and hence require that BH_1^{jm} , $H_2^{00} - H_1^{00}$, and H_3^{jm} all be nonsingular at $r = r_H$. We will see that the field equations place further restrictions on the behavior near the horizon.

Differential equations for the various H_a^{jm} are obtained by expanding both sides of the linearized Einstein equation (6.1) in terms of spherical harmonics. Because of the Bianchi identity, as well as the symmetry of the problem, many components of the resulting equations are redundant. In particular, if we write

$$\begin{aligned}
\delta G_{tt} &= \sum_{jm} \delta G_{tt}^{jm}(r) Y_{jm}(\theta, \phi), \\
\delta G_{rr} &= \sum_{jm} \delta G_{rr}^{jm}(r) Y_{jm}(\theta, \phi), \\
\delta G_{ra} &= \sum_{jm} \delta G_{ra}^{jm}(r) \nabla_a Y_{jm}(\theta, \phi),
\end{aligned} \tag{6.15}$$

then it is sufficient to calculate

$$\begin{aligned}
\delta G_{tt}^{00} &= \frac{B}{r^2} [(1 - rB' - B)(H_1^{00} + H_2^{00}) - H_2^{00} - rB(H_2^{00})'] \\
&= -\frac{B}{r^2} (rBH_2^{00})' + O(Gm^2/e^2), \\
\delta G_{rr}^{00} &= \frac{1}{r^2 B} [rB(H_1^{00})' + H_2^{00}],
\end{aligned}$$

$$\begin{aligned}
\delta G_{rr}^{jm} &= \frac{1}{r^2 B} \left[rB(H_1^{jm})' - \left(rB + \frac{r^2 B'}{2} \right) (H_3^{jm})' + \frac{(j-1)(j+2)}{2} (H_3^{jm} - H_1^{jm}) \right], \quad j \geq 1 \\
\delta G_{ra}^{jm} &= \frac{1}{2} [(H_3^{jm})' - (H_1^{jm})'] - \frac{B'}{2B} H_1^{jm}, \quad j \geq 1.
\end{aligned} \tag{6.16}$$

In the second equality for δG_{tt}^{00} we have used the fact that, with the approximations we are making,

$$1 - rB'(r) - B(r) = O(Gm^2/e^2), \tag{6.17}$$

everywhere outside the horizon; for a Schwarzschild metric the left-hand side of this equation would vanish identically.

We start with the $j = 0$ modes, for which we can use Eq. (4.22). To leading order, the tt component of Eq. (6.1) leads to

$$(rBH_2^{00})' = \frac{8\pi G a^2}{m\sqrt{4\pi}} \left[B(f')^2 + \left(m^2 - \frac{qg}{2r^2} \right) f^2 \right]. \tag{6.18}$$

A second equation is obtained by multiplying the rr equation by $B(r)$ and the tt equation by $1/B(r)$, and then adding

these to give

$$(H_1^{00})' - (H_2^{00})' = -\frac{16\pi G a^2 (f')^2}{\sqrt{4\pi} mr}. \quad (6.19)$$

These two equations can be immediately integrated, using the boundary condition that $H_1^{00}(\infty) = H_2^{00}(\infty) = 0$, to give

$$B(r)H_2^{00}(r) = \frac{2\sqrt{4\pi}G\delta M}{r} - \frac{2\sqrt{4\pi}Ga^2}{mr} \int_r^\infty ds \left[B(s)(f'(s))^2 + \left(m^2 - \frac{qg}{2s^2}\right) f^2(s) \right], \quad (6.20)$$

$$B(r)H_1^{00}(r) = \frac{2\sqrt{4\pi}G\delta M}{r} - \frac{2\sqrt{4\pi}Ga^2}{mr} \int_r^\infty ds \left\{ \left[B(s) - \frac{2B(r)r}{s} \right] [f'(s)]^2 + \left(m^2 - \frac{qg}{2s^2}\right) f^2(s) \right\}, \quad (6.21)$$

where δM is an arbitrary constant that may be interpreted as a shift of the black hole mass. Because $f(r)$ falls exponentially fast at large distance, so do the integrals on the right-hand sides of these two equations.

Turning now to the $j \neq 0$ modes, we find that the ra components of Eq. (6.1) give

$$B(H_3^{jm})' - B(H_1^{jm})' - B'H_1^{jm} = 8\pi G a^2 \left[\frac{qg\sigma_{jm}}{j(j+1)} \right] \frac{BF_j'}{r^2}, \quad j \geq 1, \quad (6.22)$$

while the rr component leads to

$$\begin{aligned} rB(H_1^{jm})' - \left(rB + \frac{r^2 B'}{2} \right) (H_3^{jm})' + \frac{(j-1)(j+2)}{2} [H_3^{jm} - H_1^{jm}] \\ = 8\pi G a^2 \sigma_{jm} \left\{ \frac{1}{m} \left[-B(f')^2 + \left(m^2 - \frac{qg}{2r^2}\right) f^2 \right] + \frac{qg}{2r^2} \mathcal{F}_j \right\}, \quad j \geq 1. \end{aligned} \quad (6.23)$$

It was noted above that the nonsingularity of g_{tt} required that BH_1^{jm} be regular at $r = r_H$; this leaves the possibility that H_1^{jm} might be singular there. This can be ruled out by multiplying Eq. (6.22) by r and then adding the result to Eq. (6.23). The resulting equation can be solved to express H_1^{jm} in terms of H_3^{jm} and $(H_3^{jm})'$, thus showing that it is regular at the horizon. Hence, $(H_1^{jm})'$ is finite at r_H . Using this fact in Eq. (6.22), we find that

$$H_1^{jm}(r_H) = 0, \quad j \geq 1. \quad (6.24)$$

For $j = 1$, the fact that H_3^{jm} only enters Eqs. (6.22) and (6.23) through its derivative simplifies matters considerably. By using Eq. (6.23) to solve for $(H_3^{jm})'$ and then substituting into Eq. (6.22), we obtain a first-order equation involving only H_1^{jm} . This can be easily integrated, and the result then used to obtain H_3^{jm} . The two constants of integration are fixed by the boundary conditions that $H_1^{jm}(r_H) = H_3^{jm}(\infty) = 0$. The result is that

$$\begin{aligned} H_1^{1m} = -\frac{8\pi G a^2 \sigma_{1m}}{r^2 B(r)} \int_{r_H}^r ds \frac{B(s)}{B'(s)} \left\{ \frac{qg}{2s} [2B(s) + sB'(s)] \mathcal{F}_1'(s) + \frac{qg}{s^2} \mathcal{F}_1(s) \right. \\ \left. + \frac{2}{m} \left[-B(s)(f'(s))^2 + \left(m^2 - \frac{qg}{s^2}\right) f^2 \right] \right\} \end{aligned} \quad (6.25)$$

$$\begin{aligned} H_3^{1m} = \int_r^\infty ds \left\{ \frac{2}{s} H_1^{1m}(s) + \frac{8\pi G a^2 \sigma_{1m}}{sB'(s)} \left[\frac{qg}{s^3} (sB(s)\mathcal{F}_1'(s) + \mathcal{F}_1(s)) \right. \right. \\ \left. \left. + \frac{2}{ms} \left[-B(s)(f'(s))^2 + \left(m^2 - \frac{qg}{s^2}\right) f^2 \right] \right] \right\}. \end{aligned} \quad (6.26)$$

To obtain the asymptotic behavior of these expressions, we recall that $f(r)$ vanishes exponentially fast at large r , while $\mathcal{F}_1 \sim 1/r$. This behavior is just sufficient to guarantee the convergence of the integral in Eq. (6.25),

with the result that $H_1^{1m} \sim 1/r^2$. Inserting this into Eq. (6.26) and again using the asymptotic behavior of \mathcal{F}_j , we find that H_3^{1m} has the same asymptotic behavior, and in fact that the difference $H_1^{1m} - H_3^{1m} \sim 1/r^3$.

In solving for the modes with $j \geq 2$, it is useful to define

$$T_{jm} = H_1^{jm} - H_3^{jm} \quad (6.27)$$

and to then rewrite Eq. (6.22) as

$$H_1^{jm} = -\frac{B}{B'} \left[T_{jm}' + 8\pi G a^2 \left(\frac{qg\sigma_{jm}}{j(j+1)} \right) \frac{\mathcal{F}'_j}{r^2} \right]. \quad (6.28)$$

Substitution of this equation and its derivative into Eq. (6.23) leads to a second-order equation involving only T_{jm} :

$$\begin{aligned} r^2 B T_{jm}'' + 2(r^2 B)' T_{jm}' - (j-1)(j+2) T_{jm} \\ = \frac{16\pi G a^2 \sigma_{jm}}{m} [-B(f')^2 + m^2 f^2]. \end{aligned} \quad (6.29)$$

[In obtaining this result, both Eqs. (4.33) and (6.17) have been used, and terms of higher order in Gm^2/e^2 have been dropped.]

In the absence of the source term there are no nontrivial solutions for T_{jm} that are regular at both the horizon and spatial infinity. (This can be readily seen by examining the explicit solutions given in the Appendix.) Hence, the inhomogeneous Eq. (6.29) can be solved by Green's function methods similar to those used for Eq. (4.33). Let $h_j^-(r)$ and $h_j^+(r)$ be the solutions of the homogeneous equation that are regular at $r = r_H$ and $r = \infty$, respectively. At large r these behave asymptotically as r^{j-1} and $r^{-(j+2)}$. If they are normalized so that $r^{-(j-1)} h_j^-(r)$ and $r^{j+2} h_j^+(r)$ both tend to unity as $r \rightarrow \infty$, then the desired Green's function is

$$\begin{aligned} G_j^{(T)}(r, r') = -\frac{1}{2j+1} [\theta(r-r') h_j^+(r) h_j^-(r') \\ + \theta(r'-r) h_j^-(r) h_j^+(r')] \end{aligned} \quad (6.30)$$

and the only regular solution for T_{jm} is

$$\begin{aligned} T_{jm}(r) = \frac{16\pi G a^2 \sigma_{jm}}{m} \\ \times \int_{r_H}^{\infty} dr' G_j^{(T)}(r, r') (r')^2 B(r') \\ \times \{-B(r') [f'(r')]^2 + m^2 [f(r')]^2\}. \end{aligned} \quad (6.31)$$

H_1^{jm} can be obtained immediately from this equation together with Eqs. (4.34) and (6.28).

At large r the source term in Eq. (6.29) is exponentially small, so the first term in the Green's function dominates Eq. (6.31); thus

$$T_{jm} \sim \sigma_{jm} \frac{C_j}{r^{j+2}}, \quad r \rightarrow \infty \quad (6.32)$$

where

$$C_j = \frac{16\pi G a^2}{(2j+1)m} \int_{r_H}^{\infty} dr r^2 B h_j^- [B(f')^2 - m^2 f^2]. \quad (6.33)$$

By substituting this result back into Eq. (6.28) and using Eq. (4.36) we obtain the asymptotic behavior

$$H_1^{jm} \sim \frac{8\pi G a^2 \sigma_{jm}}{(2j+1)m r_H} \frac{A_j}{r^{j+1}}, \quad r \rightarrow \infty \quad (6.34)$$

where

$$\begin{aligned} A_j = \int_{r_H}^{\infty} dr \left\{ 2(j+2)r^2 B h_j^- [B(f')^2 - m^2 f^2] \right. \\ \left. + qg j \frac{g_j^- f^2}{r^2} \right\}. \end{aligned} \quad (6.35)$$

The resulting large distance behavior of g_{tt} corresponds to that one would obtain from a mass distribution with a 2^j -pole moment whose components are equal to the σ_{jm} times quantities of order $\beta^2 r_H^{j-1}/e^2 \sim \beta^2/e^2 m^{j-1}$.

VII. CONCLUDING REMARKS

In this paper we have exhibited black hole solutions with fields on the horizon that, contrary to common expectation, are not spherically symmetric. As the magnetic charge increases, the structure of these black holes becomes more detailed, with higher multipole components appearing in the long-range electromagnetic and gravitational fields. At the same time, the lower multipole moments decrease in magnitude, with the result that the solutions do begin to approach the expected spherical symmetry, if only in an averaged sense.

Although these solutions display some unusual, and perhaps unexpected, properties, there are rigorous general results on black holes that they must obey. We consider here two of these, the zeroth and second laws of black hole dynamics, that are concerned with the properties of the black hole horizon. At constant t , the horizon can be described as the two-dimensional surface

$$r = r_H + \Delta(\theta, \phi), \quad (7.1)$$

where r_H is the horizon radius of the unperturbed metric. To leading order, the vanishing of g_{tt} on this surface gives

$$\Delta(\theta, \phi) = \frac{B'(r_H)}{B(r_H)} \sum_{jm} H_1^{jm}(r_H) Y_{jm}(\theta, \phi). \quad (7.2)$$

Equation (6.24) implies that the terms in the sum with $j \geq 1$ all vanish, so that Δ is a constant independent of angle. Using the solution in Eq. (6.21) for H_1^{00} (with $\delta M = 0$), we find

$$\begin{aligned} \Delta = -\frac{2G a^2}{m r_H B'(r_H)} \int_{r_H}^{\infty} ds [B(f')^2 + (m^2 - \frac{qg}{s^2}) f^2] \\ = 2a^2 \beta^2 G m \end{aligned} \quad (7.3)$$

where the second equality follows from the eigenvalue and normalization Eqs. (4.10) and (4.11), as well as Eq. (6.17). Because the integral in the first line is it-

self of order β^2 , our result for Δ is proportional to β^4 , rather than the β^2 one might have expected. Since the higher order corrections to W_μ and $\delta F_{\mu\nu}$ could also shift the horizon by a distance of order β^4 , our result is really that Δ vanishes to leading order.

The zeroth law of black hole dynamics states that the surface gravity, which corresponds quantum mechanically to the black hole temperature, is constant over the horizon. For a stationary black hole the surface gravity κ is given by [14]

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu) \quad (7.4)$$

where the right-hand side is to be evaluated on the horizon and χ_μ is a Killing vector that is orthogonal to the horizon. If the metric is actually static, and is written in a manifestly t -independent form with vanishing time-space components $g_{tj} = 0$, then the only nonzero component of this Killing vector is $\chi_t = g_{tt}$ and Eq. (7.4) reduces to

$$\kappa^2 = -\frac{1}{4}g^{tt}g^{ij}(\partial_i g_{tt})(\partial_j g_{tt}) . \quad (7.5)$$

Expanding this equation to first order in $h_{\mu\nu}$ and taking into account the fact that the horizon has been shifted by an amount Δ gives

$$\begin{aligned} \kappa &= \frac{B'}{2} + \frac{B''\Delta}{2} - \partial_r h_{tt} + \frac{B'}{2}(B^{-1}h_{tt} - Bh_{rr}) \\ &= \frac{B'}{2} + \frac{B''\Delta}{2} - \frac{1}{2}\sum_{jm}(BH_1^{jm})'Y_{jm} + \frac{B'}{4}Y_{00}(H_1^{00} - H_2^{00}) , \end{aligned} \quad (7.6)$$

where all quantities are to be evaluated at $r = r_H$. Because of Eq. (6.24), all terms in the sum with $j \geq 1$ vanish. Hence, κ is independent of angle and therefore constant over the horizon, as required.

The second law of black hole dynamics is the statement that the area of a black hole horizon never decreases. Let us apply this to the case of an unstable Reissner-Nordström black hole that is perturbed and eventually evolves into a static black hole with hair. By making the perturbation sufficiently weak, we can arrange that the mass decrease from radiation be negligible, so that the final state will be a black hole with hair that has the same value for M as the original Reissner-Nordström solution. (Indeed, the area law can be used to place an upper limit on the mass loss from radiation.) The area of its horizon is

$$A = \int d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \quad (7.7)$$

where the integration is over the surface $r = r_H + \Delta$. The angular components h_{ab} of the metric perturbation only have terms involving spherical harmonics with $j \geq 1$. The contributions linear in these vanish after the integration over angles, and so for calculating the leading correction to the area we can replace g_{ab} by the unperturbed metric \bar{g}_{ab} and obtain

$$A = 4\pi(r_H^2 + 2r_H\Delta) . \quad (7.8)$$

From this, together with Eq. (7.3), we see that the second law is verified, to leading order.

The methods we have used to construct our solutions have limited us to the case where the horizon radius is close to the critical radius for the instability of the Reissner-Nordström solution. However, there seems to be every reason to expect that the solutions with smaller horizons will display similar behavior. The construction and study of such solutions, which include new extremal black holes, remains an open problem.

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APPENDIX

The Green's functions used to solve Eqs. (4.33) and (6.29) were constructed from the solutions of the corresponding homogeneous solutions. In the former case, the solutions obey

$$B(r)g''(r) + B'(r)g'(r) - \frac{j(j+1)}{r^2}g(r) = 0 . \quad (A1)$$

As we explain in Sec. VI, in the approximation to which we are working $B(r)$ may be replaced by the Schwarzschild metric function $1 - r_H/r$. It is convenient to define a variable $x = r/r_H$ and rewrite Eq. (A1) is

$$(x^2 - x)\frac{d^2g}{dx^2} + \frac{dg}{dx} - j(j+1)g = 0 . \quad (A2)$$

This has a polynomial solution of the form

$$g_j^-(x) = \frac{(j-1)!(j+1)!}{(2j)!}x^2P_{j-1}^{(0,2)}(2x-1) \quad (A3)$$

where $P_{j-1}^{(0,2)}(z)$ is a Jacobi polynomial [15], and the normalization has been chosen so that $x^{-(j+1)}g_j^-(x)$ tends to unity as $x \rightarrow \infty$. Its value at $x = 1$ (i.e., the horizon) is

$$g_j^-(1) = \frac{(j-1)!(j+1)!}{(2j)!} . \quad (A4)$$

To find the other independent solution, we first note that

any two solutions g and f of Eq. (A1) obey

$$B(r)[g'(r)f(r) - g(r)f'(r)] = c, \quad (\text{A5})$$

where c is a constant. In particular, if we write

$$g_j^+(x) = g_j^-(x) \ln \left(1 - \frac{1}{x} \right) + k(x), \quad (\text{A6})$$

then $k(x)$ obeys

$$\left(\frac{x-1}{x} \right) \left[g_j^- \frac{dk}{dx} - k \frac{dg_j^-}{dx} \right] + \frac{(g_j^-)^2}{x^2} = c. \quad (\text{A7})$$

From the fact that $g_j^-(x)$ is equal to x^2 times a polynomial of order $(j-1)$, it follows that this last equation has a solution for $k(x)$ as a polynomial of order j .

Explicit forms for low values of j are

$$\begin{aligned} g_1^-(x) &= x^2, \\ g_2^-(x) &= x^2 \left(x - \frac{3}{4} \right), \\ g_3^-(x) &= x^2 \left(x^2 - \frac{4}{3}x + \frac{2}{5} \right), \end{aligned} \quad (\text{A8})$$

and

$$\begin{aligned} g_1^+(x) &= -3x^2 \ln \left(1 - \frac{1}{x} \right) - 3x - \frac{3}{2}, \\ g_2^+(x) &= -80x^2 \left(x - \frac{3}{4} \right) \ln \left(1 - \frac{1}{x} \right) - 80x^2 + 20x \\ &\quad + \frac{10}{3}, \\ g_3^+(x) &= -1575x^2 \left(x^2 - \frac{4}{3}x + \frac{2}{5} \right) \ln \left(1 - \frac{1}{x} \right) - 1575x^3 \\ &\quad + \frac{2625}{2}x^2 - 105x - \frac{35}{4}, \end{aligned} \quad (\text{A9})$$

where the $g_j^+(x)$ have been normalized so that $x^j g_j^+(x)$ tends to unity as $x \rightarrow \infty$.

With the same approximation for B , the homogeneous

equation corresponding to Eq. (6.29) becomes

$$(x^2 - x) \frac{d^2 h}{dx^2} + (4x - 2) \frac{dh}{dx} - (j-1)(j+2)h = 0. \quad (\text{A10})$$

This has a polynomial solution of the form

$$h_j^-(x) = \frac{(j-1)!(j+1)!}{(2j)!} P_{j-1}^{(1,1)}(2x-1) \quad (\text{A11})$$

whose value at the horizon is

$$h_j^-(1) = \frac{j!(j+1)!}{(2j)!}. \quad (\text{A12})$$

By methods similar to those used to find $g_j^+(x)$, one finds that the solution that is regular as $x \rightarrow \infty$ is of the form

$$h_j^+(x) = h_j^-(x) \ln \left(1 - \frac{1}{x} \right) + \frac{l(x)}{x(x-1)}, \quad (\text{A13})$$

where $l(x)$ is a j th order polynomial.

Explicit forms for low values of j are

$$h_2^-(x) = x - \frac{1}{2}, \quad (\text{A14})$$

$$h_3^-(x) = x^2 - x + \frac{1}{5},$$

and

$$h_2^+(x) = 60(2x-1) \ln \left(1 - \frac{1}{x} \right) + 120 - \frac{10}{x(x-1)}, \quad (\text{A15})$$

$$\begin{aligned} h_3^+(x) &= 35 \left\{ 12(5x^2 - 5x + 1) \ln \left(1 - \frac{1}{x} \right) \right. \\ &\quad \left. + 60x - 30 + \frac{2x-1}{x(x-1)} \right\}, \end{aligned}$$

where the normalization conventions are analogous to those used for $g_j^\pm(x)$.

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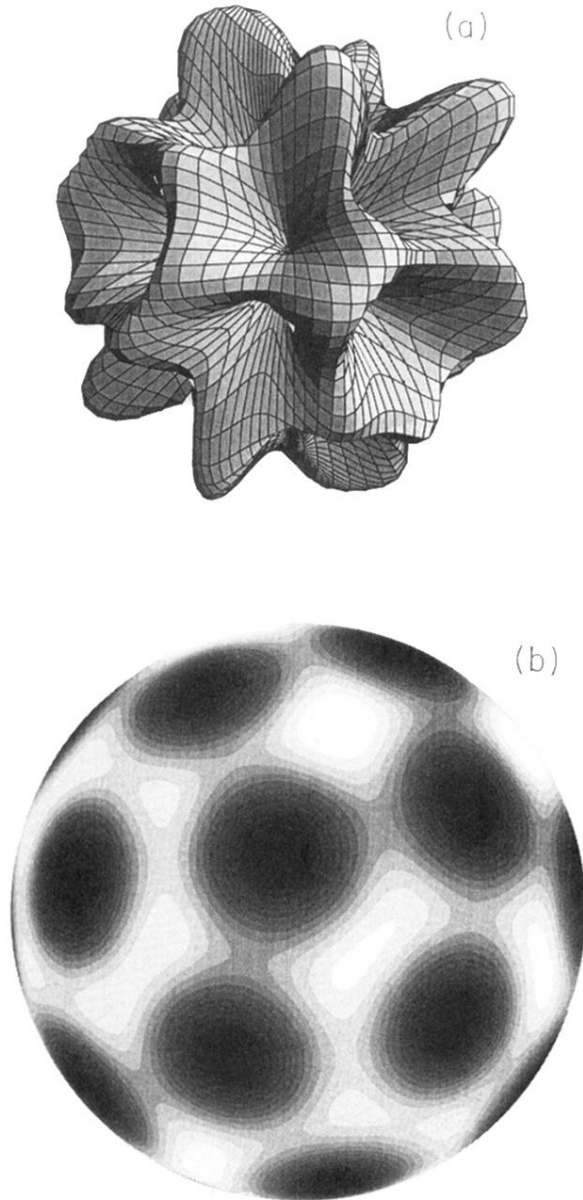


FIG. 3. (a) A spherical plot, similar to those in Figs. 1 and 2, for a solution with $q = 12$. There is no apparent symmetry. (b) Another presentation of the same solution. The value of the function $r^2 \Phi_\mu^* \Phi^\mu$ on the unit hemisphere ($0 \leq \phi \leq \pi$) is represented by one of 16 gray levels, with black being the minimum (zero) with white being the maximum. One can see the fairly even distribution of the zeros.