Isgur-Wise function in a relativistic model for a $q\bar{Q}$ system

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We use the Dirac equation with a "(asymptotically free) Coulomb + (Lorentz scalar) linear" potential to estimate the light quark wave function for $q\bar{Q}$ mesons in the limit $m_Q \to \infty$. We use these wave functions to calculate the Isgur-Wise function $\xi(v \cdot v')$ for orbital and radial ground states in the phenomenologically interesting range $1 \leq v \cdot v' \leq 4$. We find a simple expression for the zero-recoil slope, $\xi'(1) = -1/2 - \epsilon^2 \langle {r_q}^2 \rangle /3$, where ϵ is the energy eigenvalue of the light quark, which can be identified with the $\bar{\Lambda}$ parameter of the heavy quark effective theory. This result implies an upper bound of $-1/2$ for the slope $\xi'(1)$. Also, because for a very light quark q ($q = u, d$) the size $\sqrt{\langle {r_q}^2 \rangle}$ of the meson is determined mainly by the "confining" term in the potential ($\gamma_0 \sigma r$), the shape of $\xi_{u,d}(v \cdot v')$ is seen to be mostly sensitive to the dimensionless ratio $\bar{\Lambda}^2_{u,d}/\sigma$. We present results for the ranges of parameters 150 MeV$< \bar{\Lambda}_{u,d} <$ 600 MeV ($\bar{\Lambda}_s \approx \bar{\Lambda}_{u,d} + 100$ MeV), 0.14 GeV$^2 \leq \sigma \leq 0.25$ GeV^2 and light quark masses $m_u, m_d \approx 0$, $m_s = 175$ MeV and compare to existing experimental data and other theoretical estimates. Fits to the data give $\bar{\Lambda}^2_{u,d}/\sigma = 4.8 \pm 1.7$, $-\xi'_{u,d}(1) = 2.4 \pm 0.7$, and $|V_{cb}|\sqrt{\tau_B/1.48 \text{ ps}} = 0.050 \pm 0.008$ [ARGUS 1993]; $\bar{\Lambda}^2_{u,d}/\sigma = 3.3 \pm 1.2$, $-\xi'_{u,d}(1) = 1.8 \pm 0.5$, and $|V_{cb}|\sqrt{\tau_B/1.48 \text{ ps}} = 0.043 \pm 0.005$ [CLEO 1993]; $\bar{\Lambda}^2_{u,d}/\sigma = 2.0 \pm 0.7$, $-\xi'_{u,d}(1) = 1.3 \pm 0.3$, and $|V_{cb}|F(1) = 0.037 \pm 0.002$ [CLEO 1994] [existing theoretical estimates for $F(1)$ fall in the range $0.86 < F(1) < 1.01$]. Our model seems to favor the CLEO 1994 data set in two respects: the fits are better and the resulting ranges for the model parameters ($\bar{\Lambda}_{u,d}$, σ) are more in line with independent theoretical estimates.

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I. INTRODUCTION AND SUMMARY

During recent years, a lot of effort [1] has gone into the description of systems and processes involving at least one heavy quark ($m_Q \gg \Lambda_{\rm QCD}$) via a systematic expansion in the small parameters ($\Lambda_{\rm QCD}/m_Q$) and $\alpha_s(m_Q^2)$, taking advantage of "heavy quark symmetries." The usefulness of this approach is in that in the heavy quark limit $m_Q \to \infty$, all the physics can be expressed in terms of a small number of form factors, which depend on the light quark and gluon dynamics only. These "universal" functions can be used as a means for comparison among different theoretical models (such as nonrelativistic and relativistic potential models, lattice QCD calculations, etc.). Comparisons with experiment or with methods such as QCD sum rules require in general introducing model-dependent $O(\frac{\Lambda_{\rm QCD}}{m_Q})$ corrections [except at $v \cdot v' = 1$, where corrections start at order $(\Lambda^2_{\rm QCD}/m_Q^2)$ [2]], as well as calculable perturbative QCD corrections.

In this paper we limit ourselves to the leading order in the heavy quark expansion. We calculate the Isgur-Wise function [3] $\xi(v \cdot v')$ for radial and orbital ground state mesons using a relativistic model for the light quark. This function in our case (radial and orbital ground state), for a given flavor of light quark, fully describes the transition $M_q \to M'_q$ in which a local operator transforms the heavy quark Q in the initial meson M_q (of four-velocity v, $J = 0$ or 1) into the heavy quark Q' in the meson M'_q (of four-velocity v', $J = 0$ or 1).

We assume that the light quark wave function obeys a Dirac equation with a spherically symmetric potential in the reference frame in which the heavy quark is stationary at the origin. We also assume that the potential has the form $V(|\vec{x}|) = V_c(|\vec{x}|) + c_0 + \gamma^0 \sigma |\vec{x}|$, where $V_c(|\vec{x}|)$ is an asymptotically free Coulomb term, c_0 is a constant, and the last term is a Lorentz scalar confining term. Thus, the spatial wave function for the light quark q, of mass m_q, is assumed to obey the time-independent Dirac equation

$$\left[\vec{\alpha} \cdot (-i\vec{\nabla}_{\vec{x}}) + V_c(|\vec{x}|) + c_0 + \gamma^0(\sigma|\vec{x}| + m_q)\right] \Psi(\vec{x}) = \epsilon \Psi(\vec{x}) \,. \quad (1)$$

Because we are investigating the meson system in the heavy quark limit ($\Lambda_{\rm QCD}/m_Q \to 0$), the energy eigenvalue ϵ of the Dirac equation can be identified with the "inertia" parameter $\bar{\Lambda}_q$ often introduced in heavy quark effective theory (HQET) [1]:

$$\epsilon \approx \bar{\Lambda}_q \equiv \lim_{M_Q \to \infty} (M_{(q\bar{Q})_{\rm meson}} - M_Q). \quad (2)$$

In our investigation, we allow the $\bar{\Lambda}$ parameters to vary over the range $0.15 \leq \bar{\Lambda}_{u,d} \leq 0.6$ GeV ($\bar{\Lambda}_s \approx \bar{\Lambda}_{u,d} + 100$ MeV), obtained from recent lattice gauge theory studies [4] and other theoretical estimates [5, 6]. As will be seen below, the parameter $\bar{\Lambda}$ plays an important role in the calculation of the Isgur-Wise function. However, since the additive constant c_0 in the potential is indeterminate, only the difference $\bar{\Lambda} - c_0 \approx \epsilon - c_0$ can be extracted from Eq. (1) so that $\bar{\Lambda}$ is effectively an independent input parameter in our model. The asymptotically free

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Coulomb term $V_c(r)$ is parametrized in terms of the QCD scale $\Lambda_{\overline{\rm MS}}$ and a saturation value for the strong coupling $\alpha_s^\infty \equiv \alpha_s(r=\infty)$, where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme. (We use $\Lambda_{\overline{\rm MS}} = 240$ MeV and $\alpha_s^\infty \approx 1$.) Phenomenological and theoretical considerations motivate our use of three different values for the string tension: $\sigma = 0.25$ GeV^2, 0.18 GeV^2, and 0.14 GeV^2. We set $m_u, m_d = 0$ and $m_s = 0.175$ GeV. Further details about the parametrization of the potential are given in Sec. III. At the end of that section we present arguments that could be used to narrow the ranges of the model parameters $(\bar\Lambda_{u,d}\ ,\ \sigma)$ to $\bar\Lambda_{u,d} = 0.5 \pm 0.1$ GeV and $\sigma \approx 0.14$ GeV^2.

We find that the shape of the Isgur-Wise function is mostly sensitive to the parameters $\bar\Lambda$ and σ. In fact, to a very good approximation $\xi_{u,d}(v\cdot v')$ depends on the dimensionless ratio $\frac{\bar\Lambda^2}{\sigma}$ only. Within our approximation, we also find a strict upper bound on the slope at zero recoil: $\xi'(1) < -1/2$. A recent paper [7] that uses the MIT bag model formalism obtains the same bound, which is stronger than the well-known Bjorken bound $\xi'(1) < -1/4$ [8]. For large values of $v\cdot v'$ $(2 < v\cdot v' < 4)$ the shape of $\xi(v\cdot v')$ is relatively insensitive to the input parameters $\bar\Lambda$, σ, and $\Lambda_{\overline{\rm MS}}$.

We present explicit results for $\xi(v\cdot v')$ in the phenomenologically interesting region $1 \le v\cdot v' \le 4$, using the above-mentioned range of parameters. We compare our results with recent experimental data [9–11] of semileptonic B decays as well as with other theoretical estimates. From fits to the ARGUS 1993 data [9] we obtain the ranges $\frac{\bar\Lambda^2_{u,d}}{\sigma} = 4.8 \pm 1.7$ [corresponding to $-\xi'_{u,d}(1) = 2.4 \pm 0.7$] and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\ \rm ps}} = 0.050 \pm 0.008$. Although the fits are of good quality, for reasonable values of the string tension σ they favor values for the inertia parameter $\bar\Lambda_{u,d}$ that are significantly above the range 0.15 GeV$< \bar\Lambda_{u,d} <$ 0.6 GeV advocated by most theoretical estimates [4–6]. The corresponding range for the slope $-\xi_{u,d}(1)$ is also above most independent theoretical estimates (see discussion in Sec. IV). The best fit to the CLEO 1993 data [10] was of significantly poorer quality. Here we found the ranges $\frac{\bar\Lambda^2_{u,d}}{\sigma} = 3.3 \pm 1.2$ [corresponding to $-\xi'_{u,d}(1) = 1.8 \pm 0.5$] and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\ \rm ps}} = 0.043 \pm 0.005$. The ranges for $\frac{\bar\Lambda^2_{u,d}}{\sigma}$ have in this case some overlap with previous theoretical estimates, but are still somewhat on the high side of these estimates. We noticed that if the data point from the CLEO 1993 set [10] corresponding to the largest value of $v\cdot v'$ is ignored, the quality of the fit greatly improves. The parameter ranges obtained in this case are $\frac{\bar\Lambda^2_{u,d}}{\sigma} = 2.0 \pm 1.4$ [corresponding to $-\xi'_{u,d}(1) = 1.3 \pm 0.6$] and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\ \rm ps}} = 0.038 \pm 0.005$. These ranges for $\frac{\bar\Lambda^2_{u,d}}{\sigma}$ and $-\xi'_{u,d}(1)$ overlap with many previous theoretical estimates, but are somewhat too wide to provide useful new information that could distinguish between these.

The recently released CLEO 1994 data analysis [11] has smaller error bars than the data mentioned above [9, 10]. Also, our model produces better quality fits to this data set than to the previous ones. These two factors contribute to help narrow the parameter ranges. The following ranges are favored: $\frac{\bar\Lambda^2_{u,d}}{\sigma} = 2.0 \pm 0.7$ [corresponding to $-\xi'_{u,d}(1) = 1.3 \pm 0.3$] and $|V_{cb}|F(1) = 0.037 \pm 0.002$. [Different theoretical estimates for the constant $F(1)$ can be found in Eq. (41), Sec. IV.] Note that the central values of these ranges are quite close to the ones found from the fits to the CLEO 1993 data set after removing the CLEO 1993 data point corresponding to the highest $v\cdot v'$. The uncertainties, however, are here reduced by a factor of about 2, which should help us to narrow down the parameter space of the underlying physics.

II. GENERAL FORMALISM

We assume that the wave function Ψ of a $q\bar Q$ meson of mass $M_{q\bar Q}$ in the limit $m_Q \to \infty$ [i.e., ignoring $O(\frac{\Lambda_{\rm QCD}}{m_Q})$ effects] can be expressed in terms of a direct product of the (free spinor) wave function of the heavy antiquark, χ, and the wave function of the light quark, ψ, in the "relative" coordinates. It is therefore convenient to define the four-vectors

$$X^\mu \equiv (t_Q, \vec r_Q)\ ,\quad x^\mu \equiv (t_q - t_Q, \vec r_q - \vec r_Q). \tag{3}$$

Of course, the kinematics of the two wave functions χ and ψ are not completely independent. They are connected by the implicit constraint that in the rest frame of the heavy quark (which is also the rest frame of the $q\bar Q$ meson) the spatial part of the light quark wave function, $\psi(\vec x)$, obeys the time-independent Dirac equation (1). In order to be able to give the meson wave function Ψ a physical meaning in a given reference frame, we have to set the "relative time" of the two constituents to zero:

$$x^0 = 0\ ,\quad t_q = t_Q \equiv t.$$

In the rest frame of the meson ($\vec P = 0$), its wave function can then be written as

$$\Psi^{\lambda,\eta}_{\vec 0}(X,x) = \sqrt{\frac{2M_{q\bar Q}}{(2\pi)^3}}\left[\chi^\lambda_{\vec 0} e^{-im_Q t_Q}\right]\otimes\left[\psi^\eta_{\vec 0}(\vec x)e^{-i\epsilon t_q}\right]\ , \tag{4}$$

where λ, η are spin indices ($\lambda,\eta = \uparrow$ or $\downarrow$ with respect to some axis), $\chi^\lambda_{\vec 0}$ is a free Dirac spinor at rest corresponding to the heavy antiquark, and $\psi^\eta_{\vec 0}(\vec x)$ obeys Eq. (1) with energy eigenvalue ϵ. The factor in front of the direct product is for normalization purposes [see Eq. (10) below]. Using our convention $t_Q = t_q = t$ we see that the rest mass $M_{q\bar Q}$ is given by

$$M_{q\bar Q} = m_Q + \epsilon, \tag{5}$$

thus validating our identification of ϵ with the parameter $\bar\Lambda_q$ commonly introduced in HQET.

We now turn to the description of the meson $q\bar Q$ as seen from a reference frame L, with respect to which the meson (rest frame L') is moving with a four-velocity $v \equiv (\gamma, \gamma\vec\beta)$. In the unprimed frame, the four-momentum of the meson is given by $P = vM$. We assume that the wave functions of the light quark and heavy antiquark transform in the standard way under a Lorentz transformation. Although this would be strictly correct only if the Dirac equation for the light quark [Eq. (1)] were in fact covariant under Lorentz transformations, we assume (and verify in some special cases later) that this procedure will not introduce important errors in our results. We can then express our wave functions in the unprimed frame in terms of those in the rest frame of the meson, L':

$$\begin{aligned} \psi_v(t_q,\vec{r}_q) &= S_v\psi_{\vec{0}}(t'_q,\vec{r}'_q), \\ \chi_v(t_Q,\vec{r}_Q) &= S_v\chi_{\vec{0}}(t'_Q,\vec{r}'_Q), \end{aligned} \tag{6}$$

where

$$\begin{aligned} t'_{q,Q} &= \gamma\left[t_{q,Q} - \vec{\beta}\cdot\vec{r}_{q,Q}\right], \\ \vec{r}'_{q,Q} &= \vec{r}_{q,Q} + (\gamma-1)(\hat{\beta}\cdot\vec{r}_{q,Q})\hat{\beta} - \gamma\vec{\beta}t_{q,Q}, \\ t_q &= t_Q \equiv t, \end{aligned} \tag{7}$$

and

$$S_v = \exp\left\{\frac{\operatorname{arctanh}|\vec{\beta}|}{2}\vec{\alpha}\cdot\hat{\beta}\right\} = \sqrt{\frac{\gamma+1}{2}}\begin{bmatrix} 1 & 0 & \frac{\gamma\beta_z}{\gamma+1} & \frac{\gamma(\beta_x-i\beta_y)}{\gamma+1} \\ 0 & 1 & \frac{\gamma(\beta_x+i\beta_y)}{\gamma+1} & -\frac{\gamma\beta_z}{\gamma+1} \\ \frac{\gamma\beta_z}{\gamma+1} & \frac{\gamma(\beta_x-i\beta_y)}{\gamma+1} & 1 & 0 \\ \frac{\gamma(\beta_x+i\beta_y)}{\gamma+1} & -\frac{\gamma\beta_z}{\gamma+1} & 0 & 1 \end{bmatrix}. \tag{8}$$

Therefore, the wave function of a meson moving with velocity $\vec{\beta}$ in our (unprimed) frame will be

$$\begin{aligned} \Psi_{\vec{v}}^{\lambda,\eta}(X,x) &= \sqrt{\frac{2M_{q\bar{Q}}}{(2\pi)^3}}\left[S_v\chi_{\vec{0}}^{\lambda}e^{-im_Qt'_Q}\right]\otimes\left[S_v\psi_{\vec{0}}^{\eta}(\vec{x}')e^{-i\epsilon t'_q}\right] \\ &= \sqrt{\frac{2M_{q\bar{Q}}}{(2\pi)^3}}e^{-iP\cdot X}\left[S_v\chi_{\vec{0}}^{\lambda}\right]\otimes\left\{S_v\psi_{\vec{0}}^{\eta}[\vec{x}+(\gamma-1)(\hat{\beta}\cdot\vec{x})\hat{\beta}]e^{i\epsilon\gamma\vec{\beta}\cdot\vec{x}}\right\}, \end{aligned} \tag{9}$$

where we have used Eq. (7) for the last step and $t_q = t_Q = t$, $\vec{r}_Q = \vec{X}$, $\vec{r}_q - \vec{r}_Q = \vec{x}$, $P = (\gamma M, \gamma\vec{\beta}M)$.

It is straightforward to check that the standard normalization for meson states,

$$\langle\Psi_{v'}^{\lambda,\eta'}|\Psi_{v}^{\lambda,\eta}\rangle = 2P^0\delta^3(\vec{P}-\vec{P}')\delta_{\lambda\lambda'}\delta_{\eta\eta'}, \tag{10}$$

is obtained provided that the usual normalization is used for the light and heavy quark wave functions:

$$\chi_{\vec{0}}^{\lambda\dagger}\chi_{\vec{0}}^{\lambda'} = \delta_{\lambda\lambda'} \text{ and } \int d^3\vec{x}\,\psi_{\vec{0}}^{\eta\dagger}(\vec{x})\psi_{\vec{0}}^{\eta'}(\vec{x}) = \delta_{\eta\eta'}. \tag{11}$$

The Isgur-Wise function $\xi(v\cdot v')$ (for orbital and radial ground states) can be extracted in a simple way [1, 3] by calculating the following matrix element between two pseudoscalar states of mass M and four-velocities v and v':

$$\xi(v\cdot v') = \frac{1}{(1+v\cdot v')M}\langle P_{v'}|\bar{h}_{v'}(0)h_v(0)|P_v\rangle, \tag{12}$$

where $|P_v\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_v^{\uparrow\downarrow}\rangle - |\Psi_v^{\downarrow\uparrow}\rangle\right)$ can be obtained from Eq. (9) and $\bar{h}_{v'}(0)$ $[h_v(0)]$ is the creation [annihilation] operator for a heavy quark of four-velocity $v'(v)$ at the origin of space-time and $v^\mu\gamma_\mu h_v = h_v$ has been used. The light quark wave functions $\psi_{\vec{0}}^{\eta}$ [see Eq. (9)] correspond in this case to the radial and orbital ground state solutions of the Dirac equation (1).

Without loss of generality we can assume that $\vec{\beta}$ and $\vec{\beta}'$ are collinear. We can then use the simple result

$$\chi_{\vec{0}}^{\dagger}S_{v'}^{\dagger}\gamma^0S_v\chi_{\vec{0}} = \sqrt{\frac{1+v\cdot v'}{2}} \tag{13}$$

to obtain the explicit expression

$$\xi(v\cdot v') = \sqrt{\frac{2}{1+v\cdot v'}}\int d^3\vec{x}\frac{1}{2}\sum_{\eta=\uparrow,\downarrow}\psi_{\vec{0}}^{\eta\dagger}(\vec{x}')S_{v'}^{\dagger}S_v\psi_{\vec{0}}^{\eta}(\vec{x}'')e^{i\epsilon(\gamma\vec{\beta}-\gamma'\vec{\beta}')\cdot\vec{x}}, \tag{14}$$

where

$$\begin{aligned} \vec{x}' &= \vec{x} + (\gamma'-1)(\hat{\beta}'\cdot\vec{x})\hat{\beta}', \\ \vec{x}'' &= \vec{x} + (\gamma-1)(\hat{\beta}\cdot\vec{x})\hat{\beta}, \end{aligned} \tag{15}$$

and $v\cdot v' = \gamma\gamma'[1-\vec{\beta}\cdot\vec{\beta}']$ ($v\cdot v'$ in $[1,\infty]$). $\vec{x}'$ and $\vec{x}''$ represent the spatial coordinates in the reference frames where the mesons of respective four-velocities v' and v are at rest.

In principle, the above integral expression (14) should depend on $v\cdot v'$ only, i.e., should be Lorentz invariant so that its value will be independent of the (unprimed) frame that is chosen to evaluate the integral in. In practice, because the Dirac equation (1) is not Lorentz covariant (due to the potential), the wave functions are not either. Therefore, the function $\xi(v\cdot v')$ does depend on the frame we choose. However, it can be easily checked that the important properties $\xi(1) = 1$ and $\mathrm{Im}[\xi(v\cdot v')] = 0$ are satisfied in any Lorentz frame. Moreover, as will be detailed at the end of this section, we found that the value of the slope at zero recoil $\xi'(1)$ is the same in three simple (but quite different) Lorentz frames. Thus, at least in the vicinity of the zero recoil point, the effects of Lorentz noninvariance of our function ξ are expected to be small.

We choose to work in the Breit [7] frame, where it is easiest to perform the calculation. In this frame the incoming and outgoing mesons are moving with equal speeds but in opposite directions ($\vec{\beta} = -\vec{\beta}'$) and therefore $v \cdot v' = 2\gamma^2 - 1$. Also, $S^{\dagger}_{v'} = S^{-1}_{v}$ so that after a change in integration variable $\xi(v \cdot v')$ can be written in a simple form [12], with $\vec{\beta}$ and $\vec{\beta}'$ in the z direction:

$$\xi_q(v \cdot v') = \frac{1}{\gamma^2} \int d^3\vec{r}\,' \frac{1}{2} \sum_{\eta=\uparrow\downarrow} \left| \psi^{\eta}_{\bar{0}}(\vec{r}\,') \right|^2 e^{2i\epsilon_q \beta z'} , \quad (16)$$

where

$$\gamma = \sqrt{\frac{1 + v \cdot v'}{2}} , \quad \beta = \sqrt{\frac{v \cdot v' - 1}{v \cdot v' + 1}} . \quad (17)$$

Because $\sum_{\eta=\uparrow\downarrow} \left| \psi^{\eta}_{\bar{0}}(\vec{x}\,') \right|^2$ is spherically symmetric, the angular integration is trivial. Also, it is easy to derive an expression for the slope at zero recoil $\xi'(1)$ in terms of the inertia parameter $\epsilon_q = \bar{\Lambda}_q$ and $\langle r_q^2 \rangle_{\bar{0}}$:

$$\left. \frac{\partial \xi_q}{\partial (v \cdot v')} \right|_{v \cdot v' = 1} = -\frac{1}{2} - \frac{1}{3} \epsilon_q^2 \langle r_q^2 \rangle_{\bar{0}} = -\frac{1}{2} - \frac{1}{3} \bar{\Lambda}_q^2 \langle r_q^2 \rangle_{\bar{0}} . \quad (18)$$

This is a general result within our formalism and not linked to any particular form of the potential used in the Dirac equation for the light quark. It tells us that $\xi'(1)$ depends only on the light quark energy eigenvalue ϵ [or equivalently, for our model, the HQET "inertia" parameter $\bar{\Lambda}_q = \lim_{m_Q \to \infty} (M_{Q\bar{q}} - m_Q)$] and on the rms distance between the light quark and the (stationary) heavy antiquark, $\sqrt{\langle r_q^2 \rangle_{\bar{0}}}$. We should point out that this same result was obtained independently in a recent paper that describes $q\bar{Q}$ mesons and qqQ baryons in the context of the MIT bag model [7], also obtained in the Breit frame.

An interesting aspect of this result is that it sets an upper bound on the slope $[\xi'(1) = -\rho^2]$

$$\xi'(1) < -\frac{1}{2}. \quad (19)$$

This bound is stronger than the well-known Bjorken bound $\xi'(1) < -1/4$ and is a result of the dynamical assumptions inherent in our treatment of the meson system in the heavy quark limit. In particular, we think that it can be traced to the fact that we describe a moving meson by boosting the light and heavy quarks independently (by the same amount). General arguments based on relativistic kinematics lead to a result similar to Eq. (19), but not as a strict upper bound on $\xi'(1)$ [13]. A simple relativistic oscillator model gives the same upper bound as Eq. (19) [14].

Because our formalism is not fully covariant (spherically symmetric Dirac equation potential is put in by hand), one may suspect that the above result for $\xi'(1)$ [Eq. (18)] is Lorentz frame dependent. We have checked explicitly that the result is unchanged if $\xi'(1)$ is calculated in the frames where either the incoming or the outgoing meson is at rest.

III. PARAMETRIZATION OF THE DIRAC EQUATION POTENTIAL

In the $M_Q \to \infty$ limit, the $q\bar{Q}$ meson system should be well described by the heavy antiquark stationary at the origin and the light quark (of mass m_q) moving in a spherically symmetric static external potential. We are of course ignoring the self-interactions of the light quark in the hope that these can be absorbed to some extent in a renormalization of the parameters of the external potential.

We take the short-distance behavior of the potential from renormalization-group-improved QCD perturbation theory and the long-distance behavior from lattice and other nonperturbative studies that ignore screening by light quark pair creation. Unfortunately, knowledge about the leading behavior of the potential in these two extreme distance regimes defines the potential only up to an additive constant.

At short distances, the usual asymptotically free Coulomb form (transforming as the zeroth component of a Lorentz four-vector) is obtained:

$$V_c(r) = -\frac{4}{3} \frac{\alpha_s(r)}{r}, \quad (20)$$

where $\alpha_s(r)$ is obtained in the leading logarithmic approximation and is parametrized as

$$\alpha_s(r) = \frac{2\pi}{(11 - \frac{2N_F}{3}) \ln[A + \frac{B}{r}]}. \quad (21)$$

The parameter A defines the "long-distance" saturation value for α_s. We use $A = 2$ which corresponds to $\alpha_s(r = \infty) = 1.0$. The parameter B is related to $\Lambda_{\overline{\rm MS}}$ by $B = (2.23\Lambda_{\overline{\rm MS}})^{-1}$ for $N_F = 3$. We use $N_F = 3$ [15] throughout because for most distance regimes relevant to our calculation the $c\bar{c}$ and $b\bar{b}$ vacuum polarization contribution should be negligible. We generally use the present experimental average $\Lambda_{\overline{\rm MS}} \approx 0.240$ GeV [16], which corresponds to $B = 1.87\ {\rm GeV}^{-1}$. As described in Sec. IV, we do vary the value of B (i.e., of $\Lambda_{\overline{\rm MS}}$) with respect to the above value for the specific purpose of studying the sensitivity of the shape of $\xi(v \cdot v')$ to this parameter. We find only a weak dependence.

To describe the long-distance behavior we use a linear term in the potential that transforms as a Lorentz scalar (masslike). Many theoretical and phenomenological arguments seem to favor this form [17–19],

$$V_L = \gamma^0 \sigma r , \quad (22)$$

where γ^0 is the usual Dirac matrix, rather than an admixture with a linearly rising zeroth component of a vector term. We do our calculations with three different values for the string tension parameter σ. The choices that we made were arrived at as follows. The experimental information available for the D and D_s systems seems to indicate that the (spin-averaged) splitting between P states and S states is in both cases approximately 0.45 GeV. We found that in order to obtain a splitting of this magnitude with our $V_c(r) + c_0 + \gamma^0 \sigma r$ as defined above, σ had to be about $\sigma \approx 0.25$ GeV^2. A previous study [17] agrees with this calculation. On the other hand, this value for σ is significantly larger than those obtained from the Regge slope data ($q\bar{q}$ systems) [18, 19]. Of course, one cannot exclude the possibility that the $2P$-$1S$ splitting of about 0.45 GeV holds for the D, D_s systems but is significantly smaller in the hypothetical limit $M_Q \to \infty$, which are the systems we are trying to describe in the present work. The difference would be due to heavy quark recoil effects in the charmed systems. In fact, preliminary evidence was reported recently for a $B^{**}(\ell = 1)$ candidate with mass $M_{B^{**}} \approx 5610$ MeV (Ref. [20] and also talk by V. Lüth at the same conference), which would indicate a $2P$-$1S$ splitting of only about 0.34 GeV. If confirmed, this would allow us to use more conventional values for σ, as extracted from the Regge slope α'. We therefore also consider this possibility. The extraction of σ from α' is somewhat model dependent. Two common relations are

$$\sigma = \frac{1}{2\pi\alpha'} \quad \text{(string model [21])} \tag{23}$$

and

$$\sigma = \frac{1}{8\alpha'} \quad \text{[two-body generalization of Klein-Gordon (KG) equation [18,19]]}. \tag{24}$$

Using $\alpha' \approx 0.9$ GeV^{-2} [21] we obtain, respectively, $\sigma \approx 0.18$ GeV^2 (string model) and $\sigma \approx 0.14$ GeV^2 (two-body KG. equation). The latter value agrees also with a recent lattice estimate [22]. We would like to remark that if the $2P$-$1S$ splitting for the B mesons (and therefore also in the $m_Q \to \infty$ limit) turns out to be similar in magnitude than the observed splitting in the D, D_s systems, as predicted in several models [23], we could still fit this splitting using the more conventional values of σ (0.18 and 0.14 GeV^2), provided that we change the $\Lambda^{(3)}_{\overline{\rm MS}}$ parameter in $V_c(r)$ [see Eq. (21) and discussion below] to $\Lambda^{(3)}_{\overline{\rm MS}} > 0.5$ GeV, i.e., $B < 0.9$ GeV^{-1}.

So far we have used as input the well-known leading short-distance and long-distance behavior of the potential. There is less theoretical knowledge about the shape of the potential in the intermediate region. We introduce an additive constant term c_0, which is clearly subleading both in the short- and long-distance regimes. Because the only role of this constant is to define the absolute scale of the light quark energy ϵ_q, this constant gets absorbed once we identify $\epsilon_q \equiv \bar{\Lambda}_q = \lim_{m_Q \to \infty}(M_{q\bar{Q}} - m_Q)$ and assign $\bar{\Lambda}_q$ some physical value.

We obtain a plausible range for the physical parameter $\bar{\Lambda}_q$ from previous theoretical works that estimated the value of the "pole mass" of the b quark, m_b. We use the relation

$$M_B \text{ (spin averaged)} \approx 5310 \text{ MeV} \approx \bar{\Lambda}_{u,d} + m_b + \frac{\langle \vec{P}_q^2 \rangle}{2m_b}\ , \tag{25}$$

where $\langle \vec{P}_Q^2 \rangle \approx \langle \vec{P}_q^2 \rangle$ has been used.

A recent lattice study finds $m_b = 4950 \pm 150$ MeV [4], which also agrees with a HQET estimate [5]. QCD sum rule estimates are lower, closer to $m_b \approx 4.6$ GeV [6]. From these values for m_b and Eq. (25) we then find the range

$$\bar{\Lambda}_{u,d} \approx 150\text{–}600 \text{ MeV}. \tag{26}$$

We do our calculations using mostly the two extreme values as well as the average value in this range, $\bar{\Lambda}_{u,d} =$ 150, 375, and 600 MeV. Notice that we have to make a distinction between $\bar{\Lambda}_{u,d}$ and $\bar{\Lambda}_s$, but this does not introduce any complications because, experimentally,

$$M_{D_s} - M_D \approx M_{B_s} - M_B \approx 100 \text{ MeV}\ , \tag{27}$$

$$M_{D_s^*} - M_{D_s} \approx M_{D^*} - M_D\ , \tag{28}$$

and

$$M_{B_s^*} - M_{B_s} \approx M_{B^*} - M_B\ . \tag{29}$$

Thus, simply taking

$$\bar{\Lambda}_s = \bar{\Lambda}_{u,d} + 100 \text{ MeV} \tag{30}$$

seems to be consistent with the heavy quark expansion to $O(\frac{1}{m_Q})$. We will adopt this prescription to compare results for mesons containing a u or d quark with those containing an s quark.

In summary, our Dirac equation potential is of the form

$$V = \frac{-8\pi}{27r \ln\left(2.0 + \frac{1.87\ (\text{GeV}^{-1})}{r}\right)} + c_0 + \gamma^0 \sigma r, \tag{31}$$

where we assumed $\alpha_s^\infty = 1$, $N_F = 3$, and $\Lambda_{\overline{\rm MS}} = 0.240$; σ takes the values 0.25, 0.18, or 0.14, and c_0 takes values such that $\epsilon_{u,d} \equiv \bar{\Lambda}_{u,d}$ is in the range 150–600 MeV. For the quark masses we use $m_u = m_d = 0$, $m_s = 0.175$ GeV. We have checked that this choice for m_s is consistent with Eq. (30).

We would like to remark that we are allowing wide ranges for the model parameters σ and $\bar{\Lambda}_{u,d}$ so that our results will be useful in spite of present theoretical uncertainties in the determination of their "physical" values. We could significantly narrow these ranges by making model-dependent assumptions. For instance, the arguments leading to Eq. (24) [18, 19] are closest to the theoretical context of the present paper, leading to $\sigma \approx 0.14$ GeV. As mentioned earlier, this value for σ also agrees well with the recent lattice estimate of Ref. [22]. Also, because the constituent mass of a light quark q in a $q\bar{Q}$

system is expected to be somewhat larger than the constituent mass of the same quark in a $q\bar{q}$ system (due to the smaller size of the former), we can obtain lower bounds on $\bar{\Lambda}$ (energy eigenvalue of the light quark in our $q\bar{Q}$ systems) that are stronger than 150 MeV [see Eq. (26)]. For example, a commonly quoted value for the constituent mass of the s quark in light baryons or mesons is 0.5 GeV. Then, a lower bound of $\bar{\Lambda}_{u,d} > 0.4$ GeV would follow through Eq. (30). We thus obtain "preferred" ranges for σ and $\bar{\Lambda}_{u,d}$:

$$\begin{aligned} &\sigma \approx 0.14 \quad \text{GeV}^2 , \\ &0.4\ \text{GeV} \ < \ \bar{\Lambda}_{u,d} \ < \ 0.6\ \text{GeV}. \end{aligned} \tag{32}$$

We however prefer to present our results for the wider (and less model-dependent) ranges given earlier (i.e., 0.14 $\text{GeV}^2 < \sigma < 0.25$ GeV^2 and 0.15 GeV $< \bar{\Lambda}_{u,d} < 0.6$ GeV).

IV. QUANTITATIVE RESULTS AND DISCUSSION

In this section we use the parametrization for the Dirac equation discussed in the previous section in order to find the light quark wave function $\psi_q(\vec{x})$. We then use the formalism developed in Sec. II, to calculate the Isgur-Wise function $\xi(v \cdot v')$ in the Breit reference frame, for different values of the parameters in the potential and for systems where the light quark is u, d ($m_u = m_d \approx 0$) or s ($m_s \approx 0.175$ GeV).

In the case of a central potential, the time-independent Dirac equation for a state with angular momentum quantum numbers j and m is reduced to radial equations by writing

$$\psi = \frac{1}{r} \begin{pmatrix} g(r)\Omega_{\kappa m}(\theta,\phi) \\ -if(r)\Omega_{-\kappa m}(\theta,\phi) \end{pmatrix} , \tag{33}$$

where $\kappa = -\ell - 1$ for $j = \ell + 1/2$ and $\kappa = \ell$ for $j = \ell - 1/2$ and $\Omega_{\kappa m}$ is a spinor with spin 1/2 coupled to orbital angular momentum ℓ. The radial equations are then

$$[V_l(r) + m]g(r) + \left[-\frac{d}{dr} + \frac{\kappa}{r}\right] f(r) = \epsilon g(r) \ , \tag{34}$$

$$\left[\frac{d}{dr} + \frac{\kappa}{r}\right] g(r) + [-m + V_s(r)]f(r) = \epsilon f(r) \ .$$

In the present calculation the potentials operating on the large and small components are parametrized as

$$\begin{aligned} V_l(r) &= \sigma r - \tfrac{8\pi}{27 r \ln[A+B/r]} + c_0 \ , \\ V_s(r) &= -\sigma r - \tfrac{8\pi}{27 r \ln[A+B/r]} + c_0 \ . \end{aligned} \tag{35}$$

The eigenvalue problem has been solved in two independent ways. The algebraic approach is to expand $g(r)$ and $f(r)$ in "basis" functions $\phi_i(r)$, $i = 1, ..., M$, and $\chi_j(r)$, $j = 1, ..., N$, leading to a matrix eigenvalue problem of dimension $(M+N)$ for ϵ. The smallest N eigenvalues correspond to hole states and the $(N+1)$st eigenvalue is the lowest particle state. In the present calculation ϕ and χ are taken to be

$$\begin{aligned} \phi_i(r) &= r^i e^{-\sigma r^2/2} \ , \quad i = 1, ..., M \ , \\ \chi_j(r) &= r^j e^{-\sigma r^2/2} \ , \quad j = 1, ..., N \ . \end{aligned} \tag{36}$$

This form is chosen to have the correct analytic behavior at large r values. Matrix elements of the kinetic energy operators and the linear potential can be evaluated analytically, but matrix elements of the "Coulomb" term are computed numerically. Satisfactory results are obtained for modest values of M and N of order 10.

The problem has also been solved fully numerically in the finite difference approximation by converting the two first-order equations to the second-order Pauli equation. Details of the method used can be found in Ref. [24]. Energies found by the two methods are identical. However, for evaluating the form factor the more accurate numerical wave functions have been employed.

We present our results for $\xi_{u,d,s}(v \cdot v')$ in graphical form for the range $1 < v \cdot v' < 4$ which is of phenomenological interest for decays of the types $B, B^* \to D, D^* + X$; $B, B^* \to K^* + X$; $D, D^* \to K^* + X$. Figures 1, 2, and 3 exhibit our results for the "inertia" parameter $\bar{\Lambda}_{u,d} = 0.15$ GeV, 0.375 GeV, and 0.6 GeV ($\bar{\Lambda}_s = 0.25$ GeV, 0.475 GeV, and 0.7 GeV), respectively. For each value of $\bar{\Lambda}_{u,d}$ we show the result $\xi_{u,d}(v \cdot v')$ for $\sigma = 0.25$ GeV^2, 0.18 GeV^2, and 0.14 GeV^2 while for each $\bar{\Lambda}_s$ we give $\xi_s(v \cdot v')$ with $\sigma = 0.18$ GeV^2 for comparison with $\xi_{u,d}(v \cdot v')$.

In Table I we present the values of the zero recoil slope, $\xi'(v \cdot v')|_{v \cdot v'=1}$ for the same values of the parameters $\bar{\Lambda}_{u,d}$, $\bar{\Lambda}_s$, and σ as used in Figs. 1–3.

We observe that for $\bar{\Lambda}_{u,d} \approx 0.15$, $\xi_{u,d}(v \cdot v')$ is almost independent of the parameter σ. It is clear from Eqs. (16)–(18) that if $|\bar{\Lambda}| (= |\epsilon_q|)$ is very small, $\xi(v \cdot v')$ is controlled mostly by purely kinematic factors, the shape of the wave function becomes unimportant (provided that it is properly normalized). We observe also that $\xi_{u,d}(v \cdot v')$ and $\xi_s(v \cdot v')$ are quite close ($\sigma = 0.18$, $\bar{\Lambda}_s \approx \bar{\Lambda}_{u,d} + 100$ MeV) for all values of $\bar{\Lambda}_{u,d}$. This confirms that the strange quark can be treated as a light quark and that $SU(3)_F$ is only softly broken for the processes considered here, once the (phenomenologically imposed) shift in the value of $\bar{\Lambda}$ has been considered.

Although it is not obvious from Figs. 1–3, we checked (by using different values of σ and $\bar{\Lambda}_{u,d}$ and keeping

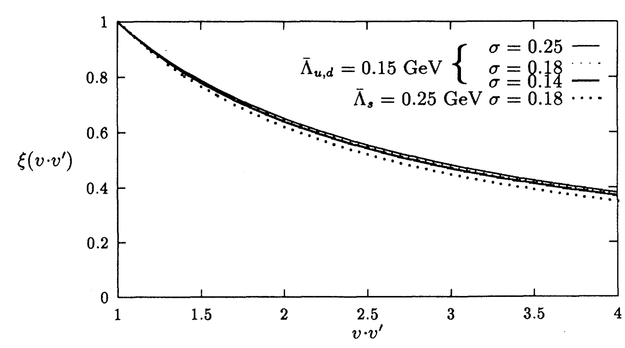

FIG. 1. The Isgur-Wise function $\xi_{u,d}$ (ξ_s) for $\bar{\Lambda}_{u,d} = 0.15$ GeV ($\bar{\Lambda}_s = 0.25$ GeV) and $\sigma = 0.25$, 0.18, and 0.14 GeV^2 ($\sigma = 0.18$ GeV^2).

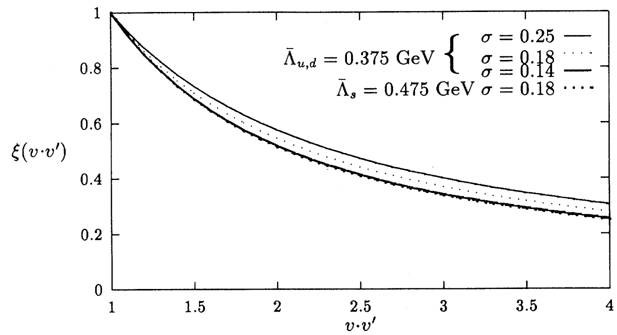

FIG. 2. The Isgur-Wise function $\xi_{u,d}$ (ξ_s) for $\bar{\Lambda}_{u,d} = 0.375$ GeV ($\bar{\Lambda}_s = 0.475$ GeV) and $\sigma = 0.25$, 0.18, and 0.14 GeV^2 ($\sigma = 0.18$ GeV^2).

TABLE I. The magnitude of slope of the Isgur-Wise function at zero recoil for various values of the parameters $\bar{\Lambda}$ and σ.

	$\sigma = 0.25$	$\sigma = 0.18$	$\sigma = 0.14$
$\bar{\Lambda}_{u,d} = 0.15$ $\bar{\Lambda}_s = 0.25$	$\rho^2_{u,d} = 0.53$ $\rho^2_s = 0.58$	$\rho^2_{u,d} = 0.55$ $\rho^2_s = 0.61$	$\rho^2_{u,d} = 0.56$ $\rho^2_s = 0.64$
$\bar{\Lambda}_{u,d} = 0.375$ $\bar{\Lambda}_s = 0.475$	$\rho^2_{u,d} = 0.72$ $\rho^2_s = 0.80$	$\rho^2_{u,d} = 0.81$ $\rho^2_s = 0.90$	$\rho^2_{u,d} = 0.89$ $\rho^2_s = 0.99$
$\bar{\Lambda}_{u,d} = 0.6$ $\bar{\Lambda}_s = 0.7$	$\rho^2_{u,d} = 1.06$ $\rho^2_s = 1.16$	$\rho^2_{u,d} = 1.28$ $\rho^2_s = 1.37$	$\rho^2_{u,d} = 1.50$ $\rho^2_s = 1.56$

$\bar{\Lambda}^2_{u,d}/\sigma$ constant) that $\xi_{u,d}(v{\cdot}v')$ in the range $1 < v{\cdot}v' < 4$ depends on the ratio $\bar{\Lambda}^2_{u,d}/\sigma$ only, to a precision better than 2%. This "scaling" is not completely unexpected since σ and $\bar{\Lambda}_{u,d}$ are the main dimensionful parameters entering the problem, and $\xi(v \cdot v')$ is dimensionless. The other dimensionful parameter is $\Lambda_{\overline{\rm MS}}$ in V_c, which we are keeping constant. This affects V_c only logarithmically, and, for massless quarks, does not much affect the shape of the wave function far from the origin. (Recall that a pure vectorlike potential does not produce bound states for massless quarks, and we are using $m_u \approx m_d \approx 0$.)

Figure 4 illustrates this scaling effect by giving the values of the zero recoil slope $\rho^2 \equiv -\xi'(1)$ for different values of $\frac{\bar{\Lambda}^2_{u,d}}{\sigma}$. The points were obtained from Table I using the three values of σ and $\bar{\Lambda}_{u,d}$ quoted. It turns out that all the points satisfy to a very good approximation the empirical linear relation

$$\rho^2_{u,d} \equiv -\xi'_{u,d}(1) \approx \frac{1}{2} + 0.39\frac{\bar{\Lambda}^2_{u,d}}{\sigma}, \tag{37}$$

where the numerical coefficient of $\frac{\bar{\Lambda}^2_{u,d}}{\sigma}$ varies only by ± 0.01 within our range of $\bar{\Lambda}_{u,d}$ and σ. This equation can then be used to find the zero recoil slope within our model for an arbitrary value of the imput parameters $\bar{\Lambda}_{u,d}$ and σ.

We checked whether this approximate dependence of $\xi(v{\cdot}v')$ on the ratio $\bar{\Lambda}^2_{u,d}/\sigma$ only also holds for the system with a strange (light) quark ($m_s \approx 0.175$ GeV). We found that $\xi_s(v{\cdot}v')$ does depend on σ and $\bar{\Lambda}_s$ independently, i.e., not only through the ratio $\bar{\Lambda}^2_s/\sigma$. What causes the different behavior in this case is that there is an additional dimensionful parameter in the problem, m_s, so that we can form two independent dimensionless ratios $\bar{\Lambda}/m_s$ and σ/m^2_s. Regarding $\rho^2_s = -\xi'_s(1)$, we notice from Table I that consistently $\rho^2_s > \rho^2_{u,d}$; i.e., the Isgur-Wise function has (exhibits) a more rapid decrease for $q\bar{Q}$ mesons where the light quark is an s quark. This result was obtained as well in the calculations of Refs. [7, 25].

We also studied the effects of changing the value of $\Lambda_{\overline{\rm MS}}$ (B parameter) in V_c [see Eqs. (20), (21) and the discussion below] on $\xi(v \cdot v')$, keeping σ and $\bar{\Lambda}$ fixed. We found that the behavior of $\xi(v{\cdot}v')$ is not very sensitive to changes in $\Lambda_{\overline{\rm MS}}$. For instance, using $\sigma = 0.18$ GeV^2 and $\bar{\Lambda}_{u,d} = 0.375$ GeV ($\bar{\Lambda}_s = 0.475$ GeV), a change of $\Lambda_{\overline{\rm MS}}$ by 50% either way from our central value $\Lambda_{\overline{\rm MS}} \approx 0.24$ GeV changes the zero recoil slope $\xi'(1)$ by only about 2.2% (3%). The relative changes in $[1-\xi(v{\cdot}v')]$ are of the same order of magnitude for the whole range $1 < v \cdot v' < 4$.

Finally, we compare our results for $\xi_{u,d}(v \cdot v')$ with the existing experimental data and other theoretical estimates.

As mentioned above, we observed that $\xi_{u,d}(v{\cdot}v')$ in our model depends (to a very good approximation) only on the ratio $\bar{\Lambda}^2_{u,d}/\sigma$. Hence, we adopt a fitting procedure to the ARGUS 1993 [9], CLEO 1993 [10], and CLEO 1994 [11] data in which the parameters of the potential, $A = 2$,

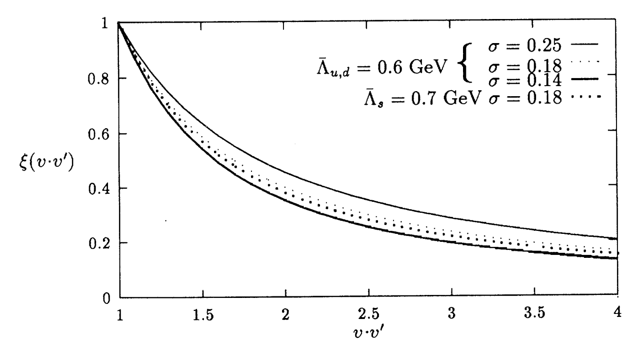

FIG. 3. The Isgur-Wise function $\xi_{u,d}$ (ξ_s) for $\bar{\Lambda}_{u,d} = 0.6$ GeV ($\bar{\Lambda}_s = 0.7$ GeV) and $\sigma = 0.25$, 0.18, and 0.14 GeV^2 ($\sigma = 0.18$ GeV^2).

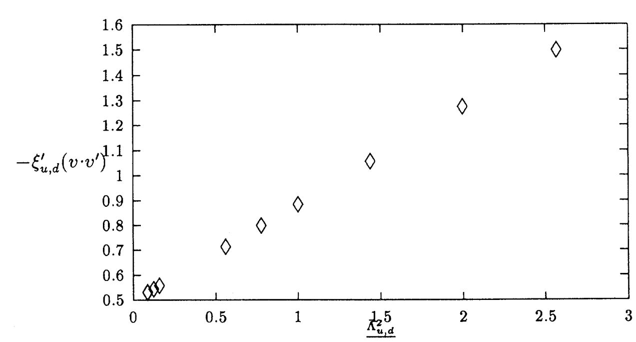

FIG. 4. The slope of the Isgur-Wise function at zero recoil $\rho^2_{u,d} = -\xi'_{u,d}(1)$ for different values of the parameter $\frac{\bar{\Lambda}^2_{u,d}}{\sigma}$.

$B = 1.87$ GeV^{-1} and $\sigma = 0.18$ GeV^2, are kept fixed, while $\bar{\Lambda}_{u,d}$ and the physical observable $|V_{cb}|\mu$ are varied in such a way that χ^2, defined in standard fashion [26] as being the total square deviation from the data (weighted by one standard deviation experimental uncertainty for each point), is minimized:

$$\chi^2 \equiv \sum_{i=1}^{N} \left(|V_{cb}|\mu\xi[(v\cdot v')_i] - f_i\right)^2/\sigma_i^2 \, , \tag{38}$$

where

$$\mu \equiv \sqrt{\frac{\tau_B}{1.48\text{ ps}}} \quad \text{(fits to ARGUS 1993 and CLEO 1993)} \tag{39}$$

and

$$\mu \equiv F(1) \approx \eta_A \quad \text{in the notation of [11]} \quad \text{(fits to CLEO 1994).} \tag{40}$$

We will use $\chi_0^2/N_{\rm DF}$ as a measure of the "quality" of the fit, where χ_0^2 corresponds to the minimum of the function $\chi^2(\bar{\Lambda}_{u,d}\ ,\ |V_{cb}|\mu)$ and $N_{\rm DF} = N - 2$ is the number of degrees of freedom for a particular data set (N experimental points, two fitting parameters) [26]. The factor $F(1) \approx \eta_A$ required to extract an estimate of $|V_{cb}|$ from the CLEO 1994 data [11] has been estimated theoretically by several authors:

$$\begin{aligned} &F(1) = 0.97 \pm 0.04 \quad [1,27],\\ &F(1) = 0.96 \pm 0.03 \quad [28],\\ &F(1) < 0.94\ ,\ F(1) \approx 0.89 \pm 0.03 \quad [29]\,. \end{aligned} \tag{41}$$

For each data set, we obtain the one standard deviation (68.3% confidence level) ranges for our two fitting parameters, $\bar{\Lambda}_{u,d}$ and $|V_{cb}|\mu$ [26]. Because of the scaling behavior described above Eq. (37), the ranges that we obtain for the parameter $\bar{\Lambda}_{u,d}$ and fixed $\sigma = 0.18$ GeV^2 can be translated into ranges for the dimensionless ratio $\bar{\Lambda}_{u,d}^2/\sigma$ where σ is allowed to take values other than 0.18 GeV^2.

We would like to add a cautionary note before giving the results of our fits. Because the Isgur-Wise function that we calculate in our model and use for these fits does *not* take into account finite-M_Q corrections [$O(\Lambda_{\rm QCD}^2/M_Q^2)$ at $v\cdot v' = 1$ and $O(\Lambda_{\rm QCD}/M_Q)$ elsewhere], we have to be aware that direct comparison of our model with experiment (via our χ^2 function) can be reliable to *leading* order in $\Lambda_{\rm QCD}/M_Q$ only. At $v\cdot v' = 1$, this uncertainty is of order $\Lambda_{\rm QCD}^2/M_Q^2$ and can be absorbed into the parameter μ which multiplies $|V_{cb}|$ [see Eqs. (39)–(41)]. However, away from $v\cdot v' = 1$ there is an "intrinsic" uncertainty in the comparison of our model to experiment of expected relative magnitude of order $(v\cdot v'-1)\Lambda_{\rm QCD}/M_Q$.

The best fit to the ARGUS 1993 data (eight points) [9] gives $\chi_0^2/N_{\rm DF} = 0.54$. We obtain the following ranges for the fitting parameters: $\frac{\bar{\Lambda}_{u,d}^2}{\sigma} = 4.8 \pm 1.7$ (corresponding to [see Eq. (37)] $\rho_{u,d}^2 = -\xi_{u,d}'(1) = 2.4 \pm 0.7$) and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\text{ ps}}} = 0.050 \pm 0.008$. We note that even if a small value for the string tension, $\sigma = 0.14$ GeV^2 were used, the range for the inertia parameter would be $\bar{\Lambda}_{u,d} \approx 0.81 \pm 0.15$ GeV, which is significantly above most theoretical estimates [see Eq. (26)]. The corresponding range for $\rho_{u,d}^2$ overlaps with some of the theoretical estimates (see Table II) but is centered above most of the predicted ranges.

The best fit to the CLEO 1993 data (seven points) [10] is poorer with $\chi_0^2/N_{\rm DF} \approx 1.19$. Here we find the ranges $\frac{\bar{\Lambda}_{u,d}^2}{\sigma} = 3.3 \pm 1.2$ (corresponding to [see Eq. (37)] $\rho_{u,d}^2 = -\xi_{u,d}'(1) = 1.8 \pm 0.5$) and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\text{ ps}}} = 0.043 \pm 0.005$. There is some overlap between the resulting range for $\bar{\Lambda}_{u,d}$ (e.g., $\bar{\Lambda}_{u,d} = 0.66 \pm 0.13$ for $\sigma = 0.14$ GeV^2) and independent theoretical estimates [see Eq. (26)]. Also, the range for the slope $\rho_{u,d}^2$ significantly overlaps with several previous theoretical predictions (see Table II). We would like to remark that if we ignore the CLEO 1993 data point corresponding to highest recoil ($v\cdot v' \approx 1.5$), the fit to the remaining six points is greatly improved. We obtain in this case $\chi_0^2/N_{\rm DF} = 0.62$ and ranges $\frac{\bar{\Lambda}_{u,d}^2}{\sigma} = 2.0 \pm 1.4$ (corresponding to $\rho_{u,d}^2 = 1.3 \pm 0.6$) and $|V_{cb}|\sqrt{\frac{\tau_B}{1.48\text{ ps}}} = 0.038 \pm 0.005$. For commonly used values for the string tension ($\sigma = 0.14$–0.18 GeV^2) the acceptable range for $\bar{\Lambda}_{u,d}$ in this case is centered well within the range of previous theoretical estimates [see Eq. (26)]. The range for the zero recoil slope $\rho_{u,d}^2 = -\xi_{u,d}'(1)$ is very similar to the ranges obtained in recent lattice estimates [34, 35] as well as from several other theoretical calculations (see Table II).

The best fit of our model to the recent CLEO 1994 data analysis (seven points) [11] gives $\chi_0^2/N_{\rm DF} = 0.50$ which is the lowest $\chi_0^2/N_{\rm DF}$ of all our fits, in spite of the smaller experimental error bars. This means that within a region of its parameter space, our model agrees well with this data set. In turn, we expect that these data with smaller error bars will be useful in selecting a relatively narrow region for our model parameters (mainly $\bar{\Lambda}_{u,d}^2/\sigma$)

TABLE II. Various theoretical estimates of $\rho_{u,d}^2 = -\xi'(1)$.

Bjorken [8]	$> \frac{1}{4}$
Isgur *et al.* [30]	0.63(0.33)
Rosner [31]	1.44 ± 0.41
Mannel *et al.* [32]	1.77 ± 0.74
Neubert [33]	1.28 ± 0.25
Bernard *et al.* [34]	$1.41 \pm 0.19 \pm 0.41$
UKQCD Collaboration [35]	1.2^{+7}_{-8}
Radyushkin [36]	∞
Karanikas and Ktorides [37]	0
Sadzikowski and Zalewski [7]	1.24
Kugo *et al.* [38]	1.8–2.0
Ivanov *et al.* [39]	0.43
Ivanov and Mizutani [40]	0.42–0.82
Narison [6]	0.52–0.92
Blok and Shifman [41]	0.5–0.8
Close and Wambach [13]	1.19 ± 0.03

as well as for the standard model flavor-mixing parameter $|V_{cb}|$ [or at least the product $|V_{cb}|F(1)$]. The parameter ranges obtained are $\frac{\bar{\Lambda}^2_{u,d}}{\sigma} = 2.0 \pm 0.7$ [corresponding to $\rho^2_{u,d} = -\xi'_{u,d}(1) = 1.3 \pm 0.3$] and $|V_{cb}|F(1) = 0.037 \pm 0.002$. As expected, the smaller experimental error bars lead to a better determination of our model parameter $\bar{\Lambda}^2_{u,d}/\sigma$ [and correspondingly, via Eq. (37), of the slope at zero recoil $\rho^2_{u,d}$] as well as of the physically interesting product $|V_{cb}|F(1)$. We note that for the commonly used values for the string tension, $\sigma = 0.14$–0.18 GeV^2, the resulting range for $\bar{\Lambda}_{u,d}$ (0.42 GeV$< \bar{\Lambda}_{u,d} <$ 0.69 GeV) overlaps significantly with the upper half of the range given in Eq. (26), which was obtained from recent theoretical estimates of m_b [4–6]. In this respect, the larger values of σ (e.g., $\sigma = 0.25$ GeV^2 [see discussion below Eq. (22)]), seem to be less favored. We would like to remark here that an estimate of the pseudoscalar decay constants f_D and f_B in the context of the relativistic model used here [42], when compared with recent estimates of these constants with lattice and QCD sum rule methods, favors the lower values for σ as well.

Because the Isgur-Wise function is normalized at zero recoil, $\xi(1) \equiv 1$, the slope at zero recoil $\xi'(1)$ determines to a good approximation the value of the function close to $v \cdot v' = 1$. Therefore, at least close to $v \cdot v' = 1$ (all the existing data is in the interval [1,1.5]) the slope $\xi'(1) \equiv -\rho^2$ is a reliable tool for comparison of the different theoretical estimates of $\xi(v \cdot v')$. Our original model parameter ranges 0.15 GeV$< \bar{\Lambda}_{u,d} <$ 0.6 GeV and 0.14 GeV^2 $< \sigma <$ 0.25 GeV^2 [see discussion between Eqs. (22) and (26)] give [through our result in Eq. (37)] a wide range $0.54 < \rho^2_{u,d} < 1.5$, which overlaps with many different theoretical estimates (see Table II and Refs. [6–8,13,30–41]). The only exceptions are Refs. [36–39]. On the other hand, our "preferred" range for the model parameters, $\sigma \approx 0.14$ GeV^2 and 0.4 GeV$< \bar{\Lambda}_{u,d} <$ 0.6 GeV [see Eq. (32) and discussion preceding it as well as comments at the end of the last paragraph], leads to the narrower range $0.94 < \rho^2_{u,d} < 1.5$. This range is very similar to the one favored by our fit to the CLEO 1994 data [11] ($\rho^2_{u,d} = 1.3 \pm 0.3$) and overlaps with the predictions in Refs. [7,8,13,31–35] only, although it is not far from the range advocated in Ref. [6] (see Table II). Both the lattice gauge theory calculations of [34,35] contain the range favored by the fit of our model to the CLEO 1994 data within their predicted ranges.

The extraction of a precise value for the standard model parameter $|V_{cb}|$ from the range $|V_{cb}|F(1) = 0.037 \pm 0.002$ favored by our fit to the CLEO 1994 data is partially hindered by the present uncertainty in the theoretical determination of $F(1)$. For example, if we use the whole range of values for $F(1)$ given in Eq. (41), we would obtain $0.034 < |V_{cb}| < 0.046$.

It is interesting to note that our fits to the CLEO 1994 data (seven points) produce similar central values for $\bar{\Lambda}^2_{u,d}/\sigma$, $\rho^2_{u,d}$, and $|V_{cb}|$ [for $F(1) \lesssim 1$] as do our fits to the CLEO 1993 data with the point corresponding to the largest $v \cdot v'$ omitted from the CLEO 1993 set (i.e., six points). The resulting ranges for these quantities are, however, significantly narrower (by a factor of about 2) for the CLEO 1994 data set. We should also point out that the above-mentioned central values for $\rho^2_{u,d}$ and $|V_{cb}|F(1)$ that we obtained from the fits of our model to the CLEO 1994 data set (Fig. 12 in Ref. [11]) are both somewhat larger than the central values for these quantities obtained in the CLEO analysis carried out in Ref. [11] (in their notation $a^2 = \rho^2_{u,d}$). Once the uncertainties are included, however, our results for these quantities are compatible.

Eventually, when the experimental data become more precise and the $1/M_Q$ corrections can be incorporated with fewer uncertainties, one should be able to further narrow the allowed range for our main model input, $\bar{\Lambda}^2_{u,d}/\sigma$, as well as for the flavor-mixing parameter $|V_{cb}|$.

Note added. After this paper was submitted for publication, we became aware of recent data by the ALEPH Collaboration on $\bar{B}^0 \to D^{*+}\ell^-\bar{\nu}$ decays [43]. For completeness we give below the results of a fit of our model to their data set (six points). We obtain $\chi^2_0/N_{\rm DF} = 0.67$ and the ranges $\bar{\Lambda}^2_{u,d}/\sigma = 1 \pm 1$ [corresponding to $\rho^2_{u,d} = -\xi'_{u,d}(1) = 0.9 \pm 0.4$] and $|V_{cb}|F(1)\sqrt{\frac{\tau_B}{1.5\ {\rm ps}}} = 0.039 \pm 0.004$. This fit is of poorer quality than our fit to the CLEO 1994 data [11] described earlier. The resulting ranges for the parameters are wider than and significantly overlapping with the ranges obtained in our fit to the CLEO 1994 data [44]. An approximate method for solving the Dirac equation for this type of problem has been described previously by Franklin and Intemann [45].

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