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Extending the Euler-Heisenberg Lagrangian to some nonperturbative and inhomogeneous field configurations

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In this paper we discuss the Euler-Heisenberg effective Lagrangian in nonperturbative and inhomogeneous field configurations characterized by a strong magnetic field of the form $B_x = B_y = 0$, $B_z = B(x,y)$. Our treatment exploits some interesting properties of the second-order Dirac Hamiltonian describing the electronic motion in the transverse plane (x,y). In particular, we take advantage of the existence of an energy gap separating the excited states from the lowest-lying modes. The latter are exactly calculable and give the leading nontrivial contribution to the effective Lagrangian. Our results show that the presence of gradients can be accounted for by introducing an effective magnetic field strength defined as the sum of the square moduli of the ground state wave functions. This surprisingly simple conclusion is mainly due to a quantum-mechanical supersymmetry of the problem, that of the second-order Dirac Hamiltonian in the transverse plane (x,y). We recall that the same supersymmetry plays an important role for the spontaneous mass generation in the Nambu–Jona-Lasinio model interacting with an external magnetic field. As a simple application of our effective Lagrangian, we discuss the asymptotic behavior of the dielectric permeability tensor of the vacuum as a function of the external field configuration. No anomalous enhancement of the effective electromagnetic coupling is observed. Implications of this result for the GSI peaks are briefly considered.

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I. INTRODUCTION

The behavior of QED in the presence of strong background fields represents a nonperturbative problem which is not completely and satisfactorily understood. This interesting sector of quantum electrodynamics has attracted much attention in recent years thanks to the observation of pulsars and heavy-ion collision experiments [1]. In both cases, the relevant field strength can well exceed the so-called critical value $f_{\text{cr}} = m^2/e$, which is, roughly speaking, the border separating the perturbative domain from the nonperturbative one (m is the electron mass and e stands for the elementary charge). With the notable exception of Coulomb systems [2–4], our knowledge about overcritical electromagnetic fields seems to be restricted to the case of constant fields, in which the most important Green's functions can be expressed by means of exact and quite simple integral representations [5–8]. A powerful tool to investigate QED in these simple configurations is the Euler-Heisenberg Lagrangian [9–12], which is an effective Lagrangian obtained by integrating over the fermionic degrees of freedom appearing in the generating functional of QED. In this framework the effects of fermion loops with an arbitrary number of external photon legs are described by a complicated self-interaction for the $F_{\mu\nu}$ field, that is, by nonlinear corrections to Maxwell's equations. The purpose of this paper is to extend the Euler-Heisenberg effective Lagrangian to a wide class of inhomogeneous configurations, characterized by a strong magnetic field of the form

$$B_x = B_y = 0, \quad B_z = B(x,y), \tag{1a}$$

where we assume that $B(x,y)$ is bounded from below by a positive constant much greater than the critical field:

$$B(x,y) \gg \frac{m^2}{e}. \tag{1b}$$

This last condition may appear very restrictive, since it implies that our field is strong everywhere. Actually, for any point of the space, the effective Lagrangian receives important contributions only from a limited region extending over some Compton wavelength around that point. Therefore, the conclusions we shall derive in the following sections can be safely applied to those configurations for which inequality (1b) is satisfied only in a finite region, provided that its size is sufficiently large with respect to a Compton wavelength. As a matter of fact, one of the main advantages of assuming (1b) for all points (x,y) consists in suppressing from the very beginning the inessential, long-distance contributions to the effective Lagrangian.

We shall proceed in strict analogy with Ref. [13], where we discussed the Nambu–Jona-Lasinio model in similar conditions. In particular, our treatment will exploit a remarkable property of the Dirac equation in the presence of magnetic fields with a constant direction. As shown by Aharonov and Casher [14] and Jackiw [15], it is always possible to exactly calculate the lowest-energy eigenfunctions of a charged Dirac particle, provided that it interacts with a field of type (1a). Furthermore, the associated eigenvalues turn out to be zero, independently

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of $B(x,y)$, whereas the excited levels are separated from the ground state by an energy gap which is expected to scale with $\sqrt{eB}$, B standing for the order of magnitude of the applied field. Actually, this latter property does not hold in general. In fact, for magnetic fields with a finite flux the separation between zero-energy modes and excited states does not exist (see example in Ref. [16], Sec. IV). However, the presence of an energy gap is ensured [15] for magnetic fields with an infinite flux as those we are dealing with [recall (1b)]. We can readily verify that these statements hold for a homogeneous field. This simple problem can be solved in terms of an equivalent harmonic oscillator and the spectrum is given by [17]

$$w^2 = m^2 + p_z^2 + eB(2n - \sigma + 1), \tag{2}$$

where p_z is the momentum in the direction of $\mathbf{B}$, $\sigma(=\pm 1)$ corresponds to the spin projection along the same direction, and n takes integer values. For $n = 0$ and $\sigma = 1$, we see that the energy is independent of B, since there is a cancellation between the spin contribution and the zero-point energy of the equivalent harmonic oscillator. By contrast, for any other state we have $w^2 \geq eB$. In the next section we shall verify that a similar gap exists under quite natural assumptions about the function $B(x,y)$ describing our unidirectional magnetic field. Keeping this in mind, we are in a position to understand the main idea underlying the present work. Let us consider processes whose energy is well below the scale $\sqrt{eB}$, which we suppose to be large with respect to the electron mass (strong field regime $\sqrt{eB}/m \gg 1$). Under these assumptions, the states with $w^2 \sim eB$ are excited with a vanishingly small amplitude. Therefore, we can describe the fermionic dynamics by using only the lowest-energy wave functions, which are known for any configuration of type (1). With this approximation we greatly simplify the calculation of the dominant terms appearing in the strong field limit of the Euler-Heisenberg effective Lagrangian, to which the next section is devoted.

II. EFFECTIVE LAGRANGIAN

From the path integral approach to quantum field theory or from "proper time" arguments [10], we know that the effective Lagrangian density for QED can be written as

$$\mathcal{L}' = \mathcal{L}_0 + \delta\mathcal{L}, \tag{3a}$$

where $\mathcal{L}_0$ is the free-field Lagrangian and $\delta\mathcal{L}$ is given by

$$\delta\mathcal{L} = \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-is(m^2 - i0)} \langle x, \alpha | e^{is[(P-eA)^2 + e\sigma_{\mu\nu}F^{\mu\nu}/2]} - e^{isP^2} | x, \alpha \rangle. \tag{3b}$$

In Eq. (3b), A_μ is the prescribed four-vector potential, the $|x,\alpha\rangle$ are classical spinors, and a sum over the discrete indices α is understood. In the following, the components $A_{x,y}$ will be functions of the transverse coordinates (x,y) and we shall impose

$$A_0 = 0, \quad A_z = -Et. \tag{4}$$

The expression for A_z accounts for a constant electric field pointing in the z direction. This field has been introduced for later convenience as well as for the sake of generality. The first steps required in the actual computation of $\delta\mathcal{L}$ are extremely conventional and analogous to those relevant for homogeneous configurations. Proceeding as in [10], we cast Eq. (3b) in the form

$$\delta\mathcal{L} = \frac{i}{8\pi^2}\int_0^\infty \frac{ds}{s} e^{-is(m^2 - i0)} \left[eE \coth(eEs) \sum_{\nu\sigma} e^{-is\varepsilon_{\nu\sigma}} 2\pi |\Psi_{\nu\sigma}(x,y)|^2 + \frac{i}{s^2} \right], \tag{5}$$

where the $\Psi_{\nu\sigma}$ are eigenfunctions of the second order Dirac Hamiltonian in the transverse plane

$$[(p_x - eA_x)^2 + (p_y - eA_y)^2 - e\sigma B_z(x,y)]\Psi_{\nu\sigma}(x,y) = \varepsilon_{\nu\sigma}\Psi_{\nu\sigma}(x,y), \tag{6}$$

and $\sigma(=\pm 1)$ corresponds to the spin projection along the z axis as in Eq. (2). In general, we cannot exactly solve problem (6). Nevertheless, from the works by Aharonov and Casher [14] and Jackiw [15], we know that the ground state is exactly calculable and its energy is zero with an infinite degeneracy in the case of infinite flux. The actual form of the corresponding wave functions and their proper normalization depend on the gauge choice. For the moment we denote the Aharonov-Casher-Jackiw states by the symbol ψ_κ, without specifying the nature of the quantum number(s) κ. Later on, in discussing several examples, we shall make a definite choice both for the gauge and the indices κ. With this in mind, let us now consider the strong field regime we are interested in, namely, $eB/m^2 \gg 1$. In this condition we can consider processes whose energy is well below the large scale $\sqrt{eB}$. This constraint enables us to introduce a small-s cutoff $\bar{s} \sim 1/(eB)$ as a lower bound of the integration appearing in Eq. (5). This procedure suppresses the contribution from excitations with $\varepsilon \gtrsim eB$. Therefore, it suffices to include the zero-energy states ψ_κ in expansion (5). By this way we obtain

$$\delta\mathcal{L} = \frac{i}{8\pi^2}\int_{\bar{s}}^\infty \frac{ds}{s} e^{-is(m^2 - i0)} \times \left[eE \coth(eEs) \sum_\kappa 2\pi |\psi_\kappa(x,y)|^2 + \frac{i}{s^2} \right]. \tag{7}$$

For a constant magnetic field of strength B, the sum over κ yields the simple factor eB. To see this, let us introduce the ground state wave functions in the form [18]

$$\psi_{p_x}(x,y) = \frac{1}{\sqrt{2\pi}} e^{ip_x x} \left(\frac{eB}{\pi}\right)^{1/4} \exp\left[-\frac{eB}{2}\left(y + \frac{p_x}{eB}\right)^2\right], \tag{8a}$$

corresponding to the gauge choice

$$A_x(y) = -By, \quad A_y = 0. \tag{8b}$$

In this case the sum over κ is nothing but an integration over the diagonalized momentum p_x and we have

$$\sum_\kappa 2\pi|\psi_\kappa(x,y)|^2 = \int_{-\infty}^{+\infty} dp_x \sqrt{\frac{eB}{\pi}} \times \exp\left[-eB\left(y + \frac{p_x}{eB}\right)^2\right] = eB, \tag{9}$$

as anticipated. It is then quite natural to cast Eq. (7) in the form

$$\delta\mathcal{L} = -\frac{1}{8\pi^2}\int_{\bar{s}}^{\infty} \frac{ds}{s} e^{-sm^2}\left[eE\,\cot(eEs) eB_{\text{eff}}(x,y) - \frac{1}{s^2}\right], \tag{10}$$

where we have rotated the contour of integration ($s \to -is$) and

$$eB_{\text{eff}}(x,y) = \sum_\kappa 2\pi|\psi_\kappa(x,y)|^2. \tag{11}$$

We note that the effect of gradients can be accounted for by introducing an effective field strength B_{eff} which exhibits an extremely simple connection with Aharonov-Casher-Jackiw states ψ_κ. From the physical point of view, B_{eff} is nothing but an appropriate average of the magnetic field strength in the neighborhood of the point (x,y). The appearance of such an average is rather natural, and it is due to the fermionic fluctuations which probe the magnetic field in different points of the space. Simple dimensional arguments can provide us useful information about the scaling properties of B_{eff}. These properties, in turn, will strengthen our interpretation (11). Suppose that the function $B(x,y)$ is given in the form

$$B(x,y) = B_0 F(x,y;\lambda), \tag{12}$$

where B_0 is the magnetic field at the origin and λ represents the typical length of the background configuration. Since B_{eff} does not depend on the electron mass, we can write

$$B_{\text{eff}}(x,y) = B_0\mathcal{F}(x\sqrt{eB_0}, y\sqrt{eB_0}, \lambda\sqrt{eB_0}), \tag{13}$$

$\mathcal{F}$ being a dimensionless function. Thus, for the effective field strength at the origin, we obtain

$$B_{\text{eff}}(0,0) = B_0\mathcal{F}\left(0,0,\frac{1}{\tau}\right), \quad \tau = \frac{1}{\lambda\sqrt{eB_0}}. \tag{14a}$$

The dimensionless parameter τ has a simple physical meaning: It gives the relative variation of B_z along a distance equal to the natural length scale $1/\sqrt{eB_0}$ (the uncertainty in the electron position). Equation (14a) means that B_{eff} exactly scales with B_0, provided that we increase B_0 while keeping fixed τ, that is, increasing also the gradients. Obviously, if λ is kept fixed and B_0 is increased, then we recover the homogeneous field case ($\tau \to 0$) and $B_{\text{eff}} = B_0$. Similar dimensional arguments enable us to verify the presence of the large gap separating the zero-energy modes of the spectrum from its higher levels. Indeed, for infinite flux fields we know that a gap exists [15] and we can write it in the form

$$w^2 = eB_0 G(\tau), \quad \tau = \frac{1}{\lambda\sqrt{eB_0}}, \tag{14b}$$

where w^2 is the lower bound for the excited levels and G is a dimensionless function of the previously introduced parameter τ. Once again, we see that w^2 exactly scales with B_0 when τ is fixed, while $w^2 \to eB_0$ if λ is fixed and B_0 is increased ($\tau \to 0$).

In order to avoid misunderstandings about Eqs. (11) and (14a), some further comments are appropriate. In the case of homogeneous magnetic fields, it is well known that the degeneracy of the ground state has a simple interpretation, different zero-energy wave functions corresponding to different centers of rotation for the Landau orbits. Moreover, if we quantize the transverse motion in a finite region, then the number of degenerate levels is proportional to B [18]. This fact may suggest that the introduction of an effective field strength is sensible only for those configurations whose ground state degeneracy resembles the constant field case. Actually, our previous dimensional arguments do not rely on specific assumptions about the background magnetic fields and interpretation (11) is therefore of much wider validity. As an example of this, it is interesting to consider in some detail a magnetic field of the form

$$B(x,y) = B_0\left(1 + \frac{r^2}{\lambda^2}\right), \quad r^2 = x^2 + y^2, \tag{15}$$

where on the right-hand side (RHS) B_0 and λ are constants. For this configuration the results obtained in Ref. [19] show that the zero-energy wave functions can be parametrized by a point in a $\mathbb{R}^4$ space. Therefore, the lowest-energy manifold now has a structure which is completely different from the uniform case (where the degeneracy goes as $\mathbb{R}^2$). Following Aharonov, Casher, and Jackiw, we derive $B(x,y)$ via the two-dimensional Laplacian of an appropriate "potential" $\phi(x,y)$:

$$\phi(r) = \frac{B_0}{4}\left(r^2 + \frac{r^4}{4\lambda^2}\right), \tag{16a}$$

$$A_x = -\partial_y\phi, \quad A_y = \partial_x\phi, \tag{16b}$$

$$(\partial_x^2 + \partial_y^2)\phi = B(x,y). \tag{16c}$$

A set of orthogonal ground states is then given by [15]

$$\psi_\kappa(r) = C_\kappa e^{i\kappa\theta} r^\kappa e^{-e\phi(r)}, \quad \kappa = 0, 1, \ldots, \tag{17}$$

where the C_κ are normalization factors which can be expressed in terms of parabolic cylinder functions (see formula 3.462;1 of [20]):

$$C_\kappa^2 = \frac{eB_0}{2^{\kappa+1}\pi\lambda^{2k}} \frac{e^{-1/4\tau^2}}{\Gamma(\kappa+1)U(\kappa+1/2, 1/\tau)\tau^{\kappa-1}}, \tag{18}$$

with τ as in Eq. (14) (here U is the parabolic cylinder function defined as in [21]). For large κ, the reader can easily check that $C_\kappa^2 < (\text{const})^\kappa \kappa^{-\kappa/2}$, so that the RHS of Eq. (11) gives rise to a power series which is convergent for any r. If we focus our attention on the fields at the origin, then only the contribution with $\kappa = 0$ survives in the definition of B_{eff} and we obtain

$$B_{\text{eff}}(x=0, y=0) = 2\pi \frac{C_0^2}{e} = B_0 \frac{\tau e^{-1/4\tau^2}}{U(1/2, 1/\tau)}, \tag{19}$$

which is evidently of the form (14a). In passing, we note that the constant field result $B_{\text{eff}} = B_0$ is easily reproduced by recalling the asymptotic behavior of the parabolic cylinder function appearing in Eq. (19).

Other explicit expressions for B_{eff}, can be obtained in the case of magnetic fields varying in one direction only. To be specific let us consider the configuration discussed in Ref. [13], namely, $B_x = B_y = 0$, $B_z = B(y)$, with a vector potential given by

$$A_x(y) = -\int_0^y dy' B(y'). \tag{20}$$

Here the Aharonov-Casher-Jackiw states can be replaced by [13]

$$\psi_{p_x}(x, y) = \frac{1}{\sqrt{2\pi}} e^{ip_x x} \chi(p_x, y), \tag{21}$$

with

$$\chi(p_x, y) = N \exp\left[-\int_0^y dy'[p_x - eA_x(y')]\right], \tag{22}$$

N being a normalization factor. The effective field is now given by the simpler expression

$$eB_{\text{eff}}(y) = \int_{-\infty}^{+\infty} dp_x \chi^2(p_x, y). \tag{23}$$

The actual computation of $B_{\text{eff}}(y)$ is much simplified if we replace the exact wave functions (22) by appropriate Gaussians whose parameters are obtained by expanding around its maximum the exponent appearing in the $\chi(y)$:

$$\chi(p_x, y) \cong \left(\frac{\omega(p_x)}{\pi}\right)^{1/4} \exp\left[-\frac{\omega(p_x)}{2}[y - y_M(p_x)]^2\right], \tag{24}$$

where $y_M(p_x)$ solves the equation

$$p_x - eA_x(y_M) = 0 \tag{25}$$

and $\omega(p_x) = eB[y_M(p_x)]$. By substituting (24) into (23), we obtain

$$eB_{\text{eff}}(y) = \int_{-\infty}^{+\infty} dp_x \sqrt{\frac{\omega(p_x)}{\pi}} \exp\{-\omega(p_x)[y - y_M(p_x)]^2\}. \tag{26}$$

With the change of variable $p_x \to u = y_M(p_x)$ and noting that its Jacobian is eB [from the derivative of (25)], the last equation can be cast in a simpler form which does not require the numerical solution of Eq. (25),

$$eB_{\text{eff}}(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} du [eB(u)]^{3/2} e^{-eB(u)(y-u)^2}, \tag{27}$$

which has the simple functional derivative

$$\frac{\delta B_{\text{eff}}(y)}{\delta B(u)} = \sqrt{\frac{eB(u)}{\pi}} e^{-eB(u)(y-u)^2} \left[\frac{3}{2} - eB(u)(y-u)^2\right]. \tag{28}$$

Once again, it is reassuring to note that Eq. (27) gives the correct answer, $B_{\text{eff}} = B$, for the constant field case. Furthermore, it is worthwhile to stress the presence of the factor $\exp[-eB(u)(y-u)^2]$ which weights the contributions of the local field $B(u)$ to the effective field $B_{\text{eff}}(y)$. As expected from very simple arguments, the large distance contributions turn out to be exponentially suppressed. The numerical estimates of Ref. [13] confirm the reliability of (24) up to $\tau \sim 1$. Even for such strong gradients, $B_{\text{eff}}(y)$ is very close to $B(y)$. More precisely, we have found that $(B_{\text{eff}} - B)/B \lesssim 2\%$ for the particular configuration

$$B(y) = B_0[1 + \tfrac{1}{2}\tanh(2\sqrt{eB_0}y)], \tag{29}$$

to which we can assign $\tau = 1$ at the origin; see [13] for details.

Up to now we have assumed that $\mathbf{E}$ and $\mathbf{B}$ point in the same direction and this appears as a severe restriction. Actually, in the particular domain we are interested in, $eB/m^2 \gg 1$, the presence of a field $\mathbf{E}_\perp$ perpendicular to $\mathbf{B}$ gives subleading contributions to Eq. (10). This can be verified by means of an explicit calculation based on the "standard" Euler-Heisenberg Lagrangian; otherwise, we can resort to simple physical arguments as the one we sketch below. Let $\phi = -\mathbf{E}_\perp \cdot \mathbf{x}$ be the potential associated with the electric field $\mathbf{E}_\perp$. In the transverse plane (x, y), the charged particle motion is confined in a region of order $1/\sqrt{eB}$, so that the variations of ϕ are $\delta\phi \sim E_\perp/\sqrt{eB}$, which becomes smaller and smaller as B increases. In a similar fashion we can consider the effects induced by a $\mathbf{B}_\perp$ field superimposed on our initial configuration (1). As far as the direction of $\mathbf{B}' = \mathbf{B} + \mathbf{B}_\perp$ is concerned, the presence of $\mathbf{B}_\perp$ causes tilts from the z axis by an amount $\theta \sim B_\perp/B$. In order to resolve the actual direction from the z axis, the electron should travel along $\mathbf{B}'$ by an amount δz such that $(B_\perp/B)\delta z \sim 1/\sqrt{eB}$; this gives $\delta z \sim \sqrt{eB}/(eB_\perp)$, which certainly becomes unphysical as B increases. Moreover, the correction to the magnitude of the magnetic field is $O(B_\perp^2/B)$. The above considerations show two important things: (i) The quantity E appearing in Eq. (10) can be interpreted as $\mathbf{E}\cdot\mathbf{n}\equiv E_{||}$, where $\mathbf{n}$ is the unit vector along $\mathbf{B}$ and $\mathbf{E}$ has now an arbitrary direction; (ii) if we introduce an additional magnetic field of arbitrary shape, then we are allowed to consider only its component along $\mathbf{n}$. As a consequence, functional derivatives such as (28) may be used to correct B_{eff}.

Before considering a specific application of Eq. (10), it is appropriate to address the issue of renormalization. It is well known that Eq. (3b) gives rise to an unrenormalized effective Lagrangian containing an ultraviolet-divergent term proportional to the free-field Lagrangian $(E^2-B^2)/2$. This divergence has to be subtracted via the introduction of a suitable counterterm. The latter has a universal meaning, since it corresponds to the charge renormalization [11]. Therefore, we can correct our effective Lagrangian (10) with the same counterterm used for homogeneous configurations:

$$\mathcal{L}_{\text{ct}}=-\frac{e^2}{24\pi^2}(\mathbf{E}^2-\mathbf{B}^2)\int_{\bar{s}}^{\infty}\frac{ds}{s}e^{-sm^2}. \qquad (30)$$

Adding $\mathcal{L}_{\text{ct}}$ and $\mathcal{L}_0$ to the RHS of Eq. (10), we obtain our complete effective Lagrangian $\mathcal{L}'$:

$$\mathcal{L}'=\frac{\mathbf{E}^2-\mathbf{B}^2}{2}-\frac{e^2}{24\pi^2}(\mathbf{E}^2-\mathbf{B}^2)\ln\frac{eB}{m^2} -\frac{1}{8\pi^2}eB_{\text{eff}}\int_{\bar{s}}^{\infty}\frac{ds}{s}e^{-sm^2}eE_{||}\cot(eE_{||}s)+O(B^2) \qquad (31a)$$

and, for perturbative electric fields,

$$\mathcal{L}'\cong\frac{\mathbf{E}^2-\mathbf{B}^2}{2}-\frac{\alpha}{6\pi}(\mathbf{E}^2-\mathbf{B}^2)\ln\frac{eB}{m^2}+\frac{\alpha}{6\pi}\frac{eB_{\text{eff}}}{m^2}E_{||}^2. \qquad (31b)$$

Some comments are now appropriate to understand the role played by the various terms appearing in Eq. (31). As conjectured in [9] (Chap. 9), the leading contribution to the effective Lagrangian $\mathcal{L}'$ is of order $O(B^2\ln(B))$ and comes from the counterterm $\mathcal{L}_{\text{ct}}$ given by Eq. (30). Here the arbitrariness in the cutoff $\bar{s}$ affects only subleading corrections of order $O(B^2)$, as the one coming from the $1/s^3$ term in Eq. (10). Strictly speaking, the contribution connected with the Aharonov-Casher-Jackiw states gives a subleading correction to $\mathcal{L}'$. However, the term proportional to B_{eff} is qualitatively different from the others, since it is the only one which survives after a derivation with respect to the $\mathbf{E}$ components. In the next section we shall verify that this property is essential for a satisfactory understanding of QED in the presence of strong background fields.

III. EFFECTIVE COUPLINGS IN NONHOMOGENEOUS MAGNETIC FIELDS

The simplest application of our Lagrangian $\mathcal{L}'$ is the computation of the dielectric permeability tensor of the vacuum. There exists a simple relation [11] connecting the polarization $\mathbf{P}$ of the vacuum with the correction $\delta\mathcal{L}$, namely, $P_i=\partial\delta\mathcal{L}/\partial E_i$. If we expand $\delta\mathcal{L}$ in powers of $\mathbf{E}$ and keep only the lowest-order terms as in Eq. (31b), then $\mathbf{P}$ is a linear function of $\mathcal{E}$, $P_i=\chi_{ij}(\mathbf{B})E_j$, and we can define the dielectric permeability tensor of the vacuum as $\varepsilon_{ij}(\mathbf{B})=\delta_{ij}+\chi_{ij}(\mathbf{B})$. Starting from Eq. (31b), a simple algebra gives (omitting subleading contributions)

$$\varepsilon_{ij}=\left[1-\frac{\alpha}{3\pi}\ln\frac{eB}{m^2}\right]\delta_{ij}+\frac{\alpha}{3\pi}\frac{eB_{\text{eff}}}{m^2}n_in_j \qquad (32)$$

(recall that $\mathbf{n}$ is the unit vector in the direction of $\mathbf{B}$). The two eigenvalues of ε_{ij} are

$$\varepsilon_{\perp}=1-\frac{\alpha}{3\pi}\ln\frac{eB}{m^2},\quad \varepsilon_{||}=1-\frac{\alpha}{3\pi}\ln\frac{eB}{m^2}+\frac{\alpha}{3\pi}\frac{eB_{\text{eff}}}{m^2}, \qquad (33)$$

corresponding to eigenvectors perpendicular and parallel to the applied field, respectively. The deviations of ε_{ij} from the identity tensor can be interpreted as corrections to the (effective) fine structure constant α. Actually, because of the anisotropy introduced by the external field, it is more appropriate to state that the fine structure constant splits in two effective couplings $\alpha_{\perp}$ and $\alpha_{||}$ given by

$$\alpha_{\perp}\equiv\frac{\alpha}{\varepsilon_{\perp}}=\frac{\alpha}{1-\frac{\alpha}{3\pi}\ln\frac{eB}{m^2}},$$
$$\alpha_{||}\equiv\frac{\alpha}{\varepsilon_{||}}=\frac{\alpha}{1-\frac{\alpha}{3\pi}\ln\frac{eB}{m^2}+\frac{\alpha}{3\pi}\frac{eB_{\text{eff}}}{m^2}}. \qquad (34)$$

At the end of the 1980s, the strong field dependence of the fine structure constant was discussed in connection with the anomalous production of e^+e^- pairs observed in heavy-ion collisions at GSI [22,23]. As claimed by some authors [24–27] (see also [28] for a critical review and a somehow different perspective), the intriguing data collected by the EPOS and ORANGE Collaborations [22,23] could be explained by a mechanism according to which a strong external field drives the effective fine structure constant from the perturbative regime $\alpha\sim\frac{1}{137}$ to the strong-coupling one $\alpha\sim 1$, where QED is likely to have a transition toward a new confining phase, characterized by a different mass scale and by new electron-positron bound states. The effectiveness of the above-mentioned mechanism was discussed in Refs. [24,29] for homogeneous background configurations (see also [8,30]). Equations (34) confirm and generalize the results obtained in those works. As for constant external fields, we observe a logarithmic increase of $\alpha_{\perp}$, but the effect is too small to trigger the required phase transition. The longitudinal coupling $\alpha_{||}$ is much more sensitive to the external field strength; however, the result is disappointing with respect to explaining the GSI peaks. In fact, $\alpha_{||}$ is a decreasing function of the external field strength when $eB/m^2\gg 1$, and the role played by the gradients is simply to replace B by B_{eff} in the nonlogarithmic term that dictates the asymptotic behavior of $\alpha_{||}$. It goes without saying that the actual fields involved in a heavy-ion collision are much more complicated than in our model (1). Therefore, Eqs. (34) do not rule out the new phase of QED: They simply mean that one should investigate highly nontrivial background configurations, where the fixed direction hypothesis (1a) has to be released. Moreover, a realistic description of a heavy-ion collision should also take into account the presence of strong electric fields, whose treatment introduces a further difficulty, namely, the instability of the vacuum with the associated spontaneous production of electron-positron pairs.

IV. SUMMARY AND OUTLOOK

Starting from the path integral which defines QED and integrating over the fermionic degrees of freedom, one obtains an effective Lagrangian for the electromagnetic field A_μ. If the corresponding $F_{\mu\nu}$ can be considered as slowly varying, then the functional integration can be performed in a closed form and the result is the so-called Euler-Heisenberg Lagrangian, from which one can extract nonperturbative Green's functions with a fermion loop and an arbitrary number of photon legs. In this paper we have extended the Euler-Heisenberg Lagrangian to a wide class of inhomogeneous external field configurations, characterized by a strong magnetic field with a fixed direction, say, the z axis, but with a magnitude arbitrarily varying in orthogonal directions. For such configurations, the electronic motion in the transverse plane (x, y) has a number of extremely interesting properties. First, the lowest-energy modes and the excited states are separated by a gap which is expected to increase with $B^{1/2}$ (the energy scale associated with the external field). Second, the lowest-lying wave functions are exactly calculable and the corresponding eigenvalues are rigorously zero. For sufficiently large fields (large gaps), these degenerate ground states give the leading-nontrivial contribution to the functional integration, so that its actual computation is enormously simplified. Our results show that the effects of gradients can be taken into account by introducing an effective field B_{eff}, which is surprisingly close to the local magnetic field. Scaling properties of B_{eff} have been briefly discussed, and a simple approximation to B_{eff} has been given, provided that the external magnetic field varies in one direction only. As a simple application of our effective Lagrangian, we have discussed the asymptotic properties of the dielectric permeability tensor of the vacuum as a function of the applied external field; no anomalous enhancement of the electromagnetic effective couplings has been observed.

The important role played by the Aharonov-Casher-Jackiw states deserve a further comment. As far as the present work is concerned, we have verified that they determine the strong field dependence of the Euler-Heisenberg Lagrangian. In a similar fashion, they are responsible for the dynamical mass generation in the Nambu–Jona-Lasinio model minimally coupled to an external magnetic field of the form (1) [13,31,32]. As pointed out by Jackiw in [15], the occurrence of the zero-energy modes is intimately connected with a quantum mechanical supersymmetry of the problem, namely, the supersymmetry of the second-order Dirac Hamiltonian appearing in the LHS of Eq. (6). Consequently, if we consider an arbitrary external field (for which the separation of variables is no longer possible), then we lose not only translational invariance, but also supersymmetry. From this point of view, the background configuration we have studied, the homogeneous case included, appears as quite an exceptional situation. Therefore, we believe that it is extremely important to reconsider the question of effective electromagnetic couplings and spontaneous mass generation under more general assumptions, where new effects and new trends are likely to appear.

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