

Time-symmetric initial data sets in four-dimensional dilaton gravity

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I study the time-symmetric initial-data problem in theories with a massless scalar field (dilaton), free or coupled to a Maxwell field in the stringy way, finding different initial-data sets describing an arbitrary number of black holes with arbitrary masses, charges, and asymptotic value of the dilaton. The presence of the scalar field gives rise to a number of interesting effects. The mass and charges of a single black hole are different in its two asymptotically flat regions across the Einstein-Rosen bridge. The same happens to the value of the dilaton at infinity. This forbids the identification of these asymptotic regions in order to build (Misner) wormholes in the most naive way. Using different techniques, I find regular initial data for stringy wormholes. The price paid is the existence singularities in the dilaton field. The presence of a single-valued scalar seems to constrain strongly the allowed topologies of the initial spacelike surface. Other kinds of scalar fields (taking values on a circle or being defined up to an additive constant) are also briefly considered.

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I. INTRODUCTION

The usual procedure of getting exact solutions in general relativity (i.e., imposing some symmetries on the solutions, substituting an appropriate ansatz, and solving the differential equations) has an important drawback: One does not know what physical system a solution is going to describe until one actually gets it. In the simplest cases one can expect a black hole solution, a cosmological solution, etc., but in more complex cases, following that recipe, one might never find a solution describing the evolution of the system one is interested in.

If one wants to describe the time evolution of a system consisting, for example, of two black holes subjected to their mutual attraction, the initial-value formulation of the Einstein equations¹ is far more appropriate. Obtaining exact (complete) solutions is still probably hopeless but, at least, if one has the right initial data, one knows which system one is working with and one knows that there exists such a solution. Then it makes sense to use numerical methods to evolve the initial data. Many interesting results have been obtained in this way.

The problem of finding the *right* initial data remains, but it is a much more tractable one. In general relativity (as in any other theory with gauge freedom) the initial data cannot be chosen arbitrarily but have to satisfy certain constraints. Solving these constraints is what is called the *initial-data problem*. Solving the initial-data problem is interesting, not only to get something whose evolution can be studied. An initial-data set contains a great deal of information about the system since it already solves part of the Einstein equations (the constraints).

My goal in this paper is to find initial-data sets which solve a simple case of the initial-data problem: the time-symmetric case [2, 3]. The solutions to the time-symmetric initial-data problem will describe an arbitrary number of nonrotating black holes which are momentarily at rest, that is, at the moment at which they “bounce.” I will also look for (Misner) wormhole initial data.

Solutions to this problem are already known for the vacuum and electrovacuum cases [2, 4–6]. Here I will work with two different theories with a scalar field (dilaton): Einstein plus a dilaton (also called the Einstein-Higgs system in the literature) and Einstein-Maxwell plus a dilaton which couples to the Maxwell field in the stringy way. In this paper I present solutions analogous to those in Refs. [2, 4, 5] for the former cases, with a number of peculiar features. Perhaps the most important one is that in most of these solutions the dilaton field has different asymptotic values in different asymptotic regions. Since the zero mode of the dilaton is physically meaningful (in string theory it is the coupling constant), those regions are physically different universes.

Another consequence of the existence of different asymptotic values of the dilaton arises when one tries to find wormhole initial data. Naively, one would build a wormhole by identifying two different asymptotic regions linked by an Einstein-Rosen bridge [7] and the fields defined in them. In this case one would get a multivalued dilaton field which is unacceptable unless, for some reason, it is assumed that the scalar field takes values in a circle or its zero mode has no physical meaning; it is “pure gauge.” Solutions of this kind will be presented in this paper for the Einstein-Maxwell-dilaton and Einstein-dilaton cases. Although it is extremely hard to find a wormhole solution for the Einstein-Maxwell-dilaton system with a single-valued dilaton, one solution of this kind will also be presented. This solution is not smooth. The string-frame metric is smooth (actually, it is exactly equal to the metric of a Reissner-Nordström wormhole in the

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¹See, for instance, Ref. [1] for a comprehensive presentation whose main lines I will follow in Sec. II.

Einstein frame) but the dilaton field still has many singularities.

One “experimentally” observes that there is no way to build wormhole initial data without introducing at the same time unwanted singularities in the metric, in the dilaton field, or in both, or without assuming that the dilaton field “lives” in a circle or that it is defined up to a constant. One can consider other kinds of scalar fields, different from the string theory dilaton field, with those properties. For them, finding wormhole solutions is easier and the resulting configurations are very interesting. In the last section, I will discuss the role a single-valued scalar field seems to play in limiting the possible topologies of the initial Cauchy surface.

What can be learned from all these solutions? First of all, the mere existence of some of them is interesting *per se*, as I am going to explain. Second, a number of issues can be investigated using them: no-hair theorems, area theorems, cosmic censorship (first studied in Ref. [8]), critical behavior in the gravitational collapse [9, 10] as analyzed in Ref. [11], etc.

The purely scalar case has special interest since no static black-hole solutions with a nontrivial scalar field exist. Different no-hair theorems forbid its existence² If solutions to the initial-data problem describing many black holes with nontrivial scalar hair exist, two processes must take place: First, all their scalar hair must be radiated away to infinity, and second, the black holes must merge. The end point of these processes should be a single black hole of the Kerr family, according to the Carter-Israel conjecture, and, in our case in which there is no angular momentum, a Schwarzschild black hole. The area of its event horizon must be bigger than the sum of those of the black holes one started with, and its mass must be smaller. Then one can compare initial and final states and give bounds on the energy radiated away by the system [8]. This provides a strong test of many ideas widely believed to hold in general relativity. I will not investigate these issues in this paper, though, and I will limit myself to the search and identification of the sought-for initial data, leaving that investigation for further publications.

In the usual Einstein-Maxwell plus dilaton case in which the dilaton is not coupled to the Maxwell field the no-hair theorems state that the only black-hole-type solution is the Reissner-Nordström solution, with trivial (constant, zero charge) dilaton. However, when the dilaton couples to the Maxwell field (that is, in the low-energy string theory case), a nontrivial dilaton field whose charge is determined by the electric charge is required in order to get the dilaton black holes of Ref. [16]. These are the only black-hole-type solutions of this theory according to the unicity theorem of Ref. [17]. In some sense the situation is analogous to that of the purely scalar case: There is a family of solutions for which the

dilaton charge is a free parameter (found in Ref. [12] for the purely scalar case and in Refs. [14, 15] for this case) but only for a determined value of the dilaton charge is the solution not singular and describes a black hole (zero for the purely scalar case and $-e^{-2\phi_0}Q^2/2M$ for this case).

It is reasonable to expect that the end point of a non-rotating low-energy string theory black hole will be a static dilaton black hole and that the dilaton charge will evolve (increasing or decreasing) in such a way that, in the end, it will have the *right* value, no singularities will be present, and cosmic censorship will be enforced. In this process the area must not decrease and the mass must not increase. Observe that, if all the scalar hair disappears, as one expects to happen in the Einstein-dilaton case, the end point would exhibit naked singularities. Why the dilaton behaves so differently and whether it really does or not (violating cosmic censorship) are two important questions that can be addressed in this framework. The initial-data sets that I will present are most suited for investigating these issues.

The structure of this article is the following.

In Sec. II, I set up the time-symmetric initial-data problem for the theories considered in this paper, define the charges and asymptotic values of the fields, and explain some conventions.

In Sec. III, I present the *Ansätze*, substitute them into the constraints, and reduce them to differential equations with well-known solutions.

In Sec. IV, I solve those equations to find black-hole solutions. A technique for generating solutions of the Einstein-Maxwell plus dilaton initial-data problem starting from solutions of the Einstein-Maxwell initial data is developed in order to find the initial data of the known charged dilaton black holes [16].

In Sec. V, I analyze the spherically symmetric black-hole solutions found in the previous section.

In Sec. VI, I solve the equations found in Sec. III to find wormhole initial data. This turns out to be a complicated problem and it will be necessary to use the solution-generating technique developed in Sec. IV to solve it, although the result will not be completely satisfactory for the reason mentioned above.

In Sec. VII, I discuss the results and present the conclusions.

II. THE TIME-SYMMETRIC INITIAL-DATA PROBLEM

A set of initial data for a space-time $(\mathcal{M}, g_{\mu\nu})$ in general relativity consists of (1) a spacelike hypersurface Σ determined by its normal unit³ vector n^μ , $n^2 = +1$, (2) the induced metric on Σ , $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, and (3) the

²The closest to a black hole with scalar hair is the solution given in Ref. [12]. For references on no-hair theorems see, for instance, the recent paper Ref. [13].

³We follow the conventions in Ref. [18]. In particular the signature is $(+---)$, although I will use the signature $(+++)$ for the intrinsic three-metric on Σ . Greek indices go from 0 to 3 and Latin indices from 1 to 3. The dual of the Maxwell tensor is $*F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ with $\epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}} = +i$.

extrinsic curvature $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}$ of Σ .

If, in addition to $g_{\mu\nu}$, there are more fundamental fields, one has to consider also their initial data. Both

$h_{\mu\nu}$ and $K_{\mu\nu}$ can be expressed as the intrinsic metric ${}^{(3)}g_{ij}$ and tensor ${}^{(3)}K_{ij}$. It is also convenient to use the covariant derivative D_μ associated with $h_{\mu\nu}$:

$$D_\mu T_{\nu_1 \dots \nu_n}{}^{\rho_1 \dots \rho_m} \equiv h_\mu{}^\alpha h_{\nu_1}{}^{\beta_1} \dots h_{\nu_n}{}^{\beta_n} h^{\rho_1}{}_{\gamma_1} \dots h^{\rho_m}{}_{\gamma_m} \nabla_\alpha T_{\beta_1 \dots \beta_n}{}^{\gamma_1 \dots \gamma_m}, \quad (1)$$

which is equivalent to the covariant derivative ${}^{(3)}\nabla_i$ associated with the intrinsic metric ${}^{(3)}g_{ij}$.

The projections $n^\mu(G_{\mu\nu} - T_{\mu\nu}) = 0$ of the Einstein equations contain only first time derivatives of the metric. Therefore, they are not dynamical equations, but constraints on the metric and its first time derivatives which have to be satisfied, in particular, by the initial data on Σ . There are two sets of constraints: $n^\mu n^\nu(G_{\mu\nu} - T_{\mu\nu}) = 0$ and $n^\mu h_\rho{}^\nu(G_{\mu\nu} - T_{\mu\nu}) = 0$. Both result in equations for h and K or their intrinsic counterparts.

The reason for the existence of these constraints is the gauge freedom of general relativity. If there are more fields with gauge freedom, some projections of their equations of motion will typically be constraints for their initial data, and the equations for h and K will have to be supplemented with them.

In the special case in which Σ is a surface of time symmetry (i.e., invariant under time reflection with respect to Σ itself [2]) K vanishes and the equations take a very simple form

$${}^{(3)}R - 2n^\mu n^\nu T_{\mu\nu} = 0,$$

$$n^\mu h_\rho{}^\nu T_{\mu\nu} = 0. \quad (2)$$

All the theories I want to consider can be described by different truncations of the action [18]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \{-R + 2(\partial\phi)^2 - e^{-2\phi} F^2\}, \quad (3)$$

which is itself a truncation of the low-energy string theory effective action in four dimensions written in the Einstein frame. I will consider different cases: the full theory, the Einstein-Maxwell case by setting $\phi = 0$ everywhere and ignoring its equation of motion, the purely scalar case by setting $F_{\mu\nu} = 0$ and ignoring its equations of motion, and, in the obvious way, the vacuum case. I will not consider the Einstein-Maxwell plus uncoupled dilaton case.

In string theory, the dilaton ϕ is a scalar field that takes values in R . This is the kind of scalar field I will work with, unless I explicitly indicate otherwise. However, in some situations, it will be interesting to consider different scalar fields, taking values in S^1 or defined up to an additive constant.

Sometimes I will also be interested in the string-frame metric

$$g_{\mu\nu}^{\text{string}} = e^{2\phi} g_{\mu\nu} \quad (4)$$

or other metrics that can be obtained by rescaling the Einstein-frame metric with certain powers of e^ϕ .

A complete initial-data set for the theory described by the action Eq. (3) consists of the initial Cauchy surface Σ , its induced metric $h_{\mu\nu}$, and its extrinsic curvature $K_{\mu\nu}$, the values of ϕ and $n^\mu \nabla_\mu \phi$ on Σ , and the electric and magnetic fields on Σ , defined by

$$E_\mu = n^\nu F_{\nu\mu}, \quad B_\mu = -i n^\nu {}^* F_{\nu\mu}. \quad (5)$$

If Σ is a surface of time symmetry, then

$$n^\mu \nabla_\mu \phi = 0 \quad (6)$$

on Σ .

Now, what it is needed to study the initial-data problem of the theory given by the action Eq. (3) is the following.

(i) The energy-momentum tensor, which is most conveniently written as

$$T_{\mu\nu} = e^{-2\phi} [F_{\mu\rho} F_{\nu}{}^\rho - {}^* F_{\mu\rho} {}^* F_{\nu}{}^\rho] - 2[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2], \quad (7)$$

and which has to be substituted in Eqs. (2).

(ii) The equations of motion of ϕ and $F_{\mu\nu}$ to find the possible constraints contained in them. These are

$$\nabla^2 \phi - \frac{1}{2} e^{-2\phi} F^2 = 0, \quad (8)$$

$$\nabla_\mu (e^{-2\phi} F^{\mu\nu}) = 0, \quad (9)$$

$$\nabla_\mu {}^* F^{\mu\nu} = 0. \quad (10)$$

The equation of motion of the dilaton, Eq. (8), leads to no constraint on Σ . This was expected since there is no gauge invariance associated with ϕ , which is a fundamental physical field⁴. We have included the Bianchi identity

⁴It should be stressed that the zero mode of the dilaton is physically meaningful. The symmetry $\phi \rightarrow \phi' = \phi + \text{const}$ of the action Eq. (3) means that given a solution of its equations of motion, another solution can be obtained by shifting the value of the dilaton zero mode. However, this is not a gauge symmetry and both solutions have to be regarded as physically inequivalent. Exactly the opposite happens to the electrostatic potential which is not a physical field. Shifting the value of its zero mode is not just a symmetry, but it is part of a gauge symmetry. The zero mode of the electrostatic potential is physically meaningless. This will be important later, when trying to build wormhole solutions. On the other hand, for the same reasons, the dilaton charge is not a conserved charge while the electric charge is. In spite of this fact, the dilaton charge is a useful parameter that I will use to describe the solutions found.

of F Eq. (10) because, although it is not an equation of motion of the vector field A , it constrains F on the initial surface Σ . These constraints are

$$\begin{aligned} n^\mu \nabla^\nu (e^{-2\phi} F_{\mu\nu}) &= 0, \\ n^\mu \nabla^{\nu*} F_{\mu\nu} &= 0. \end{aligned} \quad (11)$$

The complete set of equations that have to be solved consists of Eqs. (2) plus Eqs. (11), taking into account Eq. (6). It is convenient to express them in terms of the intrinsic geometric objects of Σ . One gets the following set of equations:

$$\begin{aligned} {}^{(3)}R - 2 {}^{(3)}g^{ij} e^{-2\phi} (E_i E_j + B_i B_j) \\ - 2 {}^{(3)}g^{ij} {}^{(3)}\nabla_i \phi {}^{(3)}\nabla_j \phi &= 0, \\ {}^{(3)}\nabla_i ({}^{(3)}g^{ij} e^{-2\phi} E_j) &= 0, \\ {}^{(3)}\nabla_i ({}^{(3)}g^{ij} B_j) &= 0, \end{aligned} \quad (12)$$

which define the time-symmetric initial-data problem for the above theory. In what follows I will omit the indices (3) in most places for simplicity.

The kind of solutions of these equations that I am looking for are asymptotically flat and are determined by the mass (M), the electric magnetic charges, and dilaton charges (Q , P , and Σ , respectively) and the asymptotic value of the dilaton at infinity ϕ_∞ . They are defined in the limit $r = |\vec{x}| \rightarrow \infty$ as

$$\begin{aligned} {}^{(3)}g_{rr} &\sim 1 + \frac{2M}{r}, \\ E_i &\sim Q \frac{x_i}{r^3}, \\ B_i &\sim P \frac{x_i}{r^3}, \\ \phi &\sim \phi_\infty + \frac{\Sigma}{r}, \end{aligned} \quad (13)$$

and so they coincide with those of the known exact static solutions [16, 18]. When there is more than one asymptotically flat region, one has to indentify first the coordinate that plays the same role as r in that region and then one can define as before the charges that observers in that region would measure. As we will see they are different in general.

In the following sections I am going to find different families of solutions to Eqs. (12) describing black holes and wormholes.

III. ANSÄTZE

To find solutions to the equations of the previous section I am going to use a combination of “tricks” previously used in the literature. First I make the following *ansatz* for the three-metric $dl^2 = {}^{(3)}g_{ij} dx^i dx^j$,

$$dl^2 = W(\vec{x}) dl_\xi^2, \quad (14)$$

where dl_ξ^2 is another three-metric which, later, I will choose to be either the flat Euclidean metric

$$dl_b^2 = d\vec{x}^2 \quad (15)$$

or the metric of an $S^1 \times S^2$ “doughnut”:

$$dl_D^2 = d\mu^2 + d\theta^2 + \sin^2 \theta d\phi^2. \quad (16)$$

In terms of the curvature and covariant derivative of the ξ metric, the curvature of the three-metric I want to find is given by

$${}^{(3)}R = W^{-1} \left[{}^{(3)}R_\xi - 2\nabla_\xi^2 \ln W - \frac{1}{2} (\nabla_\xi \ln W)^2 \right]. \quad (17)$$

The value of ${}^{(3)}R_\xi$ in the two cases I am interested in is ${}^{(3)}R_b = 0$ and ${}^{(3)}R_D = 2$.

I am going to restrict myself to the purely electric case $B_i = 0$. The purely magnetic one can be obtained through the duality transformation

$$\begin{aligned} e^{-2\phi} E_i &\rightarrow B_i, \\ B_i &\rightarrow -e^{-2\phi} E_i, \\ \phi &\rightarrow -\phi. \end{aligned} \quad (18)$$

In general it does not make sense to perform a continuous duality rotation since the truncation of $N = 4$, $d = 4$ supergravity Eq. (3) (in which the axion field is absent) would not be consistent.

It is also easier to work with the electrostatic potential Z defined by

$$E_i = -\partial_i Z. \quad (19)$$

Substituting all this into Eqs. (12) one arrives at

$$\begin{aligned} \nabla_\xi^2 \ln W + \frac{1}{4} (\nabla_\xi \ln W)^2 + e^{-2\phi} (\nabla_\xi Z)^2 \\ + (\nabla_\xi \phi)^2 - \frac{1}{2} {}^{(3)}R_\xi &= 0, \end{aligned} \quad (20)$$

$$\nabla_\xi (W^{\frac{1}{2}} e^{-2\phi} \nabla_\xi Z) = 0. \quad (21)$$

It is impossible to give a more specific unique *ansatz* for the different situations I am going to consider here. I will make *Ansätze* case by case, including also the well-known vacuum and electrovacuum cases for completeness.

A. Vacuum case

The equation that one has to solve is Eq. (20) with $Z = \phi = 0$

$$\nabla_\xi^2 \ln W + \frac{1}{4} (\nabla_\xi \ln W)^2 - \frac{1}{2} {}^{(3)}R_\xi = 0. \quad (22)$$

Now a further *Ansatz* can be made: The function W is a power of a function χ such that there are no terms proportional to the square of the derivative of χ in the above equation. This implies

$$W = \chi^4, \quad (23)$$

where χ satisfies the equation

$$(\nabla_{\xi}^2 - \frac{1}{8}{}^{(3)}R_{\xi})\chi = 0. \quad (24)$$

As is well known, this equation is easy to solve for the two ξ metrics I am going to consider here, Eqs. (15),(16).

B. Einstein-Maxwell case

The equations that have to be solved are Eqs. (20),(21) with $\phi = 0$:

$$\nabla_{\xi}^2 \ln W + \frac{1}{4}(\nabla_{\xi} \ln W)^2 + (\nabla_{\xi} Z)^2 - \frac{1}{2}{}^{(3)}R_{\xi} = 0, \quad (25)$$

$$\nabla_{\xi}(W^{\frac{1}{2}} \nabla_{\xi} Z) = 0. \quad (26)$$

Again I make a further *Ansatz*: the functions W and Z can be expressed in terms of two new functions χ and ψ as

$$W = \psi^{\delta} \chi^{\gamma},$$

$$Z = \alpha \ln \psi + \beta \ln \chi, \quad (27)$$

where the constants $\alpha, \beta, \gamma, \delta$ will be adjusted to cancel crossed terms, etc., in the equations. The result is

$$W = (\psi\chi)^2,$$

$$Z = C \pm \ln(\psi/\chi), \quad (28)$$

where the functions ψ and χ satisfy

$$(\nabla_{\xi}^2 - \frac{1}{8}{}^{(3)}R_{\xi})\chi = 0,$$

$$(\nabla_{\xi}^2 - \frac{1}{8}{}^{(3)}R_{\xi})\psi = 0. \quad (29)$$

This is again the same type of equation that was obtained in the vacuum case. Here C is an arbitrary integration constant. Its value can be changed by a gauge transformation and, thus, solutions with different values of C are, in fact, physically equivalent.

C. Einstein-dilaton case

The equation that has to be solved is Eq. (20) with

$$\nabla_{\xi}^2 \ln W + \frac{1}{4}(\nabla_{\xi} \ln W)^2 + (\nabla_{\xi} \phi)^2 - \frac{1}{2}{}^{(3)}R_{\xi} = 0. \quad (30)$$

This equation is exactly the same as Eq. (25) of the Einstein-Maxwell initial-data problem with ϕ playing now the role of the electrostatic potential Z . The difference with the Einstein-Maxwell case is that now there is no constraint analogous to Eq. (26), and so Eq. (30) is the only constraint on the initial data of this system. It is safe to try the same *Ansatz* as in the Einstein-Maxwell case:

$$W = \psi^{\delta} \chi^{\gamma},$$

$$\phi = \phi_0 + \alpha \ln \psi + \beta \ln \chi. \quad (31)$$

The result is

$$W = (\psi\chi)^2 (\psi/\chi)^{2a},$$

$$\phi = \phi_0 \pm \sqrt{1 - a^2} \ln(\psi/\chi), \quad (32)$$

where the functions ψ and χ satisfy again Eqs. (29) and the constant a takes values in the interval $[-1, 1]$. ϕ_0 is a constant which will coincide, in general, with the value of the dilaton at infinity if the functions ψ and χ are properly normalized. When there is more than one asymptotically flat region there is no reason why one should expect the value of the dilaton at the corresponding infinities to be the same. In fact, as will be shown later on, the asymptotic values of the dilaton are in general different in different asymptotic regions and ϕ_0 will be just one of them.

The constant ϕ_0 can be fixed arbitrarily as was the case with the integration constant C in the Einstein-Maxwell system; i.e., there is a solution for each ϕ_0 chosen. However, as has already been explained, these solutions are not physically equivalent.

In the Einstein-Maxwell case, the constant a was forced to be equal to zero by Eq. (26). This equation is nothing but Gauss' law, and it enforces the condition of the absence of electric charge on the initial surface Σ , which is assumed to be regular everywhere. The same equation with ϕ playing the role of Z would enforce the absence of dilaton charge on the initial surface. Thus, for all cases with $a \neq 0$ it is reasonable to expect the solutions to have net dilaton charge. Then they will be very different from all known solutions in which the charges are either located at a singularity or the effect of nontrivial topology (Wheeler's "charge without charge").

As a matter of fact, Eq. (30) could be regarded as the equation for the Einstein-Maxwell plus (unspecified) charged matter initial-data problem, just by changing ϕ by Z . The charge density would then be given by

$$\rho = {}^{(3)}\nabla^2 Z. \quad (33)$$

The analogy is not perfect because this hypothetical matter should contribute to the density of energy on Σ , $n^{\mu}n^{\nu}T_{\mu\nu}$, and no contribution of this kind appears in Eq. (30).

D. Einstein-Maxwell-dilaton case

Now one has to solve the complete Eqs. (20),(21). To get something different from the previous cases one has to use a slightly different *Ansatz*:⁵

⁵If we try again with Z proportional to the logarithms of ψ and χ , one finds that the solutions have either a trivial dilaton or trivial electrostatic potential, recovering the solutions found in the two later sections.

$$\begin{aligned}
W &= \psi^\delta \chi^\gamma, \\
\phi &= \phi_0 + \epsilon \ln \psi + \mu \ln \chi, \\
Z &= B \psi^\alpha \chi^\beta.
\end{aligned} \tag{34}$$

As one might have expected after all the examples studied so far the result is

$$\begin{aligned}
W &= (\psi \chi)^2 (\psi / \chi)^{+2b}, \\
e^{-2\phi} &= e^{-2\phi_0} (\psi / \chi)^{-2b}, \\
Z &= C \pm e^{+\phi_0} \frac{\sqrt{1-2b^2}}{b} (\psi / \chi)^{+b},
\end{aligned} \tag{35}$$

where ψ and χ satisfy, yet again, Eqs. (29) and b is a constant that takes values in the interval $[-1/\sqrt{2}, +1/\sqrt{2}]$.

It is possible to find more solutions following a different procedure that will be explained later in Sec. IV D.

IV. BLACK-HOLE INITIAL DATA

One can get black-hole initial data for all the cases studied in the previous section by choosing the ξ metric to be the flat Euclidean metric, which has ${}^{(3)}R_b = 0$. The three-metric one gets is conformally flat. Then, with all the *Ansätze* made one finds that only two functions ψ and χ , harmonic in flat three-space, are needed to build solutions for all the different cases:

$$\partial_i \partial_i \psi = \partial_i \partial_i \chi = 0. \tag{36}$$

These functions will be normalized to 1 at infinity and will correspond to pointlike sources \vec{x}_i :

$$\begin{aligned}
\psi &= 1 + \sum_{i=1}^N \frac{\psi_i}{|\vec{x} - \vec{x}_i|}, \\
\chi &= 1 + \sum_{i=1}^N \frac{\chi_i}{|\vec{x} - \vec{x}_i|}.
\end{aligned} \tag{37}$$

For the three-metrics corresponding to these ψ and χ to be regular everywhere (except at the points \vec{x}_i which have to be erased from Σ) the constants ψ_i and χ_i have to be strictly positive.

The i th black hole corresponds to an Einstein-Rosen-like bridge between two asymptotically flat regions or *sheets*: the $|\vec{x}| \rightarrow \infty$ region and the $|\vec{x} - \vec{x}_i| \rightarrow 0$ region.⁶ Therefore Σ is a set of $N + 1$ sheets, one of them with N holes cut and the remaining N with a single hole cut and pasted to one of the N holes of the *largest* sheet forming

throats or *necks* (Einstein-Rosen bridges). In the largest sheet there will be N black holes.

The corresponding solutions for the vacuum and Einstein-Maxwell cases were found in Refs. [2, 4, 5] and I will not discuss them here. I will write them down, however, for completeness and later use.

A. Vacuum case

The solution is given by the metric

$$dl^2 = \chi^4 d\vec{x}^2, \tag{38}$$

with χ given by Eq. (37), and describes N Schwarzschild (i.e., no charge or angular momentum) black holes. This solution was first found by Misner and Wheeler in Ref. [2].

For a single black hole, this metric is nothing but the spatial part of Schwarzschild's in isotropic coordinates. For more than one black hole the corresponding exact solution cannot be static and is not known.

B. Einstein-Maxwell case

The solution is given by the metric and electrostatic potential

$$\begin{aligned}
dl^2 &= (\psi \chi)^2 d\vec{x}^2, \\
Z &= C \pm \ln(\psi / \chi),
\end{aligned} \tag{39}$$

with the functions ψ and χ given by Eqs. (37), and describes N electric Reissner-Nordström (i.e., no angular momentum or scalar hair) black holes. This solution was first found by Brill and Lindquist in Ref. [4].

For a single black hole this solution coincides with the spatial part of the Reissner-Nordström solution in isotropic coordinates. If $\psi = 1$ or $\chi = 1$, one has the spatial part of the static solution that describes N *extreme* Reissner-Nordström black holes in equilibrium. The remaining situations do not correspond to systems in static equilibrium and no known solution describes them.

C. Einstein-dilaton case

The metric and dilaton are given by

$$\begin{aligned}
dl^2 &= \psi^{2+2a} \chi^{2-2a} d\vec{x}^2, \\
e^\phi &= e^{\phi_0} (\psi / \chi)^{\pm \sqrt{1-a^2}},
\end{aligned} \tag{40}$$

with ψ and χ given by Eqs. (37).

First of all, observe that by setting $a = 1$ or $\psi = \chi$ the vacuum solutions Eq. (38) are recovered. If one sets $a = 0$, one recovers the metric of the Einstein-Maxwell solution Eqs. (39) and, upon rescaling by $\exp\{\frac{-2a}{\sqrt{1-a^2}}\phi\}$, one always recovers that metric.

The choice

$$\begin{aligned}
\psi &= 1 - \rho_0 / 2r, \\
\chi &= 1 + \rho_0 / 2r, \\
a &= M / \rho_0,
\end{aligned} \tag{41}$$

⁶In each case one has to prove first that the limit $|\vec{x} - \vec{x}_i| \rightarrow 0$ indeed corresponds to another asymptotically flat region. I will do so later.

with $r = |\vec{x}|$ and $\rho_0 = \sqrt{M^2 + \Sigma^2}$, corresponds to the surface of the time symmetry of the family of singular static solutions of Ref. [12], which are not black holes and have naked singularities, in agreement with the no-hair theorems of Ref. [13] and the unicity theorem of Ref. [17].

Observe that, if one is looking everywhere for regular initial data, one would choose $\psi = 1 + \rho_0/2r$ instead of ψ as in Eq. (41). However, this does not lead to an exact static solution of the full set of Einstein's equations.

The solution in Eqs. (40) is clearly asymptotically flat in the limit $|\vec{x}| \rightarrow \infty$ (*upper sheet*). The metric is also regular everywhere if $|a| < 1$ and $\psi > 0$, $\chi > 0$ everywhere (i.e. $\psi_i > 0$, $\chi_i > 0$ for all i). Now I want to prove that it is also asymptotically flat in the i th *lower sheet* $|\vec{x} - \vec{x}_i| \equiv r_i \rightarrow 0$. In this limit the metric looks like

$$dl^2 \sim \left(1 + \frac{\psi'_i}{r'_i}\right)^{2+2a} \left(1 + \frac{\chi'_i}{r'_i}\right)^{2-2a} (dr_i'^2 + r_i'^2 d\Omega^2), \quad (42)$$

where

$$\begin{aligned} r_i' &= \psi_i^{1+a} \chi_i^{1-a} r_i^{-1}, \\ \psi_i' &= \chi_i (\psi_i / \chi_i)^a \left(1 + \sum_{j \neq i}^N \frac{\psi_j}{r_{ij}}\right), \\ \chi_i' &= \psi_i (\psi_i / \chi_i)^a \left(1 + \sum_{j \neq i}^N \frac{\chi_j}{r_{ij}}\right), \\ r_{ij} &= |\vec{x}_i - \vec{x}_j|. \end{aligned} \quad (43)$$

Equation (42) proves that there is one asymptotic region in each limit $r_i \rightarrow 0$ and that this solution describes N black holes represented by their Einstein-Rosen bridges at the moment of time symmetry. The dilaton field is nontrivial and, thus, these black holes have scalar hair. There is no exact static solution describing any of these objects [13, 12].

D. Einstein-Maxwell-dilaton case

This solution is given by

$$\begin{aligned} dl^2 &= \psi^{2+2b} \chi^{2-2b} d\vec{x}^2, \\ e^{-2\phi} &= e^{-2\phi_0} (\psi/\chi)^{-2b}, \\ Z &= C \pm e^{+\phi_0} \frac{\sqrt{1-2b^2}}{b} (\psi/\chi)^{+b}. \end{aligned} \quad (44)$$

Taking the limit $r_i = |\vec{x} - \vec{x}_i| \rightarrow 0$ in the metric one gets Eq. (42) with a replaced by b , and this implies again that this solution has N asymptotically flat regions at the other side of N Einstein-Rosen bridges; so this family of solutions must indeed describe N charged dilaton black holes.

Among this family of solutions I could identify only the case $b = -1/2$, $\chi = 1$ ($W = \psi$) as the spatial part

of the exact static solution describing N extreme electric dilaton black holes in equilibrium: [16, 18]

$$\begin{aligned} ds^2 &= \psi^{-1} dt^2 - \psi d\vec{x}^2, \\ e^{-2\phi} &= e^{-2\phi_0} \psi, \\ F_{i0} &= \pm \frac{e^{\phi_0}}{\sqrt{2}} \partial_i \psi^{-1}. \end{aligned} \quad (45)$$

The nonextreme dilaton black holes of Refs. [16, 18] also possess surfaces of time symmetry. However, none of them is described by the initial-data sets of Eqs. (44). To find the family of initial data which describes them one has to follow a different procedure.

Let (W, Z) be a set of time-symmetric initial data of the Einstein-Maxwell system. Then one can build out of it a set of time-symmetric initial data of the Einstein-Maxwell-dilaton system $(\tilde{W}, \tilde{\phi}, \tilde{Z})$ according to the rules

$$\begin{aligned} \tilde{W} &= p(t')^{2r} W, \\ e^{-2\tilde{\phi}} &= q(t')^{-(1+r)}, \\ \tilde{Z} &= t(Z), \end{aligned} \quad (46)$$

where p, q, r are arbitrary constants and $t(Z)$ is a function of Z satisfying the differential equation

$$\begin{aligned} q(t')^{-(1+r)} + 2r \frac{t''' t' - (t'')^2}{(t')^2} \\ + \frac{(1+r)^2 + 4r^2}{4} \left(\frac{t''}{t'}\right)^2 - 1 = 0. \end{aligned} \quad (47)$$

Here t' denotes the derivative of $t(Z)$ with respect to its argument Z .

Finding the most general solution of this equation is a very difficult problem. It turns out that there are several choices of the constants p, q, r which simplify enormously the problem and that those choices give, too, the solutions I am after. In particular, for $r = 1$ the solution is simply

$$t(Z) = D \left(A e^{\sqrt{\frac{1-a}{2}} Z} - B e^{-\sqrt{\frac{1-a}{2}} Z} \right). \quad (48)$$

Although it is not obvious, for $q = \frac{1}{2}$, $D = \pm \sqrt{2} e^{\phi_0}$, $p = 2e^{-2\phi_0} (A+B)^2 = 1$, this is exactly the transformation that takes the Reissner-Nordström initial data of Eq. (39) into the wanted dilaton black-hole initial data:

$$\begin{aligned} dl^2 &= \left[A(\psi/\chi)^{1/2} + B(\psi/\chi)^{-1/2} \right]^2, \\ e^{-2\phi} &= e^{-2\phi_0} \left[A(\psi/\chi)^{1/2} + B(\psi/\chi)^{-1/2} \right]^{-2}, \\ Z &= C \pm \sqrt{2} e^{\phi_0} \left[A(\psi/\chi)^{1/2} - B(\psi/\chi)^{-1/2} \right]. \end{aligned} \quad (49)$$

To see that this is indeed true, these initial data, for a single black hole, have to be compared with the (single, static, spherically symmetric) dilaton black-hole solution of Refs. [16], which can be rewritten as

$$ds^2 = V dt^2 - W d\vec{x}^2,$$

$$V = \frac{1}{(A-B)^2} \left[A(\psi/\chi)^{1/2} - B(\psi/\chi)^{-1/2} \right]^2,$$

$$W = (\psi\chi)^2 \left[A(\psi/\chi)^{1/2} + B(\psi/\chi)^{-1/2} \right]^2,$$

$$e^{-2\phi} = e^{-2\phi_0} \left[A(\psi/\chi)^{1/2} + B(\psi/\chi)^{-1/2} \right]^{-2},$$

$$F_{it} = \pm \frac{e^{\phi_0}}{\sqrt{2}(A-B)} \partial_i \left[A(\psi/\chi)^{1/2} + B(\psi/\chi)^{-1/2} \right]^2, \quad (50)$$

where

$$\psi = 1 + \frac{M - \Sigma - \sqrt{-4M\Sigma}}{2r},$$

$$\chi = 1 + \frac{M - \Sigma + \sqrt{-4M\Sigma}}{2r}, \quad (51)$$

and

$$\begin{aligned} A + B &= 1, \\ A - B &= \sqrt{\frac{-\Sigma}{M}}. \end{aligned} \quad (52)$$

Obviously, the initial data of Eq. (49) for more than one black hole cannot be compared with any known exact static solution.

By allowing more general values of q in Eq. (48) one gets the more general family of initial data:

$$\begin{aligned} dl^2 &= (\psi/\chi)^2 \left[A(\psi/\chi)^c + B(\psi/\chi)^{-c} \right]^2 d\vec{x}^2, \\ e^{-2\phi} &= e^{-2\phi_0} \left[A(\psi/\chi)^c + B(\psi/\chi)^{-c} \right]^{-2}, \\ Z &= C \pm \frac{\sqrt{1-2c^2}}{c} e^{\phi_0} \left[A(\psi/\chi)^c - B(\psi/\chi)^{-c} \right], \end{aligned} \quad (53)$$

where the constant c takes values in the interval $[-1/\sqrt{2}, +1/\sqrt{2}]$. This family of initial data includes that of Eqs. (44) ($c = \pm b$ and $A = 1$ or $B = 1$) and Eqs. (49) [$c = 1/2$ and $(A+B)^2 = 1$]. More general solutions based on the transformation Eq. (48) are possible if one allows the constant c to be imaginary. I will not treat them here, but some of them will be important when we study the wormhole initial-data problem in Sec. VI. Nevertheless, I would like to make the following observation here: If one performs the duality transformation Eq. (18), then the rescaled string-frame metric $e^{2\phi} dl^2$ is the same as the Einstein-Maxwell one, Eq. (39).

It is easy to show again that in each of the limits $r_i \equiv |\vec{x} - \vec{x}_i| \rightarrow 0$ there is another asymptotically flat region. In this limit the metric looks like

$$\begin{aligned} dl^2 &\sim \left[(1 + \psi'_i/r'_i)/(1 + \chi'_i/r'_i) \right]^2 \\ &\quad \times \{ A' [(1 + \psi'_i/r'_i)/(1 + \chi'_i/r'_i)]^c \\ &\quad + B' [(1 + \psi'_i/r'_i)/(1 + \chi'_i/r'_i)]^{-c} \} \\ &\quad \times [dr'^2_i + r'^2_i d\Omega^2], \end{aligned} \quad (54)$$

where

$$r'_i = [A(\psi_i/\chi_i)^c + B(\psi_i/\chi_i)^{-c}] / r_i^{-1},$$

$$A' = A(\psi_i/\chi_i)^c / [A(\psi_i/\chi_i)^c + B(\psi_i/\chi_i)^{-c}],$$

$$B' = B(\psi_i/\chi_i)^{-c} / [A(\psi_i/\chi_i)^c + B(\psi_i/\chi_i)^{-c}],$$

$$\psi'_i = \chi_i [A(\psi_i/\chi_i)^c + B(\psi_i/\chi_i)^{-c}] \left(1 + \sum_{j \neq i}^N \frac{\psi_j}{r_{ij}} \right),$$

$$\chi'_i = \psi_i [A(\psi_i/\chi_i)^c + B(\psi_i/\chi_i)^{-c}] \left(1 + \sum_{j \neq i}^N \frac{\chi_j}{r_{ij}} \right),$$

$$r_{ij} = |\vec{x}_i - \vec{x}_j|. \quad (55)$$

For the metric in Eq. (53) to be regular the positivity of the constants ψ_i and χ_i is not sufficient. One has to impose certain restrictions on the values of the constants A and B as well. When both have the same sign, the metric is always regular, but when they have opposite signs one needs to study in detail the problem. I will do it for the case of a single black-hole in Sec. VB.

The conclusion is that for any N , if the constants A and B are carefully chosen, the initial data of Eq. (53) [and hence those of Eq. (49)] describe several charged dilaton black holes of the same type at the moment of time symmetry.

It seems that the initial data of Eqs. (49) are those sought for. However, the initial data in Eqs. (53) *also* describe charged dilaton black holes. And the possibilities of generating new solutions are not exhausted. It seems that now there are too many solutions. For given physical parameters Q , P , M , and Σ , how many different solutions are there now? Why are they different? I will address this problem, already present in the Einstein-dilaton system, in the next section by studying the simplest solutions in these families: the spherically symmetric, which describe a single black hole.

V. SPHERICALLY SYMMETRIC CASE

Throughout this section I will take the harmonic functions ψ, χ to be

$$\begin{aligned} \psi &= 1 + \frac{E}{r}, \\ \chi &= 1 + \frac{F}{r}. \end{aligned} \quad (56)$$

In most cases, the regularity of the solutions implies that E and F are strictly positive constants and, then, all the initial-data sets found in the previous section describe a single spherically symmetric black hole. In this section I want to identify the physical meaning of the constants E and F in the different asymptotic regions, expressing E and F in terms of the mass and charges. My goal is to find out if there is more than one initial-data set describing spherically symmetric black-hole solutions with fixed mass and charges, and whether there are any extra degrees of freedom.

A. Einstein-dilaton case

In the upper sheet ($r \rightarrow \infty$), the mass, the dilaton charge, and the asymptotic value of the dilaton are

$$\begin{aligned} M &= (1+a)E + (1-a)F, \\ \Sigma &= \pm \sqrt{1-a^2}(E-F), \\ \phi_\infty &= \phi_0. \end{aligned} \quad (57)$$

Observe that the strict positivity of E and F and the fact that $|a| < 1$ ensure the strict positivity of the mass.

It is convenient to have the expressions of E and F in terms of the physical constants M and Σ , and the parameter a :

$$\begin{aligned} E &= \frac{1}{2} \left(M \pm \sqrt{\frac{1-a}{1+a}} \Sigma \right), \\ F &= \frac{1}{2} \left(M \mp \sqrt{\frac{1+a}{1-a}} \Sigma \right). \end{aligned} \quad (58)$$

These constants are positive and, therefore, the mass obeys the bounds

$$M > \mp \sqrt{\frac{1-a}{1+a}} \Sigma, \quad M > \pm \sqrt{\frac{1+a}{1-a}} \Sigma, \quad (59)$$

which means that, for a fixed value of the dilaton charge, there is always a value of the parameter a such that the mass is as small as we please and it is only bounded by zero in this family of initial data.

The radius of the minimal surface (there is only one) is given in terms of the integration constants E, F, a by

$$r_{\min} = \frac{1}{2} \left\{ a(E-F) + \sqrt{a^2(E-F)^2 + 4EF} \right\} \quad (60)$$

and, in terms of the physical constants and the parameter a , by

$$r_{\min} = \frac{1}{2} \left\{ \pm \frac{a}{\sqrt{1-a^2}} \Sigma + \sqrt{\left(M \mp \frac{a\Sigma}{\sqrt{1-a^2}} \right)^2 - \Sigma^2} \right\}. \quad (61)$$

It is easy to see in Eq. (60) that the positivity of E and F implies that r_{\min} is always real and positive; i.e., there

is always an apparent horizon. From the expression of r_{\min} in terms of the physical constants and the fact that it is real and positive one sees that the charges obey a number of bounds which reduce to those in Eq. (59).

The area of the minimal surfaces can be readily found, but it is a complicated and not very enlightening expression. For small dilaton charge one finds

$$A(r_{\min}) = 16\pi(M^2 - \frac{1}{2}\Sigma^2) + O(\Sigma^3), \quad (62)$$

which does not depend on the parameter a to this order and coincides with the area of the event horizon of a Reissner-Nordström black hole with electric charge $Q = \Sigma$.

From these expressions it is easy to see that this is a one-parameter family of initial data describing a black hole of fixed mass and dilaton charge. Which additional degree of freedom is a describing? To answer this question one has to calculate first the values of the mass and dilaton charge as seen from the second asymptotic region.

To study the “lower sheet” one has to perform first the coordinate transformation $r \rightarrow r' = E^{1+a}F^{1-a}/r$. The metric and the dilaton field are, in these new coordinates,

$$\begin{aligned} dl^2 &= \left(1 + \frac{E^a F^{1-a}}{r'} \right)^{2+2a} \left(1 + \frac{E^{1+a} F^{-a}}{r'} \right)^{2-2a} \\ &\quad \times (dr'^2 + r'^2 d\Omega^2), \\ \phi &= \phi_0 \pm \sqrt{1-a^2} \log(E/F) \\ &\quad \pm \ln \left[\left(1 + \frac{E^a F^{1-a}}{r'} \right) / \left(1 + \frac{E^{1+a} F^{-a}}{r'} \right) \right]. \end{aligned} \quad (63)$$

One immediately gets, in the limit $r' \rightarrow \infty$,

$$\begin{aligned} M' &= (1+a)E^a F^{1-a} + (1-a)E^{1+a} F^{-a}, \\ \Sigma' &= \pm \sqrt{1-a^2}(E^a F^{1-a} - E^{1+a} F^{-a}) = -(E/F)^a \Sigma, \\ \phi'_\infty &= \phi_0 \pm \sqrt{1+a^2} \ln(E/F). \end{aligned} \quad (64)$$

When $a \neq 0$ all the physical constants are different in the upper and lower sheets. These results may seem surprising at first (they are certainly unusual) but, for M and Σ , they are simple consequences of the absence of Gauss’ law for the dilaton charge, as I am going to explain.

In the Reissner-Nordström initial-data problem Gauss’ law says that there is no charge on the initial surface Σ . The nontrivial topology of Σ allows the electric force lines to get *trapped* and go from the upper sheet to the lower sheet across the Einstein-Rosen bridge. The electric flux through an asymptotic sphere in each separate sheet does not vanish, but the total flux does, in agreement with Gauss’ law. The effect is that observers in both sheets measure the same amount of charge but with opposite signs. I will call this type of charge the *topological charge*.

In the present case there is no Gauss’ law for ϕ . The presence of a certain amount of *physical* scalar charge Σ_p on the initial surface is allowed. It corresponds to the charge density

$$\rho = {}^{(3)}\nabla^2 \phi = \partial_i (W^{\frac{1}{2}} \partial_i \phi). \quad (65)$$

Integrating ρ on Σ and applying Gauss’ theorem,

$$\begin{aligned} \frac{1}{4\pi} \int_{\Sigma} d^3x \rho &= \frac{1}{4\pi} \int_{S_{\text{up}}^2} dS^{\hat{t}(3)} \nabla_{\hat{i}} \phi + \frac{1}{4\pi} \int_{S_{\text{low}}^2} dS^{\hat{t}(3)} \nabla_{\hat{i}} \phi \\ &\equiv 2\Sigma_p, \end{aligned} \quad (66)$$

where S_{up}^2 and S_{low}^2 are two two-spheres in the limits $r \rightarrow \infty$ and $r \rightarrow 0$, respectively.

If there were only this charge, one would measure exactly the same charge Σ_p in both sheets (with the same signs). As one can see in Eq. (64), this is not the case, in general. The force lines of the field ϕ can also get trapped in the throat.⁷ This effect contributes with different signs to the flux of ϕ in both sheets: $\pm\Sigma_t$. The result is that the flux of ϕ in the upper sheet is $\Sigma = \Sigma_p + \Sigma_t$ and in the lower sheet is $\Sigma' = \Sigma_p - \Sigma_t$:

$$\begin{aligned} \Sigma &= \frac{1}{4\pi} \int_{S_{\text{up}}^2} dS^{\hat{t}(3)} \nabla_{\hat{i}} \phi = \Sigma_p + \Sigma_t, \\ \Sigma' &= \frac{1}{4\pi} \int_{S_{\text{low}}^2} dS^{\hat{t}(3)} \nabla_{\hat{i}} \phi = \Sigma_p - \Sigma_t. \end{aligned} \quad (67)$$

This explains the physical meaning of the parameter a as well: It measures the relative value of Σ_p versus Σ_t which was the physical degree of freedom that we failed to identify before:

$$\Sigma_p/\Sigma = \frac{1}{2}(\Sigma + \Sigma')/\Sigma = \frac{1}{2}[1 - (E/F)^a] < \frac{1}{2}. \quad (68)$$

Σ_p is always different from Σ except when $\Sigma = 0$; i.e., there is always some topological and some physical dilaton charge except in two cases: when there is no dilaton charge at all and when $a = 0$, the Reissner-Nordström-like case in which all the charge is “topological.”

This justifies the difference between Σ and Σ' . The difference between M and M' must correspond to the difference in the matter contents (scalar charge) observed in both regions.

Finally, observe that most of these effects disappear if one rescales the metric by $\exp\{\frac{-2a}{\sqrt{1-a^2}}\phi\}$: The mass and dilaton charge in the upper and lower sheets are the same, but the asymptotic value of the dilaton, ϕ_{∞} , is still different.

B. Einstein-Maxwell-dilaton case

As was pointed out in Sec. IV D, the regularity of the metric in Eqs. (53), (56) is not guaranteed by the simple positivity of the constants E and F , and additional conditions have to be imposed on the constants A and B . The first thing to do in order to study these initial data is to find these conditions.

In what follows I will assume that $A > 0$ with no loss of

generality. If $B > 0$ too, then the metric is obviously regular. If $B < 0$, the metric vanishes when the coordinate r takes the value

$$r_{\text{sing}} = \frac{E - (-B/A)^{1/2c}}{(-B/A)^{1/2c} - 1}, \quad (69)$$

and therefore it will be regular if $r_{\text{sing}} < 0$. A careful analysis leads to the identification of the following five cases in which the metric is regular: (1) $B \geq 0$, $A + B = +1$; (2) $B < 0$, $A + B = +1$, $(E/F)^{2c} > 1$; (3) $B < 0$, $A + B = +1$, $(E/F)^{2c} < 1$, $A < 1/[1 - (E/F)^{2c}]$; (4) $B < 0$, $A + B = -1$, $(E/F)^{2c} > 1$, $A < -1/[1 - (E/F)^{2c}]$; (5) $B < 0$, $A + B = -1$, $(E/F)^{2c} < 1$.

In principle in these five instances the positivity of the mass in the upper and lower sheets should be automatically guaranteed, but a detailed study of each case is required to prove it.

In the upper sheet the mass, the dilaton charge, the electric charge, and the asymptotic value of the dilaton are

$$M = \left[1 + c \frac{(A - B)}{(A + B)}\right] E + \left[1 - c \frac{(A - B)}{(A + B)}\right] F,$$

$$\Sigma = c \frac{(A - B)}{(A + B)} (E - F),$$

$$Q = \pm(A + B)e^{\phi_0} \sqrt{1 - 2c^2(E - F)},$$

$$\phi_{\infty} = \phi_0. \quad (70)$$

In the first case above it is easy to prove that $M > 0$. One simply has to observe that $A > 0$, $B \geq 0$, $A + B = 1$ implies $|(A - B)/(A + B)| \leq 1$. This together with $|c| \leq 1/\sqrt{2}$ implies $|c(A - B)/(A + B)| \leq 1/\sqrt{2} < 1$ and the factors that multiply E and F in the mass formula are strictly positive. In the second case it is also easy to prove that $M > 0$ by rewriting the mass formula as

$$M = (E + F) + c \frac{(A - B)}{(A + B)} (E - F). \quad (71)$$

Now the product $c(E - F) > 0$ in this case and all the terms in the above expression are manifestly positive.

The third case is more complicated. One has to study two distinct possibilities: (i) $E > F$, $c < 0$ and (ii) $E < F$, $c > 0$. One has to prove that $A < 1/[1 - (E/F)^{2c}]$ implies $A < \frac{1}{2} \left[1 - \frac{1}{c} \left(\frac{E+F}{E-F}\right)\right]$. In case (i), if $c < -1/2$, it follows from the inequality

$$\frac{1}{1 - (F/E)^{-2c}} < \frac{1}{1 - (F/E)}, \quad (72)$$

and, if $c > -1/2$, it follows from

$$\frac{1}{1 - (F/E)^{-2c}} < \frac{-1}{2c} \left[\frac{1}{1 - (F/E)} \right]. \quad (73)$$

Case (ii) follows from (i) with the interchange $E \leftrightarrow F$, $c \rightarrow -c$. The remaining two cases go through in a similar fashion and I will omit the details.

These mass and charge formulas can be inverted to

⁷It should be stressed that this is simply a good way of describing the system. There is no conserved charge of any kind associated with the dilaton. I will keep on using the terms “charge” and “topological” charge although it should be clear that in this system there is no conserved dilaton charge of any kind.

express the integration constants in terms of the physical constants

$$\begin{aligned} E &= \frac{1}{2} \left(M - \Sigma \pm \frac{e^{-\phi_\infty}}{\sqrt{1-2c^2}} Q \right), \\ F &= \frac{1}{2} \left(M - \Sigma \mp \frac{e^{-\phi_\infty}}{\sqrt{1-2c^2}} Q \right), \\ c(A-B) &= \pm \sqrt{1-2c^2} e^{-\phi_\infty} \frac{\Sigma}{Q}. \end{aligned} \quad (74)$$

Again the number of independent integration constants is the same as the number of physical constants plus one. It is clear that there is again another degree of freedom described by the extra integration constant. If one calculates now the value of $M, \Sigma, Q, \phi_\infty$ in the lower sheet, one would find different values for the four of them. This seems to suggest that there is one less integration constant than necessary. This confusion arises because the “electric” charge to look for is $\tilde{Q} = e^{-2\phi_\infty} Q$ for which there is a Gauss’ law ${}^{(3)}\nabla_i(e^{-2\phi} E^i) = 0$, enforcing its absence on the initial surface. This equation implies that all the \tilde{Q} charge is “topological” and that it takes equal values with different signs in the upper and lower sheets in agreement with the arguments of the previous section. This can be readily checked.

The integration constant c measures the ratio between topological and physical dilaton charge for which there is no Gauss’ law whatsoever.

The radii r_{\min} of the minimal surfaces are given by the zeros of the function

$$\begin{aligned} f(r) &= A(r+E)^{2c}(r-r_+)(r-r_-) \\ &\quad + B(r+F)^{2c}(r+r_+)(r+r_-), \end{aligned} \quad (75)$$

where

$$r_\pm = \frac{1}{2} \left[-c(F-E) \pm \sqrt{c^2(F-E)^2 + 4EF} \right]. \quad (76)$$

Finding an analytical expression for r_{\min} is out of the question. Nevertheless, observing that the first term of $f(r)$ only vanishes at $r = r_+$ and the second term only vanishes at $r = r_-$ it is possible to say the following about it: (1) If $\text{sgn}(A) = \text{sgn}(B)$, $c(F-E) > 0$, then $r_{\min} \in [r_+, -r_-]$; (2) if $\text{sgn}(A) = \text{sgn}(B)$, $c(F-E) < 0$, then $r_{\min} \in [-r_-, r_+]$; (3) if $\text{sgn}(A) = -\text{sgn}(B)$, $c(F-E) > 0$, then $r_{\min} \in (0, r_+] \cup [-r_-, \infty)$; (4) if $\text{sgn}(A) = -\text{sgn}(B)$, $c(F-E) < 0$, then $r_{\min} \in (0, -r_-] \cup [r_+, \infty)$.

VI. WORMHOLE INITIAL DATA

To find wormhole initial data I will follow Misner [6]. I will take the ξ metric to be the metric of the “doughnut” $S^1 \times S^2$, Eq. (16). The effect of the conformal factor W on this metric is to blow up one side of the doughnut and transform it in an asymptotically flat region. This metric has constant curvature ${}^{(3)}R_D = 2$. Then, the kind of equation one has to solve for all the *Ansätze* of Sec. III is

$$(\nabla_D^2 - \frac{1}{4})\chi = 0. \quad (77)$$

Misner observed that, although this equation looks complicated, there is one simple solution. In fact, the metric of flat three-space in bispherical coordinates is

$$dl_b^2 = \frac{k^2}{(\cosh \mu - \cos \theta)^2} dl_D^2, \quad (78)$$

and must be a solution of the initial-data constraint equations in vacuum. Therefore the functions

$$f_d(\theta, \mu) = k^{-1/2} [\cosh(\mu + d) - \cos \theta]^{-1/2}, \quad (79)$$

where d is any constant, must solve Eq. (77). Now, of course, this fact can be used not just for the vacuum case, but for all the cases in which the ξ metric is the doughnut metric of Eq. (16) and one has been able to reduce the initial-data problem to equations like Eq. (77). One just has to build χ and ψ as linear combinations of functions f_d , obeying the boundary conditions required in each case.

Essentially the boundary conditions can be described as follows: In wormhole space the coordinate μ crosses the wormhole and parametrizes S^1 . Therefore μ is periodic with period⁸ μ_0 . The metric, the electric field, and the dilaton (which are physical fields) must be single valued around the wormhole. That is, all those fields must be periodic in the variable μ with period μ_0 . The electrostatic potential is not physical and can be multivalued around the wormhole.

To construct these periodic fields it is convenient to define first the functions $f_n(\theta, \mu) = f_0(\theta, \mu + n\mu_0)$ where n is an integer number. The series

$$\sum_{n \in \mathbb{Z}} c_n f_n(\theta, \mu) \quad (80)$$

is a solution of the linear equation (77) with different periodicity properties depending on the choice of the constants c_n . If one chooses all the $c_n = 1$, the series is periodic with period μ_0 (assuming it converges). For $c_n = \pm(-1)^n$ the series is antiperiodic. Many more choices are possible, leading to different periodicity properties, and some of them will be used later. Now I will study each case separately.

A. Vacuum (Misner) wormholes

The single valuedness of $W = \chi^4$ implies $c_n \sim e^{i\frac{\pi n}{2}}$. Reality and regularity of W imply $c_n = 1$ for all integers n , and so

$$\chi = k^{-1/2} \sum_{n \in \mathbb{Z}} [\cosh(\mu + n\mu_0) - \cos \theta]^{-1/2}. \quad (81)$$

which is the solution found by Misner in Ref. [6]. For an external observer the wormhole’s two mouths are just

⁸Observe that our convention differs slightly from that of Ref. [6] and the subsequent literature in which μ has period $2\mu_0$.

two Schwarzschild black holes. The fact that there are no more asymptotically flat regions (“universes”) as in the initial data found in the previous section seems untestable for that observer since the wormhole’s throat (or the Einstein-Rosen bridge between universes) collapses before any signal crosses it. These initial data were used by Smarr in Ref. [19] to study the gravitational radiation produced in a head-on collision of two Schwarzschild black holes. The evolution of these initial data has also been studied in a different limit by Tomimatsu in Ref. [20].

B. Einstein-Maxwell wormholes

The single valuedness of W in Eq. (28) requires that, if the functions χ and ψ get a factor K when moving from μ to $\mu + \mu_0$, then

$$\begin{aligned}\psi(\theta, \mu + \mu_0) &= K\psi(\theta, \mu), \\ \chi(\theta, \mu + \mu_0) &= K^{-1}\chi(\theta, \mu).\end{aligned}\quad (82)$$

Regularity and reality of W require K to be a positive real number which is customarily written as $K = e^{-\lambda}$. When $\lambda = 0$ one is back into the vacuum case.

Functions with the required monodromy properties can be built as the series Eq. (80) with coefficients $c_n = e^{n\lambda}$ and $c_n = e^{-n\lambda}$, respectively; that is,

$$\begin{aligned}\psi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{n\lambda} [\cosh \mu - \cos \theta]^{-1/2}, \\ \chi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{-n\lambda} [\cosh \mu - \cos \theta]^{-1/2}.\end{aligned}\quad (83)$$

Observe that with this choice of monodromy of χ and ψ around μ , the electrostatic potential Z is not single valued,

$$Z(\theta, \mu + \mu_0) = Z(\theta, \mu) \mp 2\lambda, \quad (84)$$

but the physically meaningful quantity, the electric field, is single valued. These are the electrically charged wormhole initial data found by Lindquist by the method of images in Ref. [5].

C. Einstein-dilaton wormholes

The single valuedness of the metric in Eq. (35) requires now

$$\begin{aligned}\psi(\theta, \mu + \mu_0) &= K^{\frac{1}{2(1+a)}} \psi(\theta, \mu), \\ \chi(\theta, \mu + \mu_0) &= K^{\frac{-1}{2(1-a)}} \chi(\theta, \mu).\end{aligned}\quad (85)$$

I will write $K = e^{-2\lambda}$ to recover the previous case when $a = 0$. The functions ψ and χ are the series Eq. (80) with coefficients $c_n = e^{\frac{n\lambda}{1+a}}$ and $c_n = e^{\frac{-n\lambda}{1-a}}$, respectively:

$$\begin{aligned}\psi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{\frac{n}{1+a}\lambda} [\cosh(\mu + n\mu_0) - \cos \theta]^{-1/2}, \\ \chi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{\frac{-n}{1-a}\lambda} [\cosh(\mu + n\mu_0) - \cos \theta]^{-1/2}.\end{aligned}\quad (86)$$

Now, analyzing the monodromy properties of the dilaton field, one finds that

$$\phi(\theta, \mu + \mu_0) = \phi(\theta, \mu) \mp \frac{2\lambda}{\sqrt{1+a^2}}, \quad (87)$$

that is, the dilaton is not single valued around the wormhole, but its value changes by a constant each time one goes around the the wormhole, just as happened with the electrostatic potential. Since the zero mode of the dilaton is physically meaningful, one concludes that this is not a good solution of the initial-data problem for a dilaton field such as the string theory one.

One could have anticipated this result because, roughly speaking, to build a wormhole one has to identify the two asymptotic regions of an Einstein-Rosen bridge and in Sec. V we found that the asymptotic value of the dilaton in both regions is, in general, different and cannot be identified.

Were one considering a different kind of field, for instance, an scalar taking values on a circle whose length is a submultiple of $\frac{2\lambda}{\sqrt{1+a^2}}$, the solution would be perfectly valid. If the length of this circle is $\frac{2\lambda}{n\sqrt{1+a^2}}$, $n \in \mathbb{Z}^+$, then one is identifying ϕ with $\phi + m\frac{2\lambda}{n\sqrt{1+a^2}}$, for all integers m . This scalar field is a map from one circle parametrized by μ to another circle (the target), such that going around the first circle once means going around the target circle $\pm n$ times, and one can consider this number as the winding number of the map. This is a topological invariant of ϕ that cannot change if the initial surface Σ is deformed in a continuous fashion. Therefore, as long as Σ does not become singular, the winding number of the field configuration ϕ will not change in the time evolution of the initial data.

Another case would be that of an axion field a . One might consider that what has physical meaning is not a but ∂a . The zero mode being meaningless, the solution would be valid as well.

D. Einstein-Maxwell-dilaton wormholes

The single valuedness of the metric in our *Ansatz* Eq. (35) requires

$$\begin{aligned}\psi(\theta, \mu + \mu_0) &= K^{\frac{1}{2(1+b)}} \psi(\theta, \mu), \\ \chi(\theta, \mu + \mu_0) &= K^{\frac{-1}{2(1-b)}} \chi(\theta, \mu).\end{aligned}\quad (88)$$

Writing again $K = e^{-2\lambda}$ we get

$$\begin{aligned}\psi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{\frac{n}{1+b}\lambda} [\cosh(\mu + n\mu_0) - \cos \theta]^{-1/2}, \\ \chi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{\frac{-n}{1-b}\lambda} [\cosh(\mu + n\mu_0) - \cos \theta]^{-1/2}.\end{aligned}\quad (89)$$

Now let us examine the monodromy properties of the dilaton and the electrostatic potential. Going around the wormhole once,

$$\begin{aligned}\phi(\theta, \mu + \mu_0) &= \phi(\theta, \mu) \mp \frac{2\lambda}{\sqrt{1+b^2}}, \\ Z(\theta, \mu + \mu_0) &= e^{\frac{-2b\lambda}{1-b^2}} Z(\theta, \mu).\end{aligned}\quad (90)$$

The electrostatic potential changes by a factor, and so does the electric field. This means that this is not a valid solution even accepting the multivaluedness of the dilaton field.

Fortunately there is another procedure to get a wormhole solution for the Einstein-Maxwell-dilaton system. As I showed in Sec. IV D there is a way of mapping initial data of the Einstein-Maxwell system into initial data of the Einstein-Maxwell-dilaton system. One could start from the Reissner-Nordström wormhole, Eqs. (39),(83), and transform it by means of a function $t(Z)$. According to Eq. (84), the electrostatic potential Z of the Reissner-Nordström wormhole is not a single-valued function in the initial surface. Its value changes by an amount of $T = \pm 2\lambda$. Since $t(Z)$ is the new electrostatic potential, it is clear that one simply needs a function $t(Z)$ periodic in Z with a period which is a submultiple of T and satisfies the differential equation (47). This would also guarantee the single valuedness of the dilaton field which is a power of dt/dZ .

The surprising thing is that the function needed for this problem was already found in Sec. IV D. It is given by Eq. (48) with the constant q given by

$$q = 1 + \frac{32\pi^2\lambda}{n^2}, \quad n = 1, 2, \dots, \quad (91)$$

and appropriate choices of the constants A, B, D ; so $t(Z)$ is simply

$$\begin{aligned}t(Z) &= M \sin(\omega Z + \gamma_0), \\ \omega &= \frac{4\pi\lambda}{n}.\end{aligned}\quad (92)$$

I rewrite below for convenience the whole wormhole initial-data family for the Einstein-Maxwell-dilaton system:

$$\begin{aligned}dl^2 &= \tilde{W}(d\mu^2 + d\theta^2 + \sin^2\theta d\phi^2), \\ \tilde{W} &= pM^2\omega^2 \cos^2[\omega \ln(\psi/\chi) + \gamma_0](\psi\chi)^2, \\ e^{-2\tilde{\phi}} &= \frac{1+2\omega^2}{M^2\omega^2} \cos^{-2}[\omega \ln(\psi/\chi) + \gamma_0], \\ \tilde{Z} &= M \sin[\omega \ln(\psi/\chi) + \gamma_0](\psi\chi)^2, \\ \psi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{n\lambda} [\cosh \mu - \cos \theta]^{-1/2}, \\ \chi &= k^{1/2} \sum_{n \in \mathbb{Z}} e^{-n\lambda} [\cosh \mu - \cos \theta]^{-1/2}, \\ \omega &= \frac{4\pi\lambda}{n}.\end{aligned}\quad (93)$$

Although this initial-data set has the required mon-

odromy properties, it is far from being a regular initial-data set. The metric function \tilde{W} vanishes in many places, and in those places the dilaton field blows up.

Observe that, again, after performing an electric-magnetic duality transformation and shifting the dilaton by an appropriate constant, the string-frame metric is just that of the Reissner-Nordström wormhole, Eqs. (39),(83), perfectly regular. The dilaton field ϕ still blows up in many places. This, or so it seems, is the price one has to pay for being able to find a single-valued dilaton field in a space with wormhole topology.

VII. CONCLUSIONS

In this paper I have found several families of time-symmetric initial-data sets for theories with a massless scalar (dilaton) which takes values in R or in a circle S^1 . These families depend on a certain number of parameters. For certain values of the parameters, these solutions describe several black holes (Einstein-Rosen-like bridges connecting different asymptotically flat regions) in the instant in which they “bounce.” Some solutions describe two black holes connected by a “wormhole.”

In the case of a single black hole it is possible to prove analytically that the same choice of values of the parameters ensures the regularity of the solution, the positivity of the mass in the two asymptotically flat regions, and the existence of an apparent horizon.

The presence of a scalar field has many interesting effects. Perhaps the most unusual one is that it is more difficult to “build” initial surfaces with nontrivial topologies on which the initial data are regular. If one had a gauge field on that surface, it would be easier to find solutions in different topologically trivial patches and then glue them together because the gauge fields of two overlapping patches do not have to match exactly in the overlap: They only have to match up to a *gauge transformation*. In the dilaton case there is no gauge invariance available and the solutions have to match exactly.

Most fields usually considered are not scalars and have gauge invariances. This is true for the metric, vector fields, and axion two-form (and higher-order differential forms). It seems that in theories containing this kind of fields there are more possible classical configurations than in theories containing scalars.

In some sense a scalar seems to play the role of a *topological censor*. Of course, more work is necessary to determine to which extent this is so and which kind of topologies are not allowed if the absence of singularities and single valuedness are required.

These results can be generalized and extended to more complex cases: theories with many scalars (nonlinear σ models), with scalar potential, scalar masses, etc. Also, time-symmetric initial-data sets for other interesting systems besides black holes can be studied. Particularly interesting in this context are black strings and black membranes.

Finally, these initial-data sets can be used as the starting point for investigations on cosmic censorship along the lines of Ref. [8] and, perhaps, critical behavior in the

gravitational collapse of a scalar field.⁹ Work on these issues is in progress.

⁹Observe that, at the critical value of the parameter that measures the energy of the imploding scalar wave, a “point-like” black hole is formed and explodes again. It is reasonable to expect that the instant of time at which the black hole exists is a moment of time symmetry.

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