

Hadron scattering lengths in lattice QCD

M. Fukugita

*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan
and Institute for Advanced Study, Princeton, New Jersey 08540*

Y. Kuramashi and M. Okawa

National Laboratory for High Energy Physics(KEK), Tsukuba, Ibaraki 305, Japan

H. Mino

Faculty of Engineering, Yamanashi University, Kofu 404, Japan

A. Ukawa

Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan

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A lattice QCD calculation of s -wave hadron scattering lengths in the channels π - π , π - N , K - N , \bar{K} - N , and N - N is carried out in the quenched approximation at $\beta = 6/g^2 = 5.7$. A variant of the method of wall source is developed for this purpose, which reduces the computer time by a factor L^3 on an $L^3 \times T$ lattice compared to the conventional point source method and avoids the Fierz mixing problem. A version of the method in which gauge configurations are not fixed to any gauge can be extended to calculate disconnected quark loop contributions in hadron two- and three-point functions. An analytical estimate of statistical errors for this method is worked out, and the magnitude of errors without and with gauge fixing is compared for the case of π - π four-point functions calculated with the Kogut-Susskind quark action. For π - π scattering both $I = 0$ and 2 scattering lengths are evaluated using the Kogut-Susskind and Wilson quark actions on a $12^3 \times 20$ lattice. For the same size lattice, π - N , K - N , and \bar{K} - N scattering lengths are calculated with the Wilson quark action. For the π - π and π - N cases simulation results are consistent with the predictions of current algebra and PCAC within 1–2 standard deviations up to quite heavy quark masses corresponding to $m_\pi/m_\rho \approx 0.74$, while for the K - N and \bar{K} - N cases the agreement is within a factor of 2. For N - N scattering a phenomenological study with one-boson exchange potentials indicate that the deuteron becomes unbound if the quark mass is increased beyond 30–40% of the physical value. Simulations with the Wilson action on a 20^4 lattice with heavy quarks with $m_\pi/m_\rho \approx 0.74$ –0.95 show that the nucleon-nucleon force is attractive for both spin triplet and singlet channels, and that the scattering lengths are substantially larger compared to those for the π - π and π - N cases even for such heavy quarks. The problem of statistical errors, which has to be overcome toward a more realistic calculation of hadron scattering lengths, is discussed.

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I. INTRODUCTION

Calculation of scattering lengths of hadrons presents a variety of problems of physical interest in the lattice QCD approach to strong interactions. For the case of π - π and π - N scattering, the experimental s -wave scattering lengths are of order 0.1 – 0.4 fm which are quite small compared to the sizes of the pion and nucleon (see Table I for experimental values). It is well known that the small values can be understood as a result of soft pion theorems [2] that follow from the approximate chiral symmetry of up and down quarks realized in a spontaneously broken mode; scattering amplitudes involving pions vanish at threshold for zero pion mass, and the s -wave scattering lengths predicted for the experimental pion mass are in reasonable agreement with experiment as recapitulated in the last column of Table I. Whether a dynamical calculation directly based on the QCD Lagrangian suc-

cessfully explains experiment therefore provides a useful testing ground of our theoretical understanding of chiral symmetry based on lattice QCD.

The situation is quite different for nucleon-nucleon

TABLE I. Experimental s -wave scattering lengths [1] and current algebra predictions in units of fm.

		Experiment	Current algebra
π - π	$I = 0$	+0.37(7)	+0.222
	$I = 2$	-0.040(17)	-0.0635
π - N	$I = 1/2$	+0.245(4)	+0.221
	$I = 3/2$	-0.143(6)	-0.111
N - N	3S_1	-5.432(5)	
	1S_0	+20.1(4)	
K - N	$I = 0$	-0.0075	0
	$I = 1$	-0.225	-0.399
\bar{K} - N	$I = 0$	-1.16 + i 0.49	+0.598
	$I = 1$	+0.17 + i 0.41	+0.199

scattering. Experimentally the scattering lengths are very large, being of the order of 10 fm (see Table I). Since chiral symmetry places no constraint on the low energy behavior, the large scattering lengths are purely dynamical phenomena. So far theoretical approach to the low energy nucleon-nucleon scattering has remained at a phenomenological level in which meson exchange models with tunable coupling constants are utilized to reproduce the low energy phase shifts [3]. Deriving the large scattering lengths from first principles of QCD represents an important challenge to lattice QCD.

Calculation of scattering lengths poses several problems from the technical point of view. One of the problems is that Euclidean hadron Green's functions amenable to Monte Carlo evaluation methods of lattice QCD are generally only indirectly connected to real time scattering amplitudes. An elegant solution to this problem is the formula [4,5] relating scattering phase shifts, and hence also scattering lengths, to the energy of two hadron states on a finite spatial lattice which can be extracted from the exponential decay of Euclidean hadron four-point functions in time [6].

Another problem, which is computationally quite troublesome, is the calculation of hadron four-point functions itself. For hadrons in a definite momentum state, such a calculation generally requires quark propagators connecting two arbitrary space-time sites. With the conventional method of point source the necessary number of quark matrix inversions equals the space-time lattice volume, which would require a prohibitively large amount of computer time. This is a novel situation quite different from a calculation of connected three-point functions of hadrons, which can be reduced to that of connected two-point functions through source methods [7]. We note that a similar difficulty arises even for two- or three-point functions when one attempts to calculate contributions involving disconnected quark loops. Typical examples are the two-quark loop amplitude for the flavor singlet η' propagator and flavor singlet nucleon matrix elements such as the π - N σ term.

We found that a variant of the wall source technique [8,9] allows to overcome the problem with a modest cost of computing power [10–16]. Particularly interesting is a modified version of the original proposal [8] which does not fix gauge configurations to any gauge. This version can be applied to calculate disconnected quark loop contributions in two- and three-point functions as well as hadron four-point functions. Our chief findings obtained with the method for the π - π , π - N , and N - N scattering lengths have been briefly reported in Refs. [10–12]. In this paper we present the full details of the method and analyses of our calculation of hadron scattering lengths. We also report additional results for K - N and \bar{K} - N scattering. For applications of the method to calculate disconnected quark loop contributions in hadron two- and three-point functions, we refer to our work on the η' meson mass [13], the π - N σ term [14,15], and the axial vector matrix elements of the proton [15,16].

Historically the first attempt toward a lattice calculation of hadron scattering lengths was made by Guagnelli, Marinari, and Parisi [17] for the π - π and π - N cases

using the Kogut-Susskind and Wilson quark actions in quenched QCD. They found a finite volume shift of the energy of two-hadron states and examined the volume dependence. However, they calculated only gluon exchange diagrams, and thus the results cannot be compared with physical scattering lengths. Gupta, Kilcup, Patel, and Sharpe [18,19] developed an analysis of the physical π - π scattering length for the $I = 2$ channel which can be calculated with the conventional wall source method. Their studies were made in quenched QCD using the Kogut-Susskind [18] and Wilson [19] quark actions, and lattice results for scattering lengths were found to be consistent with the predictions of current algebra and PCAC (partial conservation of axial vector current) for the $I = 2$ channel up to quite large quark masses.

The focus of this paper consists in an extension of the previous $I = 2$ results for the π - π scattering length to the $I = 0$ channel and also to the π - N ($I = 1/2, 3/2$), K - N and \bar{K} - N ($I = 0, 1$) scattering. These extensions require the modified version of the wall source method developed here. We also carry out an exploratory study of the nucleon-nucleon scattering lengths. Our simulations have been made within quenched QCD at $\beta = 5.7$ on a $12^3 \times 20$ lattice for the π - π , π - N , K - N , and \bar{K} - N cases and on a 20^4 lattice for the N - N case. For π - π scattering we employed both the Kogut-Susskind and the Wilson quark action, while other scattering lengths are studied with the Wilson quark action.

This paper is organized as follows. In Sec. II we describe the formalism for calculation of hadron scattering lengths including the Lüscher's formula [5] and our calculational technique of modified wall sources for construction of hadron four-point functions. In Sec. III we summarize our data sets. The results for π - π scattering lengths are presented in Sec. IV, where we also compare Kogut-Susskind results with and without gauge fixing, and discuss the question of infrared singularities in quenched QCD associated with the degeneracy of η' and π . Results for π - N , K - N , and \bar{K} - N scattering lengths are given in Sec. V, while those for N - N scattering lengths are presented in Sec. VI. In Sec. VII we discuss the issue of statistical errors in relation to a fully realistic lattice QCD calculation of hadron scattering lengths in the future. Our conclusions are summarized in Sec. VIII.

Throughout this paper we employ lattice units for expressing physical quantities and suppress the lattice spacing a unless necessary.

II. FORMALISM AND METHOD OF MEASUREMENT

A. Scattering length from two-particle energy in a finite box

The basic formula for a calculation of scattering lengths was derived by Lüscher [5] who related the s -wave scattering length a_0 between the two hadrons h_1 and h_2 to the energy shift of the two hadron state at zero relative momentum confined in a finite periodic spatial box of a size L^3 . The formula is given by

$$E_{h_1 h_2} - (m_{h_1} + m_{h_2}) = -\frac{2\pi(m_{h_1} + m_{h_2})a_0}{m_{h_1} m_{h_2} L^3} \left[1 + c_1 \frac{a_0}{L} + c_2 \left(\frac{a_0}{L} \right)^2 \right] + O(L^{-6}) \quad (1)$$

$$C_{h_1 h_2}(t_4, t_3, t_2, t_1) = \left\langle \sum_{\vec{x}_4} \mathcal{O}_{h_2}(\vec{x}_4, t_4) \sum_{\vec{x}_3} \mathcal{O}_{h_1}(\vec{x}_3, t_3) \sum_{\vec{x}_2} \mathcal{O}_{h_2}^\dagger(\vec{x}_2, t_2) \sum_{\vec{x}_1} \mathcal{O}_{h_1}^\dagger(\vec{x}_1, t_1) \right\rangle. \quad (2)$$

In order to avoid Fierz rearrangement of quark lines to be discussed in Sec. IID, we choose $t_4 = t+1$, $t_3 = t$, $t_2 = 1$, and $t_1 = 0$. In this case the large time behavior reads

$$C_{h_1 h_2}(t+1, t, 1, 0) = Z_{h_1 h_2} \exp(-E_{h_1 h_2} t) + \dots, \quad (3)$$

where the ellipsis denotes contributions of higher excited states.

The hadron masses m_{h_1} and m_{h_2} needed to obtain the energy shift $\delta E = E_{h_1 h_2} - (m_{h_1} + m_{h_2})$ are calculated in the usual manner from the two-point functions

$$C_{h_1}(t, 0) = \left\langle \sum_{\vec{x}_3} \mathcal{O}_{h_1}(\vec{x}_3, t) \sum_{\vec{x}_1} \mathcal{O}_{h_1}^\dagger(\vec{x}_1, 0) \right\rangle = Z_{h_1} \exp(-m_{h_1} t) + \dots, \quad (4)$$

$$C_{h_2}(t+1, 1) = \left\langle \sum_{\vec{x}_4} \mathcal{O}_{h_2}(\vec{x}_4, t+1) \sum_{\vec{x}_2} \mathcal{O}_{h_2}^\dagger(\vec{x}_2, 1) \right\rangle = Z_{h_2} \exp(-m_{h_2} t) + \dots. \quad (5)$$

We extract the energy shift δE from the ratio

$$R(t) = \frac{C_{h_1 h_2}(t+1, t, 1, 0)}{C_{h_1}(t, 0) C_{h_2}(t+1, 1)} = \frac{Z_{h_1 h_2}}{Z_{h_1} Z_{h_2}} \exp(-\delta E t) + \dots. \quad (6)$$

We should note that the Lüscher's formula (1) allows a reliable extraction of scattering lengths only if the expansion parameter a_0/L is small. Since the magnitude of a_0 is *a priori* not known, this point has to be examined with results for a_0 themselves, which we shall make with our data. A full analysis to confirm the size dependence of the energy shift employing a set of sizes, however, is beyond the scope of this paper since this will require a substantial amount of computing resources.

Another point to note is that formula (1) cannot be applied to the lowest two-hadron state if it is a bound state. In such a case the lowest scattering state orthogonal to the bound state has to be constructed to apply (1). This is pertinent for N - N scattering lengths since the spin triplet channel has the deuteron bound state. We shall discuss this point further in Sec. VI.

B. Hadron four-point functions

Consider s -wave π - π scattering, for which isospin 0 and 2 channels are allowed due to Bose symmetry. We con-

with $c_1 = -2.837297$, $c_2 = 6.375183$.

The energy $E_{h_1 h_2}$ is extracted from the large time behavior of the hadron four-point function with the hadrons in the zero momentum state defined by

struct the two-pion operators for these isospin eigenchannels as

$$\mathcal{O}_{I=0}^{\pi\pi}(t) = \frac{1}{\sqrt{3}} [\mathcal{O}_{\pi^+}(t) \mathcal{O}_{\pi^-}(t+1) - \mathcal{O}_{\pi^0}(t) \mathcal{O}_{\pi^0}(t+1) + \mathcal{O}_{\pi^-}(t) \mathcal{O}_{\pi^+}(t+1)], \quad (7)$$

$$\mathcal{O}_{I=2}^{\pi\pi}(t) = \mathcal{O}_{\pi^+}(t) \mathcal{O}_{\pi^+}(t+1), \quad (8)$$

with the pion operators defined by

$$\mathcal{O}_{\pi^+}(t) = -\sum_{\vec{x}} \bar{d}(\vec{x}, t) \gamma_5 u(\vec{x}, t), \quad (9)$$

$$\mathcal{O}_{\pi^-}(t) = \sum_{\vec{x}} \bar{u}(\vec{x}, t) \gamma_5 d(\vec{x}, t), \quad (10)$$

$$\mathcal{O}_{\pi^0}(t) = \frac{1}{\sqrt{2}} \sum_{\vec{x}} [\bar{u}(\vec{x}, t) \gamma_5 u(\vec{x}, t) - \bar{d}(\vec{x}, t) \gamma_5 d(\vec{x}, t)]. \quad (11)$$

In Fig. 1 we display quark line diagrams contributing to the π - π four-point functions, denoting them as direct (D), crossed (C), rectangular (R), and vacuum (V) diagrams (meaning of circles and bars will be explained in Sec. IIC). The ratio $R(t)$ for isospin eigenchannels can be expressed in terms of the four amplitudes in the combination

$$R_{I=0}^{\pi\pi}(t) = R^D(t) + \frac{1}{2} R^C(t) - 3R^R(t) + \frac{3}{2} R^V(t), \quad (12)$$

$$R_{I=2}^{\pi\pi}(t) = R^D(t) - R^C(t). \quad (13)$$

For π - N scattering the $I = \frac{1}{2}$ and $\frac{3}{2}$ operators may be taken as

$$\mathcal{O}_{I=1/2}^{\pi N}(t) = \sqrt{\frac{2}{3}} \mathcal{O}_{\pi^+}(t+1) \mathcal{O}_n(t) - \frac{1}{\sqrt{3}} \mathcal{O}_{\pi^0}(t+1) \mathcal{O}_p(t), \quad (14)$$

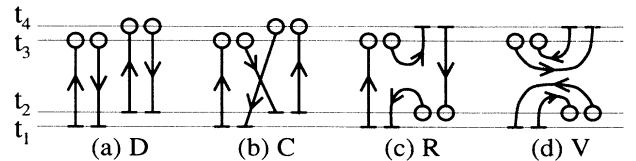


FIG. 1. Diagrams contributing to π - π four-point functions. Short bars represent wall sources. Open circles are sinks for local pion operator.

$$\mathcal{O}_{I=3/2}^{\pi N}(t) = \mathcal{O}_{\pi^+}(t+1)\mathcal{O}_p(t). \quad (15)$$

For the pion operators we use (9)–(11). Relativistic op-

$$\mathcal{O}_p(t) = \sum_{\vec{x}} \varepsilon_{abc} \{ [{}^t u^a(\vec{x}, t) C^{-1} \gamma_5 d^b(\vec{x}, t)] u^c(\vec{x}, t) - [{}^t d^a(\vec{x}, t) C^{-1} \gamma_5 u^b(\vec{x}, t)] u^c(\vec{x}, t) \}, \quad (16)$$

$$\mathcal{O}_n(t) = \sum_{\vec{x}} \varepsilon_{abc} \{ [{}^t u^a(\vec{x}, t) C^{-1} \gamma_5 d^b(\vec{x}, t)] d^c(\vec{x}, t) - [{}^t d^a(\vec{x}, t) C^{-1} \gamma_5 u^b(\vec{x}, t)] d^c(\vec{x}, t) \}. \quad (17)$$

In terms of $\mathcal{O}_I^{\pi N}$ we construct the π - N four-point function according to

$$C_I^{\pi N}(t) = \frac{1}{2} \text{Tr} \left[\frac{1 + \gamma_4}{2} \langle \mathcal{O}_I^{\pi N}(t) \mathcal{O}_I^{\pi N \dagger}(0) \rangle \right], \quad (18)$$

where trace refers to Dirac components.

Topologically four types of diagrams contribute to π - N four-point functions, direct (D), crossed (C), rectangular (R), and crossed rectangular (CR), as shown in Fig. 2. In contrast to the π - π case, these diagrams have 6, 18, 18, and 36 members having the same quark line topology but different quark contractions. All four types of diagrams contribute to both isospin eigenchannels. Weights of the members are listed in the Appendix.

For $I = 0$ and 1 operators for K - N and \bar{K} - N scattering we take

$$\mathcal{O}_{I=0}^{KN}(t) = \frac{1}{\sqrt{2}} [\mathcal{O}_{K^+}(t+1)\mathcal{O}_n(t) - \mathcal{O}_{K^0}(t+1)\mathcal{O}_p(t)], \quad (19)$$

$$\mathcal{O}_{I=1}^{KN}(t) = \mathcal{O}_{K^+}(t+1)\mathcal{O}_p(t), \quad (20)$$

$$\mathcal{O}_{I=0}^{\bar{K}N}(t) = \frac{1}{\sqrt{2}} (\mathcal{O}_{\bar{K}^0}(t+1)\mathcal{O}_n(t) - \mathcal{O}_{K^-}(t+1)\mathcal{O}_p(t)), \quad (21)$$

$$\mathcal{O}_{I=1}^{\bar{K}N}(t) = \mathcal{O}_{\bar{K}^0}(t+1)\mathcal{O}_p(t), \quad (22)$$

where the nucleon operators \mathcal{O}_p and \mathcal{O}_n are given in (16) and (17) and the K meson operators are defined by

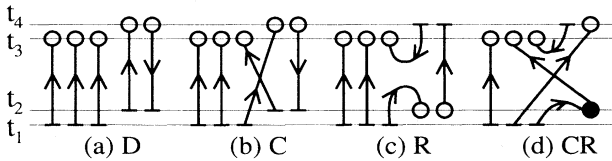


FIG. 2. Diagrams contributing to π - N four-point functions. Short bars represent wall sources. Open circles are sinks for local pion or nucleon operator. Solid circle for the CR diagram means use of source method for calculating the product of quark propagators connected by it.

erators for the proton (p) and neutron (n) are defined with the charge conjugation matrix $C = \gamma_4 \gamma_2$ taking account of the symmetries for isospin and nonrelativistic spin representations:

$$\mathcal{O}_{K^+}(t) = \sum_{\vec{x}} \bar{s}(\vec{x}, t) \gamma_5 u(\vec{x}, t), \quad (23)$$

$$\mathcal{O}_{K^0}(t) = \sum_{\vec{x}} \bar{s}(\vec{x}, t) \gamma_5 d(\vec{x}, t), \quad (24)$$

$$\mathcal{O}_{\bar{K}^0}(t) = - \sum_{\vec{x}} \bar{d}(\vec{x}, t) \gamma_5 s(\vec{x}, t), \quad (25)$$

$$\mathcal{O}_{K^-}(t) = \sum_{\vec{x}} \bar{u}(\vec{x}, t) \gamma_5 s(\vec{x}, t). \quad (26)$$

In contrast to the π - N case, K - N scattering receives contributions only from direct (D) and crossed (C) type diagrams and \bar{K} - N scattering only from direct and rectangular (R) ones. Weights of various quark contractions to isospin eigen amplitudes are given in the Appendix.

In s -wave N - N scattering spin triplet 3S_1 and singlet 1S_0 states choose isospin 0 and 1, respectively, due to Fermi statistics. For these eigenchannels we use the operators

$$\mathcal{O}({}^3S_1)(t) = \frac{1}{\sqrt{2}} [\mathcal{O}_p(t)\mathcal{O}_n(t+1) - \mathcal{O}_n(t)\mathcal{O}_p(t+1)], \quad (27)$$

$$\mathcal{O}({}^1S_0)(t) = \mathcal{O}_p(t)\mathcal{O}_p(t+1), \quad (28)$$

where the upper two Dirac components of the nucleon operators \mathcal{O}_p and \mathcal{O}_n as defined in (16) and (17) are combined to form the respective spin states. N - N four-point functions require only two types of diagrams shown in Fig. 3. The direct (D) diagram has 36 members with different quark contractions. For the crossed (C) dia-

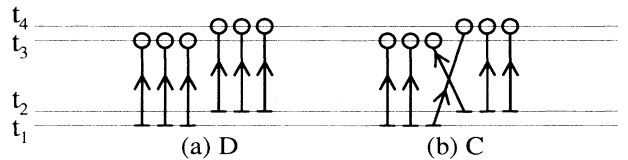


FIG. 3. Diagrams contributing to N - N four-point functions. Short bars represent wall sources. Open circles are sinks for local nucleon operator.

gram, which has 324 members, diagrams with a single quark-antiquark exchange and a double quark-antiquark exchange are equivalent because the two nucleon operators are summed over all spatial sites for s -wave states. Since quark contractions for N - N four-point functions are very complicated and tedious, we employed symbolic manipulations on computers to work out the weights of amplitudes.

C. Wall source method without gauge fixing

Let us consider the numerical procedure for a calculation of hadron four-point functions. For π - π scattering shown in Fig. 1 the direct and crossed diagrams can be easily calculated because we need only two quark propagators with wall sources placed at the fixed time slices t_1 and t_2 [18,19] in order to construct the corresponding four-point amplitudes for arbitrary values of t_3 and t_4 (the N - N case is similar). This does not apply to the

case of rectangular and vacuum diagrams which require additional quark propagators connecting the time slices t_3 and t_4 . The same difficulty also exists in the π - N , K - N , and \bar{K} - N cases for the rectangular and crossed-rectangular types of diagrams.

We handle this problem by calculating T quark propagators on an $L^3 \times T$ lattice, each propagator corresponding to a wall source placed at the time slice $t = 0, \dots, T-1$, which are defined by

$$\sum_{n''} D_{n',n''} G_t(n'') = \sum_{\vec{x}} \delta_{n',(\vec{x},t)}, \quad 0 \leq t \leq T-1, \quad (29)$$

where D denotes the quark matrix for the Wilson or the Kogut-Susskind quark action. The combination of $G_t(n)$ that we employ for hadron four-point functions is displayed in Figs. 1–3, where short bars represent the position of wall source and circles the sink. For example, the π - π rectangular diagram in Fig. 1(c) corresponds to

$$C^R(t_4, t_3, t_2, t_1) = \sum_{\vec{x}_2, \vec{x}_3} \langle \text{ReTr}[G_{t_1}^\dagger(\vec{x}_2, t_2) G_{t_4}(\vec{x}_2, t_2) G_{t_4}^\dagger(\vec{x}_3, t_3) G_{t_1}(\vec{x}_3, t_3)] \rangle, \quad (30)$$

where daggers mean conjugation by γ_5 for the Wilson quark action and by the even-odd parity $(-1)^n$ for the Kogut-Susskind quark action.

Using the relation $G_t(n'') = \sum_{\vec{x}} D_{n'',(\vec{x},t)}^{-1}$, we see that this prescription yields four-point amplitudes corresponding to nonlocal and nongauge invariant hadron operators at the time slices where two wall sources or one wall and one sink are placed [e.g., at the time slices t_1 and t_4 in (30)]. These terms create gauge-variant noise. One way to suppress the noise is to fix gauge configurations to some gauge as is done in all recent work using wall sources [9]. A potential drawback is that gauge noninvariant states may contaminate the four-point function.

Alternatively one can choose not to fix gauge configurations to any gauge since gauge dependent fluctuations should cancel out in the ensemble average. This is the wall source version of the original proposal of extended sources [8]. One might worry if signal stands out among noise. For example, if two wall sources are placed at the same time slice, there are $O((L^3)^2)$ gauge-dependent nonlocal terms relative to $O(L^3)$ local gauge invariant ones, and hence naively the magnitude of the noise $O(\sqrt{(L^3)^2})$ would be comparable to that of the signal. However, a generalization [20] of the well-known argument [21] can be used to show that the noise is in fact smaller than the signal by a factor $L^{3/2}$ for sufficiently large L .

Let us illustrate this point with a simpler example of a two-point function $C_\Gamma(t)$ for the meson operator $\bar{q}\Gamma q$ for the Wilson quark action. We evaluate this quantity as

$$C_\Gamma(t) = \frac{1}{L^3} \left\langle \sum_{\vec{x}} \text{Tr}[G_0^\dagger(\vec{x}, t) \Gamma G_0(\vec{x}, t) \Gamma] \right\rangle, \quad (31)$$

where the factor $1/L^3$ is inserted to ensure a finite limit as $L \rightarrow \infty$ and the trace is taken over Dirac indices and color indices. Substituting the expression

$$G_0(\vec{x}, t) = \sum_{\vec{y}} G[(\vec{x}, t), (\vec{y}, 0)] \quad (32)$$

for the wall-source quark propagator $G_0(n)$ in terms of the point-to-point quark propagator $G(n, m)$, we can rewrite (31) as

$$C_\Gamma(t) = \left\langle \sum_{\vec{y}, \vec{z}} \text{Tr}[G_\Gamma(\vec{y}, \vec{z}) \Gamma] \right\rangle, \quad (33)$$

where

$$G_\Gamma(\vec{y}, \vec{z}) = \frac{1}{L^3} \sum_{\vec{x}} G[(\vec{y}, 0), (\vec{x}, t)] \Gamma G[(\vec{x}, t), (\vec{z}, 0)]. \quad (34)$$

The dispersion of the two-point function $C_\Gamma(t)$ is given by

$$\begin{aligned}
\sigma^2 &= \left\langle \left| \sum_{\vec{y}, \vec{z}} \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma] \right|^2 \right\rangle - \left\langle \sum_{\vec{y}, \vec{z}} \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma] \right\rangle^2 \\
&= \left\langle \sum_{\vec{y}, \vec{z}} \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma] \sum_{\vec{z}', \vec{y}'} \text{Tr} [G_\Gamma(\vec{z}', \vec{y}')\Gamma] \right\rangle - \left\langle \sum_{\vec{y}, \vec{z}} \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma] \right\rangle \left\langle \sum_{\vec{z}', \vec{y}'} \text{Tr} [G_\Gamma(\vec{z}', \vec{y}')\Gamma] \right\rangle, \tag{35}
\end{aligned}$$

where we have used the relation $G^\dagger(n, m) = \gamma_5 G(m, n) \gamma_5$ and assumed that $\gamma_5 \Gamma \gamma_5 = \pm \Gamma$. We recognize the first term on the right-hand side to be the expression for the four-point function with a quark operator placed at \vec{z} and \vec{y}' and an antiquark operator at \vec{y} and \vec{z}' of the time slice $t = 0$. Since gauge configurations are not fixed to any gauge, nonvanishing results are obtained only if $\vec{z} = \vec{y}$ and $\vec{y}' = \vec{z}'$ or $\vec{z} = \vec{z}'$ and $\vec{y}' = \vec{y}$. We thus find that

$$\sigma^2 = \sigma_D^2 + \sigma_C^2, \tag{36}$$

with

$$\sigma_D^2 = \sum_{\vec{y}, \vec{z}} \langle \text{Tr} [G_\Gamma(\vec{y}, \vec{y})\Gamma] \text{Tr} [G_\Gamma(\vec{z}, \vec{z})\Gamma] \rangle - \sum_{\vec{y}, \vec{z}} \langle \text{Tr} [G_\Gamma(\vec{y}, \vec{y})\Gamma] \rangle \langle \text{Tr} [G_\Gamma(\vec{z}, \vec{z})\Gamma] \rangle, \tag{37}$$

$$\begin{aligned}
\sigma_C^2 &= \sum_{\vec{y}, \vec{z}} \langle \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma] \text{Tr} [G_\Gamma(\vec{z}, \vec{y})\Gamma] \rangle \\
&= \frac{1}{12} \sum_{i=S, V, T, A, P} \sum_{\vec{y}, \vec{z}} \langle \text{Tr} [G_\Gamma(\vec{y}, \vec{z})\Gamma \Gamma_i G_\Gamma(\vec{z}, \vec{y})\Gamma \Gamma_i] \rangle, \tag{38}
\end{aligned}$$

where for σ_C^2 we used the Fierz transformation for Dirac and color indices to combine the two traces into a single trace and ignored color nonsinglet combinations. Substituting (34) it is easy to see that σ_D^2 equals the direct amplitude for the meson-meson scattering in the channel $\Gamma + \Gamma \rightarrow \Gamma + \Gamma$ with all meson at rest, multiplied by a factor $1/L^3$. Similarly σ_C^2 equals the crossed amplitude for the channel $\Gamma \Gamma_i + \Gamma \Gamma_i \rightarrow \Gamma + \Gamma$ multiplied by $1/L^3$. We thus find that the statistical error for the two-point function evaluated with N_{conf} independent gauge configurations is given by

$$\begin{aligned}
\delta C_\Gamma(t) &= \sqrt{\frac{\sigma^2}{N_{\text{conf}}}} \\
&= \sqrt{\frac{1}{N_{\text{conf}} L^3} \left(C_{\Gamma+\Gamma \rightarrow \Gamma+\Gamma}^D(t) + \frac{1}{12} \sum_{i=S, V, T, A, P} C_{\Gamma \Gamma_i + \Gamma \Gamma_i \rightarrow \Gamma+\Gamma}^C(t) \right)}. \tag{39}
\end{aligned}$$

This formula immediately shows that the statistical error decreases as $L^{-3/2}$ relative to the signal. The direct amplitude decreases as $\exp(-2m_\Gamma t)$ up to polynomials in t with m_Γ the mass of the meson in the channel Γ . For the crossed amplitude we expect the dominant term in the sum over i to come from the channel $\pi + \pi \rightarrow \Gamma + \Gamma$ having the lightest energy. For a sufficiently large spatial size L the crossed amplitude in this channel may be approximately calculated by a convolution of free propagation of two pions from the time slice $t = 0$ to t' ($0 < t' < t$), a local $\pi\pi\Gamma\Gamma$ coupling at $t = t'$ and free propagation of two mesons in the channel Γ from $t = t'$ to t . As an estimate of the relative error we then find

$$\frac{\delta C_\Gamma(t)}{C_\Gamma(t)} \propto \sqrt{\frac{1}{N_{\text{conf}} L^3}} \exp[(m_\Gamma - m_\pi)t]. \tag{40}$$

It is straightforward to generalize the argument to hadron four-point functions $C_{h_1 h_2}$. The relative error is given by

$$\begin{aligned}
&\frac{\delta C_{h_1 h_2}(t)}{C_{h_1 h_2}(t)} \\
&\propto \sqrt{\frac{1}{N_{\text{conf}} L^3}} \exp\{[m_{h_1} + m_{h_2} - (n_{h_1} + n_{h_2})m_\pi]t\}, \tag{41}
\end{aligned}$$

where $n_h = 1$ for mesons and $3/2$ for baryons.

We should note that the magnitude of the proportionality constant in (40) and (41) is not known, and hence the magnitude of errors may vary depending on the channel. With a practically manageable statistics of a few hundred configurations, we found that statistical errors are small for pion two- and four-point functions, whereas the signal deteriorates for the ρ and nucleon. For the nucleon, in particular, errors are significant even for the propagator. We, therefore, use the Coulomb gauge fixing for the nucleon source for calculation of scattering lengths involving nucleon.

D. Avoidance of Fierz contaminations

Our method of calculating quark propagators for all possible temporal positions of wall sources has another advantage. Suppose that we only have the quark propagator corresponding to a single wall source placed at the time slice t . Calculation of meson four-point functions in this case necessarily involve two meson operators $\sum_{\vec{x}, \vec{y}, \alpha} \bar{q}(\vec{x}, t)^\alpha \Gamma q(\vec{y}, t)^\alpha$ and $\sum_{\vec{x}', \vec{y}', b} \bar{q}(\vec{x}', t)^b \Gamma' q(\vec{y}', t)^b$ placed at the same time slice t . It is easy to see that there are two contributions, one in which each of the pairs $(\bar{q}(\vec{x}, t)^\alpha, q(\vec{y}, t)^\alpha)$ and $(\bar{q}(\vec{x}', t)^b, q(\vec{y}', t)^b)$ form color singlets, and the other in which the Fierz-rearranged pairs $(\bar{q}(\vec{x}, t)^\alpha, q(\vec{y}', t)^b)$ and $(\bar{q}(\vec{x}', t)^b, q(\vec{y}, t)^\alpha)$ form color singlets. Such a rearrangement causes a mixing of diagrams having different quark line topologies. In general it also switches the spin parity of the two hadrons in the initial state. This leads to quite a complication of analyses, especially for the Kogut-Susskind quark action, as was discussed in detail in Ref. [18]. Disentangling the two contributions is possible in principle, but difficult in practice. This, in fact, is the reason why the work of Ref. [18] could not determine the direct and crossed amplitudes separately for the π - π four-point function for the Kogut-Susskind quark action.

The problem can be trivially solved in our method since we calculate quark propagators for all possible temporal positions of wall source, and therefore hadron operators can be placed at different time slices. In practice we separate hadron operators by a unit time slice both for the initial and final pairs of hadrons.

III. DATA SETS

In Table II we list the parameters of our simulation. All of our calculations are made in the quenched approximation at $\beta = 5.7$. The inverse lattice spacing determined from the ρ meson mass in the chiral limit equals

$a^{-1} = 1.44(2)$ GeV for the Wilson quark action and $a^{-1} = 0.98(11)$ GeV for the Kogut-Susskind quark action. We mostly employed a $12^3 \times 20$ lattice for π - π , π - N , K - N , and \bar{K} - N scattering and a 20^4 lattice for N - N scattering, anticipating large scattering lengths for the latter. The lattice size must be large enough so that we can employ as weak a coupling as possible to avoid finite lattice spacing effects, yet it should not be too large so as not to spoil a detection of a small energy shift of $O(L^{-3})$ predicted by the Lüscher's relation (1); our parameters are a reasonable compromise with the present computing resources. Gauge configurations are generated with a 5-hit pseudo heat bath algorithm, discarding 1000 configurations for thermalization, and employing every 1000th configuration for hadron four-point function analyses.

For the π - π four-point function we made bulk of our calculations employing wall sources without gauge fixing. A calculation with Coulomb gauge fixing was also carried out for a subset of configurations with the Kogut-Susskind quark action in order to compare the two methods (see Sec. IV A 2). In the case of π - N scattering the pion source is treated with the wall source method without gauge fixing, while for the nucleon source placed at $t = 0$ we fixed the $t = 0$ time slice of gauge configurations to Coulomb gauge in order to enhance signal to noise ratio. The K - N and \bar{K} - N four-point functions are calculated in the same way. We used the same quark mass for strange and up-down quarks for this case. For calculations of the N - N four-point functions gauge configurations are fixed to Coulomb gauge over the entire space-time lattice as this leads to better signals.

Quark propagators are calculated with the Dirichlet boundary condition in the temporal direction in order to avoid contamination from hadrons propagating backward in time. The periodic boundary condition is used in the spatial directions. For the Kogut-Susskind quark action the standard conjugate gradient algorithm is employed for inverting the quark matrix, and for Wilson quark action the incomplete LU decomposition conjugate residual

TABLE II. Parameters of simulation. All runs are made at $\beta = 5.7$ in quenched QCD.

	Action	Quark mass	Lattice size	No. conf.	Gauge fixing
π - π	KS	$m_q = 0.01$	$8^3 \times 20$	400	None
	KS	$m_q = 0.01$	$12^3 \times 20$	160	None
	KS	$m_q = 0.01$	$12^3 \times 20$	60	Coulomb
	Wilson	$K = 0.164$	$12^3 \times 20$	70	None
π - N	Wilson	$K = 0.164$	$12^3 \times 20$	60	$t = 0$ Coulomb
	Wilson	$K = 0.1665$	$12^3 \times 20$	30	$t = 0$ Coulomb
K - N	Wilson	$K = 0.164$	$12^3 \times 20$	60	$t = 0$ Coulomb
\bar{K} - N	Wilson	$K = 0.164$	$12^3 \times 20$	60	$t = 0$ Coulomb
N - N	Wilson	$K = 0.150$	$20^3 \times 20$	20	Coulomb
	Wilson	$K = 0.160$	$20^3 \times 20$	30	Coulomb
	Wilson	$K = 0.164$	$20^3 \times 20$	20	Coulomb

TABLE III. Hadron masses and pion decay constant at $\beta = 5.7$ in quenched QCD. Also listed for comparison are results on a larger lattice from Refs. [23,24]. For the Wilson result of Ref. [24] a linear interpolation in $1/K$ is made when necessary (marked by an asterisk).

Action	Quark mass	Lattice size	m_π	m_ρ	m_N	f_π
KS	$m_q = 0.01$	$12^3 \times 20$	0.290(3)			0.132(3)
		$24^3 \times 32$ [23]	0.2876(7)	0.883(58)	1.454(26)	
Wilson	$K = 0.150$	$12^3 \times 20$	1.0758(51)	1.1302(66)		0.1900(12)
		$20^3 \times 20$			1.788(11)	
	$K = 0.160$	$12^3 \times 20$	0.6876(31)	0.8053(39)	1.2957(79)	0.1268(14)
		$20^3 \times 20$			1.302(13)	
		$24^3 \times 32$ [24]	0.6887(11)	0.8062(18)	1.3068(51)	0.1249(10)
	$K = 0.164$	$12^3 \times 20$	0.5080(37)	0.6865(53)	1.080(10)	0.1009(7)
		$20^3 \times 20$			1.093(20)	
		$24^3 \times 32$ [24]	0.5027(13)*	0.6886(28)*	1.0869(64)*	0.1000(10)*
	$K = 0.1665$	$12^3 \times 20$	0.3663(44)	0.6086(75)	0.926(13)	0.0832(14)
		$24^3 \times 32$ [24]	0.3656(17)*	0.6165(45)*	0.927(12)*	0.0848(16)*

(ILUCR) method [22]. The stopping condition is chosen to ensure a 0.1% accuracy in hadron four-point functions for each configuration.

In Table III we summarize π, ρ , and nucleon (N) masses in lattice units obtained on our ensemble of configurations through standard single exponential fits of hadron propagators over the time interval $6 \leq t \leq 12-14$. Also listed are the values for the pion decay constant obtained with the tadpole-improved renormalization factor [25] using $\alpha_{\overline{\text{MS}}}(1/a) = 0.2207$ for the coupling constant, where $\overline{\text{MS}}$ denotes the modified minimal subtraction scheme. The values listed are in agreement with the results obtained on larger lattices [23,24], which are also given in Table III. A necessary condition for the applicability of formula (1) is that the lattice size L is sufficiently large so that finite-size effects for hadron masses are negligible. The agreement provides a check, albeit indirect, on this point.

We estimate errors by the single elimination jackknife procedure for all quantities including hadron masses, energy shifts, and scattering lengths obtained by fits of two- and four-point functions.

IV. π - π SCATTERING LENGTHS

A. Kogut-Susskind quark action

1. No gauge fixing

Current algebra and PCAC predict [2] that the s -wave π - π scattering lengths, to leading order in m_π , take the values

$$a_0^{I=0} = +\frac{7}{32\pi} \frac{m_\pi}{f_\pi^2}, \quad (42)$$

$$a_0^{I=2} = -\frac{2}{32\pi} \frac{m_\pi}{f_\pi^2}. \quad (43)$$

It has been shown in Ref. [18] that these results can also be derived for the Kogut-Susskind quark action at a finite lattice spacing using Ward identities for U(1) chiral symmetry under some continuity assumptions on the pion four-point function in external momenta. Whether numerical simulations yield results in agreement with (42) and (43) therefore provides a valuable check of the lattice method for hadron four-point functions.

The use of Kogut-Susskind action has a subtle problem concerning the interpretation of the four quark flavors corresponding to each Kogut-Susskind field. Following Ref. [18] we introduce a Kogut-Susskind field for each continuum flavor, regarding the four flavors of the Kogut-Susskind action as a fourfold duplication of each continuum flavor. In this view the expression (12),(13) for the π - π four-point function in isospin eigenchannels needs to be modified to

$$R_{I=0}^{\pi\pi}(t) = R^D(t) + \frac{N_f}{2} R^C(t) - 3N_f R^R(t) + \frac{3}{2} R^V(t), \quad (44)$$

$$R_{I=2}^{\pi\pi}(t) = R^D(t) - N_f R^C(t). \quad (45)$$

where the factors $N_f = 4$ compensate a different number of Kogut-Susskind flavor traces in the four types of diagrams. For the pion operator it is most natural to take the one in the Nambu-Goldstone channel corresponding to U(1) chiral symmetry. This is the choice for which the current algebra result (42) and (43) can be derived.

We have carried out simulations on an $8^3 \times 20$ lattice and a $12^3 \times 20$ lattice, both at $\beta = 5.7$ and $m_q = 0.01$ in quenched QCD, employing the method of wall source without gauge fixing. The results on a $12^3 \times 20$ lattice have been briefly reported in Refs. [10,11]. In Fig. 4 we recapitulate the individual ratios $R^X(t)$ ($X = D, C, R$, and V) on a $12^3 \times 20$ lattice plotted as functions of t . Good signals observed up to $t \approx 12$ for the rectangular amplitude and up to $t \approx 8$ for the vacuum amplitude demonstrate the practical applicability of the method

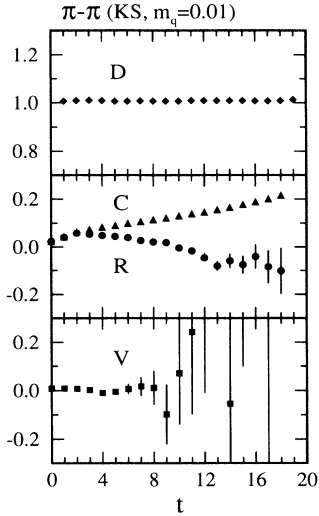


FIG. 4. Individual ratios $R^X(t)$ ($X = D, C, R,$ and V) for π - π four-point function calculated with the method of wall source without gauge fixing for Kogut-Susskind quark action on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$.

of wall source without gauge fixing. For the direct and crossed amplitudes previous results [18] employed a single wall source at $t = 0$, and could not resolve the Fierz-rearranged mixing between the two amplitudes. As we discussed in Sec. IID separate calculation of these diagrams is possible in our case since the two pion sources are placed at a unit lattice spacing apart in the time direction.

Physically the interesting features in Fig. 4 are (1) a very flat behavior of the direct amplitude showing that the interaction is weak in this channel; (2) an almost linear t dependence of the crossed and rectangular amplitudes with a slope of similar magnitude but opposite sign; and (3) a small value of the vacuum amplitude. These characteristics are in agreement with expectations from chiral perturbation theory and the empirical Okubo-Zweig-Iizuka (OZI) rule. We also note that the crossed and rectangular amplitudes have the same value at $t = 0$ as expected from the fact that their analytic expressions are identical at this value of t .

In Fig. 5 we plot the ratio $R_I(t)$ projected onto the $I = 0$ and 2 isospin channels for a $12^3 \times 20$ lattice. A decrease of the ratio for the $I = 2$ channel corresponds to a positive energy shift and hence to a repulsive interaction in this channel, while an increase of $R_{I=0}(t)$ means attraction for the $I = 0$ channel. A dip observed at $t \approx 2$ for the $I = 0$ channel becomes more pronounced on an $8^3 \times 20$ lattice as shown in Fig. 6 (note the change of vertical scale from Fig. 5). The physical origin of the dip and its size dependence is not clear to us.

Extraction of the energy shift δE_I from $R_I(t)$ and thence the s -wave scattering length require some care. Since our calculations are made in quenched QCD, rescattering effects that require sea quark loops are not properly taken into account. This effect starts with terms

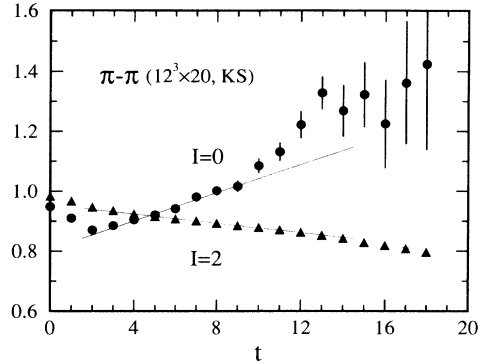


FIG. 5. $R_I(t)$ ($I = 0$ and 2) for π - π four-point function calculated without gauge fixing for the Kogut-Susskind quark action on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$. Solid lines are linear fits for $4 \leq t \leq 9$.

of $O(t^2)$ in $R_I(t)$. For the Kogut-Susskind quark action there are further complications arising from the non-degeneracy of pions in the Nambu-Goldstone and other channels at a finite lattice spacing. Briefly stating, the contribution of non-Nambu-Goldstone pions in the intermediate states is exponentially suppressed for large times due to their heavier masses compared to that of the Nambu-Goldstone pion, instead of yielding terms growing with powers of t for the degenerate case. This affects both the relation between the ratio $R_I(t)$ and the energy shift δE_I and that between δE_I and scattering lengths (1). A detailed analysis [18] shows that the $O(L^{-5})$ terms in the relation (1) are invalidated by the effect, and that $O(t^2)$ terms in $R_I(t)$ are not correct even in full QCD. Thus, a proper extraction of scattering lengths for the present case requires the spatial lattice size to be large enough so that the $O(L^{-5})$ terms are small; δE_I should also be small so that there is a range of t over which $R_I(t)$ exhibits a linear behavior.

Based on the considerations above we fit $R_I(t)$ with a

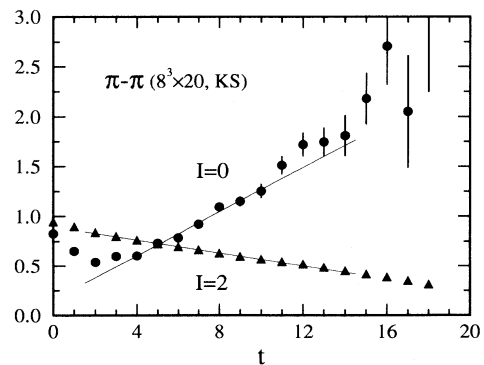


FIG. 6. $R_I(t)$ ($I = 0$ and 2) for π - π four-point function calculated without gauge fixing for the Kogut-Susskind quark action on an $8^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$. Solid lines are linear fits for $4 \leq t \leq 9$.

TABLE IV. π - π scattering lengths in lattice units for the Kogut-Susskind quark action at $m_q = 0.01$. Numbers in square brackets are current algebra predictions evaluated with the measured values of m_π and f_π listed in Table III.

Size	Gauge fixing	$I = 0$			$I = 2$		
		a_0	δE_I	Z_I	a_0	δE_I	Z_I
$8^3 \times 20$	None	4.89(53)	-0.69(21)	0.160(40)	-0.374(14)	0.0367(16)	0.897(12)
$12^3 \times 20$	None	1.57(25)	-0.0291(37)	0.807(15)	-0.301(28)	0.00813(82)	0.955(5)
	Coulomb	1.73(27)	-0.0316(40)	0.800(17)	-0.326(35)	0.0089(10)	0.948(7)
		[1.16(5)]			[-0.331(15)]		

linear form $Z_I(1 - \delta E_I t)$ with the fitting range chosen to be $4 \leq t \leq 9$, ignoring higher order terms. The fitted values of δE_I and the results for the s -wave scattering length a_0 in lattice units obtained using (1) are summarized in Table IV. Here we used pion mass given in Table III. The errors quoted for a_0 are statistical only. For a $12^3 \times 20$ lattice the $O(L^{-5})$ term contributes 10% in the $I = 0$ channel, although it is negligibly small ($< 1\%$) for $I = 2$. The $O(t^2)$ terms in $R_I(t)$, neglected in the above procedure, contribute a few percent in both isospin eigenchannels. These uncertainties are within the statistical errors of 16% for $I = 0$ and 9% for $I = 2$.

On an $8^3 \times 20$ lattice, the Z factor for the $I = 0$ channel deviate severely from unity due to a dip of $R_{I=0}(t)$ for small t , rendering the value of the extracted scattering length questionable. For the $I = 2$ channel, the value for a_0 is consistent with that from a $12^3 \times 20$ lattice.

In Table IV we also list in square brackets the values predicted by current algebra and PCAC (42),(43) substituting the value of m_π and the pion decay constant f_π in Table III. We observe that the simulation results are consistent with (42) and (43) within 1 – 1.5 standard deviations, which we find quite reasonable in view of the systematic uncertainties discussed above. Results for $a_0(I = 2)$ in agreement with (43) was previously obtained [18] with the assumption that the contribution of the direct diagram is negligible.

2. Coulomb gauge fixing

We have repeated the calculation of the π - π four-point function, fixing gauge configurations to the Coulomb gauge for all time slices for a $12^3 \times 20$ lattice. In this case the wall source and sink for pion propagators in the denominator of $R^X(t)$ have to be chosen in the same combination as for the four-point amplitude in the numerator for each type of diagrams $X = D, C, R$, and V . Without this matching the Z factors for the ratio $R^X(t)$ would be different between the diagrams, and the combination (44) and (45) would no longer project out isospin eigenchannels. This problem is absent for wall sources without gauge fixing because nonlocal terms in the wall source operator cancel out after averaging over gauge configurations.

In Fig. 7 we compare the ratio $R_I(t)$ calculated in Coulomb gauge (open symbols) and without gauge fixing (solid symbols) for the same set of 60 configurations

on a $12^3 \times 20$ lattice. We suspect that a wiggle seen up to $t \approx 4$ for the $I = 0$ channel in the Coulomb gauge calculation is due to a contribution of nonlocal ρ that can be emitted from the gauge fixed wall source. A slight discrepancy in the intercept for the $I = 2$ channel is ascribed to the difference of Z factors between the two calculations. Except for these points the two data sets are consistent. Results for scattering lengths extracted with the same procedure as for the case without gauge fixing agree with those for the case of no gauge fixing within one standard deviation (see Table IV).

An interesting question is to what extent gauge non-invariant errors are controlled in the method of wall source without gauge fixing compared to the gauge fixed case. To examine this point we plot in Fig. 8 the single-elimination jackknife errors of $R_I(t)$ for the two calculation on the same set of 60 configurations on a $12^3 \times 20$ lattice. Although the magnitude of errors is larger for the nongauge fixed case (solid symbols), the amount of increase of errors is contained at the level of a factor of 1.5 – 2 times those for the Coulomb gauge fixing (open symbols), showing that gauge variant noise does not give rise to a serious problem for calculation of π - π four-point functions.

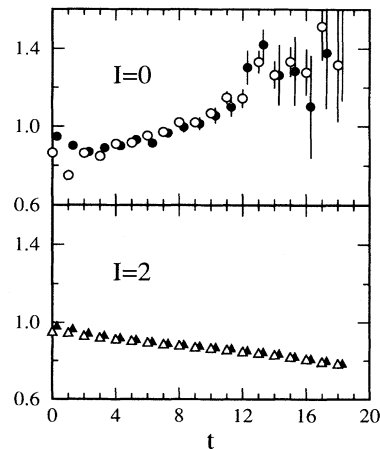


FIG. 7. Comparison of $R_I(t)$ ($I = 0$ and 2) for π - π four-point function calculated in Coulomb gauge (open symbols) and without gauge fixing (solid symbols) for the Kogut-Susskind quark action for the same set of 60 configurations on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$.

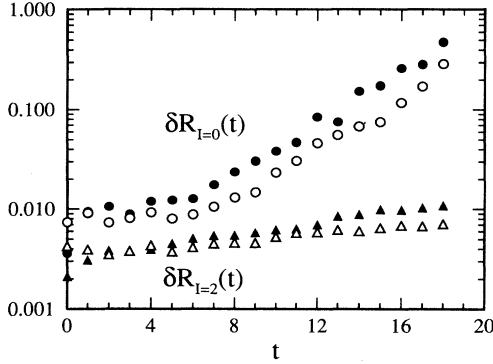


FIG. 8. Comparison of single-elimination jackknife errors of $R_I(t)$ ($I = 0$ and 2) for π - π four-point function calculated in Coulomb gauge (open symbols) and without gauge fixing (solid symbols) for the Kogut-Susskind quark action for the same set of 60 configurations on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $m_q = 0.01$.

B. Wilson quark action

We have also applied the wall source method without gauge fixing to the Wilson quark action at the hopping parameter $K = 0.164$, employing 70 configurations on a $12^3 \times 20$ lattice.

In Fig. 9 we show the ratios $R^X(t)$ for $X = D, C, R,$ and V . The direct and crossed amplitudes were previously calculated in Ref. [19] through gauge fixed wall sources placed at the same time slice. We observe in Fig. 9 that the slope for the rectangular amplitude $R^R(t)$ is quite small compared to that for the crossed ampli-

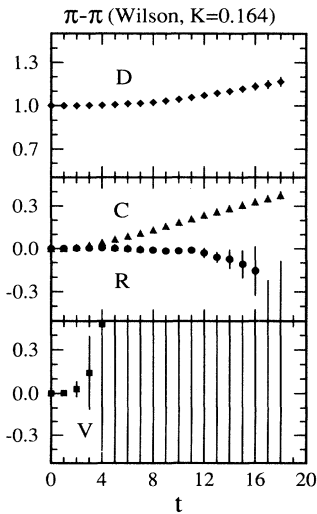


FIG. 9. Individual ratios $R^X(t)$ ($X = D, C, R,$ and V) for π - π four-point function calculated without gauge fixing for Wilson quark action on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $K = 0.164$.

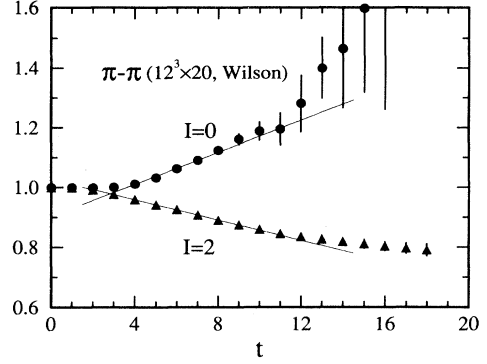


FIG. 10. $R_I(t)$ ($I = 0$ and 2) for π - π four-point function calculated without gauge fixing for the Wilson quark action on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $K = 0.164$. Solid lines are linear fits for $4 \leq t \leq 9$.

tude $R^C(t)$. This is different from the Kogut-Susskind case for which both amplitudes exhibit a similar slope (see Fig. 4). We consider that this is due to the heavy quark mass ($m_\pi/m_\rho = 0.74$) for the present calculation with the Wilson quark action compared to the small value ($m_\pi/m_\rho = 0.33$) for the Kogut-Susskind case. Another difference between the Wilson and Kogut-Susskind results is a positive curvature for the direct amplitude seen in Fig. 9 for the Wilson case, whose origin is not clear to us. Also the errors for the vacuum diagram blows up much more rapidly (compare Fig. 9 with Fig. 4). This, however, may be ascribed to a heavier pion mass for the Wilson simulation as will be discussed in Sec. VII below.

The ratios $R_I(t)$ for the isospin eigenchannels $I = 0$ and 2 are plotted in Fig. 10. To extract the energy shift δE_I for each channel, we again employ a linear form $Z_I(1 - \delta E_I t)$ with the fitting range chosen to be $4 \leq t \leq 9$. The values of the scattering lengths are listed in Table V, together with those predicted by current algebra (42) and (43) but evaluated with the measured values of m_π and f_π in Table III. We quoted only the statistical errors for the scattering lengths a_0 . Simulation results are consistent with current algebra and PCAC within 1–2 standard deviations. For the $I = 2$ channel a similar agreement was previously found in Ref. [19].

C. Question of quenched chiral divergences

All of our results are obtained in quenched QCD. A problem with quenched calculations is the uncertainty in $O(t^2)$ term of the ratio $R_I(t)$ due to lack of dynamical quark loops. We have bypassed this problem by looking only at the region of t where the ratio $R_I(t)$ shows a linear behavior. Also our data do not extend to the region of large t , where a curvature due to $O(t^2)$ terms can be established, because of increase of errors.

Another potential problem is that the π - π four-point function in quenched QCD could be affected by infrared divergences appearing in the chiral limit due to η'

TABLE V. π - π scattering lengths in lattice units for the Wilson quark action at the hopping parameter $K = 0.164$ on a $12^3 \times 20$ lattice. Numbers in square brackets are current algebra predictions evaluated with the measured values of m_π and f_π listed in Table III.

a_0	$I = 0$		$I = 2$		
	δE_I	Z_I	a_0	δE_I	Z_I
3.02(17)	-0.0297(19)	0.903(7)	-0.924(40)	0.0166(9)	1.027(5)
[3.47(5)]			[-0.993(16)]		

loops [26]. These divergences originate from the double pole $m_0^2/(p^2 + m_\pi^2)^2$ in the disconnected two quark loop amplitude of the η' propagator, where $m_0^2 = m_\eta'^2 - m_\pi^2$ represents the mass-squared splitting between the η' and pseudoscalar octet mesons in the full theory. For the π - π scattering amplitude the one-loop diagram in chiral perturbation theory formed by two double-pole propagators yields a divergent imaginary part at threshold in the s channel [27]. This implies that the vacuum amplitude $R^V(t)$ would be ill-behaved as t increases. A rapid loss of signal seen in Figs. 4 and 9 may be partly related to this point.

The same diagram viewed in the t channel gives a contribution of the form

$$\delta a_0 = \frac{1}{1536\pi^3} \frac{1}{m_\pi f_\pi^4} \left(\frac{m_0^2}{N_f} \right)^2 \quad (46)$$

to the s -wave scattering length, which diverges as $m_\pi \rightarrow 0$. In terms of the energy shift in the direct amplitude $R^D(t)$, this contribution translates to

$$\delta E_D = \frac{1}{384\pi^2} \frac{1}{m_\pi^2 f_\pi^4} \left(\frac{m_0^2}{N_f} \right)^2 \frac{1}{L^3} \quad (47)$$

with L the spatial size.

Recently we calculated [13] the parameter m_0 for quenched QCD with the Wilson quark action using the same set of configurations on a $12^3 \times 20$ lattice as are employed in this paper. At the hopping parameter $K = 0.164$ where our scattering length calculation is made we obtained $m_0/\sqrt{N_f} = 0.147(8)$. Combining this result with those of m_π and f_π in Table III we find $\delta E_D = 2.7 \times 10^{-6}$ for the expected magnitude of energy shift for the direct amplitude for the Wilson case, which is far too small to be detectable in our data for the ratio $R^D(t)$.

The singular contribution (47) diverges in the chiral limit. In order to see at what quark mass the contribution seriously begins to affect the chiral behavior we recall [13] that the values for m_0 we obtained can be fitted very well with a linear function of the quark mass $m_q = (1/K - 1/K_c)/2$:

$$\frac{m_0}{\sqrt{N_f}} = 0.299(14) - 1.57(19)m_q. \quad (48)$$

For m_π^2 , m_ρ , and f_π we find from the values reported in Ref. [24] that

$$m_\pi^2 = 2.72(10)m_q, \quad (49)$$

$$m_\rho = 0.5361(92) + 1.61(14)m_q, \quad (50)$$

$$f_\pi = 0.0680(28) + 0.338(43)m_q. \quad (51)$$

Combining (46)–(51) one can estimate the fractional change of the $I = 0$ scattering length relative to the current algebra value $a_0 = 7m_\pi/(32\pi f_\pi^2)$ expected for the Wilson action at $\beta = 5.7$. The result is plotted by a solid line in Fig. 11 as a function of m_π/m_ρ . We see that the contribution of the singular term would become significant only for realistically small values of quark masses, which are beyond our computing resources.

A similar estimate for the Kogut-Susskind case, strictly speaking, requires results for m_0 for that action which is not available. One may, however, take over (48) as a guide. Employing the data of Ref. [28] for the Kogut-Susskind quark action at $\beta = 5.7$, which yield

$$m_\pi^2 = 0.004(1) + 7.96(5)m_q, \quad (52)$$

$$m_\rho = 0.782(86) + 10.0(6.7)m_q, \quad (53)$$

$$f_\pi = 0.118(7) + 0.97m_q, \quad (54)$$

one finds the dotted line in Fig. 11 for the fractional change $\delta a_0/a_0$ for the Kogut-Susskind case. Again the singular contribution is very small $\delta a_0/a_0 \approx 0.001$ at the point $m_q = 0.01$ ($m_\pi/m_\rho = 0.33$) of the simulation. For the energy shift we find $\delta E = 3.8 \times 10^{-5}$, which may be

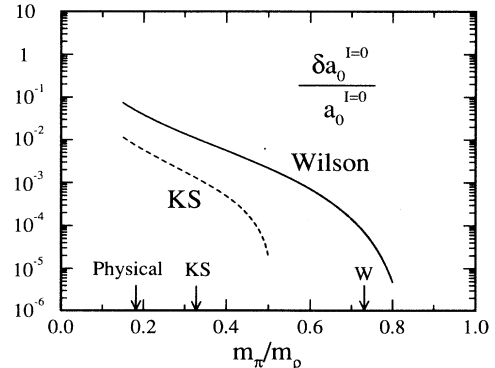


FIG. 11. Fractional change of the $I = 0$ π - π scattering length due to infrared divergences in quenched QCD normalized by the current algebra value $a_0 = 7m_\pi/(32\pi f_\pi^2)$ at $\beta = 5.7$ plotted as a function of m_π/m_ρ . Solid line is for the Wilson action and dotted line for the Kogut-Susskind action. Arrows indicate the position where our calculation of π - π four-point functions is made.

compared with the value $\delta E = 1.2(4.0) \times 10^{-4}$ extracted from the slope of the direct amplitude $R^D(t)$.

The estimate above indicates that the spurious infrared divergence and possible failure of the quenched approximation do not become manifest in π - π four-point functions unless the simulation is made with a quark mass much smaller than the value being taken in current quenched QCD studies. Of course, the problem of unphysical singularity does not arise if calculations are made on full QCD gauge configurations.

D. Summary of π - π scattering lengths

Our results for π - π scattering lengths are summarized in Fig. 12 in terms of the dimensionless ratio $32\pi f_\pi^2 a_0/m_\pi$ together with those of Refs. [18,19] for $I = 2$. We observe an agreement of lattice results with current algebra predictions up to quite heavy quark masses ($m_\pi/m_\rho \approx 0.7 - 0.8$) for both $I = 0$ and 2 channels. In detail, however, the Wilson result for the $I = 0$ scattering length obtained at a heavy quark mass is somewhat smaller than the current algebra value, while the Kogut-Susskind results at a small quark mass are larger. It would be an interesting problem to repeat the Kogut-Susskind simulation for larger quark masses in order to see if the trend seen for the Wilson action is reproduced. For the $I = 2$ channel we do not observe such a difference between the two actions.

Aside from the complications for the Kogut-Susskind case due to breaking of flavor symmetry discussed in Sec. IV A 1, possible sources of systematic errors in our results are the uncertainties in higher order $1/L$ terms of (1) and scaling violations due to a fairly large lattice spacing of our simulation ($a^{-1} \approx 1 - 1.5$ GeV at $\beta = 5.7$ determined from the ρ meson mass [23,24]). For the $I = 2$ channel the previous work [18,19] checked these points.

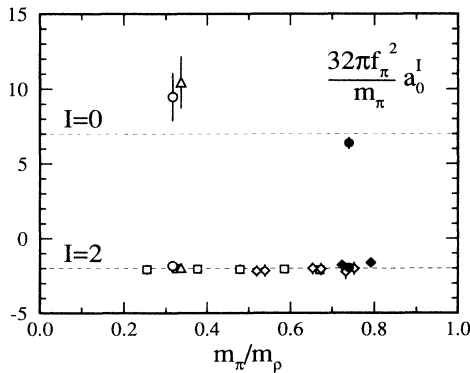


FIG. 12. $I = 0$ and 2 s -wave π - π scattering lengths a_0^I . Solid and open symbols denote Wilson and Kogut-Susskind results. Triangles are for Coulomb gauge results. Squares ($\beta = 5.7$) and diamonds ($\beta = 6.0$) for $I = 2$ are from Refs. [18,19]. Dotted lines indicate predictions of current algebra.

In particular, a comparison of results at $\beta = 5.7$ and 6.0 indicates that finite lattice spacing errors are not large, at least for the Kogut-Susskind case, for the $I = 2$ channel [compare open squares ($\beta = 5.7$) and open diamonds ($\beta = 6.0$) in Fig. 12]. Whether this applies also to the $I = 0$ channel should be examined in future work.

V. π - N , K - N , AND \bar{K} - N SCATTERING LENGTHS

A. π - N scattering lengths

We calculate the π - N scattering lengths from π - N four-point functions. Since the wall source technique without gauge fixing does not yield good signals for the nucleon, we use the Coulomb gauge fixing at the $t = 0$ time slice for the nucleon source. We employ the Wilson quark action at hopping parameters $K = 0.164$ and 0.1665, using 60 and 30 gauge configurations, respectively.

In Fig. 13 we show the ratio $R_I(t)$ of π - N amplitude for the isospin eigenchannels $I = 1/2$ and $3/2$ at the two values of the hopping parameter. The data are fitted with the linear form $R_I(t) = Z_I(1 - \delta E_I t)$ over $4 \leq t \leq 9$, and the results for the scattering length are listed in Table VI.

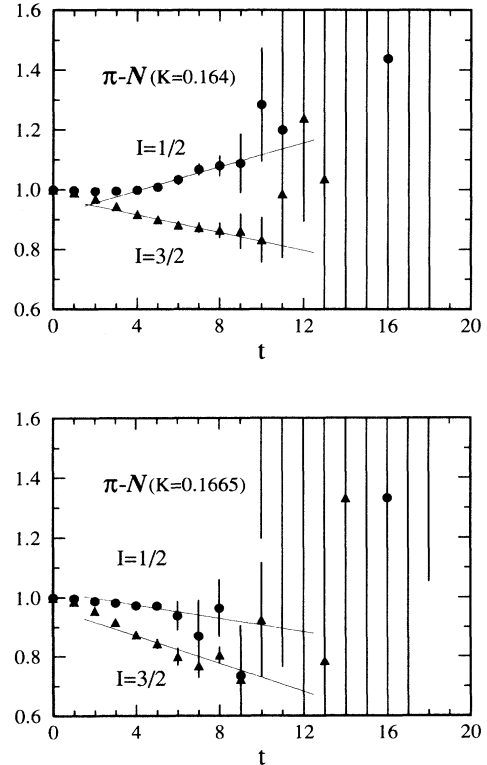


FIG. 13. $R_I(t)$ ($I = 1/2$ and $3/2$) for the π - N four-point function at (a) $K = 0.164$ and (b) 0.1665. Solid lines are linear fits for $4 \leq t \leq 9$.

TABLE VI. π - N scattering lengths in lattice units for the Wilson quark action on a $12^3 \times 20$ lattice. Numbers in square brackets are current algebra predictions evaluated with the measured values of m_π , m_N , and f_π listed in Table III.

K	$I = 1/2$			$I = 3/2$		
	a_0	δE_I	Z_I	a_0	δE_I	Z_I
0.164	3.04(66)	-0.0219(54)	0.916(22)	-1.10(20)	0.0151(35)	0.975(18)
	[2.701(41)]			[-1.350(20)]		
0.1665	-0.70(58)	0.011(11)	1.021(52)	-1.31(22)	0.0243(55)	0.964(31)
	[3.02(11)]			[-1.509(53)]		

They are compared with predictions of current algebra and PCAC (values in square brackets in Table VI), which are given by

$$a_0(I = 1/2) = +\frac{1}{4\pi} \frac{\mu_{\pi N}}{f_\pi^2}, \quad (55)$$

$$a_0(I = 3/2) = -\frac{1}{8\pi} \frac{\mu_{\pi N}}{f_\pi^2}, \quad (56)$$

where $\mu_{\pi N} = m_\pi m_N / (m_\pi + m_N)$ is the reduced mass and the right-hand sides are estimated with the measured values of m_π , m_N , and f_π in Table III. The results at $K = 0.164$ are consistent with the current algebra prediction, in spite of the fact that simulations are made with quite heavy quarks corresponding to $m_\pi/m_\rho = 0.74$. The quality of data deteriorates for a lighter quark of $K = 0.1665$; more statistics are clearly needed as Fig. 13 indicates.

Let us define $R_\pm^X(t)$ to be the contribution of X type diagrams ($X = D, C, R,$ and CR ; see Fig. 2) to crossing

even (+) and odd (-) channels, summed over fermion contractions having the quark line topology of the X diagram. In terms of the isospin eigen amplitudes $R_I(t)$, the crossing even and odd amplitudes are given by

$$R_+(t) = \frac{1}{3}R_{I=1/2}(t) + \frac{2}{3}R_{I=3/2}(t), \quad (57)$$

$$R_-(t) = \frac{1}{3}R_{I=1/2}(t) - \frac{1}{3}R_{I=3/2}(t). \quad (58)$$

The ratios $R_\pm^X(t)$ are plotted in Fig. 14 as a function of t for $K = 0.164$. We recall that the crossed (C) and rectangular (R) amplitudes are governed by current algebra; the soft pion theorem predicts that the crossing even amplitudes $R_+^R(t)$ and $R_+^C(t)$ should show a reversal of sign in the slope with respect to t . The direct (D) and crossed rectangular (CR) amplitudes do not appear in current algebra to leading order in m_π , and therefore we expect that $R_\pm^D(t)$ and $R_\pm^{CR}(t)$ are flat. Our results in Fig. 14 are roughly consistent with these expectations except that the t dependence is not very pronounced for the rectangular diagram compared to the crossed diagram and that the direct amplitude exhibits a positive curvature. Both these features are similar to what appeared in the π - π case with the Wilson quark action, perhaps originating from heavy quark mass employed ($m_\pi/m_\rho = 0.74$).

B. K - N and \bar{K} - N scattering lengths

The prediction of current algebra and PCAC for $SU(3)$ symmetry is given by

$$a_0^{KN}(I = 0) = 0, \quad (59)$$

$$a_0^{KN}(I = 1) = -\frac{1}{4\pi} \frac{\mu_{KN}}{f_K^2}, \quad (60)$$

$$a_0^{\bar{K}N}(I = 0) = \frac{3}{8\pi} \frac{\mu_{KN}}{f_K^2}, \quad (61)$$

$$a_0^{\bar{K}N}(I = 1) = \frac{1}{8\pi} \frac{\mu_{KN}}{f_K^2} \quad (62)$$

with $\mu_{KN} = m_K m_N / (m_K + m_N)$ being the reduced mass. Looking at Table I we observe that the results for the KN channel are not unreasonable compared with experiment, although the $I = 1$ scattering length is off by a factor of 2 probably due to a long extrapolation to the soft K meson limit. On the other hand, comparison of predictions (61) and (62) for the \bar{K} - N channel with the experiment requires a caution: $\pi\Lambda$ and $\pi\Sigma$ channels

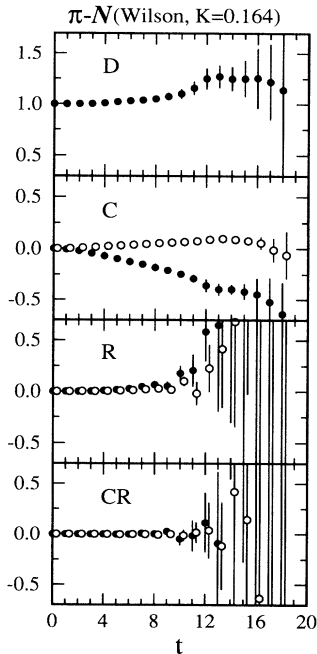


FIG. 14. Individual ratios $R_\pm^X(t)$ ($X = D, C, R,$ and CR) for π - N four-point function projected to crossing even (solid circles) and odd (open circles) channels.

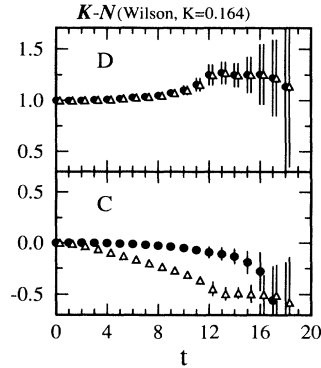


FIG. 15. Individual ratios $R_I^X(t)$ ($X = D$ and C) of the K - N amplitude for isospin eigenchannels. Solid and open symbols denote $I = 0$ and 1 results.

are already open at the $\bar{K}N$ threshold. It is interesting to note that the prediction (62) for the $I = 1$ $\bar{K}N$ scattering length still agrees with the real part of the experimental value. The fact that the real part of the $I = 0$ $\bar{K}N$ scattering length predicted by (61) has a sign opposite to that of the experiment is clearly due to the presence of $\Lambda(1405)$. We point out that the lattice results reported below can be compared better with the current algebra values, since the present lattice calculation with the quenched approximation is basically a single channel calculation without taking into account an opening of the channels below the threshold.

Our results for K - N and \bar{K} - N scattering lengths are obtained at $K = 0.164$, employing the same hopping parameter for the strange and up-down quark masses. The contribution of the direct and crossed diagrams to the K - N amplitude in the $I = 0$ and 1 isospin eigenchannels is plotted in Fig. 15. For the $I = 0$ channel (solid circles) a small positive slope of the direct amplitude is almost canceled out by the negative slope of the crossed one. The direct and rectangular contribution to the \bar{K} - N

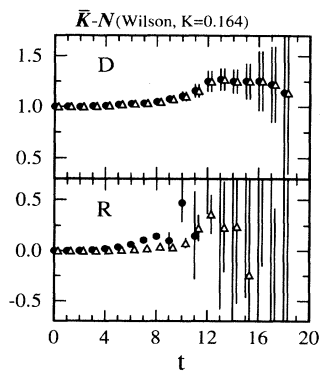


FIG. 16. Individual ratios $R_I^X(t)$ ($X = D$ and R) of the \bar{K} - N amplitude for isospin eigenchannels. Solid and open symbols denote $I = 0$ and 1 results.

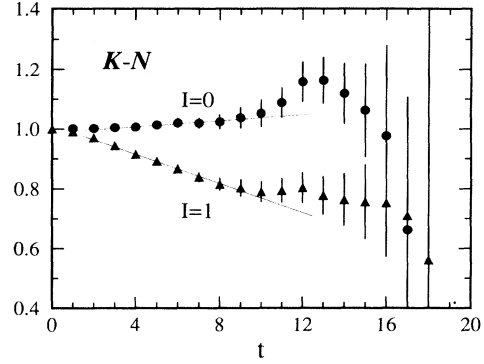


FIG. 17. $R_I(t)$ ($I = 0$ and 1) for the K - N four-point function on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $K = 0.164$. Solid lines are linear fits for $4 \leq t \leq 9$.

amplitude for both isospin eigenchannels are presented in Fig. 16.

The ratio $R_I(t)$ for the isospin eigenchannels is shown for the K - N and \bar{K} - N cases in Figs. 17 and 18, respectively. We made a linear fit $R_I(t) = Z_I(1 - \delta E_I t)$ over $4 \leq t \leq 9$ with the results given in Table VII, together with the current algebra values evaluated with the measured value of m_π , m_N , and f_π in Table III. We see that all signs agree with those of the current algebra results. For the $I = 0$ K - N scattering length for which the current algebra value is vanishing, the lattice result also yields a smaller value. For other cases, the agreement is up to a factor of 2.

VI. N - N SCATTERING LENGTHS

Calculation of N - N scattering lengths poses a number of challenging obstacles not encountered for the case of π - π and π - N scattering. First of all the experimental scat-

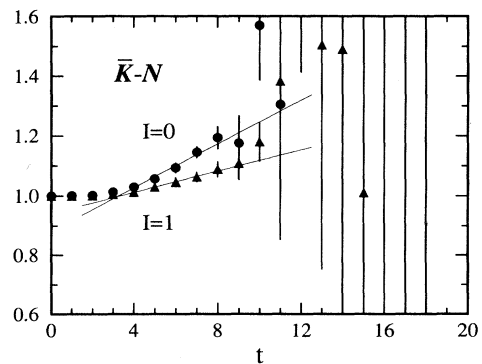


FIG. 18. $R_I(t)$ ($I = 0$ and 1) for the \bar{K} - N four-point function on a $12^3 \times 20$ lattice at $\beta = 5.7$ and $K = 0.164$. Solid lines are linear fits for $4 \leq t \leq 9$.

TABLE VII. K - N and \bar{K} - N scattering lengths in lattice units for the Wilson quark action at the hopping parameter $K = 0.164$ on a $12^3 \times 20$ lattice. Numbers in square brackets are current algebra predictions evaluated with the measured values of m_π , m_N , and f_π listed in Table III.

	$I = 0$			$I = 1$		
	a_0	δE_I	Z_I	a_0	δE_I	Z_I
$K - N$	0.55(47) [0]	-0.0051(38)	0.987(19)	-1.56(13) [-2.701(41)]	0.0240(27)	1.012(16)
$\bar{K} - N$	4.64(37) [4.051(61)]	-0.0415(64)	0.881(23)	2.63(64) [1.350(20)]	-0.0188(45)	0.941(20)

tering lengths are quite large, being of order 10 fm (see Table I). This means that lattice sizes much larger than 2–3 fm, which are accessible in current numerical simulations, will be needed to suppress $O(L^{-6})$ corrections neglected in the Lüscher's formula (1). Secondly, extraction of the scattering length in the spin triplet channel requires a calculation of the lowest scattering state orthogonal to the ground state, which is the deuteron bound state. In fact the negative sign of the s -wave spin triplet scattering length is a consequence of the existence of the deuteron as follows from Levinson's theorem. Finally statistical fluctuations in N - N four-point functions are expected to grow more rapidly than in the π - π and π - N cases toward large time separations and small quark masses as we shall discuss in detail in Sec. VII below.

A possible strategy in this situation is to start from the region of heavy quark masses, where simulations are easier, and subsequently reduce the quark mass. To follow this approach, we first examine the behavior of nucleon-nucleon forces toward larger quark masses through a phenomenological model of one-boson exchange potentials. We then present our lattice results carried out with heavy quarks corresponding to $m_\pi/m_\rho \approx 0.74 - 0.95$.

A. Phenomenological considerations

Low energy N - N scattering data are usually described by phenomenological models of one-boson exchange potentials [3]. At large nucleon-nucleon separations $r \gtrsim 2$ fm, the potential is dominated by one-pion exchange. At intermediate distances $2 \gtrsim r \gtrsim 1$ fm, the two-pion exchange provides the dominant attraction, which is modeled by an isoscalar scalar particle σ of mass ≈ 500 MeV. Heavier bosons such as ρ and ω also become important in this range of distance. The potential turns into a hard core at $r \approx 0.5$ fm whose dynamical details are not understood well.

The formation of bound states in the triplet 3S_1 channel that emerges from such a potential is subtle. The central potential, which has a depth of about -50 MeV at $r \approx 1$ fm both in the triplet 3S_1 and singlet 1S_0 channels, is insufficient for bound-state formation. For the triplet channel, however, the tensor potential provides an additional attraction of a similar depth. This leads to the binding of the deuteron with a small binding energy of 2.224 57 MeV with a sizable mixing of D wave.

Let us consider what happens if the quark mass is increased from the physical value. The mesons most affected by the increase is the pion and σ representing the two-pion exchange, whose masses increase as $\sqrt{m_q}$. Thus the attractive potential provided by these states becomes rapidly shorter ranged (while the hard core mostly given by ω changes little), and the two nucleons have to come closer to remain in the potential well. One can imagine that the resulting increase of kinetic energy may easily overwhelm the small deuteron binding energy, leading to unbinding of the deuteron state.

We have examined whether this actually takes place by employing the one-boson exchange model of Refs. [29,30] and varying the values of m_π and m_σ according to our lattice result for the slope m_π^2/m_q . We find a divergence of the scattering length for the triplet channel taking place at $m_q = 6.3$ MeV that signals unbinding of the deuteron. The fractional increase of quark mass from the value 4.9 MeV corresponding to the physical point is only 30%. We have repeated the analysis in several alternative ways, e.g., varying other meson masses and/or the nucleon mass assuming the relation $m_H = A_H + B_H m_q$, and also including a variation of the meson-nucleon coupling constants. We have found that the qualitative features are not modified: the deuteron most probably ceases to exist as a bound state when the quark mass is increased

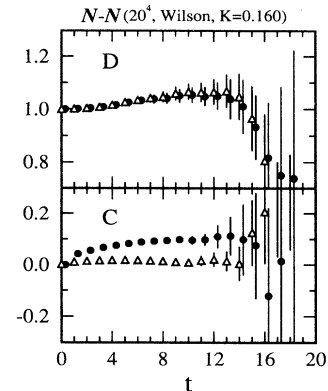


FIG. 19. Individual contributions of direct and crossed amplitudes to the ratio $R(t)$ for triplet 3S_1 (solid symbols) and singlet 1S_0 (open symbols) channels.

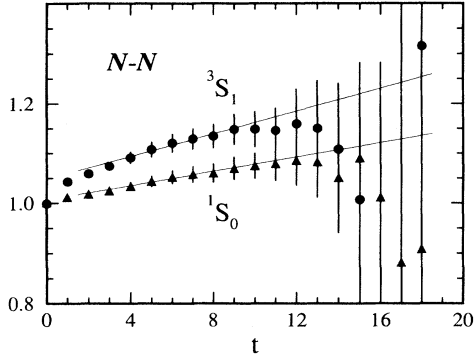


FIG. 20. $R(t)$ for N - N four-point function on a 20^4 lattice at $\beta = 5.7$ and $K = 0.160$. Solid lines are linear fits for $4 \leq t \leq 9$.

by more than 30–40% from the physical value. Another common feature is that the scattering lengths decrease to a value of the order of 1 fm, when quark mass is increased so that $m_\pi/m_\rho \approx 0.7 - 0.9$. It is reasonable to expect that the scattering length takes a value similar to the hadron size in the absence of bound-state effects.

B. Lattice results

Our lattice study is carried out with the Wilson quark action. In the calculation of the nucleon four-point function gauge configurations are fixed to the Coulomb gauge over all space-time to enhance signals. Projecting out spin singlet and triplet combinations of the two nucleon system is made by nonrelativistically combining the upper Dirac components of the nucleon operator as explained in Sec. II B. Other technical features are similar to the π - π and π - N cases.

In Fig. 19 we show the individual contributions of the direct and crossed amplitudes to the ratio $R(t)$ for the triplet 3S_1 (solid circles) and singlet 1S_0 (open triangles) channels obtained at $K = 0.160$. We find that the direct amplitudes for the two channels are virtually the same. On the other hand, the crossed amplitude in the spin triplet channel increases in contrast to a flat behavior in the spin singlet channel.

In Fig. 20 we show $R(t)$ for the spin singlet and triplet channels at $K = 0.160$, which corresponds to $m_\pi/m_\rho = 0.85$. A clear signal with a positive slope is observed for both channels, which means attraction

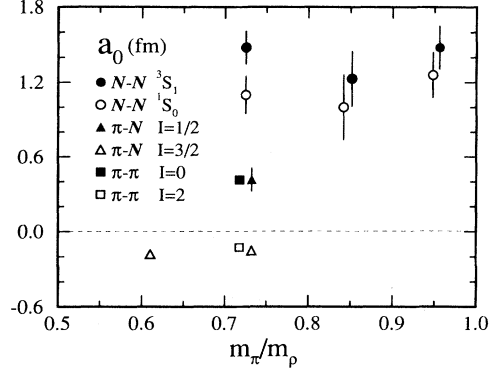


FIG. 21. N - N scattering lengths in units of fm as compared to π - N and π - π scattering lengths in quenched QCD at $\beta = 5.7$ calculated with the Wilson quark action. Conversion to physical units is made with $a = 0.137(2)$ fm determined from the ρ meson mass.

($\delta E_{NN} = E_{NN} - 2m_N < 0$). Similar results are obtained at two other values of the hopping parameter $K = 0.150$ ($m_\pi/m_\rho = 0.95$) and 0.164 ($m_\pi/m_\rho = 0.74$). We extract the energy shift δE_{NN} by fitting $R(t)$ to a linear form $R(t) = Z(1 - \delta E_{NN}t)$. The fitting range is chosen to be $4 \leq t \leq 9$ for $K = 0.150$ and 0.160 . For the case of $K = 0.164$, however, we used $2 \leq t \leq 6$ due to poor quality of our data. The fitted values of δE_{NN} are quite small (≈ 0.01), justifying the use of a linear function instead of an exponential.

From phenomenological considerations in Sec. VI A we expect that the deuteron is not a bound state at a heavy quark mass where our simulations are made. Given this expectation, Lüscher's formula (1) may be applied for both spin singlet and triplet channels. The results in lattice units are tabulated in Table VIII. We should remark that the scattering lengths we found are large enough to warrant a calculation with yet a larger lattice size, even though we used quite a large size of 20^4 : the three terms in (1) are comparable in magnitude. A finite-size analysis with larger lattice sizes is clearly necessary.

In Fig. 21 we compare the results for the N - N scattering lengths with those for π - π and π - N obtained with the Wilson quark action. Conversion to physical units is made using $a = 0.137(2)$ fm determined from the ρ meson mass. It is apparent that the N - N scattering lengths are substantially larger than the π - N and π - π scattering lengths already for a heavy quark corresponding to $m_\pi/m_\rho \approx 0.74$. Also noteworthy is the trend, albeit with sizable errors, that the values for the spin triplet channel

TABLE VIII. N - N scattering lengths in lattice units for the Wilson quark action on a 20^4 lattice.

K	a_0	3S_1	Z	a_0	1S_0	Z
		δE			δE	
0.150	10.8(1.2)	-0.0126(39)	1.012(19)	9.2(1.3)	-0.0085(27)	0.987(13)
0.160	9.0(1.6)	-0.0109(45)	1.048(25)	7.3(1.9)	-0.0072(33)	1.007(18)
0.164	10.8(9)	-0.0207(48)	1.021(12)	8.0(1.1)	-0.0102(30)	0.997(7)

are larger than those for the singlet channel, indicating a stronger attraction in the triplet channel. This is consistent with the existence of the deuteron bound state for physical quark mass.

In physical units our results correspond to $a_0(NN) \approx 1.0 - 1.5$ fm. These values are small compared to the experimental value of order 10 fm. However, the large experimental scattering lengths are a reflection of the fact that the s -wave nucleon-nucleon system is either marginally bound (triplet channel) or very close to a bound-state formation (singlet channel), which would not be the case for heavy quarks studied in our simulation. Therefore, our results are not unreasonable. We expect that lattice results will exhibit an increase as the quark mass is decreased. For the triplet channel, in particular, the scattering length should diverge at the point where the deuteron bound state is formed. According to the phenomenological consideration in the previous section, however, this will take place quite close to physical quark mass, which is not accessible in current lattice QCD simulations.

VII. PROBLEM OF STATISTICAL ERRORS

Our results for $R(t)$ for various channels share the feature that statistical errors grow rapidly for large times, examples of which are seen in Figs. 5 and 10 for $\pi\text{-}\pi$, Fig. 13 for $\pi\text{-}N$, and Fig. 20 for $N\text{-}N$ cases. This behavior can be understood from (41), which implies that the error $\delta R^X(t)$ of the ratio $R^X(t)$ for a diagram X increases as $\delta R^X(t) \propto \exp(\alpha_X t)$ with the exponent α_X given in Table IX. The t dependence of the measured errors agree very well with this formula. Some typical examples are shown in Fig. 22.

This consideration clarifies why the vacuum amplitude for the $\pi\text{-}\pi$ scattering rapidly loses signal as t increases: the error has the largest exponent $\alpha_V = 2m_\pi$ among the four amplitudes contributing to the $\pi\text{-}\pi$ four-point function. We also expect that the rate of growth of errors decreases for the $\pi\text{-}\pi$ case toward small quark mass, since the exponent is governed by the pion mass. Exploring the light quark mass region is more difficult for $\pi\text{-}N$ and $N\text{-}N$ scattering: the exponent for these cases is a difference of the nucleon mass and pion mass with some coefficient, which depends on the channel, and the exponent stays nonvanishing or even increases toward the chiral limit. We illustrate this point in Fig. 23 where we plot the largest exponent, $2m_\pi$ for $\pi\text{-}\pi$, $m_N - m_\pi/2$ for

TABLE IX. Exponent α governing the growth of error of the ratio $R^X(t)$ for a daigram of type X .

X	D	C	R	V	CR
$\pi\text{-}\pi$	0	0	m_π	$2m_\pi$	
$\pi\text{-}N$	$m_N - \frac{3}{2}m_\pi$	$m_N - \frac{3}{2}m_\pi$	$m_N - \frac{1}{2}m_\pi$		$m_N - \frac{1}{2}m_\pi$
$N\text{-}N$	$2m_N - 3m_\pi$	$2m_N - 3m_\pi$			

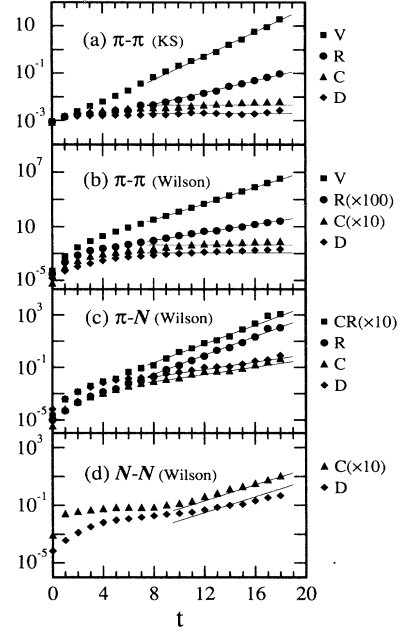


FIG. 22. Error $\delta R^X(t)$ of ratio $R^X(t)$ as a function of t at $\beta = 5.7$ in quenched QCD. Use of Kogut-Susskind or Wilson quark action is indicated in parentheses. Solid lines indicate expected slope calculated with measured values of m_N and m_π . (a) $\pi\text{-}\pi$ at $m_q = 0.01$, (b) $\pi\text{-}\pi$ at $K = 0.164$, (c) $\pi\text{-}N$ at $K = 0.164$, (d) $N\text{-}N$ in the triplet 3S_1 channel at $K = 0.160$.

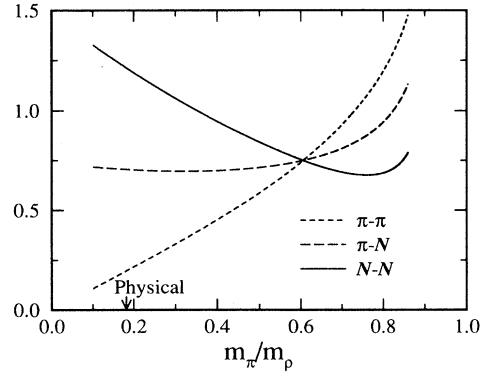


FIG. 23. Largest exponent α governing the errors of ratio $\delta R(t) \propto e^{\alpha t}$ for $\pi\text{-}\pi$ (dotted line), $\pi\text{-}N$ (broken line), and $N\text{-}N$ (solid line) cases.

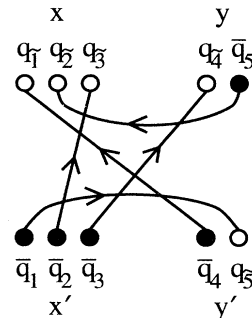


FIG. 24. Example of quark contractions contributing to $\pi\text{-}N$ four-point function.

TABLE X. Weights of direct (D) diagrams for π - N , K - N , and \bar{K} - N four-point functions projected to isospin eigenchannels.

Diagram	Contraction	π - N		K - N		\bar{K} - N	
		$I = \frac{1}{2}$	$I = \frac{3}{2}$	$I = 0$	$I = 1$	$I = 0$	$I = 1$
$D1$	12345	2	2	2	2	2	2
$D2$	13245	1	1	1	1	1	1
$D3$	21345	-2	-2	-2	-2	-2	-2
$D4$	31245	-1	-1	-1	-1	-1	-1
$D5$	32145	1	1	1	1	1	1
$D6$	23145	-1	-1	-1	-1	-1	-1

TABLE XI. Weights of crossed (C) diagrams for π - N , K - N , and \bar{K} - N four-point functions projected to isospin eigenchannels.

Diagram	Contraction	π - N		K - N		\bar{K} - N	
		$I = \frac{1}{2}$	$I = \frac{3}{2}$	$I = 0$	$I = 1$	$I = 0$	$I = 1$
$C1$	42315	1	1	1	1	0	0
$C2$	43215	3/2	0	2	0	0	0
$C3$	24315	-1	-1	-1	-1	0	0
$C4$	34215	-3/2	0	-2	0	0	0
$C5$	23415	1/2	-1	1	-1	0	0
$C6$	32415	-1/2	1	-1	1	0	0
$C7$	41325	-1	-1	-1	-1	0	0
$C8$	43125	-3/2	0	-2	0	0	0
$C9$	14325	1	1	1	1	0	0
$C10$	34125	3/2	0	2	0	0	0
$C11$	13425	-1/2	1	-1	1	0	0
$C12$	31425	1/2	-1	1	-1	0	0
$C13$	41235	1/2	-1	1	-1	0	0
$C14$	42135	-1/2	1	-1	1	0	0
$C15$	14235	-1/2	1	-1	1	0	0
$C16$	24135	1/2	-1	1	-1	0	0
$C17$	12435	-1	2	-2	2	0	0
$C18$	21435	1	-2	2	-2	0	0

TABLE XII. Weights of rectangular (R) diagrams for π - N , K - N , and \bar{K} - N four-point functions projected to isospin eigenchannels.

Diagram	Contraction	π - N		K - N		\bar{K} - N	
		$I = \frac{1}{2}$	$I = \frac{3}{2}$	$I = 0$	$I = 1$	$I = 0$	$I = 1$
$R1$	52341	1	1	0	0	1	1
$R2$	53241	-1/2	1	0	0	-1	1
$R3$	25341	-1	-1	0	0	-1	-1
$R4$	35241	1/2	-1	0	0	1	-1
$R5$	23541	-3/2	0	0	0	-2	0
$R6$	32541	3/2	0	0	0	2	0
$R7$	51342	-1	-1	0	0	-1	-1
$R8$	53142	1/2	-1	0	0	1	-1
$R9$	15342	1	1	0	0	1	1
$R10$	35142	-1/2	1	0	0	-1	1
$R11$	13542	3/2	0	0	0	2	0
$R12$	31542	-3/2	0	0	0	-2	0
$R13$	51243	-3/2	0	0	0	-2	0
$R14$	52143	3/2	0	0	0	2	0
$R15$	15243	3/2	0	0	0	2	0
$R16$	25143	-3/2	0	0	0	-2	0
$R17$	12543	3	0	0	0	4	0
$R18$	21543	-3	0	0	0	-4	0

π - N , and $2m_N - 3m_\pi$ for N - N cases, as a function of m_π/m_ρ where we used the fits (49) and (50) for m_π and m_ρ and

$$m_N = 0.741(21) + 3.80(31)m_q \quad (63)$$

for m_N obtained from the hadron mass data in Ref. [24]. We observe that the problem will be particularly severe for the N - N four-point function for which the exponent begins to increase for $m_\pi/m_\rho \lesssim 0.5$.

VIII. CONCLUSIONS

In this paper we calculated hadron scattering lengths in a variety of channels at $\beta = 5.7$ in the quenched ap-

TABLE XIII. Weights of crossed rectangular (CR) diagrams for π - N , K - N , and \bar{K} - N four-point functions projected to isospin eigenchannels.

Diagram	Contraction	π - N		K - N		\bar{K} - N	
		$I = \frac{1}{2}$	$I = \frac{3}{2}$	$I = 0$	$I = 1$	$I = 0$	$I = 1$
$CR1$	52431	-1/2	1	0	0	0	0
$CR2$	54231	-1/2	1	0	0	0	0
$CR3$	53421	-1/2	1	0	0	0	0
$CR4$	54321	1	1	0	0	0	0
$CR5$	25431	1/2	-1	0	0	0	0
$CR6$	45231	1/2	-1	0	0	0	0
$CR7$	35421	1/2	-1	0	0	0	0
$CR8$	45321	-1	-1	0	0	0	0
$CR9$	24531	0	0	0	0	0	0
$CR10$	42531	0	0	0	0	0	0
$CR11$	34521	3/2	0	0	0	0	0
$CR12$	43521	-3/2	0	0	0	0	0
$CR13$	51432	1/2	-1	0	0	0	0
$CR14$	54132	1/2	-1	0	0	0	0
$CR15$	53412	1/2	-1	0	0	0	0
$CR16$	54312	-1	-1	0	0	0	0
$CR17$	15432	-1/2	1	0	0	0	0
$CR18$	45132	-1/2	1	0	0	0	0
$CR19$	35412	-1/2	1	0	0	0	0
$CR20$	45312	1	1	0	0	0	0
$CR21$	14532	0	0	0	0	0	0
$CR22$	41532	0	0	0	0	0	0
$CR23$	34512	-3/2	0	0	0	0	0
$CR24$	43512	3/2	0	0	0	0	0
$CR25$	51423	0	0	0	0	0	0
$CR26$	54123	3/2	0	0	0	0	0
$CR27$	52413	0	0	0	0	0	0
$CR28$	54213	-3/2	0	0	0	0	0
$CR29$	15423	0	0	0	0	0	0
$CR30$	45123	-3/2	0	0	0	0	0
$CR31$	25413	0	0	0	0	0	0
$CR32$	45213	3/2	0	0	0	0	0
$CR33$	14523	3/2	0	0	0	0	0
$CR34$	41523	-3/2	0	0	0	0	0
$CR35$	24513	-3/2	0	0	0	0	0
$CR36$	42513	3/2	0	0	0	0	0

proximation. We demonstrated that Lüscher's formula that correlates scattering lengths with the energy of the two-hadron system confined in a finite box can be used to calculate any class of diagrams of scattering amplitudes, when implemented with a modified wall source method proposed by the present authors. The results are physically sensible and encouraging, being in reasonable agreement with current algebra values and the experiment for π - π and π - N cases; for N - N scattering the resulting scattering lengths are significantly larger compared to the first two cases even for heavy quarks.

In the present study dynamical effects of sea quarks are not taken into account. However, it is straightforward to apply our method once configurations are generated with dynamical quarks. Worrying toward a more realistic calculation is the exponential growth of errors for large times discussed in Sec. VII. The problem is particularly acute for N - N scattering, for which the rate is numerically large and is expected to increase as the chiral limit is approached. The anticipated large N - N scattering lengths, which henceforth requires a large lattice size, makes the problem further difficult. Advance in our understanding of the low energy nucleon-nucleon scattering from the first principle has to await not only future progress of computing power but also further innovation of calculational techniques.

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APPENDIX

In this appendix we enumerate weights of various quark contractions contributing to the π - N , K - N , and \bar{K} - N four-point functions. Let us consider the meson-baryon operators given by

$$\mathcal{O}(x, y) = \varepsilon_{abc} ({}^t q_1^a C^{-1} \gamma_5 q_2^b) q_3^c(x) \times \bar{q}_5 \gamma_5 q_4(y), \quad (\text{A1})$$

$$\begin{aligned} \mathcal{O}^\dagger(x', y') = & \varepsilon_{a'b'c'} \bar{q}_3^{c'} \left(\bar{q}_2^{b'} \{C^{-1} \gamma_5\}^\dagger \bar{q}_1^{a'} \right) (x') \\ & \times -\bar{q}_4 \gamma_5 q_5(y'), \end{aligned} \quad (\text{A2})$$

where subscripts with tildes are assigned to quark fields q and those without tildes to antiquark fields \bar{q} . Quark contractions for the operator product $\mathcal{O}(x, y)\mathcal{O}^\dagger(x', y')$ may be specified by permutations from the set $\{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}\}$ for quark fields to the set $\{1, 2, 3, 4, 5\}$ for antiquark fields. For example, the contraction with crossed rectangular topology shown in Fig. 24 corresponds to

$$\begin{array}{ccccc} \tilde{1} & \tilde{2} & \tilde{3} & \tilde{4} & \tilde{5} \\ 4 & 5 & 2 & 3 & 1. \end{array} \quad (\text{A3})$$

There are 6, 18, 18, and 36 inequivalent contractions for the direct, crossed, rectangular, and crossed-rectangular diagrams. In Tables X – XIII we summarize the weights of the corresponding amplitudes for π - N , K - N , and \bar{K} - N four-point functions projected to isospin eigenchannels. Sign factors arising from Fermi statistics are not included in the weight. For definitions of meson and nucleon operators, see Sec. II B.

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