

Pion mass effects in the large N limit of chiral perturbation theory

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We compute the large N effective action of the $O(N+1)/O(N)$ nonlinear σ model including the effect of the pion mass to order m_π^2/f_π^2 . This action is more involved than the one corresponding to the chiral limit not only due to the pion propagators but also because the chiral symmetry produces new interactions proportional to m_π^2/f_π^2 . We renormalize the action by including the appropriate infinite set of counterterms and we obtain the renormalization-group equations for the corresponding couplings. Then we study the unitarity properties of the scattering amplitudes. Finally our results are applied to the particular case of the linear σ model and also are used to fit the pion scattering phase shifts. In spite of the fact that the model has an infinite number of parameters to fit and, therefore, it does not have predictive power in a strict sense, a particularly simple choice of the couplings produces a good fit of the $J=0$ phase shifts.

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I. INTRODUCTION

Recently a lot of effort has been devoted to the so-called chiral perturbation theory (χ PT) [1,2]. This formalism provides a useful tool for the phenomenological description of the low-energy hadron dynamics, in terms of some parameters that can be fitted from the experimental data, even if it is possible that some day they will be computed from the underlying theory, which is expected to be quantum chromodynamics (QCD).

As χ PT can be applied without a precise formulation of this underlying theory, it also has been used for the parametrization of the unknown symmetry-breaking sector of the standard model [3], giving rise to a model-independent description of this sector that can be quite useful for the data analysis of the future CERN Large Hadron Collider (LHC).

However, χ PT still has some problems that should be solved in order to be completely useful for practical applications. Usually χ PT computations are done to the one-loop level, or what is the same, expanding the amplitudes to the fourth power of momenta or the pion masses (see [4] for an exception to this rule). Thus, the applicability region of χ PT is restricted to the very low-energy regime. In fact, χ PT does not satisfy exactly the unitarity conditions, but only in the perturbative sense, and this problem becomes very relevant at higher energies [5]. To avoid this limitation of χ PT, several methods have been proposed such as the introduction of new fields corresponding to resonances [6], the Padé approximants [5],

the inverse amplitude method [5,7], etc. More recently the so-called large N expansion of χ PT has also been considered (N being the number of Nambu-Goldstone bosons). This approximation is not restricted to the low-energy domain, and from the unitarity point of view it has a very good behavior. Up to now this approach has been used for the computation of the pion-scattering amplitudes [8,9] in the chiral limit, and it also has been used in [10] to study the $\gamma\gamma \rightarrow \pi^0\pi^0$ reaction. Some of the results in [8] were also reobtained in [11] but using different techniques. The work in [10] has been criticized in [12], but work is in progress in order to make a complete computation of the $\gamma\gamma \rightarrow \pi^0\pi^0$ reaction in the large N limit [13].

The main goal of this work is to extend the work in [8,9] outside the chiral limit; i.e., we study the pion effective action and the pion scattering in the large N limit taking into account the effect of the pion mass. As is well known, this is not trivial at all, since, because of the chiral symmetry, the introduction of the pion mass gives rise to new interactions.

The plan of the paper goes as follows. In Sec. II we introduce the dynamics of the $O(N+1)/O(N)$ nonlinear σ model (NLMS), and we compute the effective action up to order $1/N$ including the leading corrections coming from a nonzero pion mass. In Sec. III we obtain the renormalized elastic pion-scattering amplitude in terms of the renormalized coupling constants. In Sec. IV we study the running of those couplings and the renormalization of the pion mass and the pion decay constant. In Sec. V we discuss the unitarity properties of the amplitudes. Section VI is devoted to the particular case of the linear σ model (LSM), which is a very nice example to test the previously developed formalism. In Sec. VII we show a fit of the experimental data for the $I=J=0$ elastic scattering amplitude, and finally the conclusions are presented in Sec. VIII.

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II. THE DYNAMICS OF THE $O(N+1)/O(N) = S^N$ NONLINEAR σ MODEL

In order to define the large N limit of χ PT we start from the two-flavor chiral symmetry group $SU(2)_R \times SU(2)_L$, and then the equivalence of the coset spaces $SU(2)_L \times SU(2)_R / SU(2)_{L+R} = O(4)/O(3) = S^3$ is used to extend this symmetry pattern to $O(N+1)/O(N) = S^N$. Therefore, N is the dimension of the coset space and the number of Nambu-Goldstone bosons (NGB's), or, in other words, the pions. The NGB fields can be chosen as arbitrary coordinates on S^N . The most general $O(N+1)$ -invariant Lagrangian can be written as a derivative expansion, which is covariant with respect to both the space-time and the S^N coordinates. The lowest-order is given by

$$\mathcal{L}_0 = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b. \quad (1)$$

When using the standard coordinates the metrics is found to be

$$g_{ab} = \delta_{ab} + \frac{1}{NF^2} \frac{\pi_a \pi_b}{1 - (\pi^2/NF^2)}, \quad (2)$$

which has been obtained from the free Lagrangian for the $N+1$ fields $\pi_1, \pi_2, \dots, \pi_N, \sigma$ by imposing them to live in the S^N sphere $\pi^2 + \sigma^2 = \sum_{a=1}^N \pi_a \pi_a + \sigma^2 = NF^2$. As is well known, this Lagrangian not only contains the kinetic term but also an infinite number of interactions with an arbitrary even number of pions.

If we now want to introduce the pion mass, we have to break the $O(N+1)$ symmetry explicitly, while keeping $O(N)$. This can be achieved just considering in the Lagrangian a properly normalized term proportional to the σ component. Then the total Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} g_{ab} \partial_\mu \pi^a \partial^\mu \pi^b + NF^2 m^2 \sqrt{1 - \pi^2/NF^2}, \quad (3)$$

where m plays the role of the pion mass as can be seen when expanding the square root. In addition, we will also find another infinite set of terms that produce new pion interactions.

Therefore, the Lagrangian in Eq. (3) will be our starting point to study the pion dynamics in the large N limit. As usual the action is given by

$$S[\pi] = \int d^n x \mathcal{L}(\pi). \quad (4)$$

However, to define the quantum theory, not only do we need to know the action but also the measure in the field space, which defines the path integral representation of the Green's functions. For the case of the NLSM, the appropriate measure includes a factor that is the square root of the determinant of the coset metrics [14]. This fact contributes to the Lagrangian with a term proportional to $\delta^n(0)$. However, it is well known that using dimensional regularization [15], where $n = 4 - \epsilon$, the following rule is valid, $\delta^n(0) = 0$ or equivalently $\int d^n k = 0$, and therefore the measure term can be ignored (for a re-

cent discussion about regularization methods in χ PT see [16]).

Thus, the generating functional for the regularized Green's functions will be given by

$$e^{iW[J]} = \int [d\pi] \exp \left[i \left(S[\pi] + \int d^n x J^a \pi^a \right) \right], \quad (5)$$

with $n = 4 - \epsilon$.

As was discussed in the Introduction, we are interested in the effective action up to order $1/N$. Therefore, we need some systematic method to obtain the diagrams that contribute in this approximation. This can be done using a generalization of the technique discussed in [17] for the case of the linear model, which introduces two auxiliary fields, B_μ and ϕ . The first one is connected with the derivative interactions, whereas the second is related to those interactions proportional to m^2 . The Lagrangian including these auxiliary fields is defined as

$$\begin{aligned} \tilde{\mathcal{L}} = \mathcal{L} - \frac{1}{2} NF^2 \left(B_\mu - \frac{\pi^\alpha \partial_\mu \pi^\alpha}{NF^2 \sqrt{1 - \pi^2/NF^2}} \right)^2 \\ + \frac{1}{2} NF^2 m^2 \left\{ \phi - \sqrt{2} \left[1 - \frac{\pi^2}{2NF^2} \right. \right. \\ \left. \left. - \left(1 - \frac{\pi^2}{NF^2} \right)^{1/2} \right] \right\}^2, \end{aligned} \quad (6)$$

whose main properties are the following.

First of all, since there is no kinetic term for the new auxiliary fields, they can be easily integrated out, and then one finds

$$e^{iS[\pi]} \approx \int [dB][d\phi] \exp \left(i \int d^n x \tilde{\mathcal{L}}(\pi, B, \phi) \right). \quad (7)$$

Second, the new self-interaction terms appearing in $\tilde{\mathcal{L}}$ have been chosen so that they cancel the old ones appearing in \mathcal{L} , and thus the new action can be written as

$$\begin{aligned} \tilde{S}[\pi, B, \phi] = \int d^n x \left\{ NF^2 m^2 - \frac{1}{2} \pi^\alpha (\square + m^2) \pi^\alpha \right. \\ \left. - \frac{1}{2} NF^2 (B^2 - m^2 \phi^2) \right. \\ \left. - \frac{1}{2} NF^2 \partial_\mu B^\mu f_j \left(\frac{\pi^2}{NF^2} \right) \right. \\ \left. - \sqrt{2} NF^2 m^2 \phi g \left(\frac{\pi^2}{NF^2} \right) \right\}, \end{aligned} \quad (8)$$

where

$$f(\eta) = 2(1 - \sqrt{1 - \eta}), \quad (9)$$

$$g(\eta) = \left(1 - \frac{\eta}{2} - \sqrt{1 - \eta} \right)^{1/2}.$$

This action describes the same dynamics for the pions as that in Eqs. (3) and (4), but now, instead of self-interactions, we have B_μ - and ϕ -mediated interactions.

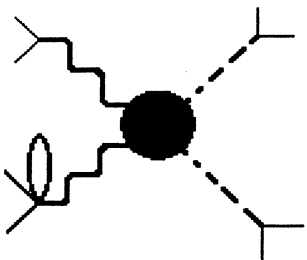


FIG. 1. One typical Green's function obtained by attaching pions (continuous line) to one B (wavy line) and ϕ (dashed line) Green's function. Note the possibility of having pion loops in the attaching point.

This fact makes the counting of the N factors for the given diagram easier and, therefore, the selection of those corresponding to the different terms in the $1/N$ expansion of the Green's functions.

From the action in Eq. (8) it is straightforward to obtain the Feynman rules. To select the relevant diagrams for the leading order in the $1/N$ expansion the most important thing is to know the N power of the different propagators and vertex involved. That the B and ϕ propagators are just constants proportional to $1/N$, the $B2\pi$ and $\phi2\pi$ vertices are of order one, the $B4\pi$ and the $\phi4\pi$ vertices are of order $1/N$, the $B6\pi$ and the $\phi6\pi$ vertices are of order $1/N^2$, and so on may be checked immediately.

The most general one-particle-irreducible (1PI) pion Green's function can be obtained by attaching pions to the legs of the most general 1PI B and ϕ Green's function (see Fig. 1). Those 1PI B and ϕ Green's functions (represented by a black dot) can be expanded in terms of other reduced B and ϕ 1PI functions (represented by a dashed dot), which are defined as the sum of all the diagrams

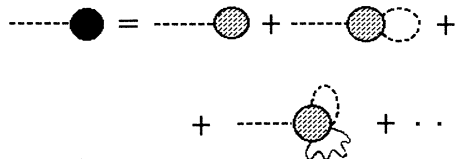


FIG. 2. Example of a ϕ Green's function (black dot) in terms of the reduced (i.e., including only pion loops) ϕ and B Green's functions (dashed dot).

containing only pion loops but not B or ϕ loops (see Fig. 2, for an example). The reduced B and ϕ Green's functions are at most of order N (see Fig. 3). In addition, the diagrams with B or ϕ loops contributing to the general B and ϕ Green's functions are suppressed in the $1/N$ expansion (because of the $1/N$ factors in the B and ϕ propagators) when compared with the first term. Thus, we find the symbolic equation shown in Fig. 4. As usual, the connected Green's functions can be expressed in terms of the 1PI ones by making trees, so that only the latter will be considered in detail.

The B and ϕ 1PI (dashed) Green's functions can be obtained from the effective action

$$\exp(i\Gamma[B, \phi]) = \int [d\pi] \exp\left(i \int d^n x \tilde{\mathcal{L}}(\pi, B, \phi)\right). \quad (10)$$

In Fig. 5 we show the diagrams contributing at leading order in $1/N$ to those 1PI Green's functions with just one or two B and ϕ legs. It is not difficult to see that diagrams with three or more B or ϕ legs do not contribute at the level of precision considered in this paper, i.e., to the leading order in the $1/N$ expansion and to the order m^2/F^2 .

Now, using standard functional techniques, it is possible to find

$$\begin{aligned} \Gamma[B, \phi] = & \int d^n x \left\{ NF^2 m^2 - \frac{1}{2} NF^2 (B^2 - m^2 \phi^2) \right\} - \sqrt{2} NF^2 m^2 g \left(\frac{I_m}{F^2} \right) \int d^n x \phi(x) \\ & - 2N m^4 g'^2 \left(\frac{I_m}{F^2} \right) \int d^n x d^n y \phi(x) K(x-y) \phi(y) - \frac{N}{4} f'^2 \left(\frac{I_m}{F^2} \right) \int d^n x d^n y B_\mu(x) G^{\mu\nu}(x-y) B_\nu(y) \\ & - i\sqrt{2} N m^2 f' \left(\frac{I_m}{F^2} \right) g' \left(\frac{I_m}{F^2} \right) \int d^n x d^n y B_\mu(x) G^\mu(x-y) \phi(y) + \dots, \end{aligned} \quad (11)$$

where the ellipsis stands for terms of $1/N^2$ order as well as terms with more than two B or ϕ fields. Notice that the above effective action is not local, and the integral kernels are defined by

$$\begin{aligned} K(x-y) &= \int d\tilde{k} e^{-ik(x-y)} I(k^2), \\ G_{\mu\nu}(x-y) &= \int d\tilde{k} e^{-ik(x-y)} k_\mu k_\nu I(k^2), \\ G_\mu(x-y) &= \int d\tilde{k} e^{-ik(x-y)} k_\mu I(k^2), \end{aligned} \quad (12)$$

where for short we have defined

$$\int d\tilde{k} \equiv \mu^\epsilon \int \frac{d^{4-\epsilon} k}{(2\pi)^{4-\epsilon}} \quad (13)$$

and the loop integrals I_m and $I(k^2)$ are given by

$$\begin{aligned} I_m &= \int d\tilde{q} \frac{i}{q^2 - m^2} \equiv -m^2 \Delta, \\ I(k^2) &= \int d\tilde{q} \frac{i}{[(k+q)^2 - m^2](q^2 - m^2)}. \end{aligned} \quad (14)$$

Using dimensional regularization, these integrals are found to be

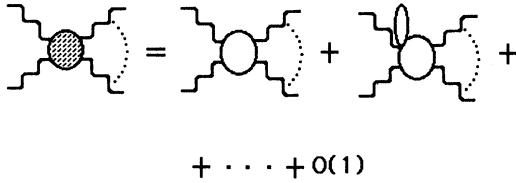


FIG. 3. Example of leading diagrams contributing to the reduced- B Green's functions in the large N limit.

$$I_m = \frac{-m^2}{(4\pi)^2} \left\{ N_\epsilon + 1 - \ln \frac{m^2}{\mu^2} \right\} \quad (15)$$

and

$$I(k^2) = \frac{-1}{(4\pi)^2} \left\{ N_\epsilon + 2 + \sqrt{1 - 4m^2/k^2} \right. \\ \left. \times \ln \frac{\sqrt{1 - 4m^2/k^2} - 1}{\sqrt{1 - 4m^2/k^2} + 1} - \ln \frac{m^2}{\mu^2} \right\}, \quad (16)$$

where, as usual,

$$N_\epsilon = \frac{2}{\epsilon} + \ln 4\pi - \gamma. \quad (17)$$

Now we couple the effective action for the B and ϕ fields to the pions by defining the action

$$\tilde{\mathcal{S}}[\pi', B, \phi] = \tilde{\mathcal{S}}[\pi', \bar{B}, \bar{\phi}] + \frac{1}{2} \int d^n x d^n y \frac{\delta^2 \tilde{\mathcal{S}}}{\delta \pi^a(x) \delta \pi^b(y)} \Big|_{\pi'=\pi} \pi'_a(x) \pi'_b(y) + \dots, \quad (20)$$

where \bar{B} , $\bar{\phi}$ and π are defined so that

$$\frac{\delta \tilde{\mathcal{S}}}{\delta B_\mu(x)} = 0, \quad \frac{\delta \tilde{\mathcal{S}}}{\delta \phi(x)} = 0, \quad \frac{\delta \tilde{\mathcal{S}}}{\delta \pi^a(x)} = 0 \quad (21)$$

for $B = \bar{B}$, $\phi = \bar{\phi}$, and $\pi' = \pi$. These equations can be used to write \bar{B} and $\bar{\phi}$ in terms of π as

$$\bar{B}_\mu(x) = - \int d^n y d\bar{k} e^{-ik(x-y)} \frac{ik_\mu}{1 + k^2 I(k)/2F^2} \frac{\pi^2(y)}{2NF^2} \left\{ 1 + \frac{I(k)m^2}{2[1 + k^2 I(k)/2F^2]F^2} \left(1 + \Delta \frac{k^2}{F^2} \right) \right\} + \dots \quad (22)$$

and

$$\bar{\phi}(x) = \frac{1}{2} \int d^n y d\bar{k} e^{-ik(x-y)} \frac{1}{1 + k^2 I(k)/2F^2} \frac{\pi^2(y)}{NF^2} + \dots \quad (23)$$

If we now integrate the B and ϕ fields to the tree level and the pion field to the one-loop level, we can write

$$\mathcal{S}_{\text{eff}}[\pi] = \tilde{\mathcal{S}}[\pi, \bar{B}, \bar{\phi}] + \frac{i}{2} \text{Tr} \ln \frac{\delta^2 \tilde{\mathcal{S}}}{\delta \pi^a(x) \delta \pi^b(y)} + \dots, \quad (24)$$

where

$$\frac{\delta^2 \tilde{\mathcal{S}}}{\delta \pi^a(x) \delta \pi^b(y)} = -(\square + m^2) \delta_{ab} \delta(x-y) + 2\partial_\mu \bar{B}^\mu(x) f'' \left(\frac{\pi^2(x)}{NF^2} \right) \frac{\pi^a(x) \pi^b(y)}{NF^2} \delta(x-y) \\ + \partial_\mu \bar{B}^\mu(x) f' \left(\frac{\pi^2(x)}{F^2} \right) \delta_{ab} \delta^2(x-y) + \dots = D_{ab}(x, y) + \Delta_{ab}(x, y) \quad (25)$$

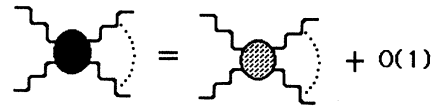


FIG. 4. Relation between the B Green's functions and the reduced- B Green's functions at leading order in the large N approximation.

$$\tilde{\mathcal{S}}[\pi, B, \phi] = \int d^n x \left\{ \frac{-1}{2} \pi^a (\square + m^2) \pi^a \right. \\ \left. - \frac{1}{2} NF^2 \partial_\mu B^\mu f \left(\frac{\pi^2}{NF^2} \right) \right. \\ \left. - \sqrt{2} NF^2 m^2 \phi g \left(\frac{\pi^2}{NF^2} \right) \right\} \\ + \Gamma[B, \phi]. \quad (18)$$

As discussed before, the pion Green's functions can be obtained by attaching pions to the B and ϕ Green's functions. However, it should be noted that it is also possible to have pion loops, such as the one shown in Fig. 1, which are proportional to the integral I_m and, therefore, to m^2 . Thus, from the formal point of view, the pion effective action can be written as

$$e^{i\mathcal{S}_{\text{eff}}[\pi]} = \int [d\pi'] [dB] [d\phi] e^{i\tilde{\mathcal{S}}[\pi', B, \phi]} \quad (19)$$

where the integration on the B and ϕ fields must be done at the tree level. Moreover, since we want to expand the pion effective action only up to order m^2/F^2 terms, we only have to perform the integration on the pion fields to the one-loop level. Therefore, to the considered degree of accuracy, we have

with

$$D_{ab}(x, y) = -(\square + m^2)\delta_{ab}\delta(x - y) . \tag{26}$$

Now it is possible to write

$$\text{Tr} \ln(D + \Delta) = \text{Tr} \ln D(1 + D^{-1}\Delta) = \text{Tr} \ln D + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (D^{-1}\Delta)^n . \tag{27}$$

In the following we will ignore the $\text{Tr} D$ term, since it does not depend on the pion fields, and therefore its contribution to the pion effective action is relevant. Moreover, it is not difficult to see that only the term $n = 1$ contributes to the effective action up to the level of approximation considered here. Then, using the above equations, it is possible to arrive at

$$\begin{aligned} \mathcal{S}_{\text{eff}}[\pi] = & \int d^4x \frac{-1}{2} \pi^a \left\{ \square + m^2 \left(1 + \frac{I_m}{2F^2} \right) \right\} \pi^a \\ & + \frac{1}{8NF^2} \int d^4x d^4y \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \pi^2(x) \pi^2(y) \frac{1}{1 + q^2 I(q)/2F^2} \left\{ q^2 - \frac{m^2(1 + \Delta q^2/F^2)}{1 + q^2 I(q)/2F^2} \right\} \\ & + O\left(\left(\frac{m^2}{F^2} \right)^2 \right) + O\left(\frac{1}{N^2} \right) , \end{aligned} \tag{28}$$

which is our final result for the regularized pion effective action to leading order in the $1/N$ expansion and to order m^2/F^2 .

III. THE SCATTERING AMPLITUDE

Following standard methods one can immediately find the scattering amplitude for the process $\pi_a \pi_b \rightarrow \pi_c \pi_d$ from the pion effective action obtained in the preceding section. This amplitude can be written as

$$T_{abcd} = \delta_{ab}\delta_{cd}A(s) + \delta_{ac}\delta_{bd}A(t) + \delta_{ad}\delta_{bc}A(u) , \tag{29}$$

where

$$A(s) = \frac{1}{NF^2} \left\{ \frac{s}{1 + \frac{sI(s)}{2F^2}} - \frac{m^2}{\left(1 + \frac{sI(s)}{2F^2}\right)^2} \left(1 + \Delta \frac{s}{F^2}\right) \right\} . \tag{30}$$

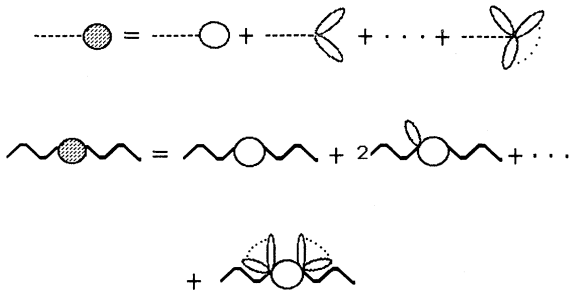


FIG. 5. (a) Diagrams contributing to the reduced-one- ϕ Green's function. (b) Diagrams contributing to the reduced-two- B Green's function. Similar diagrams contribute to the two- ϕ and the $B\phi$ reduced Green's functions.

It is not difficult to see that the above amplitude corresponds to the diagrams appearing in Fig. 6, which are those relevant to the leading order in the $1/N$ expansion and to order m^2/F^2 . However, for practical applications, this regularized amplitude should be renormalized. Moreover, the renormalization procedure is not trivial, since the NLSM is not renormalizable in the standard sense. As was discussed in [8], the renormalization of the NLSM in the large N limit requires the introduction of an infinite number of counterterms in the renormalized Lagrangian. In the case considered here, things are even more difficult, since, in addition to the derivative interactions that appear in the chiral limit, we also have new interactions proportional to m^2 .

Therefore, we will introduce the renormalized Lagrangian

$$\mathcal{L}_R = \mathcal{L} + \text{counterterms} , \tag{31}$$

where "counterterms" represents the infinite set of counterterms needed to absorb all the divergences appearing

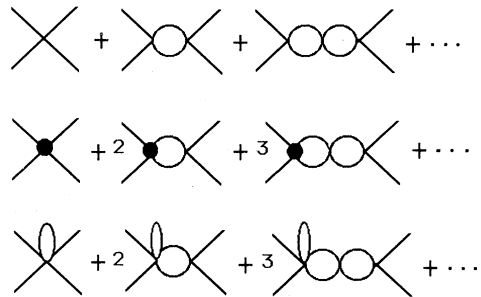


FIG. 6. Diagrams contributing to the pion scattering to leading order in the large N approximation up to order m^2/F^2 . The large black dot represents the pion coupling proportional to m^2 .

in the effective action in Eq. (28). In fact, at the level considered here, we do not need to know the precise form of these counterterms. From Fig. 6 we know that the only vertices appearing in the diagrams contributing to the regularized pion-scattering amplitude are the derivative four-pion vertex, the four-pion vertex proportional to m^2 , and the derivative six-pion vertex. The effect of the counterterms needed for the renormalization of the pion-scattering amplitude on these vertices can easily be taken into account just by making the replacements

$$\frac{q^2}{NF^2} \longrightarrow \frac{q^2}{NF^2} G \quad (32)$$

on the derivative four-pion vertex,

$$-\frac{m^2}{NF^2} \longrightarrow -\frac{m^2}{NF^2} H \quad (33)$$

on the four-pion vertex proportional to m^2 , and

$$\frac{2q^2}{(NF^2)^2} \longrightarrow \frac{2q^2}{(NF^2)^2} E \quad (34)$$

on the derivative six-pion vertex (in the above prescriptions q represents the total momentum of two of the pions involved in the vertex, although for the sake of simplicity we do not give the details of the appropriate combinations). The factors G , H , and E are arbitrary analytical functions on $s = q^2$, so that they can be written as

$$\begin{aligned} G(s) &= 1 + g_1 \frac{s}{F^2} + g_2 \left(\frac{s}{F^2}\right)^2 + \dots = \sum_{k=1}^{\infty} g_k \left(\frac{s}{F^2}\right)^k, \\ H(s) &= 1 + h_1 \frac{s}{F^2} + h_2 \left(\frac{s}{F^2}\right)^2 + \dots = \sum_{k=1}^{\infty} h_k \left(\frac{s}{F^2}\right)^k, \\ E(s) &= 1 + e_1 \frac{s}{F^2} + e_2 \left(\frac{s}{F^2}\right)^2 + \dots = \sum_{k=1}^{\infty} e_k \left(\frac{s}{F^2}\right)^k. \end{aligned} \quad (35)$$

The constants g_k , h_k , and e_k are the unrenormalized coupling constants, and they have to be renormalized to absorb the divergencies in the pion scattering amplitude. Here it is important to notice that, as can be seen in Eq. (28), the regularized action does not have any new contribution to the kinetic term, and therefore there is no need for a pion wave-function renormalization. Now we can use the above replacements to recompute the diagrams in Fig. 6. The new amplitude has the same form as the previous one with

$$\begin{aligned} A(s) &= \frac{1}{NF^2} \left\{ \frac{sG(s)}{1 + sG(s)I(s)/2F^2} \right. \\ &\quad \left. - \frac{m^2[H(s) + \Delta E(s)s/F^2]}{[1 + sG(s)I(s)/2F^2]^2} \right\} \\ &= A_0(s) + A_1(s), \end{aligned} \quad (36)$$

where $A_1(s)$ is the term proportional to m^2 , and $A_0(s) =$

$A(s) - A_1(s)$. As there is no renormalization of the pion wave function, the above amplitude should be made finite by an appropriate definition of the renormalized couplings in terms of the unrenormalized ones. This can be done as follows: First we will consider the renormalization of A_0 and A_1 separately. For the renormalization of the A_0 amplitude we follow the same procedure given in [8] for the renormalization of the pion scattering amplitude in the chiral limit. Thus, we write

$$A_0^{-1} = \frac{NF^2}{sG} - \frac{N}{2(4\pi)^2} (N_\epsilon + 2) - \frac{N}{2(4\pi)^2} T(s; \mu), \quad (37)$$

where from Eqs. (16) and (17) $T(s; \mu)$ is given by

$$T(s; \mu) \equiv \sqrt{1 - 4m^2/s} \ln \frac{\sqrt{1 - 4m^2/s} - 1}{\sqrt{1 - 4m^2/s} + 1} - \ln \frac{m^2}{\mu^2}. \quad (38)$$

Now we define G^R as

$$\frac{1}{G^R} = \frac{1}{G} - \frac{s}{2(4\pi)^2 F^2} (N_\epsilon + 2). \quad (39)$$

The G^R function can be expanded in terms of the renormalized coupling constants g_k^R ,

$$G^R = 1 + g_1^R \left(\frac{s}{F^2}\right) + g_2^R \left(\frac{s}{F^2}\right)^2 + \dots, \quad (40)$$

so that, expanding Eq. (39) in powers of s/F^2 , we can find an infinite number of equations defining the renormalized couplings g_k^R in terms of the unrenormalized ones g_k .

With the above definition for G^R , we can write

$$A_0(s) = \frac{1}{NF^2} \frac{sG^R}{1 - [sG^R/2(4\pi)^2 F^2] T(s; \mu)}. \quad (41)$$

In this form A_0 is a manifestly finite and well-defined function on s once G^R is given.

The next step is the renormalization of the second piece of the amplitude A_1 . First we note that by comparison of Eqs. (36) and (41) we can write

$$\frac{G^R}{1 - [sG^R/2(4\pi)^2 F^2] T(s; \mu)} = \frac{G}{1 + sGI(s)/2F^2} \quad (42)$$

so that A_1 is given by

$$A_1(s) = \frac{-m^2(G^R/G)^2 [H + \Delta E(s)/F^2]}{[1 - (sG^R/2F^2)T(s; \mu)]^2}. \quad (43)$$

Taking into account that the A_0 in Eq. (41) is finite, we also have to require

$$\frac{1}{G^2} \left[H' - \frac{Es}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2} \right] \quad (44)$$

to be finite, where

$$H' = H + \frac{Es}{(4\pi)^2 F^2} (N_\epsilon + 1). \quad (45)$$

In other words, the quotients H'/G^2 and E/G^2 must be finite. Therefore, we introduce the two new generating functions A^R and B^R as

$$\begin{aligned} H' &= A^R G^2, \\ E &= B^R G^2, \end{aligned} \tag{46}$$

which generate two infinite sets of renormalized constants

$$A(s) = \frac{1}{NF^2} \left\{ \frac{sG^R}{1 - [sG^R/2(4\pi)^2 F^2]T(s; \mu)} - \frac{m^2 G^{R2} [A^R - B^R [s/(4\pi)^2 F^2] \ln(m^2/\mu^2)]}{\{1 - [sG^R/2(4\pi)^2 F^2]T(s; \mu)\}^2} \right\}. \tag{48}$$

IV. RENORMALIZATION-GROUP EQUATIONS

In the preceding section we were able to write the scattering amplitude in a manifest finite form in terms of the G^R , A^R , and B^R functions. These functions generate three countable infinite sets of renormalized couplings, which implicitly depend on the renormalization scale μ . Therefore, G^R , A^R , and B^R also depend on μ . At this point it is important to notice that the set of couplings that we have considered, and therefore the set of new (counter)terms added to the original Lagrangian, is the minimum needed to absorb the divergencies appearing at the leading order in the $1/N$ expansion. Of course there are many other terms that could be added to the Lagrangian preserving the appropriate symmetry requirements (see [11] for one interesting example). However, for the sake of simplicity, we will only consider here the above-mentioned minimal terms. In the chiral limit this corresponds to the s -channel models of [11].

As we have already mentioned, there is no wavefunction renormalization in the leading order of the $1/N$ expansion, and thus we conclude that the $A(s)$ amplitude is a physical observable in the sense of the renormalization-group evolution. In other words, the explicit dependence of $A(s)$ on the renormalization scale μ must be exactly canceled by the implicit dependence through the generating functions. This can be precisely stated with the evolution equation

$$\left[\frac{\partial}{\partial \ln \mu} + \sum_k \left(\beta_k^g \frac{\partial}{\partial g_k^R} + \beta_k^a \frac{\partial}{\partial a_k^R} + \beta_k^b \frac{\partial}{\partial b_k^R} \right) \right] A(s) = 0, \tag{49}$$

where, as usual, the β functions are the derivatives of the corresponding couplings with respect to $\ln \mu$. The above equation is extremely useful, since it can be used to determine the dependence of the generating functions on μ and, finally, the dependence of all the couplings they generate on this scale. In fact, this can be done in many different ways but, in order to make the chiral limit more transparent, we will require the above equation to apply separately to A_0 and A_1 . Thus, from Eqs. (41) and (49) we can write

$$A^R = 1 + a_1^R \frac{s}{F^2} + \dots, \tag{47}$$

$$B^R = 1 + b_1^R \frac{s}{F^2} + \dots.$$

Thus, the $A(s)$ function appearing in the elastic scattering amplitude can be written in terms of functions that are finite in the $\epsilon \rightarrow 0$ limit:

$$\begin{aligned} dG^R(s; \mu) \left(1 - \frac{sG^R(s; \mu)}{2(4\pi)^2 F^2} T(s; \mu) \right) \\ + G^R(s; \mu) d \left(1 - \frac{sG^R(s; \mu)}{2(4\pi)^2 F^2} T(s; \mu) \right) = 0, \end{aligned} \tag{50}$$

and integrating this equation we find

$$G^R(s; \mu) = \frac{G^R(s; \mu_0)}{1 - [sG^R(s; \mu_0)/2(4\pi)^2 F^2] \ln(\mu^2/\mu_0^2)}. \tag{51}$$

This result describes the dependence of $G^R(s; \mu)$ on the renormalization scale μ determined by the renormalization group equations.

Now, applying Eq. (49) to A_1 , we find that the piece

$$A^R(s; \mu) - B^R(s; \mu) \frac{s}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2} \tag{52}$$

should be μ independent. For the sake of simplicity we introduce another generating function $J^R(s; \mu)$ defined as

$$\begin{aligned} J^R(s; \mu) &= A^R(s; \mu) - B^R(s; \mu) \frac{s}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2} \\ &+ \frac{s}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2}; \end{aligned} \tag{53}$$

note that, in general, the j_k^R couplings generated by J^R can depend on $\ln m^2$, but this is not the case for j_1^R . In terms of J^R , Eq. (52) reads

$$\frac{dJ^R(s; \mu)}{d(\ln \mu)} = \frac{-2s}{(4\pi)^2 F^2}. \tag{54}$$

This equation can be integrated to give

$$J^R(s; \mu) = J^R(s; \mu_0) - \frac{2s}{(4\pi)^2 F^2} \ln \frac{\mu}{\mu_0}. \tag{55}$$

Now it is possible to expand both sides of Eqs. (51) and (55) for G^R and J^R in powers of s/F^2 ,

$$\begin{aligned} G^R(s; \mu) &= \sum_{k=0}^{\infty} g_k^R(\mu) \left(\frac{s}{F^2} \right)^k, \\ J^R(s; \mu) &= \sum_{k=0}^{\infty} j_k^R(\mu) \left(\frac{s}{F^2} \right)^k, \end{aligned} \tag{56}$$

to find two infinite sets of evolution equations, which explicitly give the dependence of the renormalized couplings generated by G^R and J^R on the scale μ . For example, for the lowest couplings we find

$$g_1^R(\mu) = g_1^R(\mu_0) - \frac{1}{16\pi^2} \ln \frac{\mu}{\mu_0}, \quad (57)$$

$$j_1^R(\mu) = j_1^R(\mu_0) - \frac{1}{8\pi^2} \ln \frac{\mu}{\mu_0}.$$

In terms of G^R and J^R the $A(s)$ amplitude can be written as

$$A(s) = \frac{1}{NF^2} \frac{G^R(s; \mu)}{1 - [sG^R(s; \mu)/2(4\pi)^2 F^2]T(s; \mu)} \left\{ s - \frac{m^2 G^R(s; \mu)}{1 - [sg^R(s; \mu)/2(4\pi)^2 F^2]T(s; \mu)} \left[J^R(s; \mu) - \frac{s}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2} \right] \right\}. \quad (58)$$

From this result it is not difficult to obtain the amplitude low-energy behavior, which is given by

$$A(s) \simeq \frac{s}{NF^2} \left\{ 1 - [2g_1^R(\mu) + j_1^R(\mu)] \frac{m^2}{F^2} + 2 \frac{m^2}{(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2} \right\} - \frac{m^2}{NF^2} + O\left(\left(\frac{m^2}{F^2}\right)^2\right) + O\left(\left(\frac{s}{F^2}\right)^2\right). \quad (59)$$

This is a very useful and interesting result, since we know that at low energies $A(s)$ should go as $(s - m_\pi^2)/f_\pi^2$, where m_π is the physical pion mass and f_π is the pion decay constant. Therefore, the amplitude in Eq. (58) has the correct low-energy behavior, provided we define

$$f_\pi^2 = NF_{\text{eff}}^2 = NF^2 \left\{ 1 - \frac{m^2}{F^2} \left(\frac{1}{8\pi^2} \ln \frac{m^2}{\mu^2} - 2g_1^R(\mu) - j_1^R(\mu) \right) \right\} + O\left(\left(\frac{m^2}{F^2}\right)^2\right). \quad (60)$$

This equation also provides the correct nontrivial dependence of f_π on the logarithm of the pion mass, which is well known from chiral perturbation theory [2]. Notice that F_{eff}^2 is not depending on the scale μ , as can be shown by using Eq. (57). These facts are, therefore, a good consistency check of our results. Moreover, from Eq. (28) we see that the pion mass should also be renormalized. This can be done, for example, defining the renormalized mass as

$$m_R^2 = m^2 \left\{ 1 - \frac{m^2}{2(4\pi)^2 F^2} (N_\epsilon + 1) \right\}, \quad (61)$$

so that the effective mass appearing in the effective action can be written as

$$m_{\text{eff}}^2 = m_R^2 + \frac{m^4}{2(4\pi)^2 F^2} \ln \frac{m^2}{\mu^2}, \quad (62)$$

in good agreement with well-known results from chiral perturbation theory [2]. However, it is important to notice that our computation of the effective action and the scattering amplitude for pions is only performed up to the lowest order in m^2/F^2 . Thus, as the renormalization of the mass introduces an extra factor m^2/F^2 [$m_{\text{eff}}^2 = m^2 + O(m^2/F^2)$], we do not have to renormalize that m^2 appearing in the amplitude in Eq. (30), since otherwise it would produce extra terms proportional to

$(m^2/F^2)^2$. The same arguments apply to Eq. (58). Therefore, we can consider the m^2 appearing in Eqs. (30) and (58) as the physical pion mass squared m_π^2 .

V. PARTIAL WAVES AND UNITARITY

In order to study the unitarity properties of the amplitudes obtained in the preceding section, we will perform the standard partial wave expansion. First, we project on the isospin channels, which for the model here considered are defined as [11]

$$\begin{aligned} T_0 &= NA(s) + A(t) + A(u), \\ T_1 &= A(t) - A(u), \\ T_2 &= A(t) + A(u). \end{aligned} \quad (63)$$

The partial waves are then defined as usual:

$$a_{IJ}(s) = \frac{1}{64\pi} \int_{-1}^1 d(\cos \theta) T_I(s, \cos \theta) P_J(\cos \theta). \quad (64)$$

As is well known, the requirement of unitarity constrains the possible behavior of these partial waves. In particular, they should have a cut along the positive real axis from the threshold to infinity. The physical amplitudes are the values right on the cut, and, in addition, the condition of elastic unitarity:

$$\text{Im} a_{IJ} = \sigma |a_{IJ}|^2 \quad (65)$$

(where $\sigma = \sqrt{1 - 4m^2/E^2}$) has to be satisfied in the physical region $s = E^2 + i\epsilon$ and $E^2 > 4m^2$, where E is the center-of-mass energy. This equation is exact for energies below the next four-pion threshold, but even beyond that point it is approximately valid.

In the following we will study how well the pion-scattering amplitude satisfies this relation in the large N limit. From Eq. (63) it is obvious that the most im-

portant isospin channel at the leading order in the large N expansion is $I = 0$, which is of order 1 in the $1/N$ expansion. From Eqs. (58), (63), and (64) and for the a_{00} channel it is immediately found that

$$a_{00}(s) = \frac{1}{32\pi F^2} \left\{ \frac{sG_R(s; \mu)}{1 - [sG_R(s; \mu)/2(4\pi)^2 F^2](\tilde{T} + i\pi\sigma)} - \frac{m^2 \{J_R(s; \mu) - [s/(4\pi)^2 F^2] \ln(m^2/\mu^2)\}}{\{1 - [sG_R(s; \mu)/2(4\pi)^2 F^2](\tilde{T} + i\pi\sigma)\}^2} \right\}, \quad (66)$$

where we have defined \tilde{T} as

$$T(E^2 + i\epsilon; \mu^2) = \sigma \ln \left| \frac{\sigma - 1}{\sigma + 1} \right| + i\pi\sigma - \ln \frac{m^2}{\mu^2} \equiv \tilde{T} + i\pi\sigma, \quad (67)$$

and thus we are only considering explicitly the physical values of s . However, it is easy to see that the $a_{00}(s)$ partial-wave amplitude, when s is considered a complex variable, has the proper unitary cut mentioned above. Now the partial wave can be naturally broken down as

$$a_{00} \equiv a_{00}^{(0)} + a_{00}^{(1)}, \quad (68)$$

where $a_{00}^{(1)}$ corresponds to the part of the partial wave that is proportional to m^2 and $a_{00}^{(0)}$ is the part that is not. In terms of $a_{00}^{(0)}$ and $a_{00}^{(1)}$ the unitarity condition in Eq. (65) reads

$$\begin{aligned} \text{Im}a_{00}^{(0)} &= \sigma |a_{00}^{(0)}|^2, \\ \text{Im}a_{00}^{(1)} &= 2\sigma \text{Re}a_{00}^{(0)} a_{00}^{(1)*}, \end{aligned} \quad (69)$$

provided one neglects terms of order $(m^2/F^2)^2$. After a bit of algebra, it is not difficult to show that these equations are indeed satisfied by $a_{00}^{(0)}$ and $a_{00}^{(1)}$ as obtained from Eqs. (65) and (68). Therefore, we can finally write for the partial wave a_{00} from the amplitude in Eq. (66)

$$\text{Im}a_{00} = \sigma |a_{00}|^2 + O\left(\frac{1}{N}\right) + O\left(\left(\frac{m^2}{F^2}\right)^2\right). \quad (70)$$

As usual the right cut produces two sheets for the a_{00} function. Depending on the actual form of the generating functions, eventually some poles could appear in different places. If the poles appear in the first sheet (the physical sheet), they are not acceptable and must be understood as artifacts (ghosts) of the approximation. However, when the poles appear on the second sheet (the unphysical sheet), they are welcome and have a natural interpretation as resonances. This is for example the case of the LSM (linear σ model), which will be studied in detail in the next section.

VI. THE LINEAR σ MODEL

In this section we consider the case of the LSM, which is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - V(\phi) + \mathcal{L}_{\text{SB}} \quad (71)$$

where ϕ is an $N + 1$ vector with components $\pi_1, \pi_2, \dots, \pi_N, \pi_{N+1}$ and the potential is given by

$$V(\phi) = -\mu^2 |\phi|^2 + \lambda |\phi|^4. \quad (72)$$

The Lagrangian would be $O(N+1)$ invariant if it were not for the last symmetry-breaking piece that is only $O(N)$ invariant. This piece is given by

$$\mathcal{L}_{\text{SB}} = \sqrt{NF} m^2 \sigma, \quad (73)$$

where

$$\sigma = \pi_{N+1} = \sqrt{NF^2} + H. \quad (74)$$

This model has been considered in great detail in the last reference of [18] in the large N limit. Here we work it to the lowest order in m^2/F^2 just to see how it can be obtained as a particular case of the more general nonlinear model studied before and as a check of our results. In terms of the field H , the Lagrangian can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \pi^\alpha \partial^\mu \pi^\alpha + \frac{1}{2} \partial_\mu H \partial^\mu H + \lambda N^2 F^4 + m^2 N F^2 \\ &\quad - 4\lambda N F^2 H^2 + H \sqrt{NF^2} m^2 - \lambda(\pi^2 + H^2)^2 \\ &\quad - 4\lambda \sqrt{NF^2} H \pi^2 - 4\lambda \sqrt{NF^2} H^3. \end{aligned} \quad (75)$$

Notice that there is a linear term in the H field. In order to eliminate it, we introduce a shifted h field,

$$H = H_0 + h, \quad (76)$$

with constant H_0 . From the Lagrangian in terms of this new field h and, in particular, from the piece proportional to π^2 we find a pion mass

$$m^2 = 4\lambda H_0^2 + 8\lambda H_0 \sqrt{NF^2}. \quad (77)$$

The piece of the Lagrangian proportional to h can be written in terms of this pion mass. Then, the condition for this piece to vanish becomes

$$H_0 = -\sqrt{NF^2} + \sqrt{NF^2 + m^2/4\lambda} > 0, \quad (78)$$

so that the Lagrangian has the final form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \pi^\alpha \partial^\mu \pi^\alpha - \frac{1}{2} m^2 \pi^2 + \frac{1}{2} \partial_\mu h \partial^\mu h - \frac{1}{2} m_h^2 h^2 \\ &\quad - \lambda(\pi^2 + h^2)^2 - 4\lambda h(\pi^2 + h^2) \sqrt{NF^2 + m^2/4\lambda}, \end{aligned} \quad (79)$$

where

$$\begin{aligned} m_h^2 &= M^2 + 3m^2, \\ M^2 &= 8\lambda N F^2. \end{aligned} \quad (80)$$

Now we are able to obtain the corresponding Feynman rules. The elastic scattering amplitude can be computed in many different ways in the large N limit. Probably the easiest is the following: First, one computes the diagrams in Fig. 7(a), then one iterates the result as shown in Fig. 7(b), and, finally, one expands in powers of m^2/F^2 , while keeping both the zeroth and the first order. The

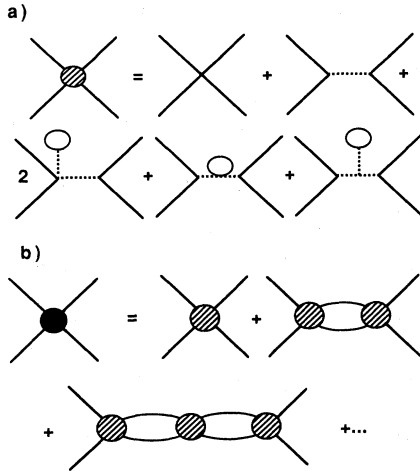


FIG. 7. Diagrams appearing in the calculation of the pion-scattering amplitude in the LSM described in the text. The continuous lines represent pions and the dashed ones the h field.

final result is given by

$$A(s) = \frac{1}{NF^2} \left\{ \frac{sG_M(s)}{1 + [sG_M(s)I(s)/2F^2]} - \frac{m^2 G_M^2(s) [1 + 2(s/M^2) + \Delta(s/F^2)]}{\{1 + [sG_M(s)I(s)/2F^2]\}^2} \right\}, \quad (81)$$

where

$$G_M(s) = \frac{1}{1 - s/M^2}. \quad (82)$$

Notice that we can consider this result a particular case of the general one given in Eq. (36), with the choices

$$\begin{aligned} G(s) &= G_M(s), \\ H(s) &= G_M^2(s) \left(1 + 2\frac{s}{M^2}\right), \\ E(s) &= G_M^2(s). \end{aligned} \quad (83)$$

The possibility of writing the LSM amplitude as a particular case of the NLSM (in the large N limit in both cases) was expected from our previous discussion, but, as far as each computation has been done in a completely different and independent way, it turns out to be a highly nontrivial check of our results. Now we can proceed with the renormalization procedure as we did in Secs. III and IV. The relevant issue here is that the LSM is renormalizable in the standard way; i.e., only a finite number of parameters have to be renormalized. This means in the large N limit here considered that only the constant M will be renormalized. In particular, we find

$$G_M^R(s; \mu) = \frac{1}{1 - s/M_R^2(\mu)}, \quad (84)$$

where

$$\frac{1}{M_R^2} = \frac{1}{M^2} + \frac{1}{2(4\pi)^2 F^2} (N_\epsilon + 2), \quad (85)$$

so that the dependence of the M_R^2 with the scale is given by

$$M_R^2(\mu) = \frac{M_R^2(\mu_0)}{1 - [M_R^2(\mu_0)/2(4\pi)^2 F^2] \ln(\mu^2/\mu_0^2)}. \quad (86)$$

The other renormalized generating function that we need is

$$J^R(s; \mu) = 1 + 2\frac{s}{M_R^2(\mu)}. \quad (87)$$

Now, using Eq. (86) we find

$$\frac{dJ_R(s; \mu)}{d(\ln \mu)} = \frac{-2s}{(4\pi)^2 F^2} \quad (88)$$

which is consistent with the general μ dependence for the generating functions found in Eqs. (51) and (55). Finally, the renormalized amplitude for the LSM is

$$A(s) = \frac{1}{NF^2} \left\{ \frac{sG^R(s; \mu)}{1 - [sG^R(s; \mu)/2(4\pi)^2 F^2] T(s; \mu)} - \frac{m^2 G^{R^2}(s; \mu) \{1 + 2[s/M_R^2(\mu)] - [s/2(4\pi)^2 F^2] \ln(m^2/\mu^2)\}}{\{1 - [sG^R(s; \mu)/2(4\pi)^2 F^2] T(s; \mu)\}^2} \right\}. \quad (89)$$

In the chiral limit, this amplitude reduces to that found in [18]. The $I = J = 0$ partial wave amplitude has a pole in the second Riemann sheet that corresponds to a physical scalar resonance. This resonance has the interesting saturation property, which means that its mass remains bounded when M_R goes to infinity (see also [19] for a general discussion). This fact is probably related with the expected triviality of the LSM.

In any case, the LSM provides a very nice example of how the general formalism developed for the computation of the χ PT large N limit works outside the chiral limit.

VII. FITTING THE PION SCATTERING

Apart from the LSM, the most interesting case to which to apply our results is the pion scattering. However, in order to use Eq. (58) to fit the experimental pion-scattering data, we have to face the problem of dealing with an infinite number of parameters, i.e., the scale μ and the values of the renormalized coupling constants $g_k^R(\mu)$ and $j_k^R(\mu)$ at that scale. This fact can be interpreted as the lack of predictive power of the model, and, in fact, it is the case in a strict sense. Still it is interesting

to know whether the model is general enough to provide a good description of the pion scattering. One simple way to deal with this question is just by considering only particular cases, where all the coupling constants except a finite set $g_1^R, g_2^R, \dots, g_r^R$ and $j_1^R, j_2^R, \dots, j_s^R$ vanish at some scale μ . These models are just defined by a finite number of parameters (μ and the $r+s$ coupling constants renormalized at this scale) and therefore can be used to fit the experimental data. In particular, one can consider the extreme case of having all the renormalized couplings equal to zero at some scale μ , i.e., $g_k^R(\mu) = j_k^R(\mu) = 0$ for all $k > 0$ (the minimal model). The model thus obtained only has one parameter (the μ scale), and it can be considered in some sense the nonlinear σ model renormalized at the leading order of the $1/N$ expansion out of the chiral limit. In fact, this minimal model can be obtained from the LSM just by taking the limit M_R^2 going to infinity.

In the following we will show how it is possible to fit the pion-scattering data with this minimal model. In any case it is important to note that our approach here is different from the more usual one consisting in the use of the LSM giving to the σ degree of freedom the interpretation of a real finite-mass resonance in the $I = J = 0$ channel.

In Fig. 8 we show the result of our fit of the $I = J = 0$ phase shift for elastic pion scattering using the above-defined minimal model. The only parameter in this especially simple case is the scale μ . The fit in Fig. 8 corresponds to a value of $\mu \simeq 775$ MeV, and, as can be seen, it describes perfectly well the data from the threshold up to 700 MeV. It is also interesting to remark that the fitted

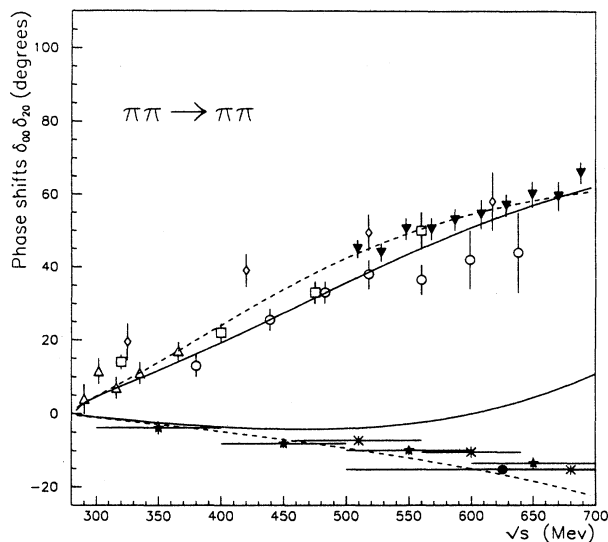


FIG. 8. Phase shift for $\pi\pi$ scattering. The results coming from the large N limit out of the chiral limit are drawn with a dashed line both for the (0,0) and the (2,0) channels. The continuous lines were taken from [7], and they are the (0,0) and (2,0) phase shifts as obtained from the standard one-loop χ PT with the parameters given in [20]. The experimental data corresponds to [21] \triangle , [22] \circ , [23] \square , [24] \diamond , [25] ∇ , [26] \star , [27] \ast , and [28] \bullet .

value for μ is the same as that found in [9]. However, as we are including here the effect of the pion mass, the fit is much better in the threshold region.

For other channels such as $I = J = 1$ or $I = 2, J = 0$ the large N leading-order prediction is that they are suppressed. This could seem quite a poor prediction, but it is not. Looking at the experimental data, we realize that the phase shifts in these channels grow very slowly with energy; for instance, at 500 MeV of center-of-mass energy we have $\delta_{11} \simeq 5^\circ$ and $\delta_{20} \simeq -7^\circ$, whereas $\delta_{00} \simeq 38^\circ$ at the same energy. In any case, one can use the results obtained in the preceding section, considering Eqs. (58), (63), and (64) with the μ value fitted for the a_{00} channels. The results are shown in Figs. 8 and 9. For the $I = 2, J = 0$ channel we see that our fit (which is, in fact, a prediction, since we only have one parameter that had already been fixed in the $I = J = 0$ channel) is quite good. However, this is not the case when $I = J = 1$. Basically this channel can be described as a noninteracting channel at low energies but a strongly interacting channel at higher energies because of the appearance of the ρ resonance. The approach here considered seems to describe the noninteracting low-energy region well, but it fails in the resonant region. One is then tempted to make an interpretation of the scale μ as some kind of cutoff signaling the range of applicability of our approach and, in fact, this interpretation had already been discussed in [8]. From this point of view, it is quite interesting to realize how close the μ fitted value ($\mu \simeq 775$ MeV) is to the ρ mass. In some way, by fitting the $I = J = 0$ channel, the model is telling us where new physics can appear in other channels such as the $I = J = 1$ (the ρ resonance) and thus setting the limits of applicability of the model. However, we would like to stress that our renormalization method is completely consistent and our results are

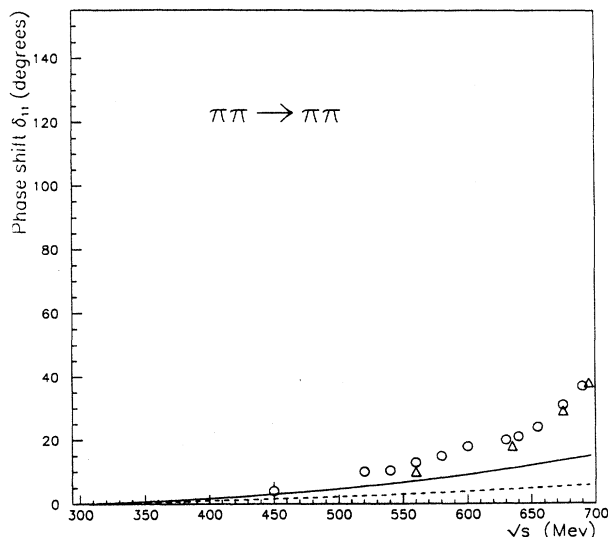


FIG. 9. (1,1) Phase shift for $\pi\pi$ scattering. The dashed line corresponds to our fit using the large N limit. The continuous line [7] is the result coming from the one-loop χ PT with the parameters proposed in [20]. The experimental data comes from [25] \circ and [29] \triangle .

formally valid at any energy, independent of how good the resulting fits are.

For the sake of comparison we have also included in Figs. 8 and 9 the results obtained with standard χ PT to one loop with the parameters proposed in [20]. It is important to note that, in spite of using more couplings, it does not do a better job than the approach followed here. The description of the $I = J = 0$ channel is good in both cases; $I = 2, J = 0$ is much better in the large N limit and the $I = J = 1$ is good at low energies (where the channel is not interacting) but it fails to describe the resonance region too.

VIII. CONCLUSIONS

In this work we have studied how the large N approximation (N being the number of Goldstone bosons) can be applied to χ PT in order to improve its unitarity behavior and enlarge its energy applicability region. In particular, we have computed the renormalized effective action at the leading order, including the effects due to the pion mass up to order m_π^2/f_π^2 . The amplitude obtained from this action has the appropriate low-energy limit and provides the right dependence of the pion mass and the pion decay constant on the chiral logs. Then we use this action to reproduce the LSM as a nontrivial check of our results. However, our model depends on an infinite number of renormalized couplings, and, in a strict sense, any phenomenological prediction requires the previous fit of these couplings, so that, in principle, the model does not have any predictive power.

Nevertheless, it is possible to consider a reduced version of the model, where an infinite number of couplings

is set to zero at some given scale. Then we use this reduced model to fit the π - π scattering. This approach is competitive with the standard one-loop one, both failing in the description of the ρ resonance. Note that we are able to fit the (0,0) and the (2,0) channel up to 700 MeV with only one parameter, while the standard one-loop computation reproduces only the (0,0) channel at this energy by using two parameters (l_1 and l_2 in the well-known notation of [1,2]).

In addition, our computations could improve a lot when next-to-leading ($1/N^2$) corrections were included. This is because of the fact that the structure of the large N amplitudes makes it possible to find poles in the second Riemann sheet that could reproduce resonances as happens in the large N description of the LSM. At present, work is also in progress in that direction [13].

In conclusion, we consider that the great success of χ PT can be enlarged by using well-defined nonperturbative techniques such as the large N expansion, even though the large N approximation, in principle, requires the introduction of an infinite number of couplings, and, in this sense, it is not a predictive theory. The most remarkable improvements can be obtained in several issues such as unitarity, the energy range of application, and by offering a picture of how χ PT works beyond the current one- or two-loop approximation.

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- [1] S. Weinberg, *Physica* **96A**, 327 (1979).
 - [2] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984); *Nucl. Phys.* **B250**, 465 (1985); **B250**, 517 (1985).
 - [3] A. Dobado and M. J. Herrero, *Phys. Lett. B* **228**, 495 (1989); **233**, 505 (1989); J. Donoghue and C. Ramirez, *ibid.* **234**, 361 (1990).
 - [4] S. Bellucci, J. Gasser, and M. E. Sainio, *Nucl. Phys.* **B423**, 80 (1994); **B431**, 413(E) (1994).
 - [5] T. N. Truong, *Phys. Rev. Lett.* **61**, 2526 (1988); A. Dobado, M. J. Herrero, and J. N. Truong, *Phys. Lett. B* **235**, 129 (1990).
 - [6] G. Ecker, J. Gasser, A. Pich, and E. de Rafael, *Nucl. Phys.* **B321**, 311 (1989); G. Ecker, J. Gasser, H. Leutwyler, A. Pirch, and E. de Rafael, *Phys. Lett. B* **223**, 425 (1989); J. F. Donoghue, C. Ramirez, and G. Valencia, *Phys. Rev. D* **39**, 1947 (1989); V. Bernard, N. Kaiser, and U. G. Meissner, *Nucl. Phys.* **B364**, 283 (1991).
 - [7] A. Dobado and J. R. Peláez, *Phys. Rev. D* **47**, 4883 (1992).
 - [8] A. Dobado and J. R. Peláez, *Phys. Lett. B* **286**, 136 (1992).
 - [9] A. Dobado, A. López, and J. Morales, *Nuovo Cimento A* **108**, 335 (1995).
 - [10] C. J. C. Im, *Phys. Lett. B* **281**, 357 (1992).
 - [11] M. J. Dugan and M. Golden, *Phys. Rev. D* **48**, 4375 (1993).
 - [12] M. R. Pennington and D. Morgan, *Phys. Lett. B* **314**, 125 (1993).
 - [13] A. Dobado and J. Morales (in preparation).
 - [14] J. Charap, *Phys. Rev. D* **2**, 1115 (1970); I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, *ibid.* **3**, 2486 (1971); J. Honerkamp, *Nucl. Phys.* **B36**, 130 (1972).
 - [15] L. Tararu, *Phys. Rev. D* **12**, 3351 (1975); T. Appelquist and C. Bernard, *ibid.* **23**, 425 (1981); J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, New York, 1989).
 - [16] D. Espriu and J. Matias, *Nucl. Phys.* **B418**, 494 (1994).
 - [17] S. Coleman, R. Jackiw, and H. D. Politzer, *Phys. Rev. D* **10**, 2491 (1974); S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985).
 - [18] R. Casalbuoni, D. Dominici, and R. Gatto, *Phys. Lett.* **147B**, 419 (1984); M. B. Einhorn, *Nucl. Phys.* **B246**, 75 (1984); R. S. Chivukula and M. Golden, *ibid.* **B372**, 44 (1992).
 - [19] A. Dobado, *Phys. Lett. B* **237**, 457 (1990); S. Willenbrock, *Phys. Rev. D* **43**, 1710 (1991).
 - [20] J. Gasser and Ulf-G. Meissner, *Nucl. Phys.* **B357**, 90 (1991); *Phys. Lett. B* **258**, 219 (1991).

- [21] L. Rosselet *et al.*, Phys. Rev. D **15**, 574 (1977).
- [22] W. Männer, in *Experimental Meson Spectroscopy*, proceedings of the Fourth International Conference, edited by D. A. Garelich, AIP Conf. Proc. No. 21 (AIP, New York, 1974).
- [23] V. Srinivasan *et al.*, Phys. Rev. D **12**, 681 (1975).
- [24] M. David *et al.* (unpublished); G. Villet *et al.* (unpublished).
- [25] P. Estabrooks and A. D. Martin, Nucl. Phys. **B79**, 301 (1974).
- [26] W. Hoogland *et al.*, Nucl. Phys. **B126**, 109 (1977).
- [27] M. J. Losty *et al.*, Nucl. Phys. **B69**, 301 (1974).
- [28] W. Hoogland *et al.*, Nucl. Phys. **B69**, 266 (1974).
- [29] Protopopescu *et al.*, Phys. Rev. D **7**, 1279 (1973).