Comment on "Current correlators in QCD at finite temperature"

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We address some criticisms by Eletsky and Ioffe on the extension of QCD sum rules to finite temperature. We argue that this extension is possible, provided the operator product expansion and QCD-hadron duality remain valid at nonzero temperature. We discuss evidence in support of this from QCD, and from the exactly solvable two-dimensional σ model O(N) in the large N limit, and the Schwinger model.

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Some time ago Bochkarev and Shaposnikov [1] proposed an extension of the QCD sum rule program to a nonzero temperature, and made an application to the two-point function involving the vector current. This application was reconsidered in [2]. Later on we discussed the axial-vector channel [3] and the nucleon channel [4] using finite energy QCD sum rules (FESR's). The results from these analyses indicate a substantial rearrangement of the hadronic spectrum with increasing temperature, and hint at the existence of a deconfining phase transition. This was later confirmed in [5], where a formalism valid even for T near the critical temperature was used. QCD sum rules [6] are based on the operator product expansion (OPE) of current correlators at short distances, suitably extended to include nonperturbative effects. The latter are parametrized in terms of a set of vacuum expectation values of the quark and gluon fields entering the basic QCD Lagrangian. Contact with the hadronic world of large distances is achieved by invoking the notion of QCD-hadron duality. The values of the vacuum condensates cannot be calculated analytically from first principles, as this would be tantamount to solving QCD exactly. Instead, they are extracted from certain channels where experimental information is available, e.g., e^+e^- annihilation, and τ decays [7]. It is also possible, in principle, to estimate them numerically from lattice QCD. An extension of this formalism of QCD sum rules to finite temperature entails the assumptions that (a) the OPE continues to be valid, except that now the vacuum condensates will develop an (*a priori*) unknown temperature dependence, and (b) the notion of QCD-hardon duality also remains valid. In analogy with the situation at $T = 0$, the thermal behavior of the vacuum condensates is not calculable analytically from first principles. Some model or approximation must be invoked, e.g., the dilute pion gas approximation, lattice QCD, etc. The quark, the gluon, and the four-quark condensates at finite temperature have thus been estimated in the literature [8–10].

In a recent paper, Eletsky and Ioffe [11] have criticized the QCD sum rule program proposed in [1], and developed in [2–5]. In this note we wish to address this criticism, and hopefully clarify the issue. We will argue that, provided the OPE and QCD-hadron duality remain valid at finite temperature, the approach of [1–5] is basically correct. We illustrate our argument with the current correlator involving the axial-vector current, although it can be trivially generalized to any local current operator. In addition, we provide further supportive evidence from two exactly solvable field theory models: the two-dimensional O(N) σ model in the large N limit, and the two-dimensional Schwinger model.

The basic object to be considered is the retarded (advanced) two-point function after appropriate Gibbs averaging:

$$\Pi(q,T) = i \int d^4x \exp(iqx)\theta(x_0)\langle\langle[J(x), J^\dagger(0)]\rangle\rangle, \quad (1)$$

where

$$\langle\langle A\cdot B\rangle\rangle = \sum_n \exp(-E_n/T)\langle n|A\cdot B|n\rangle/\mathrm{Tr}[\exp(-H/T)], \quad (2)$$

and $|n\rangle$ is a complete set of eigenstates of the (QCD) Hamiltonian. The OPE of $\Pi(q,T)$ is formally written as

$$\Pi(q,T) = C_I\langle\langle I\rangle\rangle + \sum_r C_r(q)\langle\langle O_r\rangle\rangle, \quad (3)$$

where the Wilson coefficients $C_r(q)$ depend on the Lorentz indices and quantum numbers of the external current $J(x)$, and also of the local gauge-invariant operators O_r built from the quark and gluon fields of QCD. The $C_r(q)$ could also depend on the temperature, but since this is not essential for the argument we shall ignore this dependence. The unit operator I in Eq. (3) represents the purely perturbative piece. The OPE is assumed valid, even in the presence of nonperturbative effects, for $q^2 < 0$ (spacelike), and $|q^2| \gg \Lambda^2_{\mathrm{QCD}}$. In principle, all Wilson coefficients are calculable in perturbative QCD to any desired order in the strong coupling constant. In the sequel to this paper we shall work at leading (one loop) order for simplicity. The nonperturbative effects are then buried in the vacuum condensates. Since these have dimensions, the associated Wilson coefficients fall

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off as inverse powers of $Q^2 = -q^2$.

For instance, if the current $J(x)$ in Eq. (1) is identified with the axial-vector current $A_\mu(x) = :\bar{u}(x)\gamma_\mu\gamma_5 d(x):$, then with

$$\Pi_{\mu\nu}(q,T) = -g_{\mu\nu}\Pi_1(q,T) + q_\mu q_\nu \Pi_0(q,T), \tag{4}$$

one easily finds, at $T = 0$ [6],

$$8\pi^2\Pi_0(q,T=0) = -\ln\frac{Q^2}{\mu^2} + \frac{C_4\langle 0_4\rangle}{Q^4} + \frac{C_6\langle 0_6\rangle}{Q^6} + \cdots, \tag{5}$$

where μ is a renormalization scale, and e.g., the leading vacuum condensate is given by

$$C_4\langle 0_4\rangle = \frac{\pi}{3}\langle\alpha_s G^2\rangle - 8\pi^2\bar{m}_q\langle\bar{q}q\rangle, \tag{6}$$

with $\bar{m}_q = (m_u + m_d)/2$, and $\langle\bar{q}q\rangle = \langle\bar{u}u\rangle \simeq \langle\bar{d}d\rangle$. The function $\Pi_0(q)$, Eq. (5), satisfies a dispersion relation

$$\Pi_0(Q^2) = \frac{1}{\pi}\int ds \frac{\mathrm{Im}\Pi_0(s)}{s+Q^2}, \tag{7}$$

defined in this case up to one subtraction constant, which can be disposed of by, e.g., taking the first derivative with respect to Q^2 in Eq. (7). The notion of QCD-hadron duality is implemented by calculating the left-hand side of Eq. (7) in QCD through the OPE, and parametrizing the spectral function entering the right-hand side in terms of hadronic resonances, followed by a hadronic continuum modeled by perturbative QCD. In this fashion one relates fundamental QCD parameters, such as quark masses, renormalization scales, vacuum condensates, etc., to hadronic parameters such as particle masses, widths, couplings, etc. The convergence of the Hilbert transform, Eq. (7), may be improved by considering other integral kernels. This leads to other versions of QCD sum rules, such as the Laplace transform, FESR, etc.

After this introduction we address the criticism raised in [11]. The states $|n\rangle$ entering Eq. (2) can be any complete set of states, e.g., hadronic states, quark-gluon basis, etc. Eletsky and Ioffe claim that below the critical temperature T_c the suitable set of states is the hadronic set but not the quark-gluon basis. In support of this, they argue that for $T < T_c$ the original particles forming the heat bath, being probed by the external currents, are hadrons. They go on to say that a summation over the quark-gluon basis in Eq. (2) would require taking into account the full range of their interaction, but that no account of confinement was given in [1–3].

First and foremost, we believe that the quark-gluon basis is indeed the appropriate basis to be used in QCD sum rule applications. If this were not the case, it would mean that the fruitful notion of duality would abruptly lose meaning as soon as the temperature is raised from $T = 0$ to some arbitrary small value of T. This would be a rather bizarre scenario. According to the QCD sum rule philosophy, at $T = 0$ one calculates the theoretical left-hand side of Eq. (7) through the OPE, Eq. (3), i.e., one uses quark-gluon degrees of freedom, and duality relates this QCD part to a weighted average of the hadronic spectral function. The latter arises from using hadronic degrees of freedom. At very low temperatures the hadronic spectrum will change very little, and the external current will still convert into quark-antiquark pairs. Hence, it is only reasonable to assume that nothing drastic will happen with duality. At finite temperature, though, there is a new effect coming into play; i.e., there are contributions to the QCD and hadronic spectral functions in the spacelike region (as opposed to only the timelike region at $T = 0$). However, these additional contributions vanish smoothly as T approaches zero; i.e., they do not introduce any discontinuous behavior.

In addition, the fact that the heat bath is mainly composed of hadrons at small T is not in contradiction with the use of the quark-gluon basis. For instance, quarks enter the QCD perturbative term through loops, and have any value of momentum. One should emphasize here that the currents in (1) are external objects, and hence need not be in thermal equilibrium with the heat bath which is being probed.

Next, contrary to the statement made in [11], confinement has indeed been taken into account in [1–5] in the standard way, i.e., through the nonvanishing vacuum condensates in the OPE, Eq. (3). The thermal behavior of these condensates is a separate matter. We have used in [3] the chiral perturbation theory estimates of the thermal quark and gluon condensates [8], and in [4] we used the results of [5]. Recently [10], the T dependence of four-quark condensates was obtained using soft pion techniques in conjunction with Eq. (2), where the summation was performed in the hadronic (pion) basis. In this instance we do agree that the hadronic basis is the appropriate one. However, this has nothing to do with QCD nor with duality. It is only one of many theoretical approximations to estimate the temperature dependence of the condensates.

Finally, we wish to comment on an implicit claim of Eletsky and Ioffe [11], which was stated more explicitly earlier in [12]. This is, that calculating Gibbs averages in the quark basis at small T implies dealing with soft on-shell quarks, which are usually referred to as condensates. Although the quarks are on-shell, they can have any value of momentum when circulating around loops. And, there is no possibility of confusing perturbative contributions of on-shell quarks and quark condensates. This is obvious at $T = 0$ if one computes the imaginary part of a current correlator, e.g., to one loop order, where the quark-antiquark intermediate state *is* on-mass shell. For instance, the leading perturbative contribution to the imaginary part of $\Pi_0(q, T = 0)$ in the axial-vector channel, calculated using the Cutkosky rules (on-mass shell quarks understood) gives $\mathrm{Im}\Pi_0 = 1/8\pi$. This term cannot possibly be confused with the quark condensate contribution in Eq. (5) which, first, is real and, second, it has a different Q^2 dependence. We argue next, that this is also the case at finite temperature.

With $q^2 = \omega^2 - \mathbf{q}^2$, and in the rest frame of the medium ($\mathbf{q} \to 0$), the imaginary part of $\Pi_0(\omega, T)$ in the axial-vector channel, to leading order in perturbative QCD, is [3]

$$\frac{1}{\pi}\mathrm{Im}\Pi_0^{(+)}(\omega,\mathbf{q}=0,T)=\frac{1}{8\pi^2}v(\omega)[3-v^2(\omega)]\tanh\left(\frac{\omega}{4T}\right)\times\theta(\omega^2-4m_q^2), \quad (8)$$

in the timelike region, and

$$\frac{1}{\pi}\mathrm{Im}\Pi_0^{(-)}(\omega,\mathbf{q}=0,T)=\frac{1}{8\pi^2}\delta(\omega^2)\times\int_{4m_q^2}^{\infty}dz^2v(z)[3-v^2(z)]\times 2n_F\left(\frac{z}{2T}\right), \quad (9)$$

in the spacelike region, where $v(x)=(1-4m_q^2/x^2)^{1/2}$, and n_F is the Fermi thermal factor. The nonperturbative contributions to the OPE involve the quark and gluon condensates, both of dimension $d=4$, the four-quark condensate of dimension $d=6$, etc. All these are real, and exhibit a temperature dependence very different from that in Eqs. (8) and (9). For instance, the quark condensate at low temperatures is of the form $\langle\langle\bar{q}q\rangle\rangle=\langle\bar{q}q\rangle(1-aT^2/f_\pi^2)$, while the gluon condensate is essentially independent of T [8]. In view of these differences, we see no possibility of confusing thermal quark loop contributions with thermal quark condensates.

Additional evidence can be found in the framework of the O(N) model in the large N limit, and in the Schwinger model, both in two dimensions. These models were used in the past [13] in order to verify the validity of the OPE. Since these two models can be solved exactly, one can compare the results of the exact calculation with that from the OPE. They turn out, in fact, to be identical. We obtain below the thermal Green functions in these two models, and show that the temperature corrections to the perturbative contributions cannot be shifted to the nonperturbative terms, which develop their own T dependence (calculable within the framework of these models).

We consider first the O(N) σ model in 1+1 dimensions which is characterized by the Lagrangian

$$\mathcal{L}=\frac{1}{2}[\partial_\mu\sigma^a(x)][\partial_\mu\sigma^a(x)], \quad (10)$$

where $a=1,\ldots,N$ and $\sigma^a\sigma^a=N/f$, with f being the coupling constant. In the large N limit this model can be solved exactly (for details see [13]); it is known to be asymptotically free, i.e.,

$$f(\mu)=\frac{4\pi}{\ln\mu^2/m^2}, \quad (11)$$

and despite the absence of mass parameters in Eq. (10), it exhibits dynamical mass generation. In addition, in this model there are vacuum condensates, e.g., to leading order in $1/N$,

$$\langle 0|\alpha|0\rangle=\sqrt{N}m^2, \quad (12)$$

whereas all other condensates factorize:

$$\langle 0|\alpha^k|0\rangle=(\sqrt{N}m^2)^k. \quad (13)$$

In Eqs. (12) and (13) the α field is $\alpha=f(\partial_\mu\sigma^a)^2/\sqrt{N}$. We are interested in the Green function associated with the propagation of quanta of the α field. At $T=0$, in Minkowski space, and at the one loop order, this Green function is given by [13]

$$\Gamma(Q)=-\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2-m^2}\frac{i}{(Q-k)^2-m^2}=\frac{1}{2\pi}\frac{1}{\sqrt{Q^2(Q^2-4m^2)}}\ln\frac{\sqrt{Q^2}-\sqrt{Q^2-4m^2}}{\sqrt{Q^2}+\sqrt{Q^2-4m^2}}. \quad (14)$$

Expanding Eq. (14) at short distances leads to

$$\Gamma(Q)=\frac{1}{2\pi}\frac{\ln(m^2/Q^2)}{Q^2}\times\left(1+\frac{2}{\sqrt{N}}\frac{\langle 0|\alpha|0\rangle}{Q^2}+\frac{6}{N}\frac{\langle 0|\alpha^2|0\rangle}{Q^4}+\cdots\right). \quad (15)$$

A separate calculation, based on the OPE of the current correlator involving the scalar current $J_S=f(\partial_\mu\sigma^a)^2$, gives exactly the same answer as Eq. (15) [13].

We have calculated the Green function Eq. (14) at finite temperature. Its imaginary part can be integrated analytically in closed form and is

$$\mathrm{Im}\Gamma(\omega,\mathbf{q}=0,T)=\frac{1}{2\omega^2}[1+3n_B(\omega/2T)]+\frac{1}{2}\left[\frac{2}{\sqrt{N}}\frac{\langle\langle\alpha\rangle\rangle}{\omega^4}+\frac{6}{N}\frac{\langle\langle\alpha^2\rangle\rangle}{\omega^6}+\cdots\right], \quad (16)$$

where the first term above corresponds to the perturbative contribution, the second to the nonperturbative, and n_B is the thermal Bose factor. Equation (16) is valid in the timelike region; the spacelike region counterpart vanishes in two dimensions. Since the model is exactly solvable, the thermal behavior of the vacuum condensates can also be calculated: viz.,

$$\langle\langle\alpha\rangle\rangle=\langle\alpha\rangle[1+3n_B(\omega/2T)]. \quad (17)$$

In this case the vacuum condensates contribute to the imaginary part, and as Eq. (16) shows, the thermal dependence of the perturbative piece cannot be absorbed into the condensates. Hence, no confusion should arise. We have not discussed above the effects of renormalization [13], as they are not essential to our argument.

Finally, we consider the Schwinger model in 1+1 dimensions, with the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\bar{\psi}i\gamma_\mu\mathcal{D}_\mu\psi, \quad (18)$$

where $\mathcal{D}_\mu=i\partial_\mu+eA_\mu$. This model has been solved exactly [14], and in the framework of the OPE [13]. The short distance expansion of the exact solution coincides with that from the OPE, as shown in [13]. Here, we are interested in the two-point functions

$$\Pi_{++}(x)=\langle 0|T\{j^+(x)j^+(0)\}|0\rangle, \quad (19)$$

$$\Pi_{+-}(x) = \langle 0|T\{j^+(x)j^-(0)\}|0\rangle, \tag{20}$$

where the scalar currents are

$$j^+ = \bar{\psi}_R\psi_L, \quad j^- = \bar{\psi}_L\psi_R \tag{21}$$

with $\psi_{L,R} = (1\pm\gamma_5)\psi/2$. The function $\Pi_{++}(Q)$ vanishes identically in perturbation theory, and the leading nonperturbative contribution involves a four-fermion vacuum condensate. At $T = 0$ one finds [13]

$$\Pi_{++}(Q,T=0) = \frac{16e^2}{Q^6}\langle 0|\bar{\psi}_R\partial_\mu\psi_L\bar{\psi}_R\partial_\mu\psi_L|0\rangle. \tag{22}$$

On the other hand, the function $\Pi_{+-}(Q)$ is purely perturbative. At $T = 0$ it is given by [13]

$$\Pi_{+-}(Q,T=0) = \frac{1}{4\pi}\left(ln\frac{M^2}{Q^2} + \frac{m^2}{Q^2} + \cdots\right), \tag{23}$$

where M is an ultraviolet cutoff. We have calculated the thermal behavior of these current correlators and obtain, e.g., for their imaginary parts in the timelike region (there is no spacelike contribution in two dimensions)

$$\mathrm{Im}\Pi_{++}(\omega,\mathbf{q}=0,T) = 0, \tag{24}$$

$$\mathrm{Im}\Pi_{+-}(\omega,\mathbf{q}=0,T) = \frac{1}{4}[1-2n_F(\omega/2T)]. \tag{25}$$

Hence, the choice of the fermion basis in the Gibbs average of current correlators does not imply confusing these fermions with condensates. The (perturbative) fermion loop terms and the (nonperturbative) vacuum condensates develop their own temperature dependence. In this particular example, nonperturbative terms are totally absent in $\mathrm{Im}\Pi_{+-}$, and they do not contribute to $\mathrm{Im}\Pi_{++}$ because they are real. This peculiarity makes the argument particularly transparent.

It should be mentioned, in closing, that there are some unresolved problems in finite temperature QCD sum rules, e.g., when one tries to use them to extract the thermal behavior of the vacuum condensates [15]. However, these problems are unrelated to the arguments given by Eletsky and Ioffe [11].

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