Composite gauge fields and broken symmetries

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A generalization of the non-Abelian version of the CP^{N-1} models (also known as Grassmannian models) is presented. The generalization helps accommodate a partial breaking of the non-Abelian gauge symmetry. Constituents of the composite gauge fields, in many cases, are naturally constrained to belong to an anomaly-free representation which in turn generates a composite scalar, simulating Higgs mechanism to break the gauge symmetry dynamically for large N. Two cases are studied in detail: one based on the SU(2) gauge group and the other on SO(10). Breakings such as SU(2)→U(1) or SO(10)→SU(5)×U(1) are found feasible. The properties of the composite fields and gauge boson masses are computed by doing a derivative expansion of the large N effective action.

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I. INTRODUCTION

Compositeness has been the way of nature. Some of the so-called elementary particles of earlier times have turned out to be composites of more elementary ones. Compositeness is one way of getting to a simpler theory involving usually fewer fields or fewer parameters at the fundamental level. It is also expected to soften the ultraviolet behavior. Composite gauge fields could in addition provide us with an understanding of the gauge principle. They have received considerable attention in the literature [1–4]. The generic nature of models in Ref. [1] is that some vector particles appear as composites dynamically and the gauge symmetry that is approximate becomes exact in a certain limit, for instance, when $N \to \infty$ as in Suzuki's work. In CP^{N-1} models [2] and their non-Abelian generalizations called Grassmannian models (GM's) [3], the gauge symmetry is exact and the gauge bosons introduced as auxiliary fields become physical when one-loop contributions are included. Here the gauge fields arise as composites of bosonic constituents [4]. In this paper, we are interested in these latter type of models. These models have been studied in the large N limit and it is found that the non-Abelian symmetry is either completely broken or not broken at all. Ultimately one would like to construct phenomenological models along these lines, but the phase structure of these models is not very useful for that purpose. One needs a version in which the gauge symmetry is partially broken.

Remarkably, as we will see in this article, there does exist a generalization of the GM's that allows for partial symmetry breaking. Some of the results of this paper have been briefly reported earlier [5]. The primary agent of symmetry breaking turns out to be a scalar that too is composite. This composite Higgs scalar arises naturally as a solution of the modified constraint equation. In many of the cases, it belongs to the adjoint representation of the gauge group. The constituent fields in those cases belong to an anomaly-free representation. One may recall here that that the agent of symmetry breaking in grand unified theories is usually a Higgs scalar in the adjoint representation. One may further recall that the fermions in a physical theory belong to an anomaly-free representation, and we encounter the same feature here, though in the bosonic version. Two examples are studied to illustrate the approach: one based on the gauge group SU(2) and the other based on SO(10). The SU(2) example is the simplest and best suited to illustrate the approach. Here there exists a phase where SU(2) breaks into a U(1) subgroup. The case of SO(10) studied in some detail is more interesting from the physical point of view. The phase structure is richer with symmetry breaking to various subgroups such as SU(5) or SU(5)×U(1).

The models of Refs. [2,3] and also those of this paper are renormalizable in 1+1 dimensions. We work in 3+1 dimensions and calculate various properties of the composite objects, gauge bosons and Higgs scalars, to see how far the program can be carried out and also to get some clues as to the requirements of a renormalizable model. We compute the properties of the composite fields by doing a derivative expansion of the large N effective action. The expansions available in the literature do not serve our purpose as they are, to our knowledge, also expansions in the Higgs scalar. We hence develop a suitable derivative expansion which we use to compute the kinetic terms and the mass terms for the composites in the various phases.

To start with, in the next section, we review the known models relevant to our work. First we look at the CP^{N-1} model that involves a U(1) gauge theory. Then we discuss its non-Abelian generalization, the Grassmannian model. It is based on the gauge group $U(M)$ with the scalars in the fundamental representation. In Sec. III, we introduce our model, a generalization of the GM that is capable of accommodating other gauge groups with more general scalar representations. A suitable potential responsible for a rich phase structure is then introduced. In Sec. IV, we first illustrate our approach with a simple example based on the gauge group SU(2) and then look at the interesting but more complicated case of SO(10). Section V discusses the properties of the various composites. The global symmetry with its breaking patterns and the resulting Goldstone modes are discussed in Sec. VI. Section VII concludes with a discussion of the present approach. Derivative expansion of the effective potential useful in Sec. V is carried out in the Appendix.

II. KNOWN MODELS

The known models that are relevant to our purpose, the CP^{N-1} and the Grassmannian models, are briefly reviewed in this section. The simplest model is the one that induces a U(1) gauge theory, the CP^{N-1} model. This is just a field theory involving $N > 1$ complex scalar fields,

$$Z = (Z_1, Z_2, \ldots, Z_N)\,, \tag{1}$$

satisfying the constraint $\sum_i |Z_i|^2 = 1$. The constraint sets their overall scale. A U(1) gauge invariance removes in addition an angular variable. Thus the model is in effect a field theory of $N-1$ complex scalars. For convenience in writing the various results, we have above represented the Z_i's collectively as a row vector Z. In this notation, the constraint can be rewritten as $ZZ^\dagger = 1$. The Lagrangian is

$$L = \beta N \left[\partial_\mu Z \partial_\mu Z^\dagger + \left(Z\partial_\mu Z^\dagger\right)^2\right]. \tag{2}$$

Here β is the inverse of a coupling constant. An overall multiplicative factor N is introduced for later convenience in the $1/N$ expansion. It is easy to verify the existence of a U(1) gauge invariance under which each of the Z_i's transforms with the same phase. In vector notation, this is simply $Z \to e^{i\theta}Z$ where the phase θ is space-time dependent. The constraint is clearly invariant under this symmetry. To see that it is a gauge symmetry, note first that the combination $iZ\partial_\mu Z^\dagger$ transforms as a U(1) gauge field:

$$iZ\partial_\mu Z^\dagger \to iZ\partial_\mu Z^\dagger + \partial_\mu \theta. \tag{3}$$

The last term actually has a $ZZ^\dagger$ but that drops out due to the constraint. The U(1) gauge invariance will be more explicit, if we rewrite the Lagrangian by introducing an auxiliary field $A_\mu = iZ\partial_\mu Z^\dagger$ as

$$\begin{aligned} L &= \beta N \left(\partial_\mu Z \partial_\mu Z^\dagger - 2iA_\mu Z\partial_\mu Z^\dagger + A_\mu^2\right) \\ &= \beta N \left[D_\mu Z (D_\mu Z)^\dagger\right]. \end{aligned} \tag{4}$$

Here $D_\mu Z$ is the covariant derivative $(\partial_\mu - iA_\mu)Z$. In this form, the gauge symmetry is manifest. As shown in Sec. V, the auxiliary field A_μ that transforms as a U(1) gauge field becomes dynamical and hence a genuine gauge field after quantum corrections in the large N approximation. It is a composite gauge field made of the Z fields. Thus the model under consideration can be viewed as an induced U(1) gauge theory or a theory of composite gauge fields. This model is a special case of the Grassmannian model; hence we study its phase structure below as a special case of the GM.

The Grassmannian model is a generalization of the CP^{N-1} model that induces a non-Abelian gauge theory. We now have more fields, a set of them, represented collectively by a $M \times N$ matrix Z with the elements $Z_{\alpha i}$, α labeling the rows and i labeling the columns. The column index i is an internal index or a flavor index that is essentially carried over from our previous model. The new index α, the row index, is the gauge index associated with a non-Abelian symmetry which in the present case is U(M). All our results should reduce to those of the previous model for the case of $M = 1$. The constraint is now

$$ZZ^\dagger = I_M, \tag{5}$$

where I_M is an identity matrix of order M. The present Lagrangian is of the previous form (2), but now as such it will be an $M \times M$ matrix and hence needs an overall trace to make it a number:

$$L = \beta N \mathrm{tr}\left[\partial_\mu Z \partial_\mu Z^\dagger + \left(Z\partial_\mu Z^\dagger\right)^2\right]. \tag{6}$$

It is again easy to verify that there is a U(M) gauge invariance with respect to the index α. Under this gauge symmetry, Z transforms as a set of N fundamental representations. In matrix notation, this transformation is simply $Z \to UZ$, U being an $M \times M$ space-time-dependent unitary matrix representing the gauge transformation. This transformation does not affect the i index which labels N fundamental representations. The constraint respects this symmetry. The object $A_\mu = iZ\partial_\mu Z^\dagger$ is in the adjoint representation of U(M) and transforms as a gauge field thanks to the constraint

$$A_\mu \to UA_\mu U^\dagger + iU\partial_\mu U^\dagger. \tag{7}$$

The role of the constraint here is to simplify the last term above from $iUZZ^\dagger\partial_\mu U^\dagger$ to $iU\partial_\mu U^\dagger$. The gauge symmetry becomes explicit when we rewrite the Lagrangian as in Eq. (4) with an overall trace:

$$L = \beta N \mathrm{tr}\left[D_\mu Z (D_\mu Z)^\dagger\right], \tag{8}$$

$D_\mu Z$ being the covariant derivative $(\partial_\mu - iA_\mu)Z$. As before, A_μ appears as an auxiliary field but, as can be seen at large N, it becomes dynamical and hence a genuine gauge field after quantum corrections. It is a composite gauge field with the Z fields as constituents. The constraint and the U(M) gauge invariance have the effect of suppressing M^2 degrees of freedom. The theory is thus based effectively on $M(N-M)$ scalars. Clearly, for it to be a sensible one, N is required to exceed M.

The constraint $ZZ^\dagger = I_M$ can be incorporated into the Lagrangian with the help of a Lagrangian multiplier Σ, a $M \times M$ matrix. The result is

$$L = \beta N \mathrm{tr}\left[D_\mu Z (D_\mu Z)^\dagger + \Sigma ZZ^\dagger - \Sigma\right]. \tag{9}$$

To understand symmetry breaking, and hence to identify the various phases, we need to obtain the effective potential. We will do this at large N. Because A_μ is not expected to pick up any expectation value, we will set it to zero. The classical contribution to the effective potential comes from the Lagrangian (9) by dropping the derivative terms. Because $1/N$ appears in the Lagrangian like the Planck's constant, the quantum corrections to this contribution are expected to be suppressed by a factor $1/N$. But there are N fundamental representations contributing equally and this can offset the $1/N$ suppression. The result is that at large N the effective potential for the Z and Σ fields obtained by integrating away the Z fluctuations carries a correction

$$N\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\ \ln\left(k^2 I_M+\Sigma\right). \qquad (10)$$

Here and in the rest of this article, we suppress the dependence of the momentum integrals on a cutoff Λ. The total effective potential is thus

$$V_{\text{eff}}=\beta N\mathrm{tr}\left(\Sigma ZZ^\dagger-\Sigma\right)+N\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\ \ln\left(k^2 I_M+\Sigma\right). \qquad (11)$$

To determine the various phases, we need to extremize this potential. The resulting saddle point equations (SPE's) are $\Sigma Z=0$ obtained by varying $Z^\dagger$ and

$$\beta\left(ZZ^\dagger-I_M\right)+\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 I_M+\Sigma}=0 \qquad (12)$$

coming from varying Σ. Now, let us look for solutions of the form

$$\Sigma=\begin{pmatrix}\sigma\cdot I_p & 0\\ 0 & 0\end{pmatrix},\qquad ZZ^\dagger=\begin{pmatrix}0 & 0\\ 0 & v^2\cdot I_{M-p}\end{pmatrix}, \qquad (13)$$

where I_p and I_{M-p} are two identity matrices of order p and $M-p$, respectively. Note that $\Sigma ZZ^\dagger$ is zero with this ansatz. Solutions for Z that satisfy $\Sigma Z=0$ can easily be constructed. Equation (12) leads to two equations, one in the I_p sector and the other in the I_{M-p} sector:

$$-\beta+\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\sigma}=0,$$
$$\beta(v^2-1)+\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}=0. \qquad (14)$$

For $p=0$ there is no I_p sector and hence the first equation above would be absent. Similarly for $p=M$ the second would be absent. Let us first show that p cannot lie inbetween. We will do this by showing that the two equations cannot be satisfied simultaneously. First note that σ should not be negative for the momentum integral involving it to be well defined. Hence, from the first equation, we note that β has an upper limit β_c given by

$$\beta_c=\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}=\frac{\Lambda^2}{16\pi^2}. \qquad (15)$$

But the second equation implies that β_c is also a lower limit of β. To see this, rewrite it as $v^2=1-\beta_c/\beta$ and note that v^2 cannot be negative. It thus follows that, except at the critical point $\beta=\beta_c$ where the question is irrelevant, p is either zero or M.

Thus we have two phases. For $\beta>\beta_c$, we have the broken phase ($p=0$) where $ZZ^\dagger$ has an expectation value. The solution for Z has an expectation value along all the "directions" in the fundamental representation. This breaks U(M) completely and all the gauge bosons are massive. For $\beta<\beta_c$, we have the unbroken phase ($p=M$) where the gauge symmetry is unbroken and the gauge bosons are massless. $\beta=\beta_c$ is a critical point. In fact, $\beta\leq\beta_c$ is a critical line along which all the masses vanish.

In other words, the gauge group is either completely broken or not broken at all. There are apparently no phases, at least at large N, where a partial breaking of the gauge group is possible. To obtain a richer phase structure, we invoke a generalization of these models and study them at large N in the following section.

III. GENERALIZATION

As discussed in the previous section, the Grassmannian model involves scalars in N fundamental representations of the gauge group U(M). A natural extension is to construct models for various gauge groups with the scalars transforming under different representations. To our knowledge, they have not been studied in the literature. In this section, we explore an interesting class of these models that are nontrivial generalizations offering a rich phase structure. As usual, we are concerned with symmetry breaking at large N.

A. Modifying the constraint

One approach, a straightforward one, is to choose some other gauge group G in place of U(M) but to leave the constraint $ZZ^\dagger=I_M$ unchanged. In other words, we take the Z fields to belong to an arbitrary representation R of dimension M and multiplicity N of some chosen gauge group G. We may still represent the Z fields in the form of an $M\times N$ matrix. The transformation matrix U is now in R acting on the matrix Z as before, $Z\rightarrow UZ$. The Lagrangian is still of the form we came across earlier in Eq. (8), but with the auxiliary gauge field A_μ now taking values in the Lie algebra of G. An expression for the auxiliary field and the form of the Lagrangian involving the Z fields alone can be easily derived. They are given, respectively, by Eqs. (17) and (18) given below. However, symmetry breaking at large N remains the same. This is because the gauge fields are set to zero in our discussion of the phase structure.

A more interesting generalization occurs when the constraint is modified as well. Again, we take the Z fields to be in any representation R of dimension M and multiplicity N of a gauge group G. We look for a Lagrangian that resembles (8). It is clearly gauge invariant with the auxiliary gauge field A_μ transforming as in (7). Note that the part of the Lagrangian quadratic in $A_\mu=A_\mu^a T_a$ is proportional to

$$A_\mu^a A_\mu^b \mathrm{tr}(T_a T_b Z Z^\dagger) = A_\mu^a A_\mu^b \mathrm{tr}(T_{ab} Z Z^\dagger),$$

where T_a's are the generators of the gauge group G and $T_{ab} = (T_a T_b + T_b T_a)/2$. Earlier in the GM, the constraint $ZZ^\dagger = I_M$ was responsible for rendering it quadratic in A_μ alone. This resulted in a well-defined expression for the auxiliary field A_μ as a composite of the Z fields. Now, more generally, we achieve the same goal by imposing the following constraint instead:

$$\mathrm{tr}(T_{ab} Z Z^\dagger) = l\delta_{ab}. \tag{16}$$

Here l is the Dynkin index of the representation R defined by $\mathrm{tr}(T_a T_b) = l\delta_{ab}$. Note that this new constraint also respects gauge symmetry. Using it, it is easy to obtain the following expression for the auxiliary field:

$$A_\mu^a = \frac{i}{2l}\mathrm{tr}\left[T_a(Z\partial_\mu Z^\dagger - \partial_\mu Z Z^\dagger)\right]. \tag{17}$$

Later in Sec. V, we observe that this field becomes dynamical at large N and is justly called a composite gauge field constructed out of the Z's. An expression for the Lagrangian in terms of the Z fields alone can now be obtained:

$$L = \beta N \mathrm{tr}(\partial_\mu Z \partial_\mu Z^\dagger) + \frac{1}{4l}\beta N\left\{\mathrm{tr}\left[T_a(Z\partial_\mu Z^\dagger - \partial_\mu Z Z^\dagger)\right]\right\}^2. \tag{18}$$

This derivation should ensure gauge invariance and this is easily seen to be the case. Incorporating the constraint into the Lagrangian using a Lagrange multiplier $\Sigma = \Sigma_{ab} T_{ab}$ leads to an expression that agrees with (9). The symmetry-breaking effects could be potentially different since the matrix Σ is not an arbitrary one any more.

First, we make sure that the constraint (16) is different from the earlier one $ZZ^\dagger = I_M$. $ZZ^\dagger = I_M$ clearly is a solution of (16). Fortunately, there are cases where this is not the only solution. The new constraint is in some cases weaker than the earlier one. To see this, introduce a Hermitian matrix W by $ZZ^\dagger = I_M + W$ and observe that the new constraint is equivalent to looking for a solution of $\mathrm{tr}(T_{ab} W) = 0$. Given a W that leads to a positive semidefinite $ZZ^\dagger$, Z is solvable generally as $Z = (I_M + W)^{1/2} Z_0$ for some Z_0 obeying $Z_0 Z_0^\dagger = I_M$. The earlier constraint corresponds to the trivial solution $W = 0$. That there exist cases where W is nontrivial can be seen as follows. Make the ansatz that W is in the Lie algebra itself, that is, $W = W_a T_a$. Now, the constraint $\mathrm{tr}(T_{ab} W) = 0$ simply states that the Adler-Bell-Jackiw anomaly associated with the representation R should vanish. That there exist anomaly-free representations is well known. A simple example is the doublet of the gauge group SU(2) that yields a triplet for W. We will use this example later to illustrate our approach. A more interesting example is the spinor representation **16** of the gauge group SO(10) that is well known to be anomaly-free. The ansatz gives a solution for W that is in the adjoint representation **45**. This is interesting, for it is known that an adjoint scalar is a promising candidate to break a grand unified theory based on SO(10). Its appearance here is quite unexpected.

In some instances, the above ansatz gives the most general solution. This is the case with both the examples mentioned above. This can be seen from representation theory. Regard W as belonging to the product representations $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$ and $\mathbf{16} \times \overline{\mathbf{16}} = \mathbf{1} + \mathbf{45} + \mathbf{210}$ in the SU(2) and SO(10) examples, respectively. The component representations in these decompositions should separately obey the constraint. The singlet appearing in both cases is ruled out easily. This is because the singlet component being proportional to identity gives $\mathrm{tr}(T_{ab} W) \propto \delta_{ab}$, violating the constraint. In the case of SO(10), we need to exclude the **210** as well. It is easily observed that there exist $\mathbf{16} \times \overline{\mathbf{16}}$ traceless Hermitian matrices violating the constraint. This excludes **210** because such matrices can only have components along the **45** and **210**, and the **45** alone cannot violate the constraint. It is not to be deduced, however, that the ansatz always gives the most general solution. For instance, if one were to pick a sufficiently large representation of the gauge group for R, one will easily end up with more representations that remain unsuppressed in W. But representation theory should still be applicable to solve for W in general.

Let us call the models as type-1 models when W solves identically to zero and our new constraint reduces to the old one. They are closer to the Grassmannian models discussed earlier, or rather to their generalizations mentioned in the beginning of this section involving subgroups of U(M). The other models where W can be nontrivial are referred to as type-2 models. Models based on a reducible R are quite generally of type-2. This is because the constraint does not determine some components of W, for instance, those connecting different subrepresentations in R. Note, however, that a reducible R arising from an irreducible one repeated, say, q times, though appears to be of type 2, can be recast as of type 1 by combining q and N to an overall multiplicity qN in place of N. We prefer to view them as type-1 models with multiplicity qN. Type-1 models are thus necessarily based on irreducible R's.

The effective potential at large N is again of the form (11) encountered earlier. The SPE's governing the phases are $\Sigma Z = 0$ obtained by varying $Z^\dagger$ and

$$\beta \mathrm{tr}\left[T_{ab}(ZZ^\dagger - I_M)\right] + \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left(T_{ab}\frac{1}{k^2 I_M + \Sigma}\right) = 0 \tag{19}$$

obtained by varying Σ. For type-1 models, the traces can be dropped and the equation becomes equivalent to the one we had earlier for the GM [see Eq. (12)]. Solutions can be sought in the same manner. The phase structure is governed by a critical β given by Eq. (15). It is ensured that the ansatz for Σ one makes in solving these equations is consistent with its definition $\Sigma = \Sigma_{ab}T_{ab}$. This is because the final conclusion involves either the broken phase $\Sigma = 0$ (for $\beta > \beta_c$) or the unbroken phase $\Sigma = \sigma I_M$ (for $\beta < \beta_c$). This is equivalent to $\Sigma_{ab} = 0$ or $\Sigma_{ab} = \sigma\delta_{ab}/C_2(R)$, where $C_2(R)$ is the second Casimir invariant of the irreducible representation R defined by $T_aT_a = C_2(R)I_M$, and is acceptable.

For type-2 models, the above equation can still be reduced to that of the GM, Eq. (12), but with a matrix W satisfying $\mathrm{tr}(T_{ab}W) = 0$ on the right-hand side (RHS):

$$\beta\left(ZZ^\dagger - I_M\right) + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 I_M + \Sigma} = \beta W. \tag{20}$$

The factor β on the RHS makes this equation agree with $ZZ^\dagger = I_M + W$ at the level of expectation values. Again, the solutions $\Sigma = 0$ (for $\beta > \beta_c$) and $\Sigma = \sigma I_M$ (for $\beta < \beta_c$) will satisfy this equation, for W can clearly be chosen to be zero. As before, $\Sigma = \sigma I_M$ is acceptable for irreducible R's as it follows from $\Sigma_{ab} = \sigma\delta_{ab}/C_2(R)$. In the case of reducible R's, that is, $R = \sum_i R_i$ where each R_i is irreducible, the solution for $\beta < \beta_c$ is a bit more involved. We try again the ansatz $\Sigma_{ab} = \sigma\delta_{ab}$. In this case Σ is not proportional to identity. Instead, it is a diagonal matrix taking values $C_2(R_i)\sigma$ along each representation R_i. We look for a W matrix that is also diagonal, with values w_i along R_i. Note that the constraint $\mathrm{tr}(T_{ab}W) = 0$ requires the w_i's to satisfy $\sum_i l_i w_i = 0$ where the l_i's are the indices of the representations R_i's. The SPE along R_i is

$$-\beta + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + C_2(R_i)\sigma} = \beta w_i. \tag{21}$$

The constraint on the w_i's gives

$$-\beta\sum_i l_i + \sum_i l_i \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + C_2(R_i)\sigma} = 0. \tag{22}$$

This determines σ. Individual equations simply determine the various w_i's. That this solution holds only for $\beta < \beta_c$ is easy to note.

The solutions obtained so far are not *a priori* the most general ones. There could be more solutions. This does not, however, appear to be the case with the examples mentioned earlier based on the gauge groups SU(2) or SO(10) [see Eqs. (28) and (32) and the discussions following them]. An analogous situation occurs for the gauge group E_6 with its representation **27**. This is perhaps illustrative of a generic phenomenon or suggestive of the need to look at larger representations that might lead to more solutions. We do not wish to go into those details here; rather, we find it more rewarding to consider another possibility, that of adding a potential.

B. Adding a potential

The unexpected appearance of an adjoint scalar W apparently did not help us in a partial breaking of the gauge group. The situation changes drastically when a potential is introduced, leading to a rich phase structure. The adjoint scalar, which has not played any significant role so far, plays a major one in the presence of a potential. Note that there is no simple way to incorporate a potential in the canonical GM without spoiling the global symmetries or the constraint equation. But, interestingly, the generalized models of the previous section, governed by the gauge-invariant Lagrangian of Eq. (18) constructed with scalars alone, do allow for potential terms. As we will see, the presence of a potential leads to drastically different conclusions. These models with such phase structures are relevant in model building.

Let us keep the potential quite general to begin with, $\beta N\mathrm{tr}V(ZZ^\dagger)$, where $V(\cdot)$ is an ordinary function of its argument, a polynomial, for instance. This is expected to be a nontrivial extension of type-2 models unlike the case of type-1 models in which the constraint $ZZ^\dagger = I_M$ reduces this to the addition of a constant. It is convenient to introduce a composite field variable X for $ZZ^\dagger$ and write the potential as $\beta N\mathrm{tr}V(X)$. The requirement $X = ZZ^\dagger$ can be incorporated with the help of a Lagrange multiplier Y, adding a term $\beta N\mathrm{tr}(YZZ^\dagger - YX)$ to the potential. As before, constraint (16) can be accommodated with the help of a Lagrange multiplier $\Sigma = \Sigma_{ab}T_{ab}$. Its effect is, as we know, to add a term $\beta N\mathrm{tr}(\Sigma ZZ^\dagger - \Sigma)$ to the potential. After translating Y to $Y - \Sigma$ for convenience, the total Lagrangian looks like

$$L = \beta N\mathrm{tr}\left[D_\mu Z(D_\mu Z)^\dagger + V(X) + YZZ^\dagger - YX + \Sigma X - \Sigma\right]. \tag{23}$$

The large N effective potential is now computable:

$$V_{\mathrm{eff}} = \beta N\mathrm{tr}\left[V(X) + YZZ^\dagger - YX + \Sigma X - \Sigma\right] + N\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\,\ln\left(k^2 I_M + Y\right). \tag{24}$$

The SPE's are obtained by extremizing this potential. Varying X determines Y to be $\Sigma + V'(X)$ where a prime denotes differentiation with respect to the argument. Varying Σ gives $\mathrm{tr}[T_{ab}(X - I_M)] = 0$. As before, one may look for a solution of this in the form $X = I_M + W$ where W satisfies $\mathrm{tr}(T_{ab}W) = 0$. Varying Y and using these solutions yields

$$\beta\left(ZZ^\dagger - I_M\right) + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 I_M + \Sigma + V'(I_M + W)} = \beta W. \tag{25}$$

This is to be supplemented with $YZ = [\Sigma + V'(I_M + W)]\,Z = 0$ obtained by varying $Z^\dagger$. This system of equations resembles the ones obtained earlier [see Eq. (20)], with Σ replaced by $\Sigma + V'(I_M + W)$. The presence of $V'(I_M + W)$, however, is suggestive of a different phase structure.

For type-1 models $W = 0$, and $V'(I_M + W)$ just adds a constant to Σ. This can be absorbed into Σ because these models, being based on an irreducible R, allow for the addition of a term proportional to identity to Σ. As expected in the beginning of this section, this is a trivial extension. However, this is not the case for type-2 models and we expect a rich phase structure.

An example we will use later is a potential of sixth degree in Z and $Z^\dagger$ that leads to $V'(I_M + W) = aI_M + bW + cW^2$ for some constants a, b, and c. For a model based on an irreducible R, the term aI_M can be absorbed into Σ as we have already noted. When W is in the adjoint representation, the term cW^2 can also be absorbed into Σ. As a result, the SPE's to be solved are

$$\beta\left(ZZ^\dagger - I_M\right) + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 I_M + \Sigma + bW} = \beta W, \quad (26)$$

and $[\Sigma + bW]\,Z = 0$. These equations are difficult to handle analytically and we present our numerical results for SU(2) and SO(10) below. We find that they do have solutions for a range of parameters when $\beta < \beta_c$ and $b < 0$.

Given all such solutions for $\beta < \beta_c$, the next step is to determine those that are preferred energetically. The GM solutions of Sec. II leave the gauge group unbroken for $\beta < \beta_c$ whereas those found here break it at least partially. Which one is preferred is of course determined by the effective potential. In other words, one needs to compute V_{eff} for all the solutions and pick the one (or more) that has the the lowest value. We do this along a chosen path in the parameter space of β and b that crosses all the phases. We find that some of the new solutions end up always having the lowest potential. In other words, for a range of parameters, a partial breaking of the gauge group is preferred over the unbroken case. Details are presented in the next section. The following expression for the effective potential at a saddle point (SP) is used to this end:

$$V_{\text{eff}}(\text{SP}) = -\beta N \text{tr}\left(\Sigma + bW^2/2\right) + N \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln\left(k^2 I_M + \Sigma + bW\right). \quad (27)$$

We are, as always, concerned with an adjoint W for an irreducible R. In the above expression, terms aI and cW^2 have been absorbed into Σ for convenience.

IV. TWO EXAMPLES: SU(2) AND SO(10)

A. Case of SU(2)

The example based on SU(2) with doublet Z's is the simplest and the most convenient one to illustrate the ideas presented above. The matrix $\Sigma = \Sigma_{ab} T_{ab}$ is now proportional to identity, hence chosen to be σI_2. The W scalar, being a triplet, is taken to be along the σ_3 direction, that is, $W = w\sigma_3$. First we consider the case when the Z fields develop no expectation value. The resulting SPE's in the presence of a potential are

$$-\beta + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \sigma + bw} = \beta w,$$
$$-\beta + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \sigma - bw} = -\beta w. \quad (28)$$

Note that there is no solution to these equations when $b = 0$, that is, in the absence of a potential, other than the one discussed earlier where $W = 0$ and Σ is proportional to identity. In the presence of a sixth degree potential given by $V'(I_M + W) = aI_M + bW + cW^2$, these equations do have solutions for a range of parameters. This can be seen by treating $x = (\sigma + bw)/\Lambda^2$ and $y = (\sigma - bw)/\Lambda^2$ as independent variables to determine β and w from the above two equations. Given x and y and knowing w, one obtains b from $x - y = 2bw/\Lambda^2$. The resulting equations are

$$1 - \frac{\beta}{\beta_c} = \frac{1}{2} x \ln\left(1 + \frac{1}{x}\right) + \frac{1}{2} y \ln\left(1 + \frac{1}{y}\right),$$
$$-\frac{\beta_c}{\beta} \frac{b}{\Lambda^2} = \frac{x - y}{x \ln(1 + 1/x) - y \ln(1 + 1/y)}. \quad (29)$$

For the momentum integrals to remain well defined, x and y should be positive (or zero). The region of the parameter space of β and b is obtained by letting x and y vary from zero to infinity. This falls in between the curves (a) and (b) shown in Fig. 1. There are two solutions for a given β and b in this regime, but they are related to each other by an interchange of x and y and should be

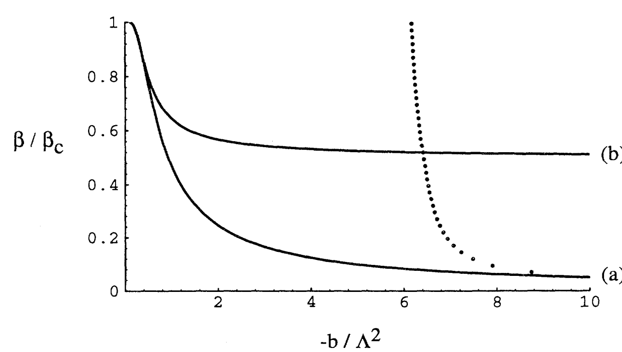

FIG. 1. The phase diagram in the case of SU(2) obtained solving Eqs. (29) and (31). The effective potential of Fig. 2 is computed along the dotted line. Parameters in the theory are β and b, and Λ is the momentum cutoff. Details are given in the text.

treated as one. Note that all these solutions yield $\beta < \beta_c$ and $b < 0$. Curve (a) has $x = y$ and is the critical line. In fact, the region below curve (a) is a critical surface where all the masses vanish. Curve (b) has one of x, y zero. Symmetry breaking involved here is from SU(2) to U(1).

There exist new solutions for a nonzero Z as well. Giving an expectation value diag$(0, v^2)$ for $ZZ^\dagger$, the second equation in (28) gets replaced by

$$\beta v^2 - \beta + \beta_c = -\beta w, \tag{30}$$

where we have set $y = \sigma - bw$ to zero to satisfy $(\Sigma + bW)Z = 0$. Treating $x = (\sigma + bw = 2bw)/\Lambda^2$ and $y' = v^2\beta/\beta_c$ as independent variables to determine the others,

$$1 - \frac{\beta}{\beta_c} = \frac{1}{2} x \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2} y',$$
$$-\frac{\beta_c}{\beta}\frac{b}{\Lambda^2} = \frac{x}{x \ln(1 + 1/x) + y'}, \tag{31}$$

one notes the presence of solutions in a parameter range for positive x and y'. Here as well, we require $\beta < \beta_c$ and a negative b. The parameter range is the one above curve (b) in Fig. 1. SU(2) symmetry is now completely broken. Giving an expectation value diag$(v^2, 0)$ for $ZZ^\dagger$ is equivalent to this case and leads to no new solutions. A nonzero $ZZ^\dagger$ of the form diag(v_1^2, v_2^2) requires $v_1^2 = v_2^2$ and coincides with the completely broken case of the GM discussed in Sec. II.

These are all the solutions. Now, consider all those for $\beta < \beta_c$. The corresponding one of the GM leaves the gauge group unbroken whereas those of this section break it at least partially. As discussed earlier, the effective potential needs to be examined to determine the preferred solution. We have chosen a path suitably fixing y crossing all the curves, shown as a dotted line in Fig. 1. Figure 2 is a plot of the effective potential. The upper curve is for the GM solutions and the lower one is for the new solutions. Note that the lower sheet ends up always having the lowest potential. In other words, for β small, a partial breaking of the gauge group is preferred over the unbroken case.

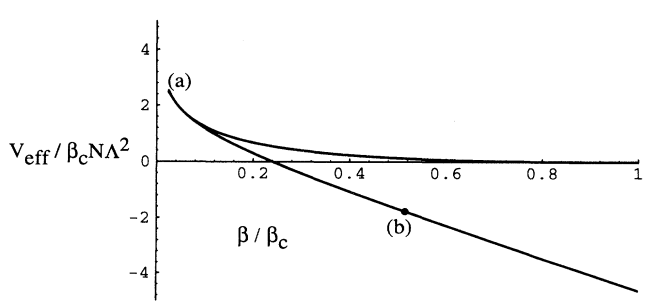

FIG. 2. A plot of the effective potential (with its zero appropriately chosen) versus β for a path shown dotted in Fig. 1 crossing all the curves. The crossings are denoted by (a) and (b). The upper curve corresponds to the unbroken case and the lower one corresponds to symmetry breaking as discussed in the text.

B. Case of SO(10)

We now come to the second example, the gauge group SO(10) with the Z's in the representation **16**. Here the possibilities for symmetry breaking are too many. We do not hope to address all of them, rather simply pick one possibility: SO(10) breaking to SU(5) or to its maximal subgroup SU(5)× U(1). Under SU(5), **16** of SO(10) decomposes to $\mathbf{10}(1) + \bar{\mathbf{5}}(-3) + \mathbf{1}(5)$ where the U(1) charges are given in parantheses. The unbroken symmetry for $Z = 0$ corresponds to those generators that commute with the ansatz for Σ. To have symmetry breaking to SU(5)×U(1), we hence choose $Z = 0$ and take $\Sigma_{ab} = \sigma\delta_{ab}$ along the SU(5) directions, ρ along the U(1), and zero otherwise. Note that the a or b index runs over the adjoint representation **45** of SO(10) that under SU(5) decomposes to $\mathbf{24} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{1}$. Our ansatz for Σ_{ab} corresponds to having it nonzero for (a, b) along $(24, 24)$ and $(1, 1)$. One could have it nonzero along $(10, \bar{10})$ and $(\bar{10}, 10)$ as well, but it turns out that this can be absorbed into σ and ρ. This means that the Σ matrix is diagonal with values $C_2(10)\sigma + \rho$, $C_2(\bar{5})\sigma + 9\rho$, and 25ρ along the representations **10**, $\bar{\mathbf{5}}$, and **1**, respectively. With $C_2(10)/C_2(\bar{5}) = 3/2$ and a suitable scaling of σ, we may take them to $3\sigma + \rho$, $2\sigma + 9\rho$, and 25ρ. The W matrix is taken to be along the U(1) direction; in other words, it is diagonal with values w, $-3w$, and $5w$. It is now straightforward to write down the SPE's in the presence of a potential:

$$-\beta + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + 3\sigma + \rho + bw} = \beta w,$$

$$-\beta + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + 2\sigma + 9\rho - 3bw} = -3\beta w,$$

$$-\beta + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + 25\rho + 5bw} = 5\beta w. \tag{32}$$

Consider first the case of no potential, that is, $b = 0$. There are no new solutions. This is because the Σ eigenvalues $3\sigma + \rho$, $2\sigma + 9\rho$, and 25ρ are either in the ascending order or in the descending order (or equal) and hence one cannot obtain the alternating signs on the RHS above. Allowing for nonvanishing Z does not improve the situation. Note that $ZZ^\dagger$ and Σ should have non-negative eigenvalues and $\Sigma ZZ^\dagger = 0$ requires at least $3\sigma + \rho$ or 25ρ to vanish to allow for a nonzero eigenvalue of $ZZ^\dagger$.

Solutions exist in the presence of a potential for a range of parameters. To see this, as in the SU(2) case, treat $x = (3\sigma + \rho + bw)/\Lambda^2$ and $y = (2\sigma + 9\rho - 3bw)/\Lambda^2$ as independent variables to determine β and w from the first two equations and $z = (25\rho + 5bw)/\Lambda^2$ from the last one. Given x and y and knowing z and w, one obtains b from $2x - 3y + z = 16bw/\Lambda^2$. Thus Eq. (32) yields

$$1-\frac{\beta}{\beta_c}=\frac{3}{4}x\,\ln\left(1+\frac{1}{x}\right)+\frac{1}{4}y\,\ln\left(1+\frac{1}{y}\right),$$
$$-\frac{\beta_c}{\beta}\frac{b}{\Lambda^2}=\frac{(2x-3y+z)/4}{x\,\ln(1+1/x)-y\,\ln(1+1/y)},\qquad(33)$$

where z is a solution of

$$z\,\ln\left(1+\frac{1}{z}\right)=2x\,\ln\left(1+\frac{1}{x}\right)-y\,\ln\left(1+\frac{1}{y}\right).\qquad(34)$$

Note that x, y, and z are required to remain positive (or zero) to keep the momentum integrals well defined. Solutions exist in a certain domain of x and y, giving rise to a range for the parameters for $\beta<\beta_c$ and $b<0$ [6]. The results of our numerical investigation are presented in Fig. 3. There are in fact two solutions for a given β and b in the region between the curves (a) and (b), and one solution between the curves (b) and (c). In other words, one of the solutions extends from curve (a) to curve (b) while the other from curve (a) to curve (c). Curve (a) has $x=y=z$ and is the critical line. Here, too, the region below curve (a) is a critical surface wherein all the masses vanish. Curve (b) has $z=0$ and curve (c) has $y=0$. The symmetry breaking involved here is from SO(10) to SU(5)×U(1) as noted before.

Solutions exist for nonzero Z. Consider giving an expectation value v^2 for $ZZ^\dagger$ along the singlet in the decomposition $\mathbf{16}=\mathbf{10}+\bar{\mathbf{5}}+\mathbf{1}$. In this case, the third equation in (32) gets replaced by

$$\beta v^2-\beta+\beta_c=5\beta w,\qquad(35)$$

where we have set $z=(25\rho+5bw)/\Lambda^2$ to zero to satisfy $(\Sigma+bW)Z=0$. Again, treat x and y as independent variables and determine the others to obtain new solutions for positive v^2 in a range of parameters:

$$1-\frac{\beta}{\beta_c}=\frac{3}{4}x\,\ln\left(1+\frac{1}{x}\right)+\frac{1}{4}y\,\ln\left(1+\frac{1}{y}\right),$$
$$-\frac{\beta_c}{\beta}\frac{b}{\Lambda^2}=\frac{(2x-3y)/4}{x\,\ln(1+1/x)-y\,\ln(1+1/y)},\qquad(36)$$

where we require

$$\frac{\beta}{\beta_c}v^2=-2x\,\ln\left(1+\frac{1}{x}\right)+y\,\ln\left(1+\frac{1}{y}\right)\geq 0.\qquad(37)$$

Here, as well, one has $\beta<\beta_c$ and $b<0$. The region of the parameter space covered by these solutions (one solution for a given β and b) is that in between the dashed curves (b) and (d) of Fig. 3. Curve (d) has $x=0$. The surviving symmetry here is SU(5) because a nonvanishing Z along the singlet in the decomposition $\mathbf{16}=\mathbf{10}+\bar{\mathbf{5}}+\mathbf{1}$ breaks the U(1) subgroup of SU(5)×U(1) as well. There are other possibilities. Giving an expectation value for $ZZ^\dagger$ along $\bar{\mathbf{5}}$ (instead of the singlet) also leads to a solution which falls above curve (c); solutions are also noted to exist for a nonzero $ZZ^\dagger$ along $\mathbf{10}$ and $\mathbf{1}$ [extending above curve (d)] or $\mathbf{10}$ and $\bar{\mathbf{5}}$ [extending beyond that of $\bar{\mathbf{5}}$]. All of these, however, break the gauge group completely.

What we have in the end is a two-sheeted cover of the parameter space above the critical curve (a) in Fig. 3. One of them (call it the upper sheet) is through the solid curves while the other one (call it the lower sheet) is through the dashed curves. They meet along curve (a). There is of course one more sheet (call it the top sheet) for the solutions of our earlier case of the unbroken gauge group covering all of the parameter space for $\beta<\beta_c$. This too meets the other two sheets along curve (a). For every point on any one of the sheets, there is a solution.

As we have noted earlier, there could be more solutions. For instance, there is the possibility that a solution breaking SO(10) to SU(3)×SU(2)×U(1) [perhaps, with an additional U(1)] exists. The number of variables and the number of equations at least match, each being 6, but the number of equations makes analysis complicated. There are more possibilities such as SO(10)→SU(4), SU(4)×U(1), etc. Each case has to be handled separately; a general treatment has eluded us.

As before, which solution is preferred is determined by the effective potential. For this, one needs to compute $V_{\rm eff}$ for all the solutions and pick the one (or more) that has the the lowest value. In the present case, this is not an easy task given the number of possibilities involved. Hence we will be content with doing this numerically for

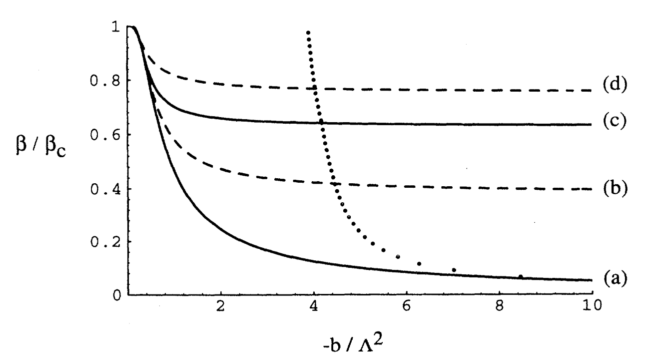

FIG. 3. The phase diagram in the case of SO(10) obtained solving Eqs. (33) and (36). The effective potential of Fig. 4 is computed along the dotted line. Parameters in the theory are β and b, and Λ is the momentum cutoff. Details are given in the text.

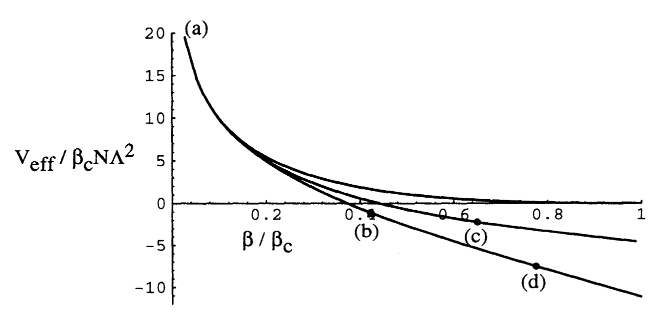

FIG. 4. A plot of the effective potential (with its zero appropriately chosen) versus β for a path shown dotted in Fig. 3 crossing all the curves. The crossings are denoted by (a), (b), (c), and (d). The uppermost curve corresponds to the unbroken case and the lower two correspond to symmetry breaking as discussed in the text.

the solutions found above. We have chosen a path suitably fixing y in the lower sheet and z in the upper sheet crossing all the curves, shown as a dotted line in Fig. 3. Figure 4 is a plot of this effective potential for the three sheets involved. The uppermost curve is for the top sheet, the middle one is for the upper sheet, and the lowermost one is for the lower sheet. Note that the lower sheet ends up always having the lowest potential. In other words, for $\beta < \beta_c$ but not close to it, a partial breaking of the gauge group is preferred over the unbroken case.

V. PROPERTIES OF THE COMPOSITES

Here we study the various composites encountered earlier, the gauge bosons and composite Higgs particles. Their properties can be read off from the effective action. Like the effective potential, the effective action at large N is obtained by integrating away the Z fields. It involves the original action plus a correction of the form

$$S_{\rm eff} = -N \ \mathrm{Tr} \ln\left(-D^2 + \Sigma\right), \tag{38}$$

where "Tr" stands for a complete trace, that is, a trace over the internal indices and an integral over the space-time coordinates. D is as before the covariant derivative $D_\mu = \partial_\mu - iA_\mu$. For simplicity, we have absorbed the bW term into Σ. To identify the kinetic terms for the various fields, a derivative expansion is needed. This is carried out in the Appendix. The result is that the kinetic terms for the Σ fields arise from

$$\frac{N}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\int_0^\infty \frac{dt}{t}e^{-tk^2}\int_0^t dt_1 \mathrm{tr}\ \mathcal{D}_\mu e^{-(t-t_1)\Sigma}\mathcal{D}_\mu e^{-t_1\Sigma}, \tag{39}$$

where $\mathcal{D}_\mu \mathcal{O}$ is $[D_\mu, \mathcal{O}]$ for any $\mathcal{O}$. At the next order in the expansion, one obtains an analogous result for the gauge fields:

$$\frac{N}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\int_0^\infty \frac{dt}{t^2}e^{-tk^2}\int_0^t dt_1 t_1(t-t_1) \mathrm{tr}\ F_{\mu\nu} e^{-(t-t_1)\Sigma}F_{\mu\nu} e^{-t_1\Sigma}. \tag{40}$$

These general results are valid for any gauge group.

The two phases of the GM, the completely broken and unbroken ones discussed in Sec. II, can be handled in this generic setup. In the unbroken phase where $\Sigma = \sigma I, Z = 0$ the kinetic terms for the gauge fields simplify to

$$\frac{1}{2g^2(\sigma/\Lambda^2)}\int d^4x \ \mathrm{tr}\ F_{\mu\nu}^2, \tag{41}$$

where g^2 is the coupling constant:

$$\begin{aligned}\frac{1}{g^2(\sigma/\Lambda^2)} &= \frac{N}{6}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\sigma)^2} \\ &= \frac{N}{96\pi^2}\left[\ln\left(1+\Lambda^2/\sigma\right) - \frac{1}{1+\sigma/\Lambda^2}\right].\end{aligned} \tag{42}$$

This also follows from the well-known contribution to the running gauge coupling constant at one-loop order in the presence of N fundamental scalars. In the broken phase where $\Sigma = 0, ZZ^\dagger = (1-\beta_c/\beta)I_M$, the above computation of the induced kinetic terms for the gauge fields suffers from an infrared divergence. Introducing an infrared cutoff μ for k^2/Λ^2, we obtain a result that coincides with the above one with σ/Λ^2 replaced by μ. The mass terms arise from the kinetic terms for the Z fields that read

$$\beta N \mathrm{tr}\left(A_\mu^2 ZZ^\dagger\right) = N(\beta-\beta_c)\mathrm{tr}\left(A_\mu^2\right). \tag{43}$$

The mass squared for A_μ is then $N(\beta-\beta_c)g^2(\mu)$. As expected, it is positive for $\beta > \beta_c$ and vanishes at the critical point $\beta = \beta_c$.

We now turn our attention to type-2 models, in particular to the models based on SU(2) and SO(10). The mass-squared results we obtain are all expressible in units of Λ^2. All our results for the kinetic terms and mass terms involve the expectation values of the scalar field Σ through x and y [and in the case of SO(10) z as well] parameters. In the phase where $ZZ^\dagger \neq 0$, the expectation value of $ZZ^\dagger$ also appears in the expressions through v^2. As we have seen in the previous section, all these parameters are expressible in terms the basic ones β/β_c and b/Λ^2 by solving a set of equations. When solving the equations, however, we found it convenient to treat x and y as independent variables to determine β/β_c and b/Λ^2. This helped us to discover multiple solutions; but once we have chosen the energetically preferred solution, the relation between the x, y parameters and the basic ones is one to one and is hence invertible.

A. Case of SU(2)

First we consider SU(2) and compute the induced kinetic terms for the gauge fields in the U(1) phase where $\Sigma = \sigma I_2 + bw\sigma_3$ and $Z = 0$. Writing

$$F_{\mu\nu} = \frac{1}{\sqrt{2}}F_{\mu\nu}^+\sigma_+ + \frac{1}{\sqrt{2}}F_{\mu\nu}^-\sigma_- + \frac{1}{2}F_{\mu\nu}^3\sigma_3, \tag{44}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, we obtain the kinetic terms for the gauge bosons of the unbroken U(1) and for those of the broken generators from Eq. (40). We have

$$\mathrm{tr}\ F_{\mu\nu}e^{-(t-t_1)\Sigma}F_{\mu\nu}e^{-t_1\Sigma} = e^{-t\sigma}\left\{\frac{1}{2}\cosh(tbw)\left(F^3_{\mu\nu}\right)^2 + \cosh[(t-2t_1)bw]F^+_{\mu\nu}F^-_{\mu\nu}\right\}. \tag{45}$$

The kinetic term for the unbroken U(1) turns out to be

$$\frac{1}{8}\left(\frac{1}{g^2(x)}+\frac{1}{g^2(y)}\right)\int d^4x\left(F^3_{\mu\nu}\right)^2, \tag{46}$$

where g^2 is given in (42), and as in Sec. IV, $x=(\sigma+bw)/\Lambda^2, y=(\sigma-bw)/\Lambda^2$. For the broken generators, the result is

$$\frac{1}{2G^2(x,y)}\int d^4xF^+_{\mu\nu}F^-_{\mu\nu}\ , \tag{47}$$

where

$$\frac{1}{G^2(x,y)} = \frac{N}{2(bw)^2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^2+\sigma}{2bw}\ln\frac{k^2+\sigma+bw}{k^2+\sigma-bw}-1\right]. \tag{48}$$

The integration over k^2 can be done:

$$\frac{1}{G^2(x,y)} = \frac{N}{8\pi^2(x-y)^2}\left[\left(\frac{x+y}{4}\right)\frac{I(x)-I(y)}{x-y}+\frac{1}{3}\frac{J(x)-J(y)}{x-y}-\frac{1}{2}\right], \tag{49}$$

where the functions I and J are

$$\begin{aligned} I(x) &= \ln(1+x)-x^2\ln(1+1/x)+x, \\ J(x) &= \ln(1+x)+x^3\ln(1+1/x)+x/2-x^2. \end{aligned} \tag{50}$$

$G^2(x,y)$ is positive for x,y positive which is the region of interest. It is a monotonically increasing function of $x+y$ in this region. It tends to $g^2(x)$ as we approach the critical line $x=y$. This is as it should be, since the SU(2) gauge symmetry is unbroken along the critical line and the kinetic terms for the broken and unbroken generators should add up to form SU(2)-invariant kinetic terms.

The gauge bosons of the broken generators receive mass terms. This arises from Eq. (39). Taking Σ to be space-time independent and writing

$$A_\mu = \frac{1}{\sqrt{2}}A^+_\mu\sigma_+ + \frac{1}{\sqrt{2}}A^-_\mu\sigma_- + \frac{1}{2}A^3_\mu\sigma_3, \tag{51}$$

one first finds

$$\mathrm{tr}\ \mathcal{D}_\mu e^{-(t-t_1)\Sigma}\mathcal{D}_\mu e^{-t_1\Sigma} = 4e^{-t\sigma}\sinh[(t-t_1)bw]\sinh(t_1bw)A^+_\mu A^-_\mu. \tag{52}$$

This yields, for the mass terms,

$$\mathcal{V}^2(x,y)\int d^4xA^+_\mu A^-_\mu\ , \tag{53}$$

where

$$\mathcal{V}^2(x,y) = N\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^2+\sigma}{(k^2+\sigma)^2-(bw)^2}-\frac{1}{2bw}\ln\frac{k^2+\sigma+bw}{k^2+\sigma-bw}\right]. \tag{54}$$

Here too, the integral over k^2 can be done, but it is instructive to rewrite the result in terms of $G^2(x,y)$. Note that this integral can be obtained by differentiating the one in Eq. (48) with respect to σ. This results in

$$\mathcal{V}^2(x,y) = -\frac{1}{2}\Lambda^2(x-y)^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left[\frac{1}{G^2(x,y)}\right]. \tag{55}$$

It then follows that $\mathcal{V}^2(x,y)$ is positive for x,y positive, the region we are interested in. As we approach the critical line $x=y$,

$$\mathcal{V}^2(x,y) \to -\frac{1}{2}\Lambda^2(x-y)^2\frac{d}{dx}\left(\frac{1}{g^2(x)}\right). \tag{56}$$

The mass squared for $A^\pm_\mu$ is given by

$$\begin{aligned} M^2 &= G^2(x,y)\mathcal{V}^2(x,y) \\ &\to \frac{1}{2}\Lambda^2(x-y)^2\frac{d}{dx}\ln g^2(x) \quad \text{as } x\to y. \end{aligned} \tag{57}$$

It is thus expressible in terms of the running coupling constant close to the critical line and vanishes along the critical line as expected.

For the completely broken phase $\Sigma = \mathrm{diag}(x,0)\Lambda^2$ and $ZZ^\dagger = \mathrm{diag}(0,v^2)$, we still have the above results but with $y=0$. The induced kinetic term for the U(1) field now suffers from an infrared divergence and suggests introducing an infrared cutoff μ for y. The mass-squared result obtained above, though relevant with $y=0$, now receives an additional contribution from the kinetic terms for the Z fields:

$$\beta N \mathrm{tr}\left(A_\mu^2 ZZ^\dagger\right) = \frac{1}{2}\beta N v^2 A_\mu^+ A_\mu^- + \frac{1}{4}\beta N v^2 \left(A_\mu^3\right)^2 . \quad (58)$$

This makes the U(1) field A_μ massive with a mass squared $\approx \beta N v^2 g^2(\mu)/2$. The additional contribution to mass squared for $A_\mu^\pm$ is $\beta N v^2 G^2(x,0)/2$.

B. Case of SO(10)

The computations for SO(10) are along the same lines. Consider the SU(5)×U(1) phase. Here $\Sigma = \mathrm{diag}(x,y,z)\Lambda^2$ along **10**, $\bar{\mathbf{5}}$, and **1**, where, as in Sec. IV, $x = (3\sigma+\rho+bw)/\Lambda^2, y = (2\sigma+9\rho-3bw)/\Lambda^2, z = (25\rho+5bw)/\Lambda^2$. One then easily computes the kinetic terms for the SU(5) generators:

$$\frac{1}{4}\left[\frac{3}{g^2(x)}+\frac{1}{g^2(y)}\right]\int d^4x \left(F_{\mu\nu}^A\right)^2 , \quad (59)$$

where A runs over the 24 generators. The function g^2 has been defined earlier in Eq. (42). The part involving $g^2(x)$ arises from the **10**, while the one involving $g^2(y)$ from the $\bar{\mathbf{5}}$. The factor of 3 is a consequence of the fact that $\mathrm{tr}(T_aT_b)$ for the **24** T's of SU(5) is 3 times in the **10** as in the $\bar{\mathbf{5}}$. For the broken generators along the **10** and $\overline{\mathbf{10}}$ appearing in the decomposition $\mathbf{45} = \mathbf{24}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{1}$, we have

$$\frac{1}{2}\left[\frac{3}{G^2(x,y)}+\frac{1}{G^2(x,z)}\right]\int d^4x \left|F_{\mu\nu}^a\right|^2 , \quad (60)$$

where a runs over the **10** generators and G^2 is the same function defined earlier in Eq. (49). The kinetic term for the U(1) field is

$$\frac{1}{16}\left[\frac{2}{g^2(x)}+\frac{9}{g^2(y)}+\frac{5}{g^2(z)}\right]\int d^4x \left(F_{\mu\nu}^{45}\right)^2 , \quad (61)$$

where the superscript 45 denotes the U(1) direction. Note that as we approach the critical line $x=y=z$, gauge symmetry breaking disappears, $G^2 \to g^2$ and Eqs. (59), (60), and (61) add up to form SO(10)-invariant kinetic terms as expected.

It is straightforward to compute the mass terms for the **10** and $\overline{\mathbf{10}}$ gauge bosons. First we note that

$$\mathrm{tr}\, \mathcal{D}_\mu e^{-(t-t_1)\Sigma}\mathcal{D}_\mu e^{-t_1\Sigma} = 4\Big\{3e^{-t(x+y)/2}\sinh[(t-t_1)(x-y)/2]\sinh[t_1(x-y)/2] + e^{-t(x+z)/2}\sinh[(t-t_1)(x-z)/2]\sinh[t_1(x-z)/2]\Big\}\left|A_\mu^a\right|^2 , \quad (62)$$

where a Λ^2 has been absorbed into the t's on the RHS. Comparing this with Eq. (52), we get, for the mass terms,

$$\left[3\mathcal{V}^2(x,y)+\mathcal{V}^2(x,z)\right]\left|A_\mu^a\right|^2 , \quad (63)$$

where $\mathcal{V}^2$ is the same function defined earlier in Eq. (55). This gives the mass squared

$$M^2 = \left[\frac{3}{G^2(x,y)}+\frac{1}{G^2(x,z)}\right]^{-1}\left[3\mathcal{V}^2(x,y)+\mathcal{V}^2(x,z)\right] \quad (64)$$

for the **10**'s. As x, y, and z tend to be the same,

$$M^2 \to \frac{1}{8}\Lambda^2\left[3(x-y)^2+(x-z)^2\right]\frac{d}{dx}\ln g^2(x). \quad (65)$$

Here too, one obtains an expression in terms of the running coupling constant close to the critical line. It vanishes along the critical line as expected.

For the SU(5) phases $\Sigma = \mathrm{diag}(x,y,0)\Lambda^2$ and $ZZ^\dagger = \mathrm{diag}(0,0,v^2)$, the above results are still relevant but with $z=0$. As in the case of SU(2), the induced kinetic term for the U(1) field suffers from an infrared divergence and suggests introducing an infrared cutoff μ for z. The mass-squared result obtained above, though relevant with $z=0$, receives an additional contribution from the kinetic terms for the Z fields:

$$\beta N \mathrm{tr}\left(A_\mu^2 ZZ^\dagger\right) = \frac{1}{2}\beta N v^2 \left|A_\mu^a\right|^2 + \frac{5}{8}\beta N v^2\left(A_\mu^{45}\right)^2 . \quad (66)$$

This makes the U(1) field massive with a mass squared $\approx \beta N v^2 g^2(\mu)/2$. The additional contribution to the mass squared for the **10**'s is

$$\frac{1}{2}\beta N v^2\left[\frac{3}{G^2(x,y)}+\frac{1}{G^2(x,0)}\right]^{-1} . \quad (67)$$

VI. GLOBAL SYMMETRY AND THE GOLDSTONE MODES

All the models we discussed have a global U(N) symmetry. In addition to the gauge symmetry, this global symmetry could also suffer a breakdown. It remains to investigate this breaking and the resulting Goldstone bosons and other massless particles if any.

In the Grassmannian model, of the two phases, the unbroken phase retains this global symmetry. In this phase, only Σ gets an expectation value, but Σ is a singlet under the global symmetry. The Σ expectation value, however, gives mass to all the Z scalars. The model has no massless particles in this phase. In the broken phase, Z gets an expectation value breaking the global symmetry in addition to the gauge symmetry. The Σ expectation value is now zero and all of the $2NM$ real components of Z are hence massless. Some of these are the would-be Goldstone bosons, eaten by the broken gauge generators. Because the gauge symmetry U(M) is completely broken down, they are M^2 in number. It turns out that all those remaining, $2NM - M^2$ in number, are the Goldstone bosons associated with the broken generators of the global symmetry. There are no unaccounted massless particles. To see this, choose the Z expectation value to be of the form

$$Z \propto (I_M, 0_{M,N-M}) , \tag{68}$$

where I_M is an identity matrix of order M and $0_{M,N-M}$ is a zero matrix of order $M \times (N-M)$. This expectation value breaks the global symmetry from U(N) down to U($N-M$). The number of broken global generators is now easily computed; they are $2NM - M^2$ in number. Along with the would-be Goldstone bosons, they account for all the $2NM$ massless particles.

Coming to our type-2 models, the global symmetry is broken down when Z gets an expectation value. In the SU(2)$\to$U(1) phase or the SO(10)$\to$SU(5)$\times$U(1) phase of our examples, the global symmetry remains unbroken. Σ, a singlet under the global symmetry, also picks up an expectation value in this phase, making all the Z scalars massive. There are no massless states. In the other interesting phase of our examples, Z picks up an expectation value along some direction, a singlet of SU(5) in the case of SO(10). The Σ expectation value in that direction is forced to zero. This will introduce $2N$ real massless states of Z. The Z expectation value can be arranged to be of the form

$$Z \propto (v, 0_{M,N-1}) , \tag{69}$$

where v is a column vector pointing in the singlet direction. This implies that the global symmetry is broken down from U(N) to U($N-1$). There are $2N-1$ Goldstone bosons associated with this breaking. The remaining one massless state of Z is a would-be Goldstone boson eaten by the broken gauge generator. This is consistent with the fact that the Z expectation value of the above type breaks one additional gauge generator. Again, there are no unaccounted massless states.

VII. DISCUSSIONS AND CONCLUSIONS

We have presented an approach to composite gauge bosons that allows for partially broken gauge symmetries. It is a generalization of the well-known Grassmannian models that otherwise allow for either unbroken or completely broken gauge symmetries. In our approach, it is also possible to incorporate interesting potential terms, leading to a rich phase structure. Even the simplest model based on SU(2) is not amenable to analytical handling of its phases; numerical investigation is called for. For models that are physically interesting in connection with unified theories, even a numerical analysis of all the phases is a challenging endeavor.

We have illustrated our approach with an SU(2) example and analyzed in some detail an SO(10) example that could be of interest to unified models. What is remarkable in this exercise is that a set of equations governed by only two parameters gives rise to a rich set of solutions with interesting symmetry-breaking patterns. There exist regions of the parameter space where SU(2) breaks down to U(1). In the case of SO(10), symmetry breaking to SU(5) or to SU(5)$\times$U(1) or perhaps to some other subgroups is possible. These examples help realize our goal of constructing an induced gauge theory with composite gauge bosons having partial symmetry breaking.

We have computed the properties of the composite fields, the gauge bosons, and the Higgs scalars, by doing a derivative expansion of the large N effective action. Because we need an expansion that does not perturb the Higgs field, we cannot utilize the canonical expansions available in the literature. We have developed a suitable derivative expansion in the Appendix and have used it to compute the kinetic terms and the mass terms for the composites in the various phases.

We have not addressed the issue of the renormalizability of Grassmannian models or our generalized ones. It is interesting to note that the theory at large N exhibits a critical point which extends to a critical line in the presence of a potential. It is known that the critical points or lines can, and in many cases do, soften the ultraviolet behavior. This softening is probably not sufficient enough to help renormalize the theory in four dimensions and one may have to include other relevant operators in the Lagrangian. In this connection, we note that certain four-dimensional Grassmannian models of composite gauge fields have been studied on the lattice and shown to be renormalizable [4]. Their phase structures and their relation to continuum theories remain unexplored.

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APPENDIX: DERIVATIVE EXPANSION

In this appendix, we carry out a derivative expansion of the effective action

$$\begin{aligned} S_{\text{eff}} &= -N \,\mathrm{Tr}\ln\left(-D^2 + \Sigma\right) \\ &= -N \int_0^\infty \frac{dt}{t} \mathrm{Tr}\left(e^{tD^2 - t\Sigma}\right) , \end{aligned} \tag{A1}$$

where "Tr" represents an integral over space-time and a trace over the internal indices. In the second step above, we have used the Schwinger representation. It is an equivalent representation at the level of equations of motion and at all orders in the derivative expansion except the lowest one. The lowest order yielding the effective potential is handled separately in the paper. The Schwinger representation involves an exponential rather than the logarithm and is hence better suited for analysis. We thus have to compute the "trace"

$$\begin{aligned}\mathrm{Tr}\left(e^{tD^2-t\Sigma}\right) &= \int d^4x \frac{d^4k}{(2\pi)^4} e^{-ikx} \mathrm{tr}\ e^{tD^2-t\Sigma}\left(e^{ikx}\right) \\ &= \int d^4x \frac{d^4k}{(2\pi)^4} \mathrm{tr}\ e^{t(ik+D)^2-t\Sigma}(1) \\ &= \int d^4x \frac{d^4k}{(2\pi)^4} e^{-tk^2} \mathrm{tr}\ e^{-t\Sigma+t(2ikD+D^2)}(1),\end{aligned} \tag{A2}$$

where "tr" is a trace over the internal indices. One now expands the exponential inside the trace and computes different terms to obtain a series representation for the effective action. In the literature, to our knowledge, one expands the Σ term as well. This is not suited for our purpose as we intend to keep all orders in Σ. This suggests that we do a perturbation theory in $2ikD + D^2$ alone. To this end, we use the following result due to Feynman:

$$e^{-t(H+V)} = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots dt_n e^{-(t-t_1)H}(-V)e^{-(t_1-t_2)H}(-V)\cdots(-V)e^{-t_nH}. \tag{A3}$$

In our case $H = \Sigma$ and $-V = 2ikD + D^2$. This leads to an expansion in $2ikD + D^2$. Rearranging the terms one obtains a derivative expansion, that is, an expansion in D:

$$S_{\text{eff}} = -N \int_0^{\infty} \frac{dt}{t} \mathrm{Tr}\left(e^{tD^2-t\Sigma}\right) = -N \sum_{n=0}^{\infty} \int d^4x \int \frac{d^4k}{(2\pi)^4} \int_0^{\infty} \frac{dt}{t} e^{-tk^2} \mathrm{tr}\ a_n, \tag{A4}$$

where a_n is of order D^{2n}. The calculations are quite involved. The result to order D^4 is

$$\begin{aligned} a_0 &= e^{-t\Sigma}, \\ a_1 &= -\frac{1}{2}\int_0^t dt_1 \mathcal{D}_\mu e^{-(t-t_1)\Sigma} \mathcal{D}_\mu e^{-t_1\Sigma}, \\ a_2 &= -\frac{1}{2t}\int_0^t dt_1 t_1 (t-t_1) F_{\mu\nu} e^{-(t-t_1)\Sigma} F_{\mu\nu} e^{-t_1\Sigma}, \end{aligned} \tag{A5}$$

where $\mathcal{D}\mathcal{O}$ for some object $\mathcal{O}$ is $[D, \mathcal{O}]$. The result for a_2 is not complete. However, our interest is in its contribution to terms quadratic in A_μ with a space-time-independent Σ and in this respect it is complete.

A brief account of the calculations now follows. We write the expansion (A3) symbolically as

$$e^{-t(H+V)} = \sum_{n=0}^{\infty} \bullet\!\sim (-V) \sim (-V) \sim \cdots \sim (-V) \sim\!\bullet\ , \tag{A6}$$

where again $H = \Sigma$ and $-V = 2ikD + D^2$. The symbol $\sim$ denotes a "propagator" of the kind $\exp[-(t_1 - t_2)H]$. The beginning and the end of a term of the above kind are indicated by black dots. First we note that only terms with an even number of D's are relevant. Those with an odd number of D's come with an odd number of k's and their contributions vanish after the k integration. Quite often, we make use of the following reduction to simplify results:

$$f(t'\mathrm{s}) \cdots \mathcal{O}(t_{i-1}) \sim 1 \sim \mathcal{O}(t_{i+1}) \cdots = \int_{t_i}^{t_{i-1}} d\tau f(t_i \to \tau) \cdots \mathcal{O}(t_{i-1}) \sim \mathcal{O}(t_i) \cdots, \tag{A7}$$

given any function f. On the RHS, t_i is first absent and we have hence replaced t_{i+1} by t_i and so on with t's of higher indices. If, for instance, the function f were absent or is independent of t_i, this reduction introduces $t_{i-1} - t_i$ in the RHS. Another property we make use of to simplify results is the presence of an overall trace and a space-time integral that lets us rearrange terms in some expressions.

Now we come to the calculations. At the lowest order we have $a_0 = \sim = \exp(-t\Sigma)$, giving us the effective potential. At the next order,

$$a_1 = \bullet\!\sim 2ikD \sim 2ikD \sim\!\bullet + \bullet\!\sim D^2 \sim\!\bullet\ . \tag{A8}$$

The first term can be simplified:

$$\bullet\!\sim 2ikD \sim 2ikD \sim\!\bullet = -4k_\mu k_\nu \bullet\!\sim D_\mu \sim D_\nu \sim\!\bullet = -\frac{2}{t} \bullet\!\sim D \sim D \sim\!\bullet\ , \tag{A9}$$

where we have replaced $k_\mu k_\nu$ by $\delta_{\mu\nu}/(2t)$ as the two would yield identical results after k integration. If $\mathcal{D}$ represents the action $[D, \mathcal{O}]$ for any $\mathcal{O}$ immediately next to it, one easily verifies that

$$\begin{aligned} \bullet\!\!\sim D \sim D \sim\!\!\bullet &= \bullet\!\!\sim \mathcal{D} \sim D \sim\!\!\bullet + \bullet\!\!\sim 1 \sim D^2 \sim\!\!\bullet \\ &= \bullet\!\!\sim \mathcal{D} \sim D \sim\!\!\bullet + (t-t_1) \bullet\!\!\sim D^2 \sim\!\!\bullet \, . \end{aligned} \tag{A10}$$

Alternately

$$\begin{aligned} \bullet\!\!\sim D \sim D \sim\!\!\bullet &= - \bullet\!\!\sim D\mathcal{D} \sim 1 \sim\!\!\bullet + \bullet\!\!\sim D^2 \sim 1 \sim\!\!\bullet \\ &= - \bullet\!\!\sim \mathcal{D}^2 \sim 1 \sim\!\!\bullet - \bullet\!\!\sim \mathcal{D} \sim D \sim\!\!\bullet + t_1 \bullet\!\!\sim D^2 \sim\!\!\bullet . \end{aligned} \tag{A11}$$

Adding the two results,

$$2 \bullet\!\!\sim D \sim D \sim\!\!\bullet = - \bullet\!\!\sim \mathcal{D}^2 \sim 1 \sim\!\!\bullet + t \bullet\!\!\sim D^2 \sim\!\!\bullet \, . \tag{A12}$$

Putting these together, we have

$$a_1 = \frac{1}{t} \bullet\!\!\sim \mathcal{D}^2 \sim 1 \sim\!\!\bullet = \frac{1}{2} \bullet\!\!\sim \mathcal{D}^2 \sim\!\!\bullet = -\frac{1}{2} \mathcal{D} \sim \mathcal{D} \sim\!\!\bullet \, , \tag{A13}$$

which is the result quoted earlier. The second step follows from the sum of

$$\begin{aligned} &\bullet\!\!\sim \mathcal{D}^2 \sim 1 \sim\!\!\bullet = t_1 \bullet\!\!\sim \mathcal{D}^2 \sim\!\!\bullet , \\ &\bullet\!\!\sim \mathcal{D}^2 \sim 1 \sim\!\!\bullet = \bullet\!\!\sim 1 \sim \mathcal{D}^2 \sim\!\!\bullet = (t-t_1) \bullet\!\!\sim \mathcal{D}^2 \sim\!\!\bullet . \end{aligned} \tag{A14}$$

The next coefficient a_2 can be computed along similar lines. The result quoted earlier is obtained by keeping only terms quadratic in A_μ with a space-time-independent Σ. This simplifies the calculations. The leftmost and rightmost D_μ's get replaced by $-iA_\mu$. This gives two A_μ's already so that those D_μ's in the middle get replaced by ∂_μ. The contributions and their simplified results are

$$\begin{aligned} \bullet\!\!\sim D^2 \sim D^2 \sim\!\!\bullet &= \bullet\!\!\sim \partial \cdot A \sim \partial \cdot A \sim\!\!\bullet , \\ \bullet\!\!\sim D^2 \sim 2ikD \sim 2ikD \sim\!\!\bullet &= -\frac{2}{t}(t_1 - t_2) \bullet\!\!\sim \partial \cdot A \sim \partial \cdot A \sim\!\!\bullet , \\ \bullet\!\!\sim 2ikD \sim D^2 \sim 2ikD \sim\!\!\bullet &= -\frac{2}{t}(t_1 - t_2) \bullet\!\!\sim \partial_\mu A_\nu \sim \partial_\mu A_\nu \sim\!\!\bullet , \\ \bullet\!\!\sim 2ikD \sim 2ikD \sim D^2 \sim\!\!\bullet &= -\frac{2}{t}(t_1 - t_2) \bullet\!\!\sim \partial \cdot A \sim \partial \cdot A \sim\!\!\bullet , \\ \bullet\!\!\sim 2ikD \sim 2ikD \sim 2ikD \sim 2ikD \sim\!\!\bullet &= 16 k_\mu k_\nu k_\rho k_\sigma \bullet\!\!\sim A_\mu \sim i\partial_\nu \sim i\partial_\rho \sim A_\sigma \sim\!\!\bullet . \end{aligned} \tag{A15}$$

Here and in the following, a ∂ immediately left to an A as in $\partial \cdot A$ or $\partial_\mu A_\nu$ acts only on that A, that is, $\partial \cdot A = \partial_\mu (A_\mu)$, for instance. Note that the replacement

$$k_\mu k_\nu k_\rho k_\sigma \to \frac{1}{4t^2} \left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} \right) \tag{A16}$$

simplifies the last contribution to

$$\frac{2}{t^2}(t_1 - t_2)^2 \left(2 \bullet\!\!\sim \partial \cdot A \sim \partial \cdot A \sim\!\!\bullet + \bullet\!\!\sim \partial_\mu A_\nu \sim \partial_\mu A_\nu \sim\!\!\bullet \right) . \tag{A17}$$

Adding all the contributions, one gets

$$\left[1 - \frac{2}{t}(t_1 - t_2) \right]^2 \bullet\!\!\sim \partial \cdot A \sim \partial \cdot A \sim\!\!\bullet - \frac{2}{t}(t_1 - t_2) \left[1 - \frac{1}{t}(t_1 - t_2) \right] \bullet\!\!\sim \partial_\mu A_\nu \sim \partial_\mu A_\nu \sim\!\!\bullet . \tag{A18}$$

Further simplification is possible due to

$$f(t_1 - t_2) \bullet\!\!\sim \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet = t_1 f(t - t_1) \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet \tag{A19}$$

and

$$f(t_1 - t_2) \bullet\!\!\sim \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet = (t - t_1) f(t_1) \bullet\!\!\sim \mathcal{O} \sim \mathcal{O} = (t - t_1) f(t_1) \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet , \tag{A20}$$

given any object $\mathcal{O}$. In our case $f(t - t_1) = f(t_1)$ so that adding and dividing by 2, we get

$$f(t_1 - t_2) \bullet\!\!\sim \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet = \frac{t}{2} f(t_1) \mathcal{O} \sim \mathcal{O} \sim\!\!\bullet . \tag{A21}$$

This simplifies the total contribution to

$$\frac{1}{2t}(t-2t_1)^2\partial\cdot A \sim \partial\cdot A \sim\!\bullet - \frac{1}{t}t_1(t-t_1)\partial_\mu A_\nu \sim \partial_\mu A_\nu \sim\!\bullet . \tag{A22}$$

Note that if one were to work in the Lorentz gauge $\partial\cdot A = 0$ the first term will vanish and the kinetic terms for the gauge fields will arise from the second term. We will regard the second term to be a part of the gauge-invariant combination

$$a_2 = -\frac{1}{2t}\int_0^t dt_1 t_1(t-t_1)F_{\mu\nu}e^{-(t-t_1)\Sigma}F_{\mu\nu}e^{-t_1\Sigma}. \tag{A23}$$

However, the $(\partial\cdot A)^2$ term that this generates does not agree with what we obtained. This is to be expected since there are other gauge-invariant combinations, for instance, $\mathcal{D}^2e^{-(t-t_1)\Sigma}\mathcal{D}^2e^{-t_1\Sigma}$, that could generate them.

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