Factorization versus duality in nonleptonic decays: A quark model approach

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We study in a quark model the contradiction between factorization and duality found in nonleptonic decays at next to leading order in $1/N_c$, concentrating on the quark exchange mechanism. The contradiction originates in the fact that the standard factorization assumption approximates the asymptotic final states by a nonorthogonal set of states, thus leading to an overcounting of the decay probability. We consider a system with two heavy quarks treated as classical color sources with constant velocity, and two mass-degenerate antiquarks. Exploiting permutation symmetry in an adiabatic approximation, we find that the final state interaction restores duality. Three $O(1/N_c)$ effects are exhibited: (i) a proper treatment of orthogonality yields a global correction $1/N_c \to 1/2N_c$ within a generalized factorization in the manner of BSW (such a factor was present in an ansatz by Shifman); (ii) the distortion of the meson wave functions at the time of the weak decay; (iii) relative phases generated by the later evolution. The latter effect becomes dominant for light antiquarks or for a small velocity of the final mesons, and may thoroughly modify the factorization picture. For exclusive decay it may interchange the role of class I and class II final channels (but it does not influence the sum I plus II), and for semi-inclusive decay it may lead to an equal sharing of the probability between the two sets of final states. In the heavy antiquark and large velocity limit, the replacement $1/N_c \to 1/2N_c$ is the dominant correction.

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I. INTRODUCTION

While some progress has been made during these last years in our understanding of semileptonic and leptonic decay mechanisms, our understanding of nonleptonic decays is still semiquantitative. Not to speak of the $\Delta I = 1/2$ mystery, the nonleptonic decays of D and B mesons are most often studied with the help of the standard factorization assumption [1], the theoretical basis for which exists only in the $N_c \to \infty$ limit, or of the generalized factorization assumption in the manner of Bauer, Stech, and Wirbel (BSW) [2] which is a phenomenological ansatz.

The nonleptonic decay channels of heavy mesons are an important issue and will grow even more so, since they provide the cannels in which the CP asymmetries will be looked for at B factories.

A critical study of factorization assumption is an urgent task, and has indeed been started [3], at a time when increasingly accurate experimental results teach us that our present understanding of $B \to \psi K(K^*)$, based on the factorization assumption among other hypotheses, severely fails [4].

It is usually claimed that the corrections to factorization are due to a final state interaction (FSI). In a sense this statement is true, but the proper meaning of what is understood needs clarification. One of our aims in this paper is to pursue this clarification in a simple model in which the dynamics is rather transparent.

Furthermore, little is known about the validity of factorization except that it is violated in low energy $K \to \pi\pi$ and $D \to K\pi, KK, \pi\pi$ [5,6] channels where strong FSI phases are experimentally known to be present, both from the direct analysis of weak nonleptonic decays and from scattering experiments. The study grows more difficult at larger energies when more channels are coupled. This happens when multiparticle channels come in, particularly multipion channels, and also when several two-body channels communicate via a strong interaction through quark exchange or quark-pair annihilation or creation.

We will concentrate on two-body channels communicating via quark exchange. In a Tamm-Dancoff-type expansion, quark exchange is dominant since it needs no quark pair creation or annihilation. Furthermore, it has been stressed by Donoghue [7] that such a mechanism might explain the $D^0 \to \phi \overline{K}^0$ decay amplitude.

On the other hand, Shifman [8] (see also [9]) has made the interesting remark that beyond leading $1/N_c$ order the factorization assumption simply violates duality. As we shall show at length in this paper, this effect is fully related to final state interaction via quark exchange.

Indeed this comes from the fact that in color space a $qq\bar{q}\bar{q}$ color singlet can be decomposed in two ways into two $q\bar{q}$ color singlets, but the two resulting states are not orthogonal to each other [9]. At the time the weak decay of the initial meson creates a $qq\bar{q}\bar{q}$ color singlet, the four quarks interact strongly with one another. It is impossible to tell that one $q\bar{q}$ pair is in one meson and the other pair in another meson. However, the system evolves, and eventually splits into two $q\bar{q}$ color singlets that are spatially distant and hence orthogonal. During the entire period of the interaction, one given quark does not know with which antiquark it is paired. This is exactly the situation which is expressed usually by the expression "quark exchange." What is depicted by this expression is not a simple and instantaneous exchange of a quark from one meson to another, but a period of overlap of the two mesons, resulting finally in a nonvanishing amplitude for an exchange of a quark.

The factorization assumption totally overlooks this complex mechanism, as it simply computes the overlap of the $qq\bar{q}\bar{q}$ system resulting from weak decay with the final mesons. *Two nonorthogonal states are taken as an approximation of two distinct final states which are obviously orthogonal.* Once one realizes this, it is not a surprise that one encounters some problems with probability conservation.

Our aim is mainly to understand better this interaction mechanism in a simple model, to check that when the dynamics is correctly treated there is no contradiction with duality, and to identify the different effects contributing at next to leading order in $1/N_c$.

We will work in the kinematical situation considered by Shifman [8], in a Hamiltonian approach, namely, a quark model one. In Sec. II we will rephrase duality in the language of closure theorem and reexpress the contradiction between factorization and duality. In Sec. III we will present a quark model, with an adiabatic approximation and mass-degenerate antiquarks, and we will exploit the resulting permutation symmetry to compute the S matrix and the weak decay amplitude. In Sec. IV we will conclude.

II. DUALITY VERSUS FACTORIZATION; REPHRASING THE PROBLEM

Let us recall the ideal process which is studied in [8]. There are three heavy quarks A, B, C and two light antiquarks $\bar{\alpha}, \bar{\beta}$. We will assume the latter to be heavy enough to justify the use of the quark potential model which will be our tool all along.

The process under study is the weak decay

$$P_{A\bar{\alpha}} \to M_{B\bar{\alpha}} + M_{C\bar{\beta}},\ M_{B\bar{\beta}} + M_{C\bar{\alpha}},$$

where $P_{A\bar{\alpha}}$ is pseudoscalar meson composed by A and $\bar{\alpha}$ and $M_{B\bar{\alpha}}$ represents any meson composed by B and $\bar{\alpha}$.

The following relations are assumed [8]:

$$M_B = M_C \equiv M, \quad M_A = 2M + \Delta, \quad \Lambda_{\rm QCD} \ll \Delta \ll M, \tag{1}$$

to which we add

$$\Lambda_{\rm QCD} \ll m_{\bar{\alpha}}, m_{\bar{\beta}} \tag{2}$$

to justify the use of the quark model.

We cannot find a physical example of such a situation. If the s quark was heavy, the ideal situation assumed in [8] would be realized by the couple of decay channels: $B_d \to D^0\overline{K}^0, D^+K^-$.

A. Relation between duality and the closure theorem

In [8] Shifman exhibits a contradiction between duality and the standard factorization hypothesis also encountered in [9] while studying $\Delta\Gamma$ for the B_s-$\overline{B}_s$ system. We will study this issue as a contradiction between the closure theorem and the standard factorization hypothesis. Indeed, duality is related to the closure theorem in quantum mechanics as we will now recall.

Let us call generically $|n\rangle$ all hadronic states built up with quarks $B, C, \bar{\alpha}, \bar{\beta}$. Calling H_W the weak Hamiltonian, the state $H_W|P_{A\bar{\alpha}}\rangle$ is composed of the four quarks $B, C, \bar{\alpha}, \bar{\beta}$. The decay width of the $P_{A\bar{\alpha}}$ meson is given by

$$\begin{aligned}\Gamma(P_{A\bar{\alpha}}) &= \sum_n \langle P_{A\bar{\alpha}}|H_W|n\rangle\langle n|H_W|P_{A\bar{\alpha}}\rangle\delta(E_n - E_{A\bar{\alpha}}) \\ &= \sum_n \langle P_{A\bar{\alpha}}|H_W|n\rangle\delta(H - E_{A\bar{\alpha}})\langle n|H_W|P_{A\bar{\alpha}}\rangle, \end{aligned} \tag{3}$$

where $E_{A\bar{\alpha}}$ is the initial energy, E_n is the energy of the state $|n\rangle$, H is the strong Hamiltonian ($H|n\rangle = E_n|n\rangle$), and the sum is to be understood as a sum over discrete states and an integral over continuum states. The set of states $|n\rangle$ is a complete set. The sum could be expanded on any basis, and in particular on the basis of the free quarks $B, C, \bar{\alpha}, \bar{\beta}$. This is where the closure of the Hilbert space comes in.

Now, $H = H_c + V$ where H_c stands for the kinetic energy and V for the potential. Whenever the contribution from V can be neglected in front of H_c, one recovers lowest order duality, i.e., the simple parton model with no perturbative corrections or nonperturbative ones from higher dimension operators. We do not want to go into the question of when this approximation is valid, and how to get better approximations. We simply want to rephrase the contradiction between factorization and duality as a contradiction between factorization and closure, and then go into a simple model to show how the dynamics solves this.

B. Contradiction between factorization and closure

Let us consider, for example, the weak Hamiltonian

$$\begin{aligned}H_W = 2\sqrt{2}G[C_1(\overline{B}\gamma_\mu LA)(\overline{C}\gamma^\mu L\beta a) + C_2(\overline{B}\gamma_\mu L\beta a) \\ \times(\overline{C}\gamma^\mu LA)],\end{aligned} \tag{4}$$

where $L = (1-\gamma_5)/2$ and C_1 and C_2 are coefficients that we do not need to specify in this paper, although they are reminiscent of the familiar coefficients in the effective weak interaction Hamiltonian.

At the time $t = 0$ the weak Hamiltonian acts on the initial meson and produces a state that contains the quarks $C, \bar{\beta}, \bar{\alpha}, B$. Let us call this state $|f\rangle$. We find it convenient for later use to decompose $|f\rangle$ into its color part and the remainder:

$$|f\rangle \equiv H_W|P_{A\bar\alpha}\rangle = \sum_{\substack{s_C,s_{\bar\beta},c_C,c_{\bar\beta},\\ s_B,s_{\bar\alpha},c_B,c_{\bar\alpha}}} \int d\vec{p}_C d\vec{p}_{\bar\beta} d\vec{p}_B d\vec{p}_{\bar\alpha} \Psi(\vec{p}_C,s_C,\vec{p}_{\bar\beta},s_{\bar\beta},\vec{p}_B,s_B,\vec{p}_{\bar\alpha},s_{\bar\alpha})$$
$$\times \frac{1}{N_c}[C_1\delta_{c_C,c_{\bar\beta}}\delta_{c_B,c_{\bar\alpha}} + C_2\delta_{c_C,c_{\bar\alpha}}\delta_{c_B,c_{\bar\beta}}]\,|C,\vec{p}_C,s_C,c_C;\bar\beta,\vec{p}_{\bar\beta},s_{\bar\beta},c_{\bar\beta};B,\vec{p}_B,s_B,c_B;\bar\alpha,\vec{p}_{\bar\alpha},s_{\bar\alpha},c_{\bar\alpha}\rangle, \quad (5)$$

where $\vec{p}_C, s_C, c_C$ ($\vec{p}_{\bar\beta}, s_{\bar\beta}, c_{\bar\beta}$) labels the momentum, spin, and color of the (anti)quark C ($\bar\beta$) and N_c is the number of colors. The function Ψ may be computed from the wave function of $P_{A\bar\alpha}$ and the operator H_W in (4). However, we will skip this computation since the precise expression for Ψ is not relevant for our argument.

It is obvious that

$$\langle f|f\rangle = \left(C_1^2 + C_2^2 + 2\frac{C_1C_2}{N_c}\right)K, \quad (6)$$

with

$$K = \sum_{\substack{s_C,s_{\bar\beta},c_C,c_{\bar\beta},\\ s_B,s_{\bar\alpha},c_B,c_{\bar\alpha}}} \int d\vec{p}_C d\vec{p}_{\bar\beta} d\vec{p}_B d\vec{p}_{\bar\alpha} \left|\Psi(\vec{p}_C,s_C,\vec{p}_{\bar\beta},s_{\bar\beta},\vec{p}_B,s_B,\vec{p}_{\bar\alpha},s_{\bar\alpha})\right|^2. \quad (7)$$

Let us also decompose the meson wave function into a color part and the remainder:

$$|M^{(n)}_{C\bar\beta}\rangle = \sum_{s_C,s_{\bar\beta},c_C,c_{\bar\beta}} \int d\vec{p}_C d\vec{p}_{\bar\beta} \psi^{(n)}_{C\bar\beta}(\vec{p}_C,s_C,\vec{p}_{\bar\beta},s_{\bar\beta}) \frac{1}{N_c^{\frac{1}{2}}}\delta_{c_C,c_{\bar\beta}}|C,\vec{p}_C,s_C,c_C;\bar\beta,\vec{p}_{\bar\beta},s_{\bar\beta},c_{\bar\beta}\rangle \quad (8)$$

and analogously for all quark-antiquark pairs. The $\psi^{(n)}_{C\bar\beta}$'s form a complete orthonormal basis of the spin-momentum $C\bar\beta$ Hilbert space. Let us now define the spin-space overlaps:

$$K^{(n,m)}_{C\bar\beta;B\bar\alpha} = \sum_{s_C,s_{\bar\beta},s,s_{\bar\alpha}} \int d\vec{p}_C d\vec{p}_{\bar\beta} d\vec{p}_B d\vec{p}_{\bar\alpha}\ \Psi^\dagger(\vec{p}_C,s_C,\vec{p}_{\bar\beta},s_{\bar\beta},\vec{p}_B,s_B,\vec{p}_{\bar\alpha},s_{\bar\alpha})$$
$$\times \psi^{(n)}_{C\bar\beta}(\vec{p}_C,s_C,\vec{p}_{\bar\beta},s_{\bar\beta})\psi^{(m)}_{B\bar\alpha}(\vec{p}_B,s_B,\vec{p}_{\bar\alpha},s_{\bar\alpha}) \quad (9)$$

and analogously for the alternative grouping of quark-antiquark pairs: $C\bar\alpha; B\bar\beta$. Closure in the $C\bar\beta$ and $B\bar\alpha$ spin-momentum subspaces implies that

$$\sum_{n,m} K^{(n,m)*}_{C\bar\beta;B\bar\alpha} K^{(n,m)}_{C\bar\beta;B\bar\alpha} = K \quad (10)$$

and analogously

$$\sum_{n,m} K^{(n,m)*}_{B\bar\beta;C\bar\alpha} K^{(n,m)}_{B\bar\beta;C\bar\alpha} = K. \quad (11)$$

From Eqs. (5), (8), and (9) it results that

$$\langle f|M^{(n)}_{C\bar\beta}; M^{(m)}_{B\bar\alpha}\rangle = a_1 K^{(n,m)}_{C\bar\beta;B\bar\alpha},$$
$$\langle f|M^{(n)}_{B\bar\beta}; M^{(m)}_{C\bar\alpha}\rangle = a_2 K^{(n,m)}_{B\bar\beta;C\bar\alpha}, \quad (12)$$

with

$$a_1 = C_1 + \frac{C_2}{N_c}, \qquad a_2 = C_2 + \frac{C_1}{N_c}. \quad (13)$$

Up to now all equations were exact. Now we shall formulate in our formalism the factorization approximation by assuming that the decay amplitudes are well approximated by the overlaps:

$$T\left(P_{A\bar\alpha} \to M^{(n)}_{C\bar\beta} M^{(m)}_{B\bar\alpha}\right) \simeq \langle f|M^{(n)}_{C\bar\beta} M^{(m)}_{B\bar\alpha}\rangle^*,$$
$$T\left(P_{A\bar\alpha} \to M^{(n)}_{B\bar\beta}; M^{(m)}_{C\bar\alpha}\right) \simeq \langle f|M^{(n)}_{B\bar\beta} M^{(m)}_{C\bar\alpha}\rangle^*. \quad (14)$$

More precisely, the standard factorization assumption [1] uses Eqs. (12), (13), and (14) with C_1, C_2 computed from the electroweak theory complemented with QCD radiative corrections. Bauer, Stech, and Wirbel [2] have proposed a phenomenological factorization assumption that keeps Eqs. (12) and (14) but with a_1 and a_2 fitted to all known D, respectively B, decays.

Within standard factorization, summing over all two meson final states and using Eqs. (5), (7), (10), (11), (12), (13), and (14) we get

$$\sum_{n,m} |T\left(P_{A\bar\alpha} \to M^{(n)}_{C\bar\beta} M^{(m)}_{B\bar\alpha}\right)|^2 + |T\left(P_{A\bar\alpha} \to M^{(n)}_{B\bar\beta} M^{(m)}_{C\bar\alpha}\right)|^2 = \left((C_1^2 + C_2^2)(1 + \frac{1}{N_c^2}) + 4\frac{C_1 C_2}{N_c}\right) K. \qquad (15)$$

To leading order in $1/N_c$, Eq. (15) gives the same result as Eq. (6). In our present framework this reflects the well-known fact that to leading order in $1/N_c$ factorization and duality are compatible. However, the $O(1/N_c)$ corrections show a discrepancy, the contradiction stressed in [8]. As suggested in [8], this discrepancy could be cured, except for the $1/N_c^2$ terms by using a phenomenological factorization with

$$a_1 = C_1 + \frac{C_2}{2N_c}, \qquad a_2 = C_2 + \frac{C_1}{2N_c}. \qquad (16)$$

However, in our framework it is easy to trace back the origin of the discrepancy. The fact is that the set of states $|M^{(n)}_{C\bar\beta} M^{(m)}_{B\bar\alpha}\rangle \oplus |M^{(n)}_{B\bar\beta} M^{(m)}_{C\bar\alpha}\rangle$ is not an orthonormal basis of the Hilbert space. The states are normalized, but they are not orthogonal:

$$\langle M^{(n)}_{C\bar\beta} M^{(m)}_{B\bar\alpha} | M^{(n)}_{B\bar\beta} M^{(m)}_{C\bar\alpha}\rangle = O\left(\frac{1}{N_c}\right) \neq 0, \qquad (17)$$

in general. This overlap is $O(1/N_c)$ as can be easily derived from the color part in wave function (8):

$$\left(\frac{1}{N_c^{\frac{1}{2}}}\right)^4 \sum_{C_C, C_B, C_{\bar\alpha}, C_{\bar\beta}} \delta_{C_C, C_{\bar\beta}} \delta_{C_B, C_{\bar\alpha}} \delta_{C_B, C_{\bar\beta}} \delta_{C_C, C_{\bar\alpha}} = \frac{1}{N_c}. \qquad (18)$$

It is easy to check that (17) is at the origin of the discrepancy between (15) and (6).

Neither are these states eigenstates of the strong Hamiltonian. Indeed, these states are built up from two asymptotic mesons combined via a plane wave for the relative momentum between the two mesons. When the two mesons lie far apart, the simple product of their wave function is an eigenstate of the strong Hamiltonian. However, in the states we consider there is a non-negligible contribution with the two mesons overlapping in space where they strongly interact, leading to an important distortion from the simple product of asymptotic meson wave functions.

III. ADIABATIC QUARK MODEL WITH DEGENERATE ANTIQUARKS

A. S matrix

Let us first consider an oversimplified model. We will assume all quarks to be spinless and $\bar\alpha$ and $\bar\beta$ to be degenerate in mass: $m_{\bar\alpha} = m_{\bar\beta} \equiv m$. Next, B and C being very heavy, we will treat their motion as classical. They are supposed to move head-on with velocity $\vec v$. As a function of time t the spatial coordinates of B and C are

$$\vec r_B = -\vec r_C = \vec v t. \qquad (19)$$

From now on the mesons will be assumed to have their center of mass localized in configuration space, at the position of the heavy quark, and we will neglect $O(1/M)$ corrections. The fact that the heavy quarks meet at the origin, i.e., that their impact parameter is zero, means that our model will describe the S-wave channel.

Concerning the Hamiltonian for the antiquarks $\bar\beta$ and $\bar\alpha$ we will use a color potential introduced in [10]:

$$H(t) = \frac{p_{\bar\beta}^2}{2m} + \frac{p_{\bar\alpha}^2}{2m} + \sum_a \Big[\lambda^a_B \lambda^a_{\bar\alpha} V(\vec r_{\bar\alpha} - \vec v t) + \lambda^a_C \lambda^a_{\bar\alpha} V(\vec r_{\bar\alpha} + \vec v t) + \lambda^a_B \lambda^a_{\bar\beta} V(\vec r_{\bar\beta} - \vec v t)$$
$$\times \lambda^a_C \lambda^a_{\bar\beta} V(\vec r_{\bar\beta} + \vec v t) + \lambda^a_{\bar\alpha} \lambda^a_{\bar\beta} V(\vec r_{\bar\beta} - \vec r_{\bar\alpha}) + \lambda^a_C \lambda^a_B V(2\vec v t)\Big], \qquad (20)$$

where λ^a_C is the Gell-Mann SU(3) Hermitian matrix applying to quark C, etc., and where $-V(\vec r)$ is a rotation-invariant confining potential, so that color-singlet mesons are bound together (the $\lambda^a \lambda^a$ factor is negative on a singlet). The Hamiltonian is bounded from below when restricted to overall color-singlet states.

$H(t)$ in (20) is invariant for the permutation $P \equiv \bar\alpha \leftrightarrow \bar\beta$. It results that all eigenstates of H will be eigenstates of P with eigenvalue ± 1. The asymptotic states, when $T \to \pm\infty$, are built from simple products of the mesonic wave functions whose center of mass are located at $\pm\vec v T$:

$$\sqrt{2}|D^{\pm,n,m}, T\rangle \underset{T\to\pm\infty}{=} |M^{(n)}_{C\bar\beta}(-\vec v T)\rangle \otimes |M^{(m)}_{B\bar\alpha}(\vec v T)\rangle \pm |M^{(n)}_{C\bar\alpha}(-\vec v T)\rangle \otimes |M^{(m)}_{B\bar\beta}(\vec v T)\rangle, \qquad (21)$$

where the $M^{(n)}(\vec r)$ states are the mesons states defined from Eq. (8) by a Fourier transform on the center-of-mass variable. We will assume the evolution in time to be adiabatic; i.e., we assume that the state $|D^{\pm,n,m}, t\rangle$ evolves in time by remaining an eigenstate of $H(t)$ for all t. We will further assume that during the evolution the fundamental states $|D^{\pm,0,0}, t\rangle$ never cross other states. It results that the two-dimensional subspace spanned by

$|D^{\pm,0,0},t\rangle$ is stable under the action of the strong Hamiltonian, i.e., that any state within this subspace evolves into a state within this subspace at a later time. Consequently, the 2×2 restriction of the S matrix to this subspace has to be unitary. We will now restrict ourselves to the study of the strong interaction scattering process of these two fundamental states. At time t, the eigenstates verify

$$H(t)|D^{\pm,0,0},t\rangle = E^{\pm}(vt)|D^{\pm,0,0},t\rangle, \tag{22}$$

where we made use of the fact that H depends on t only through the product vt. The interaction between the two terms on the right-hand side (RHS) of (21) is $O(1/N_c)$ as already argued; see (18). Hence

$$E^+(vt) - E^-(vt) = O\left(\frac{1}{N_c}\right). \tag{23}$$

Asymptotically it is also obvious that

$$E^+(\pm\infty) = E^-(\pm\infty), \tag{24}$$

since the two mesons do not overlap, implying that the two terms in the RHS of (21) become orthogonal.

In fact the two energies differ from zero only when the two mesons overlap and the overlap falls off exponentially when $vt \to \infty$. In the basis $|D^{\pm,0,0},\pm\infty\rangle$, (21), the S matrix is diagonal and, being unitary, its general form is written as

$$S = e^{2i\delta}\begin{pmatrix} e^{2i\phi} & 0 \\ 0 & e^{-2i\phi} \end{pmatrix}, \tag{25}$$

where

$$2\delta = -\int_{-\infty}^{+\infty}\frac{dz}{v}\left[\frac{E^+(z)+E^-(z)}{2} - \frac{E^+(\infty)+E^-(\infty)}{2}\right],$$

$$2\phi = -\int_{-\infty}^{+\infty}\frac{dz}{v}\frac{E^+(z)-E^-(z)}{2}. \tag{26}$$

Indeed, the S matrix is given by

$$S = T\exp\left[-i\int_{-\infty}^{\infty} dt H_I(t)\right], \tag{27}$$

which in our case of a state, say, $|+\rangle$, which remains an eigenstate of a time-dependent Hamiltonian with energy $E^+(t)$, simplifies to

$$S_{++} = \exp\left[-i\int_{-\infty}^{\infty} dt[E^+(t) - E^+(\infty)]\right], \tag{28}$$

where the interaction Hamiltonian has been taken to be the total Hamiltonian minus the energy of two noninteracting mesons.

In the meson-meson basis,

$$\begin{aligned} &|M^{(0)}_{C\bar\beta}(-\vec v T)\rangle \otimes |M^{(0)}_{B\bar\alpha}(\vec v T)\rangle, \\ &|M^{(0)}_{C\bar\alpha}(-\vec v T)\rangle \otimes |M^{(0)}_{B\bar\beta}(\vec v T)\rangle, \end{aligned} \tag{29}$$

the S matrix is written as:

$$S = e^{2i\delta}e^{2i\phi\sigma_1} = e^{2i\delta}\begin{pmatrix} \cos 2\phi & i\sin 2\phi \\ i\sin 2\phi & \cos 2\phi \end{pmatrix}. \tag{30}$$

This matrix is, as expected, unitary and invariant for the permutation P which permutes the lines and the columns of the matrix. When the angle ϕ does not vanish, there is a quark exchange between the mesons which becomes maximal for $\phi = \pi/4$. Actually, $\phi = O(1/N_c)$ from Eq. (23). From Eq. (26), we see that $\phi \propto 1/v$. The reason for this is clear: The lower the velocity, the longer the mesons overlap and can exchange quarks. We will now return in our model to the contradiction discussed in the preceding section between duality and factorization. We work out an illustrative example of these features in the Appendix.

B. Final state interaction in our model

In this section we return to the weak interaction. We consider here the exclusive decay channels $P_{A\bar\alpha} \to M^{(0)}_{B\bar\alpha}M^{(0)}_{C\bar\beta}$ and $P_{A\bar\alpha} \to M^{(0)}_{C\bar\alpha}M^{(0)}_{B\bar\beta}$. Thanks to the statement made in the preceding section that the subspace spanned by these two final states is stable for the weak interactions, we can safely forget all other channels in our study of the FSI.

We now assume that the weak Hamiltonian acts at $t = 0$, creating the heavy quarks B and C at $\vec r = 0$. The weak interaction creates a state $|f\rangle$ as defined in (5). When $t = 0$ an additional symmetry is present in the strong Hamiltonian $H(0)$: Invariance under permutation of color labels $C_C \leftrightarrow C_B$ and consequently under $C_{\bar\beta} \leftrightarrow C_{\bar\alpha}$. It results that the $|D^{\pm,0,0},0\rangle$ states are even (odd) under the color permutation $P_c : C_C \leftrightarrow C_B$. Restricted to the color-even (color-odd) sector the Hamiltonian $H(0)$ reduces to

$$H(0) = \frac{p^2_{\bar\beta}}{2m} + \frac{p^2_{\bar\alpha}}{2m} - \frac{2(N_c \pm 2)(N_c \mp 1)}{N_c}\left[V(\vec r_{\bar\alpha}) + V(\vec r_{\bar\beta})\right] \pm \frac{2(N_c \mp 1)}{N_c}V(\vec r_{\bar\beta} - \vec r_{\bar\alpha}), \tag{31}$$

where the upper (lower) sign corresponds to color-even (color-odd) states.

It is important to notice that to leading order in N_c the Hamiltonian (31) is the same for color-even and color-odd states. It results that in the $N_c \to \infty$ limit, the two-color wave function multiplies the same spatial wave function for $t = 0$. Furthermore, to leading order in N_c the Hamiltonian (31) is equal to twice the Hamiltonian for one heavy-light meson:

$$H^{(1)} = \frac{p^2}{2m} - \frac{2(N_c-1)(N_c+1)}{N_c}V(\vec{r}), \tag{32}$$

where p and $\vec{r}$ are the light quark momentum and position. This means that to leading order in N_c the Hamiltonian (31) corresponds just to the sum of two noninteracting mesons superposed at the origin. This corresponds to the factorization assumption.

Equation (31) exhibits a symmetry for the exchange of spatial variables $\vec{r}_{\bar{\alpha}} \to \vec{r}_{\bar{\beta}}$. This symmetry is simply a product of the color permutation symmetry P_c, valid at $t=0$, and the global permutation symmetry P, valid for all t. The eigenstates of (31) are eigenstates of the spatial permutation $\vec{r}_{\bar{\alpha}} \to \vec{r}_{\bar{\beta}}$, and it is not difficult to guess that the ground states are symmetric under the latter permutation. This is illustrated in the Appendix.

Hence, restricting ourselves to the subspace spanned by the two fundamental states, which are symmetric states for the permutation $\vec{r}_{\bar{\alpha}} \to \vec{r}_{\bar{\beta}}$, we have $P_c = P$ and we project the state $|f\rangle$, (5), into the subspace

$$\mathcal{H}_0 = |D^{+,0,0},0\rangle \oplus |D^{-,0,0},0\rangle. \tag{33}$$

We have

$$\langle D^{\pm,0,0}0|f\rangle = \frac{(C_1 \pm C_2)(1 \pm \frac{1}{N_c})}{(2 \pm \frac{2}{N_c})^{\frac{1}{2}}} S^{\pm}, \tag{34}$$

where $S^{\pm}$ is the spatial overlap. As stated above $S^{\pm} = S^0 + O(1/N_c)$ where S^0 is the spatial overlap of $|f\rangle$ with the direct product of two noninteracting mesons located at the origin, i.e., $\psi^{(1)}(\vec{r}_{\bar{\alpha}})\psi^{(1)}(\vec{r}_{\bar{\beta}})$, $\psi^{(1)}(\vec{r})$ being the ground-state eigenfunction of $H^{(1)}$, (32).

The evolution forward in time of the states $|D^{\pm,0,0},t\rangle$ is obtained by replacing in (26) the $-\infty$ lower bounds of the integrals by 0. It results, thanks to time reversal, in phase shifts which are simply divided by 2: $e^{i\delta \pm i\phi}$.

The resulting T matrix for the decay of the initial meson $P_{A\bar{\alpha}}$ into the fundamental mesons is

$$\begin{aligned} T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\beta}} M^{(0)}_{B\bar{\alpha}}) &= e^{i\delta}\left\{ \frac{(C_1+C_2)(1+\frac{1}{N_c})S^+}{2(1+\frac{1}{N_c})^{\frac{1}{2}}}\, e^{i\phi} + \frac{(C_1-C_2)(1-\frac{1}{N_c})S^-}{2(1-\frac{1}{N_c})^{\frac{1}{2}}}\, e^{-i\phi} \right\}, \\ T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\alpha}} M^{(0)}_{B\bar{\beta}}) &= e^{i\delta}\left\{ \frac{(C_1+C_2)(1+\frac{1}{N_c})S^+}{2(1+\frac{1}{N_c})^{\frac{1}{2}}}\, e^{i\phi} - \frac{(C_1-C_2)(1-\frac{1}{N_c})S^-}{2(1-\frac{1}{N_c})^{\frac{1}{2}}}\, e^{-i\phi} \right\}. \end{aligned} \tag{35}$$

To perform a systematic $1/N_c$ expansion, let us first define $\Delta S^{\pm}$ by

$$S^{\pm} \equiv S^0 + \frac{\Delta S^{\pm}}{N_c}. \tag{36}$$

Then, from (35) we obtain to first order in $1/N_c$,

$$\begin{aligned} T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\beta}} M^{(0)}_{B\bar{\alpha}}) &= e^{i\delta}S^0\Big\{ \left(C_1 + \tfrac{C_2}{2N_c} + \tfrac{C_1(\Delta S^+ + \Delta S^-) + C_2(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right)\cos\phi \\ &\quad + i\left(C_2 + \tfrac{C_1}{2N_c} + \tfrac{C_2(\Delta S^+ + \Delta S^-) + C_1(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right)\sin\phi \Big\}, \\ T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\alpha}} M^{(0)}_{B\bar{\beta}}) &= e^{i\delta}S^0\Big\{ \left(C_2 + \tfrac{C_1}{2N_c} + \tfrac{C_2(\Delta S^+ + \Delta S^-) + C_1(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right)\cos\phi \\ &\quad + i\left(C_1 + \tfrac{C_2}{2N_c} + \tfrac{C_1(\Delta S^+ + \Delta S^-) + C_2(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right)\sin\phi \Big\}. \end{aligned} \tag{37}$$

Comparing (37) with Eqs. (12) and (14) we see that if we take, in (37), $\Delta S^{\pm} = \phi = 0$, we recover the factorization corrected in the manner of Shifman, i.e., with Eq. (16). The global phase δ is not relevant here. Consequently we learn that this factor 1/2 in the manner of Shifman is completed by two other effects at the same order in $1/N_c$: (i) the $\Delta S^{\pm}$ which reflect the difference, at the time of the weak decay, between the total $qq\bar{q}\bar{q}$ spatial wave function and the simple product of the two asymptotic meson wave functions; (ii) the phase ϕ which reflects strictly speaking the final state interaction. The latter phase ϕ, although $1/N_c$ suppressed, may become very large when $v \to 0$. For $\phi = \pi/2$ the factorization becomes grossly wrong since the role of the two final states is interchanged: The operator multiplying C_1 in H_W produces dominantly the $M^{(0)}_{C\bar{\alpha}} M^{(0)}_{B\bar{\beta}}$ instead of $M^{(0)}_{C\bar{\beta}} M^{(0)}_{B\bar{\alpha}}$ as suggested by factorization, and vice versa for C_2. In other words, the class II decays become dominant over the class I when $\phi = \pi/2$. Still it should be stressed that the phase ϕ disappears when summing the probabilities for the two channels in (35).

Furthermore, the dynamical origin of the phase ϕ and of the factors $\Delta S^{\pm}$ is obviously strongly dependent on the precise nature of the decay channels considered. This indicates that the effects of ϕ and $\Delta S^{\pm}$ cannot be incorporated in a phenomenological factorization in the manner of BSW, which assumes a given pair of constants a_1 and a_2 for all the decay channels of a meson.

In the Appendix we have performed an explicit calculation of $\Delta S^{\pm}$ and an estimation of ϕ for an harmonic oscillator potential. As can be seen from (A9) and (A12), ϕ is always the dominant $1/N_c$ correction when $\bar{\alpha}$ and $\bar{\beta}$ are light quarks, and when they are heavy, ϕ still dominates as long as $vmR \leq 1$.

Let us now consider a more general case by first introducing spin.

C. Final state interactions of the fundamental pseudoscalar and vector mesons

From heavy quark symmetry (HQS) we know that vector and pseudoscalar mesons are degenerate. HQS also tells us that the heavy quark spin is conserved. This is of course a trivial consequence of (20), but it is quite general. For example, in (20) we might add a term coupling the light quark spins, such as $\vec{\sigma}_{\bar{\beta}} \cdot \vec{\sigma}_{\bar{\alpha}}$, but all terms including the spins s_B and s_C are $1/M$ suppressed.

Restricting ourselves to a 0 total quark spin, the ground states combine into four possible asymptotic states: $P_{C\bar{\beta}}P_{B\bar{\alpha}}$, $V_{C\bar{\beta}}V_{B\bar{\alpha}}$, $P_{B\bar{\beta}}P_{C\bar{\alpha}}$, and $V_{B\bar{\beta}}V_{C\bar{\alpha}}$ (where P stands for pseudoscalar and V for vector).

It is then convenient to use states with a given symmetry for $P_S \equiv s_B \leftrightarrow s_C$. The relevant combinations are

$$P_S \left\{ \frac{-|P_{C\bar{\beta}}P_{B\bar{\alpha}}\rangle + |V_{C\bar{\beta}}V_{B\bar{\alpha}}\rangle}{2} \right\} = - \left\{ \frac{-|P_{C\bar{\beta}}P_{B\bar{\alpha}}\rangle + |V_{C\bar{\beta}}V_{B\bar{\alpha}}\rangle}{2} \right\},$$

$$P_S \left\{ \frac{-3|P_{C\bar{\beta}}P_{B\bar{\alpha}}\rangle - |V_{C\bar{\beta}}V_{B\bar{\alpha}}\rangle}{2} \right\} = \left\{ \frac{-3|P_{C\bar{\beta}}P_{B\bar{\alpha}}\rangle - |V_{C\bar{\beta}}V_{B\bar{\alpha}}\rangle}{2} \right\}, \quad (38)$$

where VV' stands for $V^0V'^0 - V^+V'^- - V^-V'^+$ with $0, +, -$ labeling the polarization of the vector mesons.

In fact, the first combination in (38) corresponds to $S_{BC} = 0$ (total spin of B and C) and the second to $S_{BC} = 1$. Using for large $|T|$ the notation

$$|^1D^{(0,0)}_{C\bar{\beta};B\bar{\alpha}}, T\rangle = \left\{ \frac{-|P_{C\bar{\beta}}(-\vec{v}T)P_{B\bar{\alpha}}(\vec{v}T)\rangle + |V_{C\bar{\beta}}(-\vec{v}T)V_{B\bar{\alpha}}(\vec{v}T)\rangle}{2} \right\},$$

$$|^3D^{(0,0)}_{C\bar{\beta};B\bar{\alpha}}, T\rangle = \left\{ \frac{-3|P_{C\bar{\beta}}(-\vec{v}T)P_{B\bar{\alpha}}(\vec{v}T)\rangle - |V_{C\bar{\beta}}(-\vec{v}T)V_{B\bar{\alpha}}(\vec{v}T)\rangle}{2} \right\}, \quad (39)$$

the strong Hamiltonian is diagonal in the basis where both P_S and $P \equiv \bar{\alpha} \leftrightarrow \bar{\beta}$ are diagonal:

$$\sqrt{2}|^1D^{\pm,0,0}, T\rangle \underset{T\to\pm\infty}{=} |^1D^{(0,0)}_{C\bar{\beta};B\bar{\alpha}}, T\rangle \pm |^1D^{(0,0)}_{C\bar{\alpha};B\bar{\beta}}, T\rangle,$$

$$\sqrt{2}|^3D^{\pm,0,0}, T\rangle \underset{T\to\pm\infty}{=} |^3D^{(0,0)}_{C\bar{\beta};B\bar{\alpha}}, T\rangle \pm |^3D^{(0,0)}_{C\bar{\alpha};B\bar{\beta}}, T\rangle. \quad (40)$$

The four states in (40) evolve diagonally under the strong Hamiltonian and lead to four phase shifts given by formulas similar to Eq. (26).

Next we make the assumption that $|f\rangle$ defined in (5) is odd under P_S:

$$P_S|f\rangle = -|f\rangle. \quad (41)$$

Relation (41) is a consequence of Fierz symmetry whenever H_W is built up of Fierz-invariant currents, as is the case in Eq. (4). The fact that Fierz symmetry translates into a spin antisymmetry as in (41) comes from the fact that the Fierz transformation contains an additional minus sign from fermion field commutation. It results that only the states $|^1D^{\pm,0,0}, 0\rangle$ are produced during weak decay, i.e., $S_{BC} = 0$. The arguments from the beginning of Sec. III B to Eq. (34) may be repeated, except that due to the spin asymmetry, the $|^1D^{+,0,0}, 0\rangle$ ($|^1D^{-,0,0}, 0\rangle$) is color odd (color even),

$$\langle D^{\pm,0,0}0|f\rangle = \frac{(C_1 \mp C_2)(1 \mp \frac{1}{N_c})}{(2 \mp \frac{2}{N_c})^{\frac{1}{2}}} S^{\pm}, \quad (42)$$

leading to

$$T(P_{A\bar{\alpha}} \to {}^1D^{(0,0)}_{C\bar{\beta};B\bar{\alpha}}) = e^{i\delta} \left\{ \frac{(C_1 - C_2)(1 - \frac{1}{N_c})S^+}{2(1 - \frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi} + \frac{(C_1 + C_2)(1 + \frac{1}{N_c})S^-}{2(1 + \frac{1}{N_c})^{\frac{1}{2}}} e^{-i\phi} \right\},$$
$$T(P_{A\bar{\alpha}} \to {}^1D^{(0,0)}_{C\bar{\alpha};B\bar{\beta}}) = e^{i\delta} \left\{ \frac{(C_1 - C_2)(1 - \frac{1}{N_c})S^+}{2(1 - \frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi} - \frac{(C_1 + C_2)(1 + \frac{1}{N_c})S^-}{2(1 + \frac{1}{N_c})^{\frac{1}{2}}} e^{-i\phi} \right\}, \quad (43)$$

and, to first order in $1/N_c$,

$$T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\beta}}M^{(0)}_{B\bar{\alpha}}) = \tfrac{\eta}{2} e^{i\delta} S^0 \Big\{ \left(C_1 + \frac{C_2}{2N_c} + \frac{C_1(\Delta S^+ + \Delta S^-) - C_2(\Delta S^+ - \Delta S^-)}{2N_c S^0} \right) \cos\phi$$
$$- i \left(C_2 + \frac{C_1}{2N_c} + \frac{C_2(\Delta S^+ + \Delta S^-) - C_1(\Delta S^+ - \Delta S^-)}{2N_c S^0} \right) \sin\phi \Big\},$$

$$T(P_{A\bar{\alpha}} \to M^{(0)}_{C\bar{\alpha}} M^{(0)}_{B\bar{\beta}}) = \tfrac{\eta}{2} e^{i\delta} S^0 \Big\{ - \left(C_2 + \tfrac{C_1}{2N_c} + \tfrac{C_2(\Delta S^+ + \Delta S^-) - C_1(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right) \cos\phi + i \left(C_1 + \tfrac{C_2}{2N_c} + \tfrac{C_1(\Delta S^+ + \Delta S^-) - C_2(\Delta S^+ - \Delta S^-)}{2N_c S^0}\right) \sin\phi \Big\}, \quad (44)$$

where $\eta = +1$ for longitudinal vector mesons and $\eta = -1$ for transverse vector and pseudoscalar mesons. The difference between the RHS of (43) and the RHS of (35) comes from the interchange of color symmetric with color antisymmetric combinations as apparent when comparing (34) and (42).

It is to be noted that the relation between PP and VV production amplitude is exactly given by the fact that only the ^1D combination is created. Indeed, this relation is a consequence of HQS and would only be corrected if we considered the $O(1/M)$ corrections. This relation is not a surprise when one realizes that the conditions (1) and (2) imply an S-wave-dominated decay.

D. Semi-inclusive decay

We have found important channel-dependent corrections to factorization. We may still wonder if these corrections are not washed out when we sum up on one side all the decay channels $M^{(n)}_{C\bar{\beta}} M^{(m)}_{B\bar{\alpha}}$ for all m, n, and on the other side all the channels $M^{(n)}_{B\bar{\beta}} M^{(m)}_{C\bar{\alpha}}$. This is the aim of this section.

Let us call $\mathcal{H}^+(t)$ $[\mathcal{H}^-(t)]$ the Hilbert space spanned by the set of states $|D^{+,n,m}, t\rangle, \forall n, m$ $[|D^{-,n,m}, t\rangle, \forall n, m]$ defined in Eq. (21). $\mathcal{H}^+(t)$ $[\mathcal{H}^-(t)]$ contains the even [odd] states under the permutation $P \equiv \bar{\alpha} \leftrightarrow \bar{\beta}$. The latter commuting with the Hamiltonian $H(t)$, the evolution does not mix the spaces $\mathcal{H}^+(t)$ and $\mathcal{H}^-(t)$. We shall call $U^+(t_1, t_2)$ $[U^-(t_1, t_2)]$ the evolution operator in $\mathcal{H}^+(t)$ $[\mathcal{H}^-(t)]$. $U^{\pm}(t_1, t_2)$ are unitary.

As already stated in Sec. III B, $H(0)$ is also invariant under the color permutation $P_c : C_C \leftrightarrow C_B$. For $t = 0$ the permutation $P = P_c P_r$ where $P_r \equiv p_{\bar{\alpha}}, s_{\bar{\alpha}} \leftrightarrow p_{\bar{\beta}}, s_{\bar{\beta}}$ (remember $\vec{r}_B = \vec{r}_C = 0$ for $t = 0$). We then decompose $|f\rangle$ both into eigenstates of P:

$$|f^+\rangle = \frac{(C_1 + C_2)(\delta_{c_C,c_{\bar\beta}}\delta_{c_B,c_{\bar\alpha}} + \delta_{c_C,c_{\bar\alpha}}\delta_{c_B,c_{\bar\beta}})}{2N_c}|f^S_+\rangle + \frac{(C_1 - C_2)(\delta_{c_C,c_{\bar\beta}}\delta_{c_B,c_{\bar\alpha}} - \delta_{c_C,c_{\bar\alpha}}\delta_{c_B,c_{\bar\beta}})}{2N_c}|f^S_-\rangle,$$

$$|f^-\rangle = \frac{(C_1 - C_2)(\delta_{c_C,c_{\bar\beta}}\delta_{c_B,c_{\bar\alpha}} - \delta_{c_C,c_{\bar\alpha}}\delta_{c_B,c_{\bar\beta}})}{2N_c}|f^S_+\rangle + \frac{(C_1 + C_2)(\delta_{c_C,c_{\bar\beta}}\delta_{c_B,c_{\bar\alpha}} + \delta_{c_C,c_{\bar\alpha}}\delta_{c_B,c_{\bar\beta}})}{2N_c}|f^S_-\rangle, \quad (45)$$

where $|f^{\pm}\rangle$ are eigenstates of P with eigenvalue $\pm$ and $|f_S\rangle$ contains the spin-space part of the wave function (5):

$$|f^S\rangle = \sum_{\substack{s_C, s_{\bar\beta}, c_C, c_{\bar\beta}, \\ s_B, s_{\bar\alpha}, c_B, c_{\bar\alpha}}} \int d\vec{p}_C d\vec{p}_{\bar\beta} d\vec{p}_B d\vec{p}_{\bar\alpha} \Psi(\vec{p}_C, s_C, \vec{p}_{\bar\beta}, s_{\bar\beta}, \vec{p}_B, s_B, \vec{p}_{\bar\alpha}, s_{\bar\alpha})$$

$$\times |C, \vec{p}_C, s_C, c_C; \bar\beta, \vec{p}_{\bar\beta}, s_{\bar\beta}, c_{\bar\beta}; B, \vec{p}_B, s_B, c_B; \bar\alpha, \vec{p}_{\bar\alpha}, s_{\bar\alpha}, c_{\bar\alpha}\rangle, \quad (46)$$

which is expanded into the eigenvectors of P_r, $|f^S_{\pm}\rangle$ corresponding to eigenvalues $P_r|f^S_{\pm}\rangle = \pm|f^S_{\pm}\rangle$.

The norm of $|f^{\pm}\rangle$ is

$$\langle f^{\pm}|f^{\pm}\rangle = \frac{1}{2}\left[(C_1 \pm C_2)^2 \left(1 \pm \frac{1}{N_c}\right) \langle f^S_+|f^S_+\rangle + (C_1 \mp C_2)^2 \left(1 \mp \frac{1}{N_c}\right) \langle f^S_-|f^S_-\rangle\right]. \quad (47)$$

If we define $\langle f^S_{\pm}|f^S_{\pm}\rangle \equiv K_{\pm}$, then $K_+ + K_- = K$ as defined in Eq. (7).

As stated above,

$$|f^{\pm}\rangle \in \mathcal{H}^{\pm}. \quad (48)$$

The evolution toward a large positive time T leads to

$$|f^{\pm}(T)\rangle = U^{\pm}(T, 0)|f^{\pm}\rangle. \quad (49)$$

By unitarity the norm of $|f^{\pm}(T)\rangle$ equals that of $|f^{\pm}\rangle$.

We get

$$\sqrt{2}T(P_{A\bar\alpha} \to D^{\pm,m,n}) = (C_1 \pm C_2)(1 \pm \frac{1}{N_c})^{\frac{1}{2}} K_+^{\frac{1}{2}} S_+^{\pm,m,n} e^{i\phi_+^{\pm,m,n}} + (C_1 \mp C_2)(1 \mp \frac{1}{N_c})^{\frac{1}{2}} K_-^{\frac{1}{2}} S_-^{\pm,m,n} e^{i\phi_-^{\pm,m,n}}, \quad (50)$$

where $S_{\pm}^{\pm,m,n}$ and $\phi_{\pm}^{\pm,m,n}$ are real numbers defined by

$$S_{\pm}^{\pm,m,n} e^{i\phi_{\pm}^{\pm,m,n}} = \frac{1}{[\langle f^{\pm}(T)|f^{\pm}(T)\rangle]^{\frac{1}{2}}} \langle D^{\pm,m,n}, T|f_{\pm}^{\pm}(T)\rangle, \tag{51}$$

with

$$f_{\pm}^{\pm}(T) = U^{\pm}(T,0)|f_{\pm}^{\pm}(0)\rangle, \tag{52}$$

$f_{+}^{\pm}$ ($f_{-}^{\pm}$) being the first (second) terms in the right-hand sides of (45).

From unitarity,

$$\sum_{m,n}\left[\left(S_{+}^{\pm,m,n}\right)^2 + \left(S_{-}^{\pm,m,n}\right)^2\right] = 1, \tag{53}$$

which leads to

$$\sum_{\pm,m,n} \left|T(P_{A\bar{\alpha}} \to D^{\pm,n,m})\right|^2 = \left(C_1^2 + C_2^2 + 2\frac{C_1C_2}{N_c}\right), \tag{54}$$

as expected from (6).

Finally,

$$\begin{aligned} T(P_{A\bar{\alpha}} \to M_{C\bar{\beta}}^{(m)} M_{B\bar{\alpha}}^{(n)}) = {} & K_{+}^{\frac{1}{2}} \left\{ \frac{(C_1+C_2)(1+\frac{1}{N_c})S_{+}^{+,m,n}}{2(1+\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{+}^{+,m,n}} + \frac{(C_1-C_2)(1-\frac{1}{N_c})S_{+}^{-,m,n}}{2(1-\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{+}^{-,m,n}} \right\} \\ & + K_{-}^{\frac{1}{2}} \left\{ \frac{(C_1-C_2)(1-\frac{1}{N_c})S_{-}^{+,m,n}}{2(1-\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{-}^{+,m,n}} + \frac{(C_1+C_2)(1+\frac{1}{N_c})S_{-}^{-,m,n}}{2(1+\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{-}^{-,m,n}} \right\}, \\ T(P_{A\bar{\alpha}} \to M_{C\bar{\alpha}}^{(m)} M_{B\bar{\beta}}^{(n)}) = {} & K_{+}^{\frac{1}{2}} \left\{ \frac{(C_1+C_2)(1+\frac{1}{N_c})S_{+}^{+,m,n}}{2(1+\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{+}^{+,m,n}} - \frac{(C_1-C_2)(1-\frac{1}{N_c})S_{+}^{-,m,n}}{2(1-\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{+}^{-,m,n}} \right\} \\ & + K_{-}^{\frac{1}{2}} \left\{ \frac{(C_1-C_2)(1-\frac{1}{N_c})S_{-}^{+,m,n}}{2(1-\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{-}^{+,m,n}} - \frac{(C_1+C_2)(1+\frac{1}{N_c})S_{-}^{-,m,n}}{2(1+\frac{1}{N_c})^{\frac{1}{2}}} e^{i\phi_{-}^{-,m,n}} \right\}. \end{aligned} \tag{55}$$

Duality is fully verified since, summing over all states,

$$\sum_{m,n}\left[\left|T(P_{A\bar{\alpha}} \to M_{C\bar{\beta}}^{(m)} M_{B\bar{\alpha}}^{(n)})\right|^2 + \left|T(P_{A\bar{\alpha}} \to M_{C\bar{\alpha}}^{(m)} M_{B\bar{\beta}}^{(n)})\right|^2\right] = \left(C_1^2 + C_2^2 + 2\frac{C_1C_2}{N_c}\right) K. \tag{56}$$

Let us now consider the partially inclusive sums

$$\begin{aligned} \sum_{m,n}\left|T(P_{A\bar{\alpha}} \to M_{C\bar{\beta}}^{(m)} M_{B\bar{\alpha}}^{(n)})\right|^2 &\equiv \Sigma_{C\bar{\beta};B\bar{\alpha}}, \\ \sum_{m,n}\left|T(P_{A\bar{\alpha}} \to M_{C\bar{\alpha}}^{(m)} M_{B\bar{\beta}}^{(n)})\right|^2 &\equiv \Sigma_{C\bar{\alpha};B\bar{\beta}}. \end{aligned} \tag{57}$$

Nothing general can be said. In the $N_c \to \infty$ limit,

$$S_{\pm}^{+,m,n} \simeq S_{\pm}^{-,m,n} \qquad \phi_{\pm}^{+,m,n} \simeq \phi_{\pm}^{-,m,n}. \tag{58}$$

Assuming that for *finite* N_c we keep the relations (58), we recover Shifman's ansatz:

$$\begin{aligned} \Sigma_{C\bar{\beta};B\bar{\alpha}} &= \tfrac{K}{4}\left\{(C_1-C_2)(1-\tfrac{1}{N_c})^{\frac{1}{2}} + (C_1+C_2)(1+\tfrac{1}{N_c})^{\frac{1}{2}}\right\}^2 = K\left\{C_1^2 + \tfrac{C_1C_2}{N_c} + O(\tfrac{1}{N_c^2})\right\}, \\ \Sigma_{C\bar{\alpha};B\bar{\beta}} &= \tfrac{K}{4}\left\{(C_1-C_2)(1-\tfrac{1}{N_c})^{\frac{1}{2}} - (C_1+C_2)(1+\tfrac{1}{N_c})^{\frac{1}{2}}\right\}^2 = K\left\{C_2^2 + \tfrac{C_1C_2}{N_c} + O(\tfrac{1}{N_c^2})\right\}. \end{aligned} \tag{59}$$

If we made the opposite assumption that the relative phases $\phi_{\pm}^{+,m,n} - \phi_{\pm}^{-,m,n}$ are random, which might be reasonable at small velocity, the result would be

$$\Sigma_{C\bar{\beta};B\bar{\alpha}} = \Sigma_{C\bar{\alpha};B\bar{\beta}} = K\left\{\frac{C_1^2+C_2^2}{2} + \frac{C_1C_2}{N_c}\right\}, \tag{60}$$

that is, an equal sharing of the total probability between the two sets of channels. In the latter case, even though we consider an inclusive sum, the final state interaction has a nontrivial effect: The quarks have been redistributed at random between the final mesons. Of course such a random phase equal sharing may only happen when phase space allows for many final states to add up in a random way.

As a side remark we would like to mention another, not yet published, study that we have performed on duality versus factorization. We have considered a model with nonrelativistic scalar quarks bound to color singlets by a color harmonic oscillator potential [10] without assuming heavy quarks or using an adiabatic approximation as done in the present paper. This model does not assume $M_C = M_B$ or $m_{\bar{\alpha}} = m_{\bar{\beta}}$. This model also automatically restores duality, i.e., the conservation of probability. Summing over all mesons in the limit in which the radius $R \to \infty$ one should find the free quark result. This is indeed the case:

$$\Sigma_{C\bar{\beta};B\bar{\alpha}} \propto \left[C_1 + y\frac{C_2}{N_c}\right]^2 + O\left(\frac{1}{N_c^2}\right),$$

$$\Sigma_{C\bar{\alpha};B\bar{\beta}} \propto \left[C_2 + (1-y)\frac{C_1}{N_c}\right]^2 + O\left(\frac{1}{N_c^2}\right), \qquad (61)$$

with y depending nontrivially on the masses. In the limit $M_B = M_C$ studied by Shifman one recovers his value $y = 1/2$ as can be immediately seen from a symmetry argument when $C_1 = C_2$. It is seen once more from (61) that the FSI restores automatically duality.

IV. CONCLUSIONS

We have used a quark model where the motion of heavy quarks is treated as classical and where we assume two mass-degenerate antiquarks. We have used the resulting permutation symmetry to simplify the problem. We have restricted ourselves to the $qq\bar{q}\bar{q}$ sector, and we have shown that the contradiction between standard factorization and duality stems from the nonorthogonality in color space of the two decomposition of the $qq\bar{q}\bar{q}$ singlet into two pairs of $q\bar{q}$ color singlets. Taking care to use an orthogonal basis that diagonalizes the Hamiltonian, the dynamics of this sector shows very clearly how the final state interaction corrects standard factorization such as to satisfy duality.

Shifman [8] has proposed to correct factorization by a replacement of $1/N_c$ by $1/(2N_c)$ while keeping the phenomenological factorization in the manner of BSW. We have shown that this effect is indeed present. However, we find two additional effects to same order in $1/N_c$. One is related to the spatial distortion of the meson wave functions at the time of the weak decay: The two mesons overlap in space and hence interact strongly. This has been expressed by our parameter $\Delta S^{\pm}$. The second and more important additional effect is the phase difference ϕ between the permutation-even and permutation-odd states when evolving after the decay until the mesons are spatially distant. The latter effect is $O(1/N_c)$ but also $O(1/v)$ where v is the final meson velocity in the total rest frame, and also $O(1/mR)$ where m is the light antiquarks constituent mass and R is the wave function radius. This phase shift effect should dominate in the small velocity regime for light antiquarks. Notice that the velocity is indeed small in the kinematics assumed in (1): $\Delta \ll M$. For large velocity (relaxing the condition $\Delta \ll M$) and for heavy antiquarks, the phase shift effect vanishes.

We have seen that in the exclusive case, restricting ourselves to the ground state mesons, the phase shift ϕ is the dominant $1/N_c$ correction for small velocity and light antiquarks, and it may produce a total modification of the factorization assumption, which could, for $\phi = \pi/2$, be large enough to totally interchange the amplitudes of the two channels, and lead to a dominant class II decay. In an illustrative example treated in the Appendix the $t = 0$ wave function distortion $\Delta S^{\pm}$ turns out to be small. We ignore it if this is a general feature. If it were so, it would indicate the validity of Shifman's ansatz for large velocity and rather heavy antiquarks. Furthermore, since we have restricted our work to the final state interaction generated by quark exchange, the result does not depend anymore on ϕ when we add up the decay probabilities into channels related by quark exchange, as is obvious from Eqs. (35) and (43). On such sums, factorization with Shifman's ansatz would be a fair approximation as long as the terms $\Delta S^{\pm}$ and nonquark-exchange final state interactions are small.

In the semi-inclusive case we compare the total decay probability into two sets of channels that correspond to the two possible pairing of $qq\bar{q}\bar{q}$ into two $q\bar{q}$. Again, the small phase shift, small distortion limit amounts to Shifman's ansatz, while the opposite, random phase shift limit amounts to an equal sharing of inclusive decay probability into the two sets of final channels. Again the latter situation may be reasonable in the small velocity case provided many final states are kinematically allowed.

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APPENDIX: AN ILLUSTRATIVE EXAMPLE

As an illustration of Sec. III B, let us take, for the potential V in (20) an harmonic oscillator potential,

$$V(\vec{r}) = -\frac{N_c}{4(N_c^2-1)mR^4}\vec{r}^{\,2}, \qquad (A1)$$

so that the ground-state solution of (32) is

$$\psi^{(1)}(\vec{r}) = \frac{1}{R^{\frac{3}{2}}\pi^{\frac{3}{4}}}e^{-\frac{r^2}{2R^2}}, \qquad (A2)$$

where $r = |\vec{r}|$.

For the spatial part of the wave function $|f\rangle$ we take, in configuration space,

$$\Psi(\vec{r}_C, \vec{r}_{\bar\beta}, \vec{r}_B, \vec{r}_{\bar\alpha}) = G\psi^{(1)}(\vec{r}_{\bar\alpha})\delta_3(\vec{r}_C)\delta_3(\vec{r}_B)\delta_3(\vec{r}_{\bar\beta}), \quad \text{(A3)}$$

which expresses the fact that the weak operator is local[1] and that the quark $\bar\alpha$ is a spectator coming from the $P_{A\bar\alpha}$ meson, i.e., in the ground-state wave function. G is proportional to the Fermi constant.

To compute the spatial overlaps $S^{\pm}$ in (34), we need to know the ground-state solutions of the Hamiltonian (31) with (A1) for V. Let us change variables in (31):

$$\vec{r}_{\bar\beta} = \vec{R} - \tfrac{1}{2}\vec{r}, \qquad \vec{r}_{\bar\alpha} = \vec{R} + \tfrac{1}{2}\vec{r},$$
$$\vec{p}_{\bar\beta} = \tfrac{1}{2}\vec{P} - \vec{p}, \qquad p_{\bar\alpha} = \tfrac{1}{2}\vec{P} + \vec{p}, \quad \text{(A4)}$$

leading to

$$H(0) = \frac{P^2}{m} + \frac{p^2}{4m} + \frac{(N_c \pm 2)(N_c \mp 1)}{4(N_c^2-1)mR^4}\vec{R}^{\,2} + \frac{(N_c \pm 4)(N_c \mp 1)}{(N_c^2-1)mR^4}\vec{r}^{\,2}. \quad \text{(A5)}$$

The ground-state solution, i.e., the spatial wave functions of $|D^{\pm,0,0}, 0\rangle$ is

$$\psi^{\pm,0,0}(r_{\bar\alpha}, r_{\bar\beta}) = \frac{(N_c \pm 2)^{\frac{3}{8}}(N_c \pm 4)^{\frac{3}{8}}}{(N_c \pm 1)^{\frac{3}{4}}R^3\pi^{\frac{3}{2}}} \exp\left(-\frac{(N_c \pm 2)^{\frac{1}{2}}(\vec{r}_{\bar\alpha} + \vec{r}_{\bar\beta})^2}{4(N_c \pm 1)^{\frac{1}{2}}R^2}\right) \exp\left(-\frac{(N_c \pm 4)^{\frac{1}{2}}(\vec{r}_{\bar\alpha} - \vec{r}_{\bar\beta})^2}{4(N_c \pm 1)^{\frac{1}{2}}R^2}\right). \quad \text{(A6)}$$

The overlap is given by

$$\begin{aligned} S^{\pm} &= \int d^3r_{\bar\alpha} d^3r_{\bar\beta} G\psi^{(1)\dagger}(\vec{r}_{\bar\alpha})\delta_3(\vec{r}_{\bar\beta})\psi^{\pm,0,0}(r_{\bar\alpha}, r_{\bar\beta}) \\ &= G\frac{4^{\frac{3}{2}}(N_c\pm2)^{\frac{3}{8}}(N_c\pm4)^{\frac{3}{8}}}{\left[(N_c\pm2)^{\frac{1}{2}}+(N_c\pm4)^{\frac{1}{2}}+2(N_c\pm1)^{\frac{1}{2}}\right]^{\frac{3}{2}}R^{\frac{3}{2}}\pi^{\frac{3}{4}}}, \end{aligned} \quad \text{(A7)}$$

where we have in (A3) left aside the δ functions related to the heavy quarks, since the latter are treated classically. One has also, for the factorization hypothesis result,

$$S^0 = G\frac{1}{R^{\frac{3}{2}}\pi^{\frac{3}{4}}}, \quad \text{(A8)}$$

which is the $\psi^{(1)}(0)$ coming from the overlap of $\delta(\vec{r}_{\bar\beta})$ with $\psi^1(\vec{r}_{\bar\beta})$. The overlap of $\psi^{(1)}(\vec{r}_{\bar\alpha})$ with itself gives obviously 1. S^0 in (A8) is obviously the $N_c \to \infty$ limit of $S^{\pm}$ in (A7).

To next to leading order in $1/N_c$ the calculation of $\Delta S^{\pm}$ defined in (36) is now straightforward from (A7) and (A8):

$$\frac{\Delta S^{\pm}}{N_c} = \pm\frac{3}{8N_c}, \quad \text{(A9)}$$

which turns out to be rather small, mainly because the normalization factor compensates for a large part the modification of the integral.

From (A5) we also get the ground-state energy

$$E^{\pm}(0) = \frac{3}{2mR^2}\left\{\left(\frac{N_c \pm 2}{N_c \pm 1}\right)^{\frac{1}{2}} + \left(\frac{N_c \pm 4}{N_c \pm 1}\right)^{\frac{1}{2}}\right\}, \quad \text{(A10)}$$

leading to

$$E^+(0) - E^-(0) = \frac{6}{mR^2N_c} + O\left(\frac{1}{N_c^2}\right). \quad \text{(A11)}$$

To compute ϕ we need, (26), to know $E^+(z) - E^-(z)$ for all $z \neq 0$. This is not so easy to compute. We will simply use (A11) to make an order of magnitude estimate. We will assume $E^+(z) - E^-(z)$ to be equal by $E^+(0) - E^-(0)$ as long as the hadrons overlap, i.e., for $|z| \leq cR$ where c is some number of order[2] 1. Then

$$\phi \sim \frac{3c}{mvRN_c}. \quad \text{(A12)}$$

Although purely indicative, this result, besides confirming that $\phi \propto 1/vN_c$ also teaches us that mR is the dimensionless number that gives the scale. When $\bar\alpha$ and $\bar\beta$ are light antiquarks, it is known that $mR \sim 1$. Comparing (A12) with (A9) and to the $1/(2N_c)$ correction in the manner of Shifman, we check that in the exclusive case the phase shift ϕ is the dominant $1/N_c$ contribution.

[1] Remember that we assume here spinless quarks.

[2] The wave function radius for the wave function (A2) is $\langle\vec{r}^{\,2}\rangle = 3/2R^2$. Hence one might think of taking $c \simeq 3/2$.

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