# Exact inflationary cosmologies with exit

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We find exact inflationary solutions with exit, including exact forms of the potential, by specifying the rate of expansion and using the number of $e$-foldings as the effective dynamical variable. These include solutions which are nearly exponential or power-law inflation for $t_i \le t < t_e$, and then develop smoothly towards radiationlike evolution for $t \ge t_e$.

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## I. INTRODUCTION

Inflationary cosmologies not only solve some of the problems inherent in the standard cosmologies, but also provide a mechanism for generating the perturbations that could be the seeds of structure formation [1, 2]. A combination of an inflationary era with the standard radiation and dust eras can produce very successful cosmological models. One of the issues arising in such models is the transition from the inflationary to the radiation eras, i.e., the problem of "exit" from inflation. In this paper we consider an aspect of this problem: the determination of exact solutions with exit, including exact forms of the potential.

We present in Sec. II a new method for generating exact inflationary solutions, in which the Hubble rate is chosen as a function of the scale factor, and the number of $e$-foldings is the effective dynamical variable. Two new solutions are investigated in Sec. III. These are smooth isentropic solutions in which the potential initially drives nearly exponential or power-law inflation, and then subsequently "converts" the scalar field to radiationlike dynamical behavior. Although these solutions are clearly very simplified models, and in particular cannot account for reheating and entropy production, they are consistent with the typical approximate parameters governing the inflationary era. Such exact solutions may help to broaden an understanding of the theoretical possibilities encompassed by inflationary cosmologies. In this sense, our solutions extend the body of exact inflationary solutions built up by previous papers [3–16].

In order to estimate the entropy production in models based on these solutions, in Sec. IV we construct solutions where the smooth transition from inflation to radiation is replaced by an instantaneous "jump." (Similar "jump" solutions have been previously discussed, for example in [17, 18].) Exact exponential and power-law inflation are joined to exact radiation expansion at exit time, subject to the required junction conditions. These solutions are idealized models since the actual process whereby the scalar field decays, with associated fluctuations, particle production, and reheating, is reduced to an instantaneous conversion of the field to radiation. However, they are able to account for the right order of magnitude of entropy production, given possible parameters of the inflationary era.

## II. METHOD FOR GENERATING EXACT SOLUTIONS

Consider a spatially flat Friedmann-Robertson-Walker universe

$$ds^2 = -dt^2 + a(t)^2[dx^2 + dy^2 + dz^2] ,$$

containing a minimally coupled scalar field $\phi(t)$ with self-interaction potential $V(\phi)$, energy density $\rho(t)$, and pressure $p(t)$. The conservation equations $T^{ab}{}_{;b} = 0$ are

$$\dot{\rho} + 3H(\rho + p) = 0 , \tag{1}$$

where $H = \dot{a}/a$ is the Hubble rate. Equation (1) is equivalent to the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 , \tag{2}$$

since

$$\rho = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) , \tag{3}$$

$$p = \tfrac{1}{2}\dot{\phi}^2 - V(\phi) . \tag{4}$$

Radiation (i.e., thermalized massless and ultrarelativistic particles) may be described in a scalar-field formalism with $p = \frac{1}{3}\rho$, although the field and potential no longer have the physical meaning attached to them in the inflationary era. By (1),

$$\text{radiation:} \quad \rho = 3p = \frac{\text{const}}{a^4} = AT^4 \ , \tag{5}$$

where $T$ is the temperature and $A$ is the radiation constant times the number of effectively massless species, which we take as about 100 [1].

The field equations are (in units such that $8\pi G = 1 = c$)

$$2\dot{H} + 3H^2 = -p \ , \tag{6}$$

$$3H^2 = \rho \ . \tag{7}$$

Using (3), (4) these lead to the equivalent pair of equations for $V$ and $\dot{\phi}$:

$$V = \dot{H} + 3H^2 \ , \tag{8}$$

$$\dot{\phi}^2 = -2\dot{H} \ . \tag{9}$$

Equation (9) implies a reality condition, limiting the acceleration:

$$\ddot{a} \leq aH^2 \ . \tag{10}$$

During the inflationary era the acceleration must be positive:

$$\text{inflation:} \quad \ddot{a} > 0 \ \Leftrightarrow \ p < -\tfrac{1}{3}\rho \ , \tag{11}$$

where the equivalence follows from (6), (7). In an exact radiation era, (5) together with (6)–(9) shows that

$$\text{radiation:} \quad a(t) \propto (t - t_0)^{1/2} \ , \ \ V(\phi) \propto e^{-2\phi} \ , \tag{12}$$

where $t_0$ is constant, and $V, \phi$ have only a formal mathematical meaning.

The following procedure for constructing exact inflationary models has been proposed in [6]: Specify a monotonic scale factor $a(t)$ [which directly implies $H(t)$], and derive $V(t)$ and $\phi(t)$ from (8), (9). In principle one can invert to obtain $t(\phi)$ and thus $V(\phi)$. We use this approach in Sec. III A to construct a solution that evolves from nearly power-law inflation to radiationlike expansion.

The standard approach to inflation starts from particle physics and proceeds to cosmology: One specifies $V(\phi)$ and tries to find the corresponding evolution $a(t)$ [which usually leads to approximations for solving the Klein-Gordon equation (2)]. As long as there is no decisive information from particle physics as to what the inflaton is or what the form of $V(\phi)$ should be, a complementary approach can be useful for deepening our understanding of inflation: One specifies the evolution $a(t)$ and then derives the potential $V(\phi)$ that drives such an evolution. This is the approach of [6], used and extended here. It amounts to a cosmological prediction of $V(\phi)$, if one can justify a particular form of the evolution of the universe from other observational and physical evidence or principles [6, 19].

Other complementary approaches have also been developed. The various approaches are partly distinguished by what is specified in order to find exact solutions. In [6] $a(t)$ is specified, whereas, for example, [4] specifies $p(\rho)$, [7] specifies $H(\phi)$, [10, 11] specify $\phi(t)$, and [15] specifies $V(H)$. The different approaches have different advantages and disadvantages. The advantage of [6] is that one can directly specify a desired time evolution of the universe, although one may be faced with the impossibility of inverting to find $V(\phi)$ in closed form.

Our extension of [6] shares its advantage and its possible drawback. We specify $H(a)$, rather than $a(t)$, and use $\ln a$, rather than $t$, as the dynamical variable. As in [6], a desired evolution (strictly, rate of evolution) is specified, and the potential that produces it is derived. In order to render the equations autonomous, we introduce the variable

$$\alpha = \ln\left(\frac{a}{a_i}\right) \ , \tag{13}$$

where $t_i$ is the time that inflation begins. It follows that $\alpha(t)$ is the number of $e$-foldings at time $t \geq t_i$. Now suppose a rate of evolution $H(\alpha)$ is specified. Then the potential and field follow from (8),(9) and the scale factor follows implicitly from the Hubble rate:

$$V(\alpha) = \tfrac{1}{2}\left[H^2(\alpha)\right]' + 3H^2(\alpha) \ , \tag{14}$$

$$\phi(\alpha) = \phi_i \pm \int_0^\alpha \sqrt{-2\left[\ln H(\alpha)\right]'}d\alpha \ , \tag{15}$$

$$t(a) = t_i + \int_0^{\ln(a/a_i)} \frac{d\alpha}{H(\alpha)} \ , \tag{16}$$

where a prime denotes $d/d\alpha$. In principle one can invert $\phi(\alpha)$ to obtain $V(\phi)$. By (6), (7) and (13), the energy density and pressure are, like the potential in (14), linear in $H^2$:

$$\rho(\alpha) = 3H^2(\alpha) \ , \ \ p(\alpha) = -\left[H^2(\alpha)\right]' - 3H^2(\alpha) \ .$$

The linear form (14) of the potential in terms of $H^2$, which follows because of the choice of $\alpha$ as a dynamical variable, reflects a possible advantage over the method of [6], in that our method may have a better chance of producing explicit forms $V(\phi)$. The price paid for this is that one often is unable to invert (16) and obtain $a$ as a function of $t$. This relative merit and weakness is well illustrated by the solutions in Secs. III A (derived by the method of [6]) and III B (derived by our method).

The reality condition (10) is just $H' \leq 0$. From (11), the condition for inflation also takes a simple form in the new variables:

$$\begin{aligned} &\text{inflation:}\ H' > -H \ ; \\ &\text{exit:} \qquad H' = -H \ . \end{aligned} \tag{17}$$

Some simple choices of $H$ lead to interesting solutions. Consider first a "power-law" rate of evolution,

$$H = H_i\left(\frac{a_i}{a}\right)^{1/n} = H_i e^{-\alpha/n} \ , \tag{18}$$

where $n$ is a positive constant. By (17), this is inflationary (without exit) provided $n > 1$. From (14)–(16) we see that (18) is indeed standard power-law inflation:

$$a = a_i \left[\frac{H_i}{n}(t - t_i) + 1\right]^n ,$$

$$V(\phi) = \left(3 - \frac{1}{n}\right) H_i^2 \exp\left[\mp\sqrt{\frac{2}{n}}(\phi - \phi_i)\right] .$$

Second, we try an exponentially decreasing rate of evolution, $H \propto e^{-a/k}$ ($k$ a positive constant). By (17), this is inflationary while $a < k$, so that $k = a_e$, where $t_e$ is the time of exit:

$$H = H_i \exp\left[\frac{a_i}{a_e}\left(1 - \frac{a}{a_i}\right)\right] = H_i \exp\left[\frac{a_i}{a_e}\left(1 - e^{\alpha}\right)\right] . \tag{19}$$

Then (14)–(16) imply

$$\phi = \phi_i \pm 2\sqrt{2}\left(\sqrt{\frac{a}{a_e}} - \sqrt{\frac{a_i}{a_e}}\right) , \tag{20}$$

$$V(\Phi) = H_i^2 \left(3 - \frac{a_i}{a_e}\Phi^2\right) \exp\left[2\frac{a_i}{a_e}(1 - \Phi^2)\right] , \tag{21}$$

$$\pm\Phi \equiv 1 \pm \sqrt{\frac{a_e}{8a_i}}(\phi - \phi_i) = \pm\sqrt{\frac{a}{a_i}} ,$$

$$t = t_i + \frac{1}{H_i e^{a_i/a_e}}\left[\mathrm{Ei}\left(\frac{a}{a_e}\right) - \mathrm{Ei}\left(\frac{a_i}{a_e}\right)\right] , \tag{22}$$

where Ei is the exponential integral function [20].

The potential (21) has a maximum of approximately the de Sitter value $3H_i^2$ at $\Phi = 0$ (before $t_i$), and decreases monotonically to a minimum, passing through zero:

$$V = 0 \quad \text{at} \quad a = 3a_e .$$

It then tends to zero from below as $|\Phi| \to \infty$. However, since

$$a > 3a_e \;\Rightarrow\; p > \rho ,$$

the solution is only valid for $a \leq 3a_e$. The potential at the start of inflation is slightly displaced from its maximum,

$$V_i = \left(3 - \frac{a_i}{a_e}\right) H_i^2 ,$$

and exit occurs well before the potential goes negative:

$$V_e = 2\exp\left[-2\left(1 - \frac{a_i}{a_e}\right)\right] H_i^2 .$$

By (19), the Hubble rate at exit is

$$H_e \approx \frac{H_i}{e} ,$$

since $a_i \ll a_e$. Using the expansion [20]

$$\mathrm{Ei}(x) = C + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k!k} ,$$

where $C$ is Euler's number, (22) shows that

$$t_e - t_i \approx \frac{1}{H_i}\left[\ln\frac{a_e}{a_i} + \sum_{k=1}^{\infty}\frac{1}{k!k}\right] \approx \frac{1}{H_i}\left(\ln\frac{a_e}{a_i} + 1.32\right) .$$

This is consistent with typical values for the inflationary parameters [1] [see Eqs. (25) and (39) below]. Thus (19)–(22) is a possible inflationary model with exit.

Third, consider the rate of evolution,

$$H = \frac{\text{const}}{1 + (a/k)^2} , \tag{23}$$

where $k$ is constant. The expansion is initially nearly de Sitter, inflating while $a < k = a_e$ [by (17)], and then evolves towards radiationlike expansion. The potential $V(\phi)$ may be found explicitly. We investigate this solution in Sec. III B.

## III. ISENTROPIC SOLUTIONS THAT EXIT TO RADIATION

Here we investigate two solutions arising from Sec. II that are inflationary for $t_i \leq t < t_e$, pass through a transition, and then become radiationlike for $t \gg t_e$. A scalar-field formalism is used to encompass the whole evolution. This is a mathematical model which traces the overall behavior of the evolution, without regard to the actual physical process whereby the inflaton field decays, creates particles, and transfers energy and entropy to massless and effectively massless particles. Strictly, the solutions are unphysical after exit, since a "cold" scalar field still drives the expansion rather than "hot" radiation. Physical solutions with exit require a coupling of the scalar field to radiation [21]. We find that *noninteracting* radiation which is present at the onset of inflation is unable to dominate after exit, since there is no mechanism for energy exchange.

There is little hope of finding exact physical solutions, and our exact mathematical solutions may provide a framework for comparison. We show that the solutions are consistent with the typical approximate parameters governing inflation. With $t_i \ll t_e \approx 10^{-32}$ s and about 60 $e$-folds, we find that $p/\rho$ is within $10^{-4}$ of the radiation value $\frac{1}{3}$ by $t \approx 10^{-30}$ s, and the temperature is $T \approx 10^{25}$ K. The exponent for the nearly power-law inflation case is $n \approx 13$. The solutions do not satisfy the slow-roll conditions, although the nearly power-law case does exhibit "andante" [16] behavior near the start of inflation.

### A. Power-law inflation to radiation era

Using the method of [6], we begin by modifying an ansatz that models the transition from radiation to dust [21] (see also [22]). This ansatz is of the form $a \sim t^n(t + t_0)^k$, where $t_0, k, n$ are constants. Adjusting these constants to the required asymptotic behavior and initial conditions, we find that

$$a(t) = \frac{a_e}{\sqrt{t_e}}[1 + m]^{(2n-1)/2} t^n [t + m t_e]^{(1-2n)/2} , \tag{24}$$

where

$$m = \frac{n + \sqrt{n(2n-1)}}{2n(n-1)}$$

and $\ddot{a}(t_e) = 0$. Then (24) is a smooth joining of asymptotic power-law inflation (for $n > 1$) to asymptotic radiation expansion:

$$t \ll mt_e \Rightarrow a(t) \approx a_e[1 + 1/m]^{(2n-1)/2} \left(\frac{t}{t_e}\right)^n ,$$

$$t \gg mt_e \Rightarrow a(t) \approx a_e[1 + m]^{(2n-1)/2} \left(\frac{t}{t_e}\right)^{1/2} .$$

By (24),

$$\left(\frac{t_e}{t_i}\right)^n \approx \left[1 + \frac{1}{m}\right]^{(2n-1)/2} \left(\frac{a_e}{a_i}\right) ,$$

and if we use the approximate values [1]

$$\frac{t_e}{t_i} \approx 10^3 , \quad \frac{a_e}{a_i} \approx e^{60} , \tag{25}$$

we find that

$$13 < n < 14 .$$

If the number of $e$-foldings is increased from 60 to 100, we get $n \approx 26$. However, the numerical results below do not change significantly with this change in $n$, and we will take $n \approx 13$ subsequently.

Using (9), (24) implies [20]

$$\phi(t) = \phi_i \pm \left\{ \sqrt{2n-1} \arcsin\left[\frac{\sqrt{2n-1}mt_e}{\sqrt{2n}(t + mt_e)}\right] + \ln\left[f(t) + t + 2nmt_e\right] - \sqrt{2n} \ln\left[\sqrt{2n}t^{-1}f(t) + 2n + 2nmt_et^{-1}\right] \right\} \mp k , \tag{26}$$

where

$$f(t) = \left[t^2 + 4nmt_et + 2nm^2t_e^2\right]^{1/2}$$

and $k$ equals the expression in curly brackets evaluated at $t_i$. Clearly it is impossible to invert and obtain $t = t(\phi)$, so that we are unable to give $V(\phi)$ explicitly. From (8) and (24), we get the potential $V(t)$ that governs $\phi$, hence producing the evolution (24):

$$V(t) = \frac{t^2 + 4nmt_et + 4n(3n-1)m^2t_e^2}{4t^2(t + mt_e)^2} . \tag{27}$$

For $t \gg t_e$, (26), (27) show that $V(\phi) \approx (\text{const}) \times e^{-2\phi}$ [in agreement with (12)], while, for $t \approx t_i \ll t_e$,

$$V(\phi) \approx \frac{n(3n-1)}{t_i^2} \exp\left[-\sqrt{\frac{2}{n}}(\phi - \phi_i)\right] ,$$

where the right hand side is of course the form for exact power-law inflation [10]. $V(\phi)$ is monotonically decreasing, and during inflation it is bounded by

$$\frac{1 + 4nm + 4n(3n-1)m^2}{4(1+m)^2t_e^2} = V_e < V \leq V_i \approx \frac{n(3n-1)}{t_i^2} .$$

Using (25) and $t_e \approx 10^{-32}$ s [1], this gives, with $n \approx 13$,

$$1.9 \times 10^{13}\ \text{GeV} \approx V_e^{1/4} < V^{1/4} \leq V_i^{1/4} \approx 2 \times 10^{15}\ \text{GeV} .$$

The exit value is well within the limit ($\approx 10^{16}$ GeV) necessary to avoid too much large-scale anisotropy in the microwave background radiation [23, 24].

In terms of the parameters

$$\epsilon_H = -\frac{\dot{H}}{H^2} , \quad \eta_H = -\frac{\ddot{H}}{2H\dot{H}} , \tag{28}$$

"slow roll" is characterized to first order by $\epsilon_H, |\eta_H| \ll 1$. (See [25, 26] for a precise definition.) For (14), we find that $\epsilon_H, \eta_H$ are monotonically increasing, and during inflation they are bounded by

$$\frac{1}{n} < \epsilon_H < 1 ,$$

$$\frac{1}{n} < \eta_H < \frac{2 + 12nm + 12nm^2 + 4nm^3}{1 + 6nm + 2n(4n+1)m^2 + 4n^2m^3} .$$

With $13 < n < 14$ this gives $0.076 < \eta_H < 0.83$. Thus the slow-roll conditions cannot be met, although for $t$ close to $t_i$, $\epsilon_H, \eta_H$ are moderately small, indicating moderately slow roll ("andante") behavior [16].

From (4) and (26), (27), we see that the pressure becomes positive after the time

$$t_+ = 2n\left[\sqrt{4 - 2/n} - 1\right] mt_e > t_e ,$$

so that there is a delay after exit from inflation before there is an exit from negative pressure. After $t_+$, the pressure reaches a maximum and then decays to zero. Thus, having "corrected" for inflationary behavior, the pressure "overshoots" before adopting the decay that is characteristic of an ordinary expanding fluid. We can calculate the time $t_r(\varepsilon)$ for $p/\rho$ to approach within $\varepsilon$ of exact radiation behavior. From (26), (27) and (3), (4),

$$\frac{p}{\rho} = \frac{t^2 + 4nmt_et - 4n(3n-2)m^2t_e^2}{3t^2 + 12nmt_et + 12n^2m^2t_e^2} ,$$

so that, for $\varepsilon \ll \frac{1}{3}$,

$$\frac{p}{\rho} = \tfrac{1}{3} - \varepsilon \;\Rightarrow\; \frac{t_r(\varepsilon)}{t_e} \approx \left[\tfrac{2}{3}\sqrt{6n(2n-1)}m\right] \varepsilon^{-1/2} \tag{29}$$

and thus, for $\varepsilon = 10^{-4}$,

$$13 < n < 14 \;\Rightarrow\; 293 > \frac{t_r(10^{-4})}{t_e} > 291 . \tag{30}$$

Taking $t_e \approx 10^{-32}$ s, (29) shows that the expansion is effectively behaving like radiation by about $10^{-30}$ s. We could think of the period from $t_e$ to $t_r(\varepsilon)$ as an idealized $\phi$ decay, during which the expansion is "converting" to radiation. The energy density at the onset of radiationlike behavior follows from (24) and (7) (restoring units):

$$\rho_r(\varepsilon) \approx \left(\frac{9c^2}{256\pi G}\right) \frac{\varepsilon}{n(2n-1)m^2t_e^2} .$$

With $n \approx 13$ and $t_e \approx 10^{-32}$ s, this gives

$$\rho_r(10^{-4}) \approx 4.7 \times 10^{85}\ \text{erg cm}^{-3}$$
$$\Rightarrow T_r(10^{-4}) \approx 8.9 \times 10^{24}\ \text{K} , \tag{31}$$

where we have used (5), which holds to a good approximation.

### B. Exponential inflation to radiation era

Here we return to the rate of evolution (23), which is initially de Sitter–like (since $H \approx$ constant for small $a$), and becomes radiationlike (since $H \sim 1/a^2$ for large $a$):

$$H(a) = H_i \left(\frac{a_e^2 + a_i^2}{a_e^2 + a^2}\right) = H_i \left[\frac{(a_e/a_i)^2 + 1}{(a_e/a_i)^2 + e^{2\alpha}}\right] . \qquad (32)$$

From (14)–(16), (32) implies

$$\phi = \phi_i \pm 2 \left[\operatorname{arcsinh}\left(\frac{a}{a_e}\right) - \operatorname{arcsinh}\left(\frac{a_i}{a_e}\right)\right] , \qquad (33)$$

$$V(\Phi) = (H_i^2 \cosh^2 \Phi_i) \left[\frac{3 - 2\tanh^2 \Phi}{\cosh^4 \Phi}\right] , \qquad (34)$$

$$\pm\Phi \equiv \tfrac{1}{2} \left[\phi - \phi_i \pm 2\operatorname{arcsinh}\left(\frac{a_i}{a_e}\right)\right]$$
$$= \pm\operatorname{arcsinh}\left(\frac{a}{a_e}\right) ,$$
$$t = t_e + \frac{a_e^2}{2H_i(a_i^2 + a_e^2)} \left[\ln\left(\frac{a}{a_e}\right)^2 + \left(\frac{a}{a_e}\right)^2 - 1\right] . \qquad (35)$$

It follows that $V$ has a maximum of $3H_i^2 \cosh^2 \Phi_i$ at $\Phi = 0$ (before $t_i$), and then falls away monotonically towards zero as $\Phi$ increases, or equivalently as $t$ increases. The initial field

$$|\Phi_i| = \operatorname{arcsinh}\left(\frac{a_i}{a_e}\right)$$

has a very small positive value [see (25)]. We can view this situation as the field having been shifted by fluctuations out of its unstable false vacuum $\Phi = 0$ at the onset of inflation, subsequently "rolling down" the potential.

From (34) we find the limits on the potential during inflation,

$$H_i^2 \approx V_e < V \leq V_i \approx 3H_i^2 ,$$

so that, for $H_i \approx 6 \times 10^{33}$ s$^{-1}$ [1, 23], we get

$$7.7 \times 10^{13}\ \text{GeV} \approx V_e^{1/4} < V^{1/4} \leq V_i^{1/4}$$
$$\approx 1.3 \times 10^{14}\ \text{GeV} .$$

Thus $V_e^{1/4}$ is greater than for the power-law model, but still well within the limit of about $10^{16}$ GeV.

The slow-roll parameters (28) for (32),

$$\epsilon_H = \frac{2}{1 + (a_e/a)^2} , \quad \eta_H = \frac{2 - (a_e/a)^2}{1 + (a_e/a)^2} ,$$

are monotonically increasing and bounded by

$$2\left(\frac{a_i}{a_e}\right)^2 < \epsilon_H < 1 , \quad -1 < \eta_H < \tfrac{1}{2} .$$

It follows that $\epsilon_H \ll 1$ only near the start of inflation, when $\eta_H$ is about $-1$, while $|\eta_H| \ll 1$ only near $a = a_e/\sqrt{2}$, when $\epsilon_H$ is about $\frac{2}{3}$. Thus there is no interval when the slow-roll, or even moderate-slow-roll, conditions are met.

We consider now the pressure and the question of when the expansion begins to behave like radiation. By (3), (4) and (34),

$$p = H_i^2 \cosh^2 \Phi_i \left(\frac{4\tanh^2 \Phi - 3}{\cosh^4 \Phi}\right) , \quad \frac{p}{\rho} = \tfrac{4}{3}\tanh^2 \Phi - 1 . \qquad (36)$$

It follows from (33)–(36) that the pressure is initially nearly $-3H_i^2$, and becomes positive after exit from inflation:

$$a_+ = \sqrt{3}a_e \Rightarrow t_+ = t_e + \frac{2 + \ln 3}{2H_i[1 + (a_i/a_e)^2]} .$$

Subsequently, the pressure rises to a maximum $\approx H_i^2/108$, and then falls away towards zero as $t$ increases. Once again, we can calculate the time $t_r(\varepsilon)$ at which $p/\rho$ is within $\varepsilon$ of $\frac{1}{3}$. By (36) and (34), for $\varepsilon \ll \frac{1}{3}$,

$$\frac{a_r(\varepsilon)}{a_e} \approx (\tfrac{2}{3}\sqrt{3})\varepsilon^{-1/2} , \qquad (37)$$

and then (35) and (25) give, for $\varepsilon = 10^{-4}$,

$$t_r(10^{-4}) \approx t_e + \frac{6669}{H_i} . \qquad (38)$$

If we take possible values [1, 23] of

$$H_i \approx 6 \times 10^{33}\ \text{s}^{-1} , \quad t_e \approx 10^{-32}\ \text{s} , \qquad (39)$$

then (38) gives

$$t_r(10^{-4}) \approx 1.1 \times 10^{-30}\ \text{s} ,$$

which is less than, but of the same order of magnitude as, the value (30) obtained for the power-law model.

The energy density at $t_r(\varepsilon)$ follows from (32), (7) and (34), (37) (restoring units):

$$\rho_r(\varepsilon) \approx \left(\frac{27c^2}{512\pi G}\right) H_i^2 \varepsilon^2 , \qquad (40)$$

and by (38), (39) this implies

$$\rho_r(10^{-4}) \approx 1.1 \times 10^{86}\ \text{erg cm}^{-3}$$
$$\Rightarrow T_r(10^{-4}) \approx 1.1 \times 10^{25}\ \text{K} , \qquad (41)$$

which is greater than, but of the same order of magnitude as, the power-law value (31). We see that in the exponential model, the transition to radiation is more rapid and ends with a higher temperature than in the power-law model. If we take $H_i \approx 10^{-34}$ s$^{-1}$ (i.e., 100 rather than 60 $e$-foldings), we find that the rapidity and final temperature are slightly increased:

$$t_r(10^{-4}) \approx 0.7 \times 10^{-30}\ \text{s} , \quad T_r(10^{-4}) \approx 1.4 \times 10^{25}\ \text{K} .$$

## IV. SOLUTIONS WITH ENTROPY PRODUCTION

The solutions of the previous section are characterized by a period $t_e < t < t_r(\varepsilon)$ during which the scalar field "converts" isentropically from inflationary to radiation behavior. In reality this process would involve decay of the scalar field and conversion of its energy to the particles of the radiation era, with associated reheating and entropy production. In order to estimate the entropy production due to solutions like those of Sec. III (i.e., inflation with exit to radiation), we construct idealized models which do generate entropy via decay of $\phi$.

As a step towards more realistic models, we consider exact solutions in which the decay and reheating take place instantaneously at $t_e$, with a consequent discontinuity in the pressure, which instantaneously changes from negative to positive. Such "jump" solutions have been used for rough estimates of entropy (see, for example, [23]). We derive exact formulas and compare different types of inflation.

The scale factor and Hubble rate are continuous, so that the junction conditions are satisfied [17, 18]. If $\mu$ denotes the radiation energy density, it follows from the Friedmann equation (7) that

$$\rho_e = \mu_e = 3H_e^2 \ , \tag{42}$$

while the field equation (6) shows that $\dot{H}$ shares the discontinuity of the pressure.

The instantaneous decay of $\phi$, producing a jump in $\mu$ from zero to $3H_e^2$, generates radiation entropy $S_e$ in the inflated comoving volume $a_e^3$. For $t > t_e$ the expansion is effectively isentropic, ignoring the comparatively negligible entropy production in subsequent phase transitions, decouplings, and other dissipative processes. Thus the total entropy in the comoving volume $a(t)^3$ that expands from $a_e^3$ at $t = t_e$ is conserved: $S(t) = S_e$. If $a_i$ covers the causally connected region that evolves into the present-day observable universe, then $S_e$ is the total present-day entropy $S_0$ believed to have a value [1, 27]

$$S_0 \approx 10^{88} \ . \tag{43}$$

Now $S_e = \sigma_e a_e^3$, with $\sigma$ the entropy density, given by [1]

$$\sigma = \left(\frac{4A}{3k}\right) T^3 = \left(\frac{4}{3k}\right)\left(\frac{\mu^3}{A}\right)^{1/4} \ , \tag{44}$$

where $k$ is Boltzmann's constant, and we used (5) for $\mu$. From (42), (44) it follows that (restoring units)

$$S_e = \left[\frac{Ac^6}{6\pi^3 G^3 k^4}\right]^{1/4} a_e^3 H_e^{3/2} \ , \tag{45}$$

where the relation of $a_e, H_e$ to $a_i, H_i$ depends on the type of inflation.

Consider first an exponential inflation (without radiation) during $t_i \le t \le t_e$, matched to a radiation era (without scalar field) during $t \ge t_e$. The scale factor is thus given by

$$a(t) = \begin{cases} a_i \exp[H_i(t - t_i)], & t_i \le t \le t_e \ , \\ a_e\sqrt{2H_i}\left(t - t_e + \frac{1}{2H_i}\right)^{1/2}, & t_e \le t \ , \end{cases} \tag{46}$$

where

$$a_e = a_i \exp[H_i(t_e - t_i)] \tag{47}$$

and the Hubble rate is

$$H(t) = \begin{cases} H_i, & t_i \le t \le t_e \ , \\ \frac{1}{2}\left(t - t_e + \frac{1}{2H_i}\right)^{-1}, & t_e \le t \ , \end{cases}$$

so that $H_e = H_i$. Using this together with (37) in (35), we get

$$S_e = \left[\frac{Ac^6}{6\pi^3 G^3 k^4}\right]^{1/4} a_i^3 H_i^{3/2} \exp[3H_i(t_e - t_i)] \ , \tag{48}$$

and taking the values (25) and (39), with $a_i = ct_i$, we find

$$S_e \approx 4.4 \times 10^{88} \ , \tag{49}$$

in agreement with (43).

Second, consider power-law inflation matched to radiation expansion at $t_e$. The scale factor is

$$a(t) = \begin{cases} a_i\left[(t - \zeta)/(t_i - \zeta)\right]^n, & t_i \le t \le t_e \ , \\ a_e\left[(t - \nu)/(t_e - \nu)\right]^{1/2}, & t_e \le t \ , \end{cases} \tag{50}$$

and the Hubble rate is

$$H = \begin{cases} n(t - \zeta)^{-1}, & t_i \le t \le t_e \ , \\ \frac{1}{2}(t - \nu)^{-1}, & t_e \le t \ , \end{cases} \tag{51}$$

where $n > 1$ and the matching conditions are satisfied for

$$\zeta = \frac{\omega t_i - t_e}{\omega - 1} \ , \quad \omega \equiv \left(\frac{a_e}{a_i}\right)^{1/n} \ ,$$

$$\nu = \frac{[(2n-1)\omega - 2n]t_e - \omega t_i}{2n(\omega - 1)} \ .$$

(It follows that $\zeta < t_i < \nu < t_e$.) With (50), (51) in (45), we get

$$S_e = \left[\frac{Ac^6}{6\pi^3 G^3 k^4}\right]^{1/4} a_e^3 \left(\frac{n}{t_e - t_i}\right)^{3/2} \times \left[1 - \left(\frac{a_e}{a_i}\right)^{-1/n}\right]^{3/2} \ . \tag{52}$$

Using the same values that led to (49), together with

$n \approx 13$, (52) gives

$$S_e \approx 4.4 \times 10^{87} , \quad (53)$$

which is not as close to (43) as (49), but still in reasonable agreement with it. As expected from the results of Sec. III, the exponential inflation produces greater entropy.

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[1] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990); G. Borner, *The Early Universe* (Springer, Berlin, 1992).
[2] A.R. Liddle and D.H. Lyth, Phys. Rep. **231**, 1 (1993).
[3] A.G. Muslimov, Class. Quantum Grav. **7**, 231 (1990).
[4] J.D. Barrow, Phys. Lett. B **235**, 40 (1990).
[5] J.D. Barrow and P. Saich, Phys. Lett. B **249**, 406 (1990).
[6] G.F.R. Ellis and M.S. Madsen, Class. Quantum Grav. **8**, 667 (1991).
[7] J.E. Lidsey, Class. Quantum Grav. **8**, 923 (1991).
[8] J.E. Lidsey, Phys. Lett. B **273**, 42 (1991).
[9] J.E. Lidsey, Gen. Relativ. Gravit. **25**, 399 (1993).
[10] J.D. Barrow, Phys. Rev. D **48**, 1585 (1993).
[11] J.D. Barrow, Phys. Rev. D **49**, 3055 (1994).
[12] J.D. Barrow and A.R. Liddle, Phys. Rev. D **47**, R5219 (1993).
[13] B.J. Carr and J.E. Lidsey, Phys. Rev. D **48**, 543 (1993).
[14] J.E. Lidsey, Mon. Not. R. Astron. Soc. **266**, 489 (1994).
[15] F.E. Schunck and E.W. Mielke, Phys. Rev. D **50**, 4794 (1994).
[16] J.E. Lidsey and I. Waga, Phys. Rev. D **51**, 444 (1995).
[17] G.F.R. Ellis, Class. Quantum Grav. **5**, 891 (1988).
[18] L.P. Grishchuk, Phys. Rev. D **50**, 7154 (1994).
[19] S.S. Bayin, F.I. Cooperstock, and V. Faraoni, Astrophys. J. **428**, 439 (1994).
[20] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1969).
[21] A.A. Coley and B.O.J. Tupper, Can. J. Phys. **64**, 204 (1986); J. Math. Phys. **27**, 406 (1986).
[22] A.V. Nesteruk, Class. Quantum Grav. **11**, L15 (1994); A.V. Nesteruk and A.C. Ottewill, *ibid.* **12**, 51 (1995).
[23] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993).
[24] A.R. Liddle, Phys. Rev. D **49**, 739 (1994).
[25] E.J. Copeland, E.W. Kolb, A.R. Liddle, and J.E. Lidsey, Phys. Rev. D **48**, 2529 (1994).
[26] A.R. Liddle, P. Parsons, and J.D. Barrow, Phys. Rev. D **50**, 7222 (1994).
[27] D. Goldwirth and T. Piran, Class. Quantum Grav. **8**, L155 (1991).