Fermion doubling and lattice artifacts in the energy spectrum of the Dirac propagator at finite temperature and chemical potential

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We study the Euclidean temporal structure of the lattice Dirac fermion propagator for naive fermions and show that excitations in the fourth component of the momentum associated with fermion doubling invalidate the usual correspondence between positive-negative eigenvalues of the Dirac Hamiltonian and forward-backward propagation in time. The doubler contributions to the propagator are pure lattice artifacts which do not scale in the continuum limit. The computations are carried out for finite temperature and chemical potential, and the results are compared with those obtained for Wilson fermions.

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Since Wilson proposed the lattice formulation of gauge theories [1] it has been known that a naive discretization of the fermion action gives rise to a serious problem, known as the fermion doubling problem. High-momentum excitations of the order of the inverse lattice spacing impede the lattice propagator from having the correct continuum limit. Various proposals have been made in the literature to eliminate the fermion doubling problem, of which the so-called Wilson [1] and Kogut-Susskind [2] fermions are the most popular ones. The origin of the additional fermionlike excitations ("doublers") lies in the use of a symmetric (anti-Hermitian) lattice derivative, which estimates the continuum derivative by taking the difference of the fields at lattice sites separated by two lattice spacings. In the partition function for a free Dirac gas at finite temperature and chemical potential, the "doublers," associated with excitations in the fourth component of momentum near the corner of the Brillouin zone manifest themselves in that the positive-*as well as* negative-energy contributions to the partition function involve a mixture of contributions having a form of a gas of particles *and* antiparticles [3]. As we shall demonstrate in this work, this structure is also reflected in the lattice Dirac propagator. Furthermore, we will show that the doubler contributions, associated with excitations in the fourth component of momentum, invalidate the usual correspondence between the sign of the eigenvalue of the Dirac Hamiltonian, and forward-backward propagation in Minkowski time, and that they are pure lattice artifacts. Hence the positive- and negative-energy solutions to the discretized Dirac equation cannot be associated in the usual way with particles and antiparticles, as has also been recently stressed in [4]. In what follows, we first consider the lattice propagator at zero temperature and finite chemical potential, which demonstrates the main points we wish to make. The results are then extended to finite temperature and compared to those obtained for Wilson fermions.

We are interested in studying the (Euclidean) temporal structure of the fermion lattice propagator for finite chemical potential, keeping track of the positive- and negative-energy contributions. To exhibit clearly the role played by the "doublers," we will construct the Green's function from the eigenstates of the Dirac operator $\gamma_\mu \partial_\mu + M - \mu\gamma_4$ multiplied by γ_4. Consider the eigenvalue problem

$$[(\partial_4 - \mu)] + (\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \gamma_4 M)]\psi(x) = \lambda\psi(x) \; . \tag{1}$$

Here α_i are the anti-Hermitian matrices $\gamma_4\gamma_i$, with $\gamma_\mu = \gamma_\mu^\dagger$ ($\mu = 1, \ldots, 4$) satisfying the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The corresponding (dimensionless) eigenvalue equation on the lattice takes the form

$$\sum_m K(n,m)\hat{\psi}(m) = \hat{\lambda}\hat{\psi}(n) \; , \tag{2a}$$

where

$$K(n,m) = \gamma_4 \hat{M}\delta_{nm} + \frac{1}{2}(e^{\hat{\mu}}\delta_{n+\hat{e}_4,m} - e^{-\hat{\mu}}\delta_{n-\hat{e}_4,m}) + \frac{1}{2}\sum_j \alpha_j(\delta_{n+\hat{e}_j,m} - \delta_{n-\hat{e}_j,m}) \; . \tag{2b}$$

The chemical potential has been introduced in the exponential form proposed by Hasenfratz and Karsch [5], and by Kogut *et al.* [6]. Quantities denoted with a caret are measured in lattice units. A complete set of eigenfunctions is given by

$$\hat{\psi}^{(\pm)}_{\hat{p},\sigma}(n) = \omega^{(\pm)}_\sigma(\hat{\mathbf{p}})e^{i\hat{p}\cdot n}, \quad -\pi < \hat{p}_\mu < \pi \; , \tag{3}$$

where $\omega^{(\pm)}_\sigma(\hat{p})$ ($\sigma = 1, 2$) are solutions to

$$H\omega^{(\pm)}_\sigma(\hat{\mathbf{p}}) = \pm\hat{E}(\hat{\mathbf{p}})\omega^{(\pm)}_\sigma(\hat{\mathbf{p}}), \qquad \hat{E}(\hat{\mathbf{p}}) = \left(\sum_j \sin^2\hat{p}_j + \hat{M}^2\right)^{1/2} , \tag{4a}$$

with H the lattice Dirac Hamiltonian

$$H = \sum_j i\alpha_j \sin\hat{p}_j + \gamma_4\hat{M} \; . \tag{4b}$$

The corresponding eigenvalues $\hat{\lambda}$ in (2a) are given by $\hat{\lambda}^{(\pm)}(\hat{p},\hat{\mu}) = i\sin(\hat{p}_4 - i\hat{\mu}) \pm \hat{E}(\hat{\mathbf{p}})$. Since $\hat{\lambda}^{(\pm)} = \lambda^{(\pm)}a$,

$\hat{M} = Ma$ and $\hat{\mu} = \mu a$ vanish as the lattice spacing a goes to zero, the components of the dimensionless lattice four-momenta $\hat{p}_\mu$ approach 0 or $\pm\pi$ in the continuum limit. Excitations of the latter type give rise to the fermion doubling problem. The normalized four-component spinors $w_\sigma^{(+)}(\hat{\mathbf{p}})$ and $w_\sigma^{(-)}(\hat{\mathbf{p}})$ satisfy the relations

$$\sum_\sigma w_\sigma^{(\pm)}(\hat{\mathbf{p}})\bar{w}_\sigma^{(\pm)}(\hat{\mathbf{p}}) = \pm\frac{1}{2\hat{E}}(\pm\gamma_4\hat{E} - i\gamma_j\tilde{p}_j + \hat{M}) \ , \quad (5)$$

where the spatial components of $\tilde{\mathbf{p}}$ are given by $\tilde{p}_i = \sin\hat{p}_i$. In order to keep track of the role played by the "doublers," associated with high-momentum excitations in the fourth component of momentum, it is convenient to group the eigenstates of H into sets with the fourth component of the momentum lying in the ranges $0 < |\hat{p}_4| < \pi/2$, and $\pi/2 < |\hat{p}_4| < \pi$. Equivalently we may consider the complete set of eigenfunction defined by

$$\hat{\psi}_{\hat{p},\sigma}^{(\pm)}(n) = \omega_\sigma^{(\pm)}(\hat{\mathbf{p}})e^{i\hat{p}\cdot n} \ , \quad (6a)$$

$$\tilde{\psi}_{\hat{p},\sigma}^{(\pm)} = (-1)^{n_4}\omega_\sigma^{(\pm)}(-\hat{\mathbf{p}})e^{-i\hat{p}\cdot n} \ ,$$
$$-\frac{\pi}{2} < \hat{p}_4 < \frac{\pi}{2} \ ,$$
$$-\pi < \hat{p}_i < \pi \ . \quad (6b)$$

The eigenfunctions (6b) do not possess a continuum limit which is realized for $\hat{p}_\mu \to 0$, $n \to \infty$, with $\hat{p}\cdot n = p\cdot x$ fixed. Thus, excitations in the fourth component of momentum lying in the interval $\pi/2 < |\hat{p}_4| < \pi$ lead to a doubling of the space of eigenfunctions when the eigenvalue equation (1) is discretized as in (2). Excitations in the spatial components lying in such intervals give rise to an additional eightfold degeneracy, which will however play no role in the following considerations.

The corresponding eigenvalues in (2a) for solutions (6a) and (6b) are given by

$$\hat{\lambda}^{(\pm)}(\hat{p},\hat{\mu}) = i\,\sin(\hat{p}_4 - i\hat{\mu}) \pm \hat{E}(\hat{\mathbf{p}}),$$
$$\tilde{\lambda}^{(\pm)}(\hat{p},\hat{\mu}) = \hat{\lambda}^{(\pm)}(\hat{p},-\hat{\mu}) \ . \quad (7)$$

Since $\tilde{\lambda}^{(\pm)}(\hat{p},\hat{\mu}) = \hat{\lambda}^{(\pm)}(\hat{p},-\hat{\mu})$, both, the positive- as well as negative-energy contributions to $\ln Z$, with Z the partition function of a Dirac gas (which is given by the product of the eigenvalues) will include contributions resembling those of a gas of particles and antiparticles [3].

Consider the following decomposition of the Dirac propagator:

$$S(n,m) = \int_{-\pi/2}^{\pi/2}\frac{d\hat{p}_4}{(2\pi)}\int_{-\pi}^{\pi}\frac{d^3\hat{p}}{(2\pi)^3}\sum_{\sigma,\eta=\pm}\left(\frac{\hat{\psi}_{\hat{p},\sigma}^{(\eta)}(n)\bar{\hat{\psi}}_{\hat{p},\sigma}^{(\eta)}(m)}{\hat{\lambda}^{(\eta)}(\hat{p},\hat{\mu})} + \frac{\tilde{\psi}_{\hat{p},\sigma}^{(\eta)}(n)\bar{\tilde{\psi}}_{\hat{p},\sigma}^{(\eta)}(m)}{\tilde{\lambda}^{(\eta)}(\hat{p},\hat{\mu})}\right) \ , \quad (8)$$

where $\bar{\hat{\psi}} = \hat{\psi}^\dagger\gamma_4$ and $\bar{\tilde{\psi}} = \tilde{\psi}^\dagger\gamma_4$. Define $S(n_4,m_4;\hat{\mathbf{p}})$ by

$$S(n,m) = \int_{-\pi}^{\pi}\frac{d^3\hat{p}}{(2\pi)^3}S(n_4,m_4;\hat{\mathbf{p}})e^{i(\mathbf{n}-\mathbf{m})\cdot\hat{\mathbf{p}}} \ . \quad (9)$$

Furthermore, let $S_F^{(\pm)}(n_4,m_4;\hat{\mathbf{p}})$ and $S_d^{(\pm)}(n_4,m_4;\hat{\mathbf{p}})$ denote the contributions to the propagator arising from the eigenfunctions (6a) and (6b), respectively. The subscript d on $S_d^{(\pm)}$ indicates that these contributions arise from the doubler eigenfunctions (6b), while $S_F^{(\pm)}$ are those contributions which will yield the correct fermion propagator in the continuum limit. After a simple change of variables, and making use of (5), one finds that

$$S(n_4,m_4;\hat{\mathbf{p}}) = S^{(+)}(n_4,m_4;\hat{\mathbf{p}}) + S^{(-)}(n_4,m_4;\hat{\mathbf{p}}) \ , \quad (10a)$$

where

$$S^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) = S_F^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) + S_d^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) \quad (10b)$$

and

$$S_F^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) = \frac{1}{2\hat{E}}[\pm\gamma_4\hat{E}(\hat{\mathbf{p}}) - i\gamma_j\tilde{p}_j + \hat{M}] \times D^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) \ , \quad (11a)$$

$$D^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) = \pm\int_{-\pi/2}^{\pi/2}\frac{d\hat{p}_4}{2\pi}\frac{e^{i\hat{p}_4(n_4-m_4)}}{i\,\sin(\hat{p}_4 - i\hat{\mu}) \pm \hat{E}(\hat{\mathbf{p}})} \ , \quad (11b)$$

$$S_d^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) = \frac{1}{2\hat{E}}[\pm\gamma_4\hat{E}(\hat{\mathbf{p}}) - i\gamma_j\tilde{p}_j + \hat{M}] \times\tilde{D}^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) \ , \quad (12a)$$

$$\tilde{D}^{(\pm)}(n_4,m_4;\hat{\mathbf{p}}) = (-1)^{n_4-m_4}D^{(\mp)}(n_4,m_4;\hat{\mathbf{p}}) \ . \quad (12b)$$

Clearly the integral (11b) possesses a continuum limit, while $\tilde{D}^{(\pm)}$ does not, because of the factor $(-1)^{n_4-m_4}$. Notice that, apart from this factor, the contribution of the positive- (negative-) energy eigenfunctions (6b) to the propagator have the same form as the negative- (positive-) energy contributions arising from momentum excitations lying in one-half of the Brillouin zone, with γ_4 replaced by $-\gamma_4$ (which amounts to a similarity transformation).

The contributions $S^{(\pm)}(n_4,m_4;\hat{\mathbf{p}})$, can be easily calculated. Introducing the variable $z = e^{i\hat{p}_4}$, one has that

$$D^{(\pm)}(n_4,m_4,\hat{\mathbf{p}}) = \pm\frac{e^{-\hat{\mu}}}{i\pi}\int_{C_R}dz\frac{z^{n_4-m_4}}{z^2 \pm 2\hat{E}e^{-\hat{\mu}}z - e^{-2\hat{\mu}}} \ , \quad (13)$$

where the integration is carried out in the counterclockwise sense along a contour C_R, which is a half-circle of unit radius, with center at $z = 0$, extending from $-\pi/2$ to $\pi/2$ in the right half of the complex z plane. The singularities of the integrand are located at $z_\pm = e^{-\hat{\mu}}(-\hat{E} \pm \sqrt{1+\hat{E}^2})$. A similar expression holds for $\tilde{D}^{(\pm)}(n_4, m_4, \hat{\mathbf{p}})$,

$$\tilde{D}^{(\pm)}(n_4, m_4, \hat{\mathbf{p}}) = \pm \frac{e^{-\hat{\mu}}}{i\pi} \int_{C_L} dz \frac{z^{n_4 - m_4}}{z^2 \pm 2\hat{E} e^{-\hat{\mu}} z - e^{-2\hat{\mu}}} , \tag{14}$$

where the contour C_L is a half-circle of unit radius in the left half of the z plane extending from $\pi/2$ to $-\pi/2$. Hence the sum of (11) and (12) is given by

$$S^{(\pm)}(n_4, m_4; \hat{\mathbf{p}}) = \frac{1}{2\hat{E}}[\pm\gamma_4 \hat{E}(\hat{\mathbf{p}}) - i\gamma_j \tilde{p}_j + \hat{M}] \times \mathcal{D}^{(\pm)}(n_4, m_4; \hat{\mathbf{p}}) , \tag{15a}$$

where

$$\mathcal{D}^{(\pm)}(n_4, m_4; \hat{\mathbf{p}}) = \pm \frac{e^{-\hat{\mu}}}{i\pi} \int_{C} dz \frac{z^{n_4 - m_4}}{z^2 \pm 2\hat{E} e^{-\hat{\mu}} z - e^{-2\hat{\mu}}} . \tag{15b}$$

The contour C is a now a closed contour of unit radius centered at $z = 0$. The contributions to $S^{(\pm)}$ which do not possess a continuum limit arise from the poles located on the negative z axis. One finds

$$\begin{aligned} S^{(\pm)}&(n_4, m_4, \hat{\mathbf{p}}) \\ &= \frac{1}{2\hat{E}}[\pm\gamma_4 \hat{E}(\hat{\mathbf{p}}) - i\gamma_j \tilde{p}_j + \hat{M}]\Delta^{(\pm)}(n_4, m_4; \hat{\mathbf{p}}) \\ &\quad +(-1)^{n_4 - m_4} \frac{1}{2\hat{E}}[\mp\bar{\gamma}_4 \hat{E}(\hat{\mathbf{p}}) \\ &\quad - i\gamma_j \tilde{p}_j + \hat{M}]\Delta^{(\mp)}(n_4, m_4; \hat{\mathbf{p}}) , \end{aligned} \tag{16}$$

where $\bar{\gamma}_4 = -\gamma_4$, and

$$\Delta^{(+)}(n_4, m_4; \hat{\mathbf{p}}) = \theta(n_4 - m_4) \frac{e^{-(n_4 - m_4)[\text{arcsinh}\hat{E} + \hat{\mu}]}}{\sqrt{1+\hat{E}^2}} ,$$

$$\Delta^{(-)}(n_4, m_4; \hat{\mathbf{p}}) = \theta(m_4 - n_4) \frac{e^{(n_4 - m_4)[\text{arcsinh}\hat{E} - \hat{\mu}]}}{\sqrt{1+\hat{E}^2}} ,$$

$$\text{arcsinh}\hat{E} - \hat{\mu} > 0 ,$$

$$\Delta^{(-)}(n_4, m_4; \hat{\mathbf{p}}) = -\theta(n_4 - m_4) \frac{e^{(n_4 - m_4)[\text{arcsinh}\hat{E} - \hat{\mu}]}}{\sqrt{1+\hat{E}^2}} ,$$

$$\text{arcsinh}\hat{E} - \hat{\mu} < 0 . \tag{17}$$

The first contribution appearing on the right-hand side of (16) approaches in the continuum limit the correct expression for the positive- and negative-energy contributions, respectively. The second contribution is a pure lattice artifact arising from the doubler excitations. It exhibits a reversal of the usual correspondence between forward (backward) propagation in Minkowski space of the positive- (negative-) energy solutions to the Dirac equation for $\mu = 0$, and does not scale in the continuum limit. As we have seen it arises from eigenfunctions of the Dirac Hamiltonian which, unless they are restricted to live on even or odd lattice sites, do not possess such limit. Thus particles and antiparticles cannot be identified in the usual way with positive- and negative-energy solutions to the Dirac equation, as has also been recently stressed in [4].

So far we have considered the case of vanishing temperature. We now show how the above results are modified at finite temperature. At finite temperature the lattice has a finite extension in the Euclidean time direction, and the eigenfunctions in (2a) must satisfy antiperiodic boundary conditions in n_4. The extension of the lattice in the time direction (in lattice units) will be denoted by $\hat{\beta}$, which is the inverse temperature measured in lattice units. We will take $\hat{\beta}$ to be even. In the continuum limit $\hat{\beta} \to \infty$, $\hat{\mu} \to 0$, $\hat{M} \to 0$, with $\hat{\beta}\hat{\mu} = \beta\mu$, and $\hat{\beta}\hat{M} = \beta M$ fixed. At finite temperature the integral (11b) is replaced by a sum over Matsubara frequencies $\hat{\omega}_l = (\pi/\hat{\beta})(2l+1)$ with $-\hat{\beta}/2 \le l \le \hat{\beta}/2 - 1$. Correspondingly the sum of (11b) and (12b), $\mathcal{D}^{(\pm)}(n_4, m_4; \hat{\mathbf{p}})$, appearing in (15a), is replaced by

$$\mathcal{D}^{(\pm)}(n_4, m_4; \hat{\mathbf{p}}) = \pm \frac{1}{\hat{\beta}} \sum_{l=-\hat{\beta}/2}^{\hat{\beta}/2-1} \frac{e^{i\hat{\omega}_l (n_4 - m_4)}}{i \sin(\hat{\omega}_l - i\hat{\mu}) \pm \hat{E}(\hat{\mathbf{p}})} . \tag{18}$$

The frequency sums can be converted into contour integrals [3]

$$\frac{1}{\hat{\beta}} \sum_l f(\hat{\omega}_l) = -\frac{1}{2\pi i} \int_{|z|=1+\epsilon} dz \frac{\tilde{f}(z)}{z(z^{\hat{\beta}}+1)} + \frac{1}{2\pi i} \int_{|z|=1-\epsilon} dz \frac{\tilde{f}(z)}{z(z^{\hat{\beta}}+1)} , \tag{19}$$

where $\tilde{f}$ is defined by $f(\hat{\omega}) = \tilde{f}(e^{i\hat{\omega}})$, and the integrations are performed over closed circles with radii $1+\epsilon$ and $1-\epsilon$, centered at $z = 0$, in the counterclockwise sense. The singularities of $1/[z^{\hat{\beta}}+1]$ are located on a circle of unit radius, and are enclosed by these two contours. Hence at finite temperature (15b) is replaced by

$$\begin{aligned} \mathcal{D}^{(\pm)}&(n_4, m_4, \hat{\mathbf{p}}) \\ &= \pm \frac{e^{-\hat{\mu}}}{i\pi} \int_{\tilde{C}} dz \frac{z^{n_4 - m_4}}{z^2 \pm 2\hat{E} e^{-\hat{\mu}} z - e^{-2\hat{\mu}}} \frac{1}{1+z^{\hat{\beta}}} , \end{aligned} \tag{20}$$

where the contour $\tilde{C}$ consists of the two circles with radii $1-\epsilon$ and $1+\epsilon$. Because $\hat{\beta} > |n_4 - m_4|$, the outer contour can be distorted to infinity, taking into account the poles located outside the unit circle. The integral over the inner contour is given by the residues of poles associated with the vanishing of $z^2 \pm 2\hat{E}e^{-\hat{\mu}}z - e^{-2\hat{\mu}}$, and, for $n_4 - m_4 < 0$, the residue of a pole of order $m_4 - n_2$ at $z = 0$. After some algebra one finds that expressions (17) are modified at finite temperature as follows:

$$\Delta_{\hat{\beta}}^{(+)}(n_4, m_4; \hat{\mathbf{p}}) = \Delta^{(+)}(n_4, m_4; \hat{\mathbf{p}}) - \eta_{\mathrm{FD}}(\hat{E}) \frac{e^{-(n_4-m_4)[\mathrm{arcsinh}\hat{E}+\hat{\mu}]}}{\sqrt{1+\hat{E}^2}} ,$$

$$\Delta_{\hat{\beta}}^{(-)}(n_4, m_4; \hat{\mathbf{p}}) = \Delta^{(-)}(n_4, m_4; \hat{\mathbf{p}}) - \bar{\eta}_{\mathrm{FD}}(\hat{E}) \frac{e^{(n_4-m_4)[\mathrm{arcsinh}\hat{E}-\hat{\mu}]}}{\sqrt{1+\hat{E}^2}} , \quad \mathrm{arcsinh}\hat{E} - \hat{\mu} > 0 ,$$

$$\Delta_{\hat{\beta}}^{(-)}(n_4, m_4; \hat{\mathbf{p}}) = \Delta^{(-)}(n_4, m_4; \hat{\mathbf{p}}) + \bar{\eta}_{\mathrm{FD}}(\hat{E}) e^{\hat{\beta}(\mathrm{arcsinh}\hat{E}-\hat{\mu})} \frac{e^{(n_4-m_4)[\mathrm{arcsinh}\hat{E}-\hat{\mu}]}}{\sqrt{1+\hat{E}^2}} , \quad \mathrm{arcsinh}\hat{E} - \hat{\mu} < 0 , \tag{21a}$$

where η_{FD} and $\bar{\eta}_{\mathrm{FD}}$ are the lattice Fermi-Dirac distribution functions,

$$\eta_{\mathrm{FD}}(\hat{E}) = \frac{1}{e^{\hat{\beta}(\mathrm{arcsinh}\hat{E}+\hat{\mu})} + 1} , \quad \bar{\eta}_{\mathrm{FD}}(\hat{E}) = \frac{1}{e^{\hat{\beta}(\mathrm{arcsinh}\hat{E}-\hat{\mu})} + 1} . \tag{21b}$$

How are the above results modified for Wilson fermions? For Wilson fermions (with Wilson parameter $r = 1$ and fermion mass $\hat{m}$) the spatial Fourier transform of the propagator at finite temperature and chemical potential is given by

$$S(n_4, m_4; \hat{\mathbf{p}}) = \frac{1}{\hat{\beta}} \sum_{l=-\hat{\beta}/2}^{\hat{\beta}/2-1} \frac{-i\gamma_4 \sin(\hat{\omega}_l - i\hat{\mu}) - i\sum_j \gamma_j \sin\hat{p}_j + \hat{\mathcal{M}}(\hat{\mathbf{p}}, \hat{\omega}_l, \hat{\mu})}{\sin^2(\hat{\omega}_l - i\hat{\mu}) + \sum_j \sin^2\hat{p}_j + \hat{\mathcal{M}}^2(\hat{\mathbf{p}}, \hat{\omega}_l, \hat{\mu})} e^{i\hat{\omega}_l(n_4-m_4)} ,$$

$$\hat{\mathcal{M}} = \hat{M}(\hat{\mathbf{p}}) + 1 - \cos(\hat{\omega}_l - i\hat{\mu}), \quad \hat{M}(\hat{\mathbf{p}}) = \hat{m} + 2\sum_j \sin^2\frac{\hat{p}_j}{2} , \tag{22}$$

where $\hat{\omega}_l = (2l+1)\pi/\hat{\beta}$. Introducing the variable $z_l = e^{i\hat{\omega}_l}$ one has that

$$S(n_4, m_4; \hat{\mathbf{p}}) = \frac{1}{\hat{\beta}} \sum_l \frac{\frac{1}{2}\gamma_4(z_l^2 e^{2\hat{\mu}} - 1) + \frac{1}{2}(z_l e^{\hat{\mu}} - 1)^2 + \left(i\sum_j \gamma_j \sin\hat{p}_j - \hat{M}(\hat{\mathbf{p}})\right) z_l e^{\hat{\mu}}}{[1 + \hat{M}(\hat{\mathbf{p}})] e^{2\hat{\mu}}(z_l - z_+)(z_l - z_-)} z_l^{(n_4-m_4)} ,$$

$$z_\pm = e^{-\hat{\mu}}[K \pm \sqrt{K^2 - 1}] ,$$

$$K = 1 + \frac{\hat{\mathcal{E}}^2}{2(1+\hat{M})}, \quad \hat{\mathcal{E}} = \sqrt{\sum_j \sin^2\hat{p}_j + \hat{M}(\hat{\mathbf{p}})^2} . \tag{23}$$

In the continuum limit we have that $\hat{M}(\hat{\mathbf{p}}) \to \hat{m}$, $\hat{\mathcal{E}} \to \hat{E}$, and $K \pm \sqrt{K^2-1} \to 1 \pm \hat{E}$, where $\hat{E} = \sqrt{\hat{\mathbf{p}}^2 + \hat{m}^2}$. The sum over l can be performed by making use of (19). After some algebra one finds that (23) can be cast into the form

$$S(n_4, m_4; \hat{\mathbf{p}}) = \frac{1}{2(1+\hat{M})e^{\hat{\mu}}\sqrt{K^2-1}} \sum_{\sigma=\pm} \sigma \left(\sigma\gamma_4 \sinh\tilde{\mathcal{E}}(\hat{\mathbf{p}}) + i\sum_j \gamma_j \sin\hat{p}_j - \hat{M}(\hat{\mathbf{p}}) + 2\sinh^2\frac{\tilde{\mathcal{E}}(\hat{\mathbf{p}})}{2} \right)$$
$$\times \{\theta(n_4 - m_4)\eta_\sigma(\tilde{\mathcal{E}}) + \theta(m_4 - n_4)[\eta_\sigma(\tilde{\mathcal{E}}) - 1]\} e^{(n_4-m_4)(\sigma\tilde{\mathcal{E}}-\hat{\mu})} , \tag{24a}$$

where

$$\tilde{\mathcal{E}} = \ln[K + \sqrt{K^2-1}], \quad \eta_\sigma(\tilde{\mathcal{E}}) = \frac{1}{e^{\hat{\beta}(\sigma\tilde{\mathcal{E}}-\hat{\mu})} + 1} . \tag{24b}$$

In the continuum limit $\eta_-(\tilde{\mathcal{E}}) \to 1$. To decompose the above expression into its zero- and finite-temperature parts we therefore introduce the lattice version of the Fermi-Dirac distribution function

$$\hat{\eta}_{\mathrm{FD}}(\tilde{\mathcal{E}}) = 1 - \eta_-(\tilde{\mathcal{E}}) = \frac{1}{e^{\hat{\beta}(\tilde{\mathcal{E}}+\hat{\mu})} + 1} . \tag{25}$$

What concerns η_+ we must consider separately the cases where $\hat{\mu} < \tilde{\mathcal{E}}$ and $\hat{\mu} > \tilde{\mathcal{E}}$. In the first case the appropriate distribution function which vanishes for zero temperature is η_+ itself:

$$\eta_+(\tilde{\mathcal{E}}) = \bar{\hat{\eta}}_{\mathrm{FD}}(\tilde{\mathcal{E}}) = \frac{1}{e^{\hat{\beta}(\tilde{\mathcal{E}}-\hat{\mu})} + 1} , \tag{26}$$

while for $\hat{\mu} > \tilde{\mathcal{E}}$ the appropriate distribution function is

$$1 - \eta_+(\tilde{\mathcal{E}}) = \bar{\hat{\eta}}_{\mathrm{FD}}(\tilde{\mathcal{E}}) e^{\hat{\beta}(\tilde{\mathcal{E}}-\hat{\mu})} . \tag{27}$$

One then finds that

$$S(n_4, m_4; \hat{\mathbf{p}}) = \frac{1}{2(1+\hat{M})e^{\hat{\mu}}\sqrt{K^2-1}} \sum_{\sigma=\pm} \left(\sigma\gamma_4 \sinh\tilde{\mathcal{E}}(\hat{\mathbf{p}}) - i\sum_j \gamma_j \sin\hat{p}_j + \hat{M}(\hat{\mathbf{p}}) - 2\sinh^2\frac{\tilde{\mathcal{E}}(\hat{\mathbf{p}})}{2} \right) \hat{\Delta}_\beta^{(\sigma)}(n_4, m_4, \hat{\mathbf{p}}) , \tag{28a}$$

where

$$\hat{\Delta}_\beta^{(+)}(n_4, m_4, \hat{\mathbf{p}}) = \hat{\Delta}^{(+)}(n_4, m_4, \hat{\mathbf{p}}) - \hat{\eta}_{\mathrm{FD}}(\tilde{\mathcal{E}}) e^{-(n_4-m_4)(\tilde{\mathcal{E}}+\hat{\mu})} ,$$

$$\hat{\Delta}_\beta^{(-)}(n_4, m_4, \hat{\mathbf{p}}) = \hat{\Delta}^{(-)}(n_4, m_4, \hat{\mathbf{p}}) - \bar{\hat{\eta}}_{\mathrm{FD}}(\tilde{\mathcal{E}}) e^{(n_4-m_4)(\tilde{\mathcal{E}}-\hat{\mu})} , \quad \hat{\mu} < \tilde{\mathcal{E}} ,$$

$$\hat{\Delta}_\beta^{(-)}(n_4, m_4, \hat{\mathbf{p}}) = \hat{\Delta}^{(-)}(n_4, m_4, \hat{\mathbf{p}}) + \bar{\hat{\eta}}_{\mathrm{FD}}(\tilde{\mathcal{E}}) e^{\hat{\beta}(\tilde{\mathcal{E}}-\hat{\mu})} e^{(n_4-m_4)(\tilde{\mathcal{E}}-\hat{\mu})} ,$$

$$\hat{\mu} > \tilde{\mathcal{E}} , \tag{28b}$$

and where $\hat{\Delta}^{(\pm)}$ are the zero-temperature functions analogous to (17). Their explicit form is given by (17) with the replacement

$$\exp[\mp(n_4 - m_4)(\mathrm{arcsinh}\hat{E} \pm \hat{\mu})]/\sqrt{1+\hat{E}^2} \to \exp[\mp(n_4 - m_4)(\tilde{\mathcal{E}} \pm \hat{\mu})] .$$

In the continuum limit we have that $\hat{M} \to \hat{m}$, $\sqrt{K^2-1} \to \hat{E}$, $\tilde{\mathcal{E}} \to \hat{E}$, and $\sin\hat{p}_j \to \hat{p}_j$ for finite physical momenta [only these contribute to the integral (9)]. Hence (28) approaches the spatial Fourier transform of the Dirac propagator in the continuum formulation.

Summarizing, we have studied in detail the temporal structure of the lattice Dirac propagator for naive fermions at finite temperature and chemical potential and have shown in particular that for $\mu = 0$ the contributions from the doublers invalidate the usual correspondence between positive-negative eigenvalues of the Dirac Hamiltonian and forward-backward propagation in Minkowski space. Hence particles and antiparticles cannot be identified by the sign of the eigenvalue of the Dirac Hamiltonian. The contributions of the doublers to the lattice propagator are pure lattice artifacts and, apart from the nonscaling multiplicative phase, their structure is that of the normal mode contribution with the above-mentioned correspondence reversed. As the analysis of this paper has shown, the doubler excitations cannot be associated with physical degrees of freedom. A well-known way to handle fermions with a symmetric lattice derivative is the staggered fermion formulation. For Kogut-Susskind fermions it has been shown in [4] that a Fock space of flavored lattice fermions can be constructed using the Osterwalder-Schrader reconstruction scheme, which allows one to construct the Hamiltonian and baryon number operators from the path-integral representation, with the usual correspondence holding between positive (negative) eigenvalues of the Hamiltonian and particles (antiparticles). The same correspondence was shown in this paper to hold for Wilson fermions.

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