Fluctuations of the Casimir pressure at finite temperature

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Using standard techniques of quantum field theory at finite temperature T we determine the correlation function for the photon field in the presence of two parallel conducting plates and apply it to investigate the fluctuations of the Casimir pressure. Theoretical estimates of these fluctuations are given. We study especially the cases of the low and the high temperature limits. In the high temperature limit we substitute the Planck distribution by the Rayleigh-Jeans distribution leading to a classical correlation function. This function allows a derivation of the Casimir pressure, but not of its fluctuations because these functions cannot be multiplied. Therefore, the fluctuation of the Casimir pressure in the high temperature limit has to be determined from the complete expression at finite temperature. The resulting fluctuations are rather large. For measurements over long times these fluctuations increase with T^7 whereas the corresponding Casimir pressure contains the factor T only. The leading term of the fluctuations is independent of the distance d between the two plates. It is identical to the one plate case or twice that of the free case. Only the nonleading corrections depend on d and special assumptions of the properties of the plates.

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I. INTRODUCTION

In quantum theory the physical state is characterized by the expectation values of observables and by their fluctuations. Especially interesting is the ground state, which is the vacuum state for vanishing temperature. Important observables are the electromagnetic field strength [1], pressure, and energy densities [2].

Here we consider a two-plate system with superconducting plates. We define the Casimir pressure as the difference of the pressure on the different sides of the plate. Its fluctuation and correlated problems have been studied only recently [2–4]. These investigations should be extended to the case of a ground state at finite temperature.

The correlation functions, as special Green's functions which depend on the space-time points and on the temperature, have to be determined. Therefore, we apply a real-time technique of the quantum field theory at finite temperature which at least is correct for the case of free fields [5]. We use two procedures. First, we obtain the Wightman-like functions according to standard relations from the given time-ordered functions [6]. Second, we use the operator formalism [7]. Both approaches lead to the same functions. Because of the vanishing mass of the photon field, it is possible to derive reasonably explicit analytical expressions. It turns out that the correlation functions consist of the contributions resulting from the zero-point fluctuations and the temperature-induced fluctuations. This structure influences the fluctuations of the electromagnetic field strength, the pressure, and the Casimir force.

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Our aim is to investigate the fluctuations of the Casimir pressure, initiated by Barton [2], for ground states at finite temperature. Technically, we follow the methods applied in [4]. It is important that the pressure components be already bilinear functions of the photon field, so that we have to study expectation values of four field operators. By application of Wick's theorem, we obtain products of two Green's functions so that the fluctuations contain now a multiplicative mixing between vacuum- and temperature-induced fluctuations as well as their expected additive superposition.

Specifically, we investigate measurements of the pressure fluctuations at the inner side of a plate of the two-plate system. The proposed measurement is local with respect to the position on the plate, but extended over a sufficiently long finite-time interval (long-time measurement). We determine the temperature-dependent correction terms of the pressure fluctuations in comparison with the already known vacuum contributions [2,4]. We obtain quite different correction terms for "thin" plates $\beta^{-1}\exp(-\beta\frac{\pi}{d})$ and "thick" plates β^{-3}, where β is the inverse temperature T and d is the distance between the plates. As in the vacuum case [4], the fluctuations are more sensitive to special properties of the plates, i.e., whether they are "thick" or "thin," than the pressure itself. The reason is that long-time measurements detect the low-energy part of the spectrum which itself is different for the considered plates. Note that both types of plates describe ideally conducting plates satisfying the classical boundary conditions [4]. They differ in the treatment of the mode propagating parallel to the plates.

We study the high-temperature limit of free quantum electrodynamics, performing this limit in the two-point correlation function by substituting the Planck distribution by the Rayleigh-Jeans distribution. The vacuum part of the correlation function can now be neglected and the temperature-dependent smooth part changes to a generalized function with bad multiplication properties. The classical correlation function for the electromagnetic potentials contains spacelike correlations; lightlike correlations are present, but suppressed. However, the correlations of the electromagnetic field strength contain spacelike correlations as well as δ-type lightlike correlations. This is explicitly demonstrated for free blackbody radiation.

For the two-plate system, we derive the Casimir pressure directly in this limit [8–11], but not its fluctuations. The reason is that the classical correlation functions cannot be multiplied as needed. This result is remarkable because at finite temperature these functions can be multiplied without difficulties. The vacuum part represented by a Wightman function can be generally multiplied and the multiplication of the temperature-dependent part as a smooth function also gives no difficulties. In the high-temperature limit, the correlation function appears as a generalized function, defined on the real axis only. The resultant δ-type distributions cannot be multiplied without arbitrariness. Thus fluctuations of the Casimir pressure cannot be considered using correlation functions taken in the limit. This means that a theory based on the classical two-point correlation function will run into difficulties if it is necessary to multiply Green's functions with identical arguments. These problems can be solved by going back to the theory at finite temperature, carrying out the necessary calculations, and performing the high-temperature limit as the last step.

We find that for measurements over long times the leading behavior of the fluctuations is distance independent and corresponds to the fluctuations in the presence of one plate or twice that of the free space. The nonleading terms depend on the special properties of the plates. If we compare the order of the power behavior with respect to β^{-1} for $\beta \to 0$ of the Casimir pressure $\sim (\beta)^{-1}$ with that of its fluctuations $\sim (\beta)^{-7}$, then the fluctuations are remarkably large.

We will now proceed in the second section to study the necessary Green's functions. In the following section, we will investigate the fluctuations of the Casimir pressure at low temperature. In the fourth section, we will consider the limit of very high temperature. Appendixes will contain a procedure for determining the correlation functions in the operator formalism and explicit expressions for Green's functions.

II. GREEN'S FUNCTIONS AT FINITE TEMPERATURE IN THE PRESENCE OF PLATES

A. General considerations

For the determination of temperature-dependent expectation values, we need a formulation of quantum field theory at finite temperature T. There are, in principle, two approaches, the imaginary and the real-time technique [6]. Here we are interested in the real-time technique because we expect a temperature and a time dependence of the correlation functions.

In our case we have additional difficulties due to the presence of boundary conditions. However, these problems are already solved for Feynman propagators at vanishing temperature [12]. The generalizations to nonvanishing temperature are straightforward [11].

In most cases, it is sufficient to have a technique on the level of Feynman diagrams. So it is quite natural to construct the Wightman-like functions from the propagators as known from standard quantum field theory. We apply this procedure, as well as in the case of temperature-dependent Green's functions.

For simplicity, we consider a scalar field theory with the field operator $\phi(x)$. The Wightman-like function $\langle\phi(x)\phi(x')\rangle_\beta$ can be defined with the help of time-ordered functions $\langle T\phi(x)\phi(x')\rangle_\beta$:

$$\langle\phi(x)\phi(x')\rangle_\beta = \Theta(x-x')\langle T\phi(x)\phi(x')\rangle_\beta + \Theta(x'-x)\langle T^*\phi(x)\phi(x')\rangle_\beta. \tag{1}$$

The temperature-dependent time-ordered functions can be expressed by the propagators at $T = 0$:

$$\begin{aligned}\langle T\phi(x)\phi(x')\rangle_\beta &= \frac{1}{i}D^c_\beta(x,x') \\ &= \frac{1}{i}\int\frac{dk_0}{2\pi}e^{ik_0(x_0-x'_0)}\{D^c(k_0,\vec{x},\vec{x}') + \sinh^2\theta[D^c(k_0,\vec{x},\vec{x}') - D^{c*}(k_0,\vec{x},\vec{x}')]\} \\ &= \frac{1}{i}D^c(x,x') + \frac{1}{i}\int\frac{dk_0}{2\pi}e^{ik_0(x_0-x'_0)}\sinh^2\theta[D^c(k_0,\vec{x},\vec{x}') - D^{c*}(k_0,\vec{x},\vec{x}')]\end{aligned} \tag{2}$$

and, analogously,

$$\langle T^*\phi(x)\phi(x')\rangle_\beta = -\frac{1}{i}D^{*c}_\beta(x,x') = -\frac{1}{i}D^{c*}(x,x') + \frac{1}{i}\int\frac{dk_0}{2\pi}e^{ik_0(x_0-x'_0)}\sinh^2\theta[D^c(k_0,\vec{x},\vec{x}') - D^{c*}(k_0,\vec{x},\vec{x}')]. \tag{3}$$

Here we use the notation for the propagator at vanishing temperature,

$$\langle T\phi(x)\phi(x')\rangle = \frac{1}{i}D^c(x,x'),$$
$$D^c(x,x') = \int \frac{dk_0}{2\pi} e^{ik_0(x_0-x_0')} D^c(k_0,\vec{x},\vec{x}'), \tag{4}$$

and the temperature-dependent weight factors

$$\cosh^2\theta = \frac{1}{1-e^{-\beta|k_0|}}, \quad \sinh^2\theta = \frac{1}{e^{\beta|k_0|}-1}, \tag{5}$$

where $\beta = 1/(kT)$. In this way, we get a representation consisting of two parts. The first contribution is the Wightman function for vanishing temperature. The second part is temperature dependent and does not satisfy the standard analytic properties of the Wightman functions:

$$\langle\phi(x)\phi(x')\rangle_\beta = \langle\phi(x)\phi(x')\rangle + \frac{1}{i}\int \frac{dk_0}{2\pi} e^{ik_0(x_0-x_0')} \sinh^2\theta[D^c(k_0,\vec{x},\vec{x}') - D^{c*}(k_0,\vec{x},\vec{x}')] \tag{6}$$

or

$$D^-_\beta(x,x') = D^-(x,x') + D^1_\beta(x,x'),$$
$$D^1_\beta(x,x') = \int \frac{dk_0}{2\pi} e^{ik_0(x_0-x_0')} \sinh^2\theta[D^c(k_0,\vec{x},\vec{x}') - D^{c*}(k_0,\vec{x},\vec{x}')]. \tag{7}$$

In the presence of boundary conditions, the temperature-dependent Wightman-like function ${}^sD^-_\beta(x,x')$ can be obtained by replacing in Eq. (7) the free space functions D^-, D^c, and D^{c*} by the functions ${}^sD^-$, ${}^sD^c$, and ${}^sD^{c*}$ which fulfill the necessary boundary conditions. We check that Eq. (6) is correctly constructed by an independent determination of these functions. As in the case of a vanishing temperature [4], it seems helpful to have an operator formalism [7]. Detailed considerations are given in Appendix A.

The generalization to quantum electrodynamics is not trivial. We want to take into account the boundary conditions [12] for ideal conductors and a covariant gauge condition. For ideal conductors, the boundary conditions $E_t = B_n = 0$ can be written in terms of the electromagnetic potentials A_μ by

$$\epsilon_{\mu\nu\rho\sigma} n^\rho \partial^\sigma A^\nu|_S = 0, \tag{8}$$

where n^ρ denotes the normal vector of the surface S. For the case of parallel plates perpendicular to the x_3 direction, we get the following general structure of all Green's functions [4]:

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = i\left(g_{\tilde{\mu}\tilde{\nu}} - \frac{\partial_{\tilde{\mu}}\partial_{\tilde{\nu}}}{\tilde{\partial}^2}\right){}^sD^c(\tilde{x}-\tilde{y},x_3,y_3)$$
$$+i\begin{pmatrix} \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\partial}^2} & 0 \\ 0 & -1 \end{pmatrix} D^c(x-y)$$
$$-in_\mu n_\nu \tilde{D}^c(\tilde{x}-\tilde{y}), \tag{9}$$

where ${}^sD^c(x,y)$ denotes the scalar Feynman propagator satisfying the Dirichlet boundary condition, $D^c(x-y)$ a free Feynman propagator, and $\tilde{D}^c$ a special Feynman propagator. In the physical situation, we are considering, the third direction plays a special role. Therefore, we write $\tilde{x} = (x^0,x^1,x^2)$, and $\tilde{\nu}$ runs from 0 to 2.

Now we specify the properties of the plates. We may assume that these ideal plates are infinitely "thin" or "thick." In either case, we have the same boundary condition (8), but the mode propagating parallel to the plates, not subject to the boundary conditions, can be treated as a plane wave extended over the full x_3 axis or restricted to the "inner" region between the two plates. For "thin" plates, we consider the complete three-dimensional Euclidean space, whereas for "thick" plates, the space between the two plates only. Both cases are idealized ones and do not involve the penetration depth of electromagnetic waves into the metal of the plate [13,14]. If the penetration depth is to be used, it is best used for the wave propagating parallel to the plates. Both assumptions lead mostly to the same physical results; but in special cases, differences can be seen [4].

Now to continue the explanation of (9), for "thin" plates $D^c(x-y)$ is the standard free space Feynman propagator and $\tilde{D} \equiv 0$. For the case of "thick" plates, we make sure that the free propagator is defined in the physical region correctly; i.e., it should correspond to a self-adjoint operator. For the special case of two-plate systems, we have to take into account that $\tilde{D}^c$ is a third contribution.

Formally, the same tensor representations are valid for all other types of Green's functions in electrodynamics. With the help of these formulas we can now write down immediately the corresponding expressions for the temperature-dependent Wightman-like functions:

$$\langle A_\mu(x)A_\nu(x')\rangle_\beta = i\,{}^s D^-_{\beta\,\mu\nu}(x,x')$$
$$= i\left(g_{\tilde\mu\tilde\nu} - \frac{\partial_{\tilde\mu}\partial_{\tilde\nu}}{\tilde\partial^2}\right){}^s D^-_\beta + i\begin{pmatrix} \frac{\tilde\partial_\mu\tilde\partial_\nu}{\tilde\partial^2} & 0 \\ 0 & -1 \end{pmatrix} D^-_\beta - i n_\mu n_\nu \tilde D^-_\beta . \tag{10}$$

Here we use $\tilde z = (z^0, z^1, z^2) = \tilde x - \tilde x'$. The scalar temperature-dependent Green's functions are constructed according to the rules (7) given above. In the presence of boundary conditions Eq. (7) reads

$${}^s D^-_\beta(x,x') = {}^s D^-(x,x') + \int \frac{dk_0}{2\pi} e^{ik_0(x_0 - x_0')} \sinh^2\theta[{}^s D^c(k_0,\vec x,\vec x') - {}^s D^{c*}(k_0,\vec x,\vec x')]. \tag{11}$$

More specific expressions can be found in the following subsection and Appendix B.

B. Special Green's functions

Now to study explicit expressions for temperature-dependent Green's functions, we consider the cases of free space, one plate, or two plates. Taking into account the general structure of Eq. (7), an investigation of the scalar Green's function is sufficient.

1. Free space

We use the conventions and notation of [15]:

$$D^-_\beta(x-x') = D^-(x-x') + D^1_\beta(x-x'), \tag{12}$$

where $D^-(x-x') = i\int \frac{d^4k}{(2\pi)^3}\exp[ik(x-x')]\delta(k^2)\Theta(-k_0)$. The temperature-dependent contribution D^1_β has the explicit form

$$D^1_\beta(x,x') = i\int \frac{d^4k}{(2\pi)^3} e^{ik(x-x')}\delta(k^2)\frac{e^{-\beta|k_0|}}{1-e^{-\beta|k_0|}} \tag{13}$$
$$= i\int \frac{d^3\vec k}{2k_0(2\pi)^3}(e^{ik_0(x_0-x_0')} + e^{-ik_0(x_0-x_0')})e^{i\vec k(\vec x-\vec x')}\frac{e^{-\beta k_0}}{1-e^{-\beta k_0}}. \tag{14}$$

In Eq. (14), k_0 is the positive expression $k_0 = \sqrt{\vec k^2}$. Equivalent representations are

$$D^-_\beta(z) = \frac{-i}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(z_0 - i\epsilon\delta_{n0} - in\beta)^2 - \vec z^2} \tag{15}$$
$$= \frac{i\pi}{4\pi^2 2|\vec z|\beta}\left(\coth\frac{\pi}{\beta}[|\vec z| - (z_0 - i\epsilon)] + \coth\frac{\pi}{\beta}[|\vec z| + (z_0 - i\epsilon)]\right). \tag{16}$$

The last expression will be used in studying the high-temperature limit.

2. Presence of one plate

All functions for the case of one plate can be obtained by an application of the reflection principle using the corresponding formulas of free space. For this reason we write down only one representation as an example:

$${}^s D^-_{1,\beta}(x,x') = -\frac{i}{4\pi^2}\frac{\pi}{2|\vec z|\beta}\left[\coth\frac{\pi}{\beta}(z_0 - i\epsilon - |\vec z|) - \coth\frac{\pi}{\beta}(z_0 - i\epsilon + |\vec z|)\right]$$
$$+\frac{i}{4\pi^2}\frac{\pi}{2|\vec z_1|\beta}\left[\coth\frac{\pi}{\beta}(z_0 - i\epsilon - |\vec z_1|) - \coth\frac{\pi}{\beta}(z_0 - i\epsilon + |\vec z_1|)\right], \tag{17}$$

where

$$|\vec{z}_1| = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 + x_3')^2]^{1/2}.$$

Note that in this case $\tilde{D}$ vanishes identically.

3. *Presence of two plates*

The case of two plates is more complicated. We write down the expressions valid for the space between the two plates. For the regions outside the two-plate system, the Green's functions are those for the case of one plate. According to the representation (10), we need the scalar Green's function ${}^sD^-_{2,\beta}$ satisfying the Dirichlet boundary condition at the plate positions $x_3 = 0$ and $x_3 = d$. For $T = 0$, these functions are given in [4]:

$$ {}^sD_2^-(x,x') = \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int \frac{d^2k_\perp}{2k_0} e^{-i\tilde{k}(\tilde{x}-\tilde{x}')} \sin\frac{n\pi}{d}x_3 \sin\frac{n\pi}{d}x_3' \tag{18}$$

$$= -\frac{1}{8\pi d\zeta}\left\{\frac{1}{\exp\left(\frac{i\pi}{d}(\zeta - x_3 - x_3')\right) - 1} + \frac{1}{\exp\left(\frac{i\pi}{d}(\zeta + x_3 + x_3')\right) - 1} - \frac{1}{\exp\left(\frac{i\pi}{d}(\zeta - x_3 + x_3')\right) - 1} - \frac{1}{\exp\left(\frac{i\pi}{d}(\zeta + x_3 - x_3')\right) - 1}\right\}, \tag{19}$$

$$\zeta^2 = (x_0 - x_0' - i\epsilon)^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2.$$

From these expressions follow the corresponding representations for $T \neq 0$. Here we write down a Fourier representation

$$ {}^sD^-_{2,\beta}(x,x') = \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int \frac{d^2k_\perp}{2k_0} e^{-i\vec{k}_\perp(\vec{x}_\perp - \vec{x}'_\perp)} \sin\frac{n\pi}{d}x_3 \sin\frac{n\pi}{d}x_3' \times\left[e^{-ik_0(x_0-x_0')} + (e^{-ik_0(x_0-x_0')} + e^{+ik_0(x_0-x_0')})\frac{e^{-\beta k_0}}{1-e^{-\beta k_0}}\right] \tag{20}$$

and an expression based on the reflection principle [9],

$$ {}^sD^-_{2,\beta}(x,x') = \sum_{l=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}[D^-(x_0 - x_0' + in\beta, (x-x')_\perp, x_3 - x'_3 + 2ld) - D^-(x_0 - x_0' + in\beta, (x-x')_\perp, x_3 + x_3' + 2dl)]. \tag{21}$$

The temperature-independent part in Eq. (20) corresponds to (18) and (19). It shows the presence of the x-space singularities resulting from the vacuum fluctuations contained in ${}^sD_2^-$. Such resonances appeared if the distances between the considered events correspond to a classical light signal n-times reflected at the plates. The Fourier representation (20) shows explicitly the frequency distribution. The important point is that the temperature-dependent part of the weight function of the spectrum is exponentially suppressed for very high frequencies. Consequently, this part of the correlation functions is smooth and does not contribute to singularities produced by the temperature-independent zero-point fluctuations.

In the case of two "thick" plates we have to modify the free space Green's function by introducing a periodicity condition, namely, that the period equal the distance between both plates, and by taking into account the function $\tilde{D}^-_\beta$ that contains the discrete mode propagating parallel to the plates:

$$\tilde{D}^-_\beta(x-x') = i\int \frac{d^3\tilde{k}}{(2\pi)^2}\frac{1}{2d}e^{i\tilde{k}(\tilde{x}-\tilde{x}')}\delta(\tilde{k}^2)[\Theta(-k_0) + \sinh^2\theta]. \tag{22}$$

III. PRESSURE FLUCTUATIONS

In this section, we study the fluctuations of the Casimir pressure [8,9,13,16] at low temperature. For this purpose, we determine the correlation function for the pressure components of the energy-momentum tensor. As a quantum field theoretic Green's function, it contains singularities on the light cone. Therefore the fluctuations of locally measured quantities, which are taken at the same space-time point, are divergent. Experimentally, local observables are measured over finite space and time intervals. By taking into account the measuring process, we obtain finite expressions for measurements over long (44), (45) and short times (46), (47).

For observables, we use the diagonal elements of the energy-momentum tensor $T_{\mu\mu}$ or the Casimir pressure

[9,11,17]. From the component T_{33}, the Casimir pressure on a plate located at $x_3 = a$ can be obtained as the difference of T_{33} across the plates:

$$p(x) = T_{33}(x_3 = a + \epsilon) - T_{33}(x_3 = a - \epsilon). \tag{23}$$

In our case we choose $a = 0$. For the energy-momentum tensor we use the symmetric tensor

$$-T_{\mu\nu} = F^{\rho}_{\mu} F_{\rho\nu} - \tfrac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}, \tag{24}$$

with the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The essential information for the fluctuation of an observable is contained in the expectation value

$$\begin{aligned} W(x,x') &= \langle T_{33}(x) T_{33}(x')\rangle_\beta - \langle T_{33}(x)\rangle_\beta \langle T_{33}(x')\rangle_\beta \\ &\stackrel{\text{def}}{=} \langle T_{33}(x) T_{33}(x')|\rangle'_\beta. \end{aligned} \tag{25}$$

Of course, the correlation functions need symmetrization:

$$\langle T_{33}(x) T_{33}(x')\rangle_\beta \to \frac{1}{2}[\langle T_{33}(x) T_{33}(x')\rangle_\beta + \langle T_{33}(x') T_{33}(x)\rangle_\beta]$$

which is omitted for convenience. When integrating with the function describing the measuring process, the symmetrization is realized automatically.

For the determination of the correlation function in free quantum electrodynamics exist different approaches. It is possible to apply gauge-invariant methods [9,16], relying on the Green's functions for the electromagnetic field strength, or Schwinger's source theory. These methods may be perhaps easier to apply than the standard quantum field theoretic method based on the electromagnetic potentials used here. But in our case we can apply results obtained earlier [4]. The regularization procedure we use is the point-splitting technique. We apply the Wick theorem in order to obtain [4]

$$\begin{aligned} \langle T_{\mu\mu}(x) T_{\mu\mu}(x')\rangle_\beta - \langle T_{\mu\mu}(x)\rangle_\beta \langle T_{\mu\mu}(x')\rangle_\beta = -\partial^{\overline{xy}} \partial^{\overline{x'y'}} \{ &[\bar{D}^-_{\beta+}(x,x') - \tfrac{1}{2}\tilde{D}^-(x,x')][\bar{D}^-_{\beta+}(y,y') - \tfrac{1}{2}\tilde{D}^-(y,y')] \\ &+ [D^-_\beta(x,x') + \bar{D}^-_{\beta-}(x,x') + \tfrac{1}{2}\tilde{D}^-(x,x')][D^-_\beta(y,y') + \bar{D}^-_{\beta-}(y,y') \\ &+ \tfrac{1}{2}\tilde{D}^-(y,y')]\}|_{y\to x, y'\to x'}, \end{aligned} \tag{26}$$

where we have taken into account the general structure of Green's functions in the presence of two plates (10) and the notation introduced in Appendix B [see (B12)–(B14)]. The indices $\mu\mu$ are suppressed in part, they are included in the definitions

$$\partial^{xy} = g^{\rho\lambda} \partial^x_\rho \partial^y_\lambda, \quad \partial^{\overline{xy}} = {}^{(\mu)}h^{\rho\lambda} \partial^x_\rho \partial^y_\lambda,$$

whereby the matrix ${}^{(\mu)}h_{ij}$ reads

$${}^{(\mu)}h_{\alpha\beta} = \begin{cases} -g_{\alpha\beta}\, g_{\mu\mu}, & \alpha \neq \mu \text{ or } \beta \neq \mu, \\ +g_{\alpha\beta}\, g_{\mu\mu}, & \alpha = \beta = \mu. \end{cases}$$

It is not difficult to remove the point splitting because all undefined terms are subtracted automatically according to the definition of the correlation function (25). We get a well-defined generalized function. All expressions consist of two-point correlation functions of the potentials. So all properties of these correlation functions are reflected by the corresponding properties of stress fluctuations. Moreover, the pressure fluctuations (26) contain products of Green's functions, each consisting of a temperature-independent and a temperature-dependent parts. Therefore, zero-point fluctuations are multiplied with temperature-induced fluctuations.

The fluctuation of the Casimir pressure on a plate located at $x_3 = 0$ can be reduced to the correlation function (26) due to the relation (23). One obtains, for ideally conducting plates,

$$\langle p(y) p(x')\rangle'_\beta|_{x_3=y_3=0} = \langle T_{33}(x) T_{33}(x')\rangle'_\beta|_{x_3=x'_3=0+\epsilon} + \langle T_{33}(x) T_{33}(x')\rangle'_\beta|_{x_3=x'_3=0-\epsilon}. \tag{27}$$

The reason for the absence of mixed terms originates from the fact that physical modes cannot propagate across the plates of ideal conductors. Therefore, we consider only one side of the plate. For the case of "thick" plates, this is clear. For the case of "thin" plates, difficulties may arise from the mode propagating parallel to the plates. This mode is present on both sides of the plates and the corresponding fluctuations are correlated on either side of one plate. Mathematically, this mode contributes as one point of a continuous spectrum. So it can be neglected. It follows that the fluctuations in the presence of "thin" plates are reduced in comparison with those for "thick" plates.

We give a Fourier representation for the fluctuation of the Casimir pressure on the inner side of the plate at $x_3 = 0$:

$$\begin{aligned} W_{2,\beta}(\tilde{x} - \tilde{x}', d) &\equiv \langle T_{33}(x) T_{33}(x')\rangle'_\beta|_{x_3=x'_3=0_+} \\ &= \frac{1}{8d^2} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int \frac{d^2p_\perp}{(2\pi)^2} \int \frac{d^2p'_\perp}{(2\pi)^2} \left[\left(\frac{\pi n}{d}\right)^2 \left(\frac{\pi n'}{d}\right)^2 + (\widetilde{pp'})^2 \right] \frac{1}{p_0 p'_0} \\ &\quad \times \{ e^{-i(\tilde{p}+\tilde{p}')(\tilde{x}-\tilde{x}')}[1 + E(p)][1 + E(p')] + e^{+i(\tilde{p}+\tilde{p}')(\tilde{x}-\tilde{x}')} E(p) E(p') \\ &\quad + e^{-i(\tilde{p}-\tilde{p}')(\tilde{x}-\tilde{x}')}[1 + E(p)] E(p') + e^{+i(\tilde{p}-\tilde{p}')(\tilde{x}-\tilde{x}')} E(p)[1 + E(p')] \}, \end{aligned} \tag{28}$$

where $E(p) = (e^{\beta|p_0|} - 1)^{-1}$.

Within the modifications due to temperature dependence, this result is identical to the expressions derived in [4]. For "thin" plates, the summands with $n = 0$ and $n' = 0$ have to be excluded. We note the appearance of the last two contributions in Eq. (28) containing the factors

$$e^{\pm i(\tilde{p}-\tilde{p}')(\tilde{x}-\tilde{x}')}. \tag{29}$$

Later we show that these terms essentially change the behavior of the fluctuations.

The correlation function (28) contains singularities for $(\tilde{x} - \tilde{x'})^2 \to 0$. Consequently the fluctuations of the local observable $p(x)$ diverge:

$$\begin{aligned}(\Delta p(x))^2 &= \langle [p(x) - \langle p(x)\rangle]^2\rangle = \langle p(x)p(x)\rangle - \langle p(x)\rangle^2 \\ &= W_{2,\beta}(x,x) = \infty.\end{aligned}$$

Experimentally, local observables are measured over the finite space and finite time intervals $T = \int f(x)T_{33}(x)dx$ described by the function $f(x)$. We factorize the function $f(x_0, \vec{x}_\perp)$ according to $f(x_0, \vec{x}_\perp) = g(x_0)h(\vec{x}_\perp)$. As an example we choose [2,4]

$$g(x_0) = \frac{\tau}{\pi}\frac{1}{x_0^2 + \tau^2}, \quad \int dx_0 e^{-ip_0x_0} g(x_0) = e^{-|p_0|\tau}, \quad h(\vec{x}_\perp) = \delta^{(2)}(\vec{x}_\perp).$$

This means we perform a local measurement over a finite time characterized by the parameter τ. The function $g(x_0)$ is normalized to unity; moreover, $g(x^0)$ has the property $\lim_{\tau\to 0} g(x^0) = \delta(x^0)$. With these functions, the fluctuation $(\Delta T)^2$ is expressed by means of the correlation function $W_{2,\beta}(\tilde{x}, \tilde{x}')$ as

$$(\Delta T_\tau)^2 = \int d\tilde{x}d\tilde{x}' f(\tilde{x})f(\tilde{x}')W_{2,\beta}(\tilde{x},\tilde{x}'), \quad \tilde{x} = (x_0, x_1, x_2). \tag{30}$$

Combining the foregoing equations with Eq. (28), we obtain

$$\begin{aligned}(\Delta T_\tau)^2 = \frac{1}{8d^2}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}&\int\frac{d^2p_\perp}{(2\pi)^2p_0}\int\frac{d^2p'_\perp}{(2\pi)^2p'_0}\left[(\tilde{p}\tilde{p}')^2 + \left(\frac{n\pi}{d}\right)^2\left(\frac{n'\pi}{d}\right)^2\right] \\ &\times(\exp[-2(p_0+p_0')\tau]\{[1+E(p)][1+E(p')] + E(p)E(p')\} \\ &+\exp(-2|p_0-p_0'|\tau)\{[1+E(p)]E(p') + E(p)[1+E(p')]\}),\end{aligned} \tag{31}$$

with $p_0 = \sqrt{p_\perp^2 + (\frac{n\pi}{d})^2}$ and $p'_0 = \sqrt{p'^2_\perp + (\frac{n'\pi}{d})^2}$. The fluctuations depend on the three parameters d, β, τ:

$$(\Delta T_\tau)^2 = \frac{1}{d^8} f\left(\frac{\tau}{d}, \frac{\tau}{\beta}\right). \tag{32}$$

Realistically the temperature is low and so $\tau \ll \beta$, and the characteristic time τ of a measuring process is large in comparison with the time interval necessary for a light signal to traverse the plate distance so $d \ll \tau$. These are the conditions for which we explicitly determine fluctuations. Equation (31) is treated in two parts $(\Delta T_1)^2$ and $(\Delta T_2)^2$ with

$$(\Delta T_\tau)^2 = (\Delta T_1)^2 + (\Delta T_2)^2. \tag{33}$$

We carry out the angle integration and shift the variable $p_\perp \to p_0 \equiv p$ to obtain

$$\begin{aligned}(\Delta T_1)^2 = \frac{1}{8d^2}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}&\int_{\frac{|n|\pi}{d}}^{\infty} dp\int_{\frac{|n'|\pi}{d}}^{\infty} dp' \exp[-2(p+p')\tau]\{[1+E(p)][1+E(p')] + E(p)E(p')\} \\ &\times\left\{(pp')^2 + \frac{1}{2}\left[p^2 - \left(\frac{n\pi}{d}\right)^2\right]\left[p'^2 - \left(\frac{n'\pi}{d}\right)^2\right] + \left(\frac{n\pi}{d}\right)^2\left(\frac{n'\pi}{d}\right)^2\right\}\end{aligned} \tag{34}$$

and

$$(\Delta T_2)^2 = \frac{1}{8d^2}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty}\int_{\frac{|n|\pi}{d}}^{\infty} dp \int_{\frac{|n'|\pi}{d}}^{\infty} dp' \exp(-2|p-p'|\tau)\{[1+E(p)]E(p')+E(p)[1+E(p')]\}$$
$$\times\left\{(pp')^2+\frac{1}{2}\left[p^2-\left(\frac{n\pi}{d}\right)^2\right]\left[p'^2-\left(\frac{n'\pi}{d}\right)^2\right]+\left(\frac{n\pi}{d}\right)^2\left(\frac{n'\pi}{d}\right)^2\right\}. \quad (35)$$

In the first expression the integrals factorize according to

$$(\Delta T_1)^2 = \frac{1}{(2\pi)^2}\frac{1}{2d^2}([(I_{k1}+I_{k1E})^2+I_{k1E}^2+\tfrac{1}{2}(I_{k1}-I_{n1}+I_{k1E}-I_{n1E})^2$$
$$+\tfrac{1}{2}(I_{k1E}-I_{n1E})^2+(I_{n1}+I_{n1E})^2+I_{n1E}^2]$$
$$+\tfrac{3}{8}[(I_{k0}+I_{k0E})^2+I_{k0E}^2]+\{\tfrac{3}{2}[(I_{k1}+I_{k1E})(I_{k0}+I_{k0E})+I_{k1E}I_{k0E}]$$
$$-\tfrac{1}{2}[(I_{n1}+I_{n1E})(I_{k0}+I_{k0E})+I_{n1E}I_{k0E}]\}), \quad (36)$$

with

$$I_{k1} = \sum_{n=1}^{\infty}\int_{\frac{n\pi}{d}}^{\infty} dp\, p^2 e^{-2\tau p}, \quad (37)$$

$$I_{k1E} = \sum_{n=1}^{\infty}\int_{\frac{n\pi}{d}}^{\infty} dp\, p^2 e^{-2\tau p}\frac{1}{e^{\beta p}-1}, \quad (38)$$

$$I_{n1} = \sum_{n=1}^{\infty}\int_{\frac{n\pi}{d}}^{\infty} dp \left(\frac{n\pi}{d}\right)^2 e^{-2\tau p}, \quad (39)$$

$$I_{n1E} = \sum_{n=1}^{\infty}\int_{\frac{n\pi}{d}}^{\infty} dp \left(\frac{n\pi}{d}\right)^2 e^{-2\tau p}\frac{1}{e^{\beta p}-1}, \quad (40)$$

$$I_{k0} = \int_0^{\infty} dp\, p^2 e^{-2\tau p}, \quad (41)$$

$$I_{k0E} = \int_0^{\infty} dp\, p^2 e^{-2\tau p}\frac{1}{e^{\beta p}-1}. \quad (42)$$

The integrations and sums can be performed at least approximately.

The expression (35) for $(\Delta T_2)^2$ cannot be factorized in general. However, in the limit $\beta/\tau \to \infty$, we are able to take into account asymptotic expansions of the type [22]

$$\int_0^{\infty} dx(a_0+a_1x+a_2x^2+\cdots)\frac{z}{(e^{\frac{\beta}{\tau}x}-z)}\Bigg|_{(\tau/\beta)\to\infty}$$
$$\to a_0\frac{\tau}{\beta}\sum_{k=1}^{\infty}\frac{z^k}{k}+a_1\left(\frac{\tau}{\beta}\right)^2\sum_{k=1}^{\infty}\frac{z^k}{k^2}+\cdots, \quad (43)$$

so that, in the exponential $\exp(-2|p-p'|)$, p or p' can be neglected and the integrals factorized. Finally we obtain, for "thick" plates,

$$(\Delta T_\tau)^2_{\beta\gg\tau\gg d} = \frac{3}{2^{10}\pi^2}\frac{1}{d^2}\frac{1}{\tau^6}+\frac{3}{2^5\pi^2}\frac{1}{\tau^3 d^2}\frac{1}{\beta^3}\zeta(3)+\cdots \quad (44)$$

and, for "thin" plates,

$$(\Delta T_\tau)^2_{\beta\gg\tau\gg d} = \frac{\pi^2}{4}\frac{1}{d^6}\left\{\frac{1}{2\tau^2}\exp\left(-4\tau\frac{\pi}{d}\right)+\frac{2}{2\tau\beta}\exp\left(-\frac{\pi\beta}{d}\right)\right\}+\cdots. \quad (45)$$

The β-independent terms following from $(\Delta T_1)^2$ coincide with the results of [2] and [4]. Measurements over long times are sensitive to the low-energy spectrum of the fluctuations. For "thick" plates this spectrum starts at $p_0 = 0$, whereas for "thin" plates the lowest state lays at $p_0 = \pi/d$. Consequently, we see a powerlike behavior for "thick" plates and exponential suppression for "thin" plates with respect to the parameter τ.

Particularly interesting are the contributions from $(\Delta T_2)^2$ because of the new structure $\exp(-2|p_0-p_0'|\tau)$ in Eq. (31). The contributions from the region $p_0 \sim p_0'$ to the p_0 and p_0' integrals are responsible for a power behavior with respect to τ. This means that in contrast to the case of vanishing temperature for "thin" plates appears a power behavior with respect to τ. The exponential decrease of the fluctuations with respect to the inverse temperature β is not influenced by this mechanism.

Note that both cases describe ideal conducting plates and lead to the same Casimir pressure. So it is very interesting that the fluctuations may distinguish between special properties of the plates. Also measurements over very short times of fluctuations at vanishing temperature are different for "thick" and "thin" plates. A calculation shows that, for "thick" plates,

$$(\Delta T_\tau)^2_{\beta=\infty,\tau\ll d} = \frac{6}{(2\tau)^8\pi^4}+O((2\tau)^5 d^3) \quad (46)$$

and, for "thin" plates,

$$(\Delta T_\tau)^2_{\beta=\infty,\tau\ll d} = \frac{6}{(2\tau)^8\pi^4}-\frac{2}{(2\tau)^7 d\pi^2}+\frac{3}{16(2\tau)^6 d^2\pi^2}+O((2\tau)^5 d^3). \quad (47)$$

For measurements over very short times, the leading terms coincide for both cases although the nonleading terms are different. The reason is that spatially local measurements over very short times test essentially the x-space singularity of the correlation function (28) [2,4] which is identical for both cases.

IV. HIGH-TEMPERATURE LIMIT

Finally we consider the limit of very high temperature. Of course, this problem involves unphysical issues. Nevertheless, this limit is theoretically interesting. We are able to define correlation functions in this limit. These "classical" Green's functions cannot only be used to determine fluctuations of the field strength but also the high-temperature limit of the Casimir effect. However, the classical Green's functions are generalized functions which cannot be multiplied without difficulties. Therefore, fluctuations of the Casimir pressure (which contain products of two Green's functions) are not well defined if we use the two-point Green's functions in the high-temperature limit. An investigation of these fluctuations has to start from finite temperatures. The high-temperature limit has to be performed as a last step only.

A. Free space

Let us perform the high-temperature limit $\beta \to 0$ of the scalar Green's function (16) for free space, for which we obtain

$$D_\beta^-(x,x')|_{\beta\to 0} = \frac{i}{4\pi^2}\frac{\pi}{2|\vec{z}|\beta}[\epsilon(|\vec{z}|-z_0)+\epsilon(|\vec{z}|+z_0)]. \quad (48)$$

This result is in agreement with a calculation starting from Eq. (14) in momentum space using $\frac{e^{-\beta|k_0|}}{1-e^{-\beta|k_0|}} \to \frac{1}{\beta|k_0|}$. Note that

$$[\epsilon(|\vec{z}|-z_0)+\epsilon(|\vec{z}|+z_0)] = 2\Theta(-z^2). \quad (49)$$

It seems to be that at very high temperature there are neither causal propagations nor time-dependent correlations. Therefore the resulting Green's function may correspond to a three-dimensional theory by setting $z_0 = 0$:

$$D_\beta^-(\vec{x},\vec{x'})|_{\beta\to 0} = \frac{i}{4\pi^2}\frac{\pi}{\beta}(\vec{z}^2)_+^{-1/2}. \quad (50)$$

But for time-dependent fluctuations we need a four-dimensional theory. The following question has not been answered: Is it possible to reconstruct the four-dimensional function (48) knowing the function (50) only?

Let us consider the case of field-strength fluctuations

$$\langle F_{\mu\nu}(x)F_{\mu'\nu'}(x')\rangle_\beta = O^{\rho\rho'}_{\mu\nu;\mu'\nu'}\langle A_\rho(x)A_{\rho'}(x')\rangle_\beta, \quad (51)$$

where $O^{\rho\rho'}_{\mu\nu;\mu'\nu'}$ is a differential operator. Its structure follows from the definition of the field strength. Together with the free space Green's function $\langle A_\mu(x)A_\nu(x')\rangle_\beta = ig_{\mu\nu}D_\beta^-$, we obtain for the blackbody radiation at very high temperature,

$$\langle F_{\mu\nu}(x)F_{\mu'\nu'}(x')\rangle_{\beta\to 0} = iO^{\rho\rho'}_{\mu\nu;\mu'\nu'}g_{\rho\rho'}\frac{i}{4\pi^2}\frac{\pi}{|\vec{z}|\beta}\Theta(-z^2) \quad (52)$$

$$= \frac{1}{\beta}(\delta_{nn'}\partial_0^x\partial_0^{x'} - g_{00}\partial_n^x\partial_{n'}^{x'})\int\frac{d^4k}{(2\pi)^3}e^{ik(x-x')}\frac{\delta(k^2)}{k_0}$$

$$= \frac{1}{\beta 8\pi}\left\{8\left(\delta_{nn'}|\vec{z}| - \frac{z_n z_{n'}}{|\vec{z}|}\right)\delta'(z^2) + 2\frac{3z_0z_0' - \delta_{nn'}\vec{z}^2}{|\vec{z}|^5}\Theta(-z^2) + 4\frac{\delta_{nn'}\vec{z}^2 - 2z_nz_{n'}}{|\vec{z}|^3}\delta(z^2)\right\}. \quad (53)$$

The differentiations of the Θ functions create δ functions and its derivatives. These functions are responsible for lightlike correlations. This is interesting because for finite $\beta \neq 0$ the temperature-dependent part of the correlation function does not contain singularities. The singular behavior is a consequence of the limit $\beta \to 0$. The special structure of the field-strength correlations as generalized functions does not allow their multiplication because the products $\delta(z^2)\delta(z^2)$, etc., are ill defined. This means that within the theory at infinite temperature we cannot investigate energy and pressure fluctuations without arbitrariness.

B. Presence of two plates

Next we investigate the Green's functions for the two-plate system in this limit. We discuss two representations. First, we perform the limit $\beta \to 0$ in Eq. (20) and get

$$^{s}D^{-}_{2,\beta}(x,x')|_{\beta\to 0} = \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int \frac{d^2k_\perp}{2k_0} e^{-i\vec{k}_\perp(\vec{x}_\perp-\vec{x}'_\perp)} \sin\frac{n\pi}{d}x_3 \sin\frac{n\pi}{d}x'_3 \left[(e^{-ik_0(x_0-x'_0)} + e^{+ik_0(x_0-x'_0)})\frac{1}{\beta k_0}\right]. \quad (54)$$

Second, we apply the reflection principle. Inserting the foregoing result (48) for the Green's functions of free space at infinite temperature into a partially summed version of Eq. (21) we get

$$D^{-}_{2,\beta}(x,x')|_{\beta\to 0} = \sum_{l=-\infty}^{+\infty} \frac{i}{4\pi^2}\frac{\pi}{2|\vec{z}_{-,l}|\beta}[\epsilon(|\vec{z}_{l,-}|-z_0)+\epsilon(|\vec{z}_{l,-}|+z_0)]$$
$$-\sum_{l=-\infty}^{+\infty} \frac{i}{4\pi^2}\frac{\pi}{2|\vec{z}_{+,l}|\beta}[\epsilon(|\vec{z}_{+,l}|-z_0)+\epsilon(|\vec{z}_{+,l}|+z_0)], \quad (55)$$

where

$$z_{-,l} = [z_\perp^2 + z_{3-,l}^2]^{1/2}, \quad z_{+,l} = [z_\perp^2 + z_{3+,l}^2]^{1/2}, \quad z_\perp^2 = z_1^2 + z_2^2,$$
$$\tilde{z} = \tilde{x} - \tilde{x'}, \quad z_{3-,l} = x_3 - x'_3 + 2ld, \quad z_{3+,l} = x_3 + x'_3 + 2ld.$$

This series contains timelike correlations introduced by the terms with $l \neq 0$. Because of the reflections lightlike correlations correspond to timelike distances which are the origin of long time correlations [20]:

$$D^{-}_{2,\beta}(x,x')|_{\beta\to 0} = \sum_{l=-\infty}^{+\infty}\frac{i}{4\pi^2}\frac{\pi}{\beta}\left[\frac{\Theta(-z_0^2+[z_\perp^2+(x_3-x'_3+2ld)^2])}{[z_\perp^2+(x_3-x'_3+2ld)^2]^{1/2}} - \frac{\Theta(-z_0^2+[z_\perp^2+(x_3+x'_3+2ld)^2])}{[z_\perp^2+(x_3+x'_3+2ld)^2]^{1/2}}\right]. \quad (56)$$

If we restrict ourselves to spacelike distances, we can write

$$D^{-}_{2,\beta}(x,x')|_{\beta\to 0} = \Theta(-z^2)\frac{i}{4\pi^2}\frac{\pi}{\beta}\sum_{l=-\infty}^{+\infty}\left[\frac{1}{[z_\perp^2+(x_3-x'_3+2ld)^2]^{1/2}} - \frac{1}{[z_\perp^2+(x_3+x'_3+2ld)^2]^{1/2}}\right].$$

Relying on this formula, we derive the Casimir effect in the Rayleigh-Jeans limit. We use the expression for the Casimir pressure (23), so we have to determine [11]

$$\langle T_{33}(x,x')\rangle_{\beta,x\to x'} = -i\{\partial_3^x\partial_3^{x'} + \partial_{\tilde{\rho}}^x\partial^{x'\tilde{\rho}}\}[^{s}D^{-}_{2,\beta}(x,x') + \tilde{D}^{-}_{\beta}(x,x')]|_{x\to x'}. \quad (57)$$

This equation includes a divergent part which has to be compensated by an infinite pressure acting on the other side of the plate. This subtraction is automatically performed if we restrict ourselves to the distance-dependent part of this expression. A short calculation leads to

$$p(d,\beta)|_{\frac{d}{\beta}\to\infty} = -\frac{\zeta(3)}{4\pi\beta d^3}. \quad (58)$$

This coincides with the high-temperature limit of the standard result for the Casimir effect [8,9,11].

In the case of the two-plate system, we do not discuss the expression for the field-strength fluctuations or its connections to quantum optics [1,18–21]. Instead, we consider directly the fluctuation of the Casimir pressure. Because of the bad multiplication properties of the Green's functions in the high-temperature limit, it makes no sense to insert these functions into the correlation functions for stress fluctuations (26).

For this reason, we start our calculation from the well-defined fluctuations Eq. (31), at finite temperatures. We consider measurements over long times in the limit of very high temperatures, i.e., $\tau/d \gg 1$, $\tau/\beta \gg 1$. The contributions (34) from $(\Delta T_1)^2$ can be calculated essentially in the same manner as before. It turns out that its contributions for large τ can be neglected in comparison with the contributions (35) from $(\Delta T_2)^2$. The complete expression (35) is valid for "thick" plates. By a reduction of the summation according to

$$\frac{1}{8}\sum_{n=-\infty}^{+\infty}\sum_{n'=-\infty}^{+\infty}[\,] = \frac{1}{2}\sum_{n=1}^{+\infty}\sum_{n'=1}^{+\infty}[\,] + \frac{1}{4}\left\{\sum_{n=1}^{+\infty}[\,]_{n'=0} + \sum_{n'=1}^{+\infty}[\,]_{n'=0}\right\} + \frac{1}{8}[\,]_{n=0,n'=0}, \quad (59)$$

it is split into contributions describing the fluctuations for "thin" plates (the first sum) and a remainder. To begin with, we look for the contributions corresponding to "thin" plates. To exploit the presence of a large parameter τ/β, we introduce new variables $p = \kappa_+ + \kappa_-$ and $p' = \kappa_+ - \kappa_-$ and get

$$(\Delta T_2)^2 = \frac{1}{(2\pi)^2}\frac{1}{2d^2}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} 2\int_{\frac{(n+n')\pi}{2d}}^{\infty} d\kappa_+ \int_{-(\kappa_+-\frac{n\pi}{2})}^{(\kappa_+-\frac{n'\pi}{2})} d\kappa_- e^{-4\tau|\kappa_-|}$$
$$\times\left[(p'p)^2 + \frac{1}{2}\left(p^2 - \frac{n^2\pi^2}{d^2}\right)\left(p'^2 - \frac{n'^2\pi^2}{d^2}\right) + \frac{n^2\pi^2}{d^2}\frac{n'^2\pi^2}{d^2}\right]\{E(p)+E(p')+2E(p)E(p')\}. \quad (60)$$

For large τ, the κ_- integration can be approximately performed. We take the expression inside the square brackets and $E(\kappa_+,\kappa_-)$ at $\kappa_-=0$. This leads to a factor $\frac{2}{4\tau}$. By this approximation $(E+E'+2EE')$ reduces to $2[\frac{1}{e^{\beta\kappa_+}-1} + (\frac{1}{e^{\beta\kappa_+}-1})^2]$. So we are left with the expression

$$(\Delta T_2)^2 = \frac{1}{(2\pi)^2}\frac{1}{2d^2}\frac{2}{4\tau}2\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\int_R^{\infty} d\kappa_+ 2\left\{\frac{1}{e^{\beta\kappa_+}-1} + \left(\frac{1}{e^{\beta\kappa_+}-1}\right)^2\right\}$$
$$\times\left[\kappa_+^4 + \frac{1}{2}\left(\kappa_+^2 - \frac{n^2\pi^2}{d^2}\right)\left(\kappa_+^2 - \frac{n'^2\pi^2}{d^2}\right) + \frac{n^2\pi^2}{d^2}\frac{n'^2\pi^2}{d^2}\right], \quad (61)$$

where

$$R = \begin{cases} n' > n & : \ \frac{n'\pi}{2d}; \\ n > n' & : \ \frac{n\pi}{2d}; \\ n = n' & : \ \frac{n\pi}{2d}. \end{cases}$$

This change of the lower integration boundary results from the κ_- integration where we were interested in the leading power behavior. Accordingly, we reorder the sum $\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} = \sum_{n>n'}^{\infty}\sum_{n'=1}^{\infty} + \sum_{n'>n}^{\infty}\sum_{n=1}^{\infty} + \sum_{n=n'=1}^{\infty}$ where the contributions of the first and the second sums are equal. For $n' > n$, we denote $n' = n + l$ so that we get two independent sums over n and l. In the integral, we change the variable $\kappa_+ = x + \frac{n'\pi}{d}$ so that the resulting integral runs from zero to infinity. Furthermore, we take into account the integral [22]

$$I = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{\infty} dx x^a \left\{\frac{e^{-\beta\frac{(n+l)\pi}{d}}}{e^{\beta x} - e^{-\beta\frac{(n+l)\pi}{d}}} + \frac{e^{-2\beta\frac{(n+l)\pi}{d}}}{(e^{\beta x} - e^{-\beta\frac{(n+l)\pi}{d}})^2}\right\} \quad (62)$$
$$= \frac{\Gamma(a+1)}{\beta^{a+1}}\sum_{k=1}^{\infty}\frac{1}{k^a}\frac{1}{(e^{\beta\frac{\pi}{d}k}-1)^2} \to \frac{\Gamma(a+1)}{\beta^{a+1}}\frac{d^2}{\beta^2\pi^2}\zeta(a+2) + \cdots. \quad (63)$$

The last expansion used in (63) has the character of an asymptotic expansion. It is possible to determine the correction terms as powers of $1/\beta$. Using formulas of the type (62) and (63) we obtain

$$(\Delta T_2)^2_{\sum_{n'>n=1}+\sum_{n>n'=1}} = \frac{480}{\pi^4}\zeta(6)\frac{1}{\beta^7(2\tau)} - \frac{168}{\pi^3}\zeta(5)\frac{1}{\beta^6(2\tau)d} + \frac{31}{2\pi^2}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots, \quad (64)$$
$$(\Delta T_2)^2_{\sum_{n'=n=1}} = +\frac{88}{\pi^3}\zeta(5)\frac{1}{\beta^6(2\tau)d} - \frac{9}{\pi^2}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots, \quad (65)$$
$$(\Delta T_2)^2_{\sum_{n'\equiv 0,n=1}+\sum_{n\equiv 0,n'=1}} = +\frac{80}{\pi^3}\zeta(5)\frac{1}{\beta^6(2\tau)d} - \frac{9}{\pi^2}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots, \quad (66)$$
$$(\Delta T_2)^2_{n=0,n'=0} = +\frac{9}{2\pi^2}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots. \quad (67)$$

For "thin" plates contribute the terms from the sums $\sum_{n'>n=1}^{\infty}$, $\sum_{n>n'=1}^{\infty}$, and $\sum_{n'=n=1}^{\infty}$. For "thick" plates we have to take into account additionally the contributions of $\sum_{n'\equiv 0,n=1}^{\infty}$, $\sum_{n\equiv 0,n=1}^{\infty}$, and the term with $n = n' = 0$.

Collecting these terms we obtain

$$(\Delta T_\tau)^2|_{\substack{\beta\to 0,\,\tau\gg d \\ \text{"thin"}}} = \frac{480}{\pi^4}\zeta(6)\frac{1}{\beta^7(2\tau)} - \frac{80}{\pi^3}\zeta(5)\frac{1}{\beta^6(2\tau)d} + \frac{13}{2\pi^3}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots,$$
$$(\Delta T_\tau)^2|_{\substack{\beta\to 0,\,\tau\gg d \\ \text{"thick"}}} = \frac{480}{\pi^4}\zeta(6)\frac{1}{\beta^7(2\tau)} + \frac{2}{\pi^3}\zeta(4)\frac{1}{\beta^5(2\tau)d^2} + \cdots. \quad (68)$$

For both cases we find the same leading behavior. Moreover, the distance-independent leading term is twice the result for free space or the result corresponding to one plate (at the position of the plate). It seems to be reasonable, then, that in the limit of very high temperature the nonleading terms are distance dependent and reflect special assumptions.

These fluctuations diverge as $\sim (\beta)^{-7}$ for $\beta \to 0$. This concerns the pressure fluctuations in free space as well as the Casimir pressure. If we compare the power behavior with respect to β^{-1} of the Casimir pressure $\sim (\beta)^{-1}$ with that of its fluctuations $\sim (\beta)^{-7}$ for $\beta \to 0$, then the fluctuations seem to be remarkably large.

Note that the fluctuations (44), (45) calculated in a low-temperature approximation and for mesurements over long times ($\tau \gg d$) are always finite, and its temperature-dependent corrections vanish for $\beta \to \infty$. On the contrary, measurements over very short times (46), (47) reflect the singularities of the correlation functions. They are divergent for $\tau \to 0$.

If we would try to calculate the high-temperature fluctuations (68) of the pressure directly from the Green's functions of a theory formulated at infinite temperature, then we get divergent results. We have to take into account that the time-dependent Green's functions for the electromagnetic field strength at $T \to \infty$ are generalized functions (53) which cannot be multiplied without difficulties. The additional singularities should then transform to additional powers of $1/\beta$ so that the results (68) should be recovered. This is a general problem even concerning pressure and energy fluctuations in free space. In connection with a formulation of a theory in the high-temperature limit, this has to be investigated further. Alternatively, a theory in the high-temperature limit can be obtained using the imaginary-time formulation of quantum field theory. Then the following question has to be answered: Is it possible to reconstruct the four-dimensional time-dependent theory in this limit too?

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APPENDIX A: GREEN'S FUNCTIONS IN THE OPERATOR FORMALISM

Our aim is the derivation of the representations (7) and (10) in an alternative way. The field is described by $A_{\mu a} = (A_\mu, \hat{A}_\mu)$ where A_μ is the field operator and $\hat{A}_\mu$ the "ghost" field. The mode expansion of the field operators reads

$$A_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum e^i_\mu [a_i(k) f_i^-(k,x) + a_i^\dagger(k) f_i^+(k,x)], \tag{A1}$$

where $f_i^\pm$ describes the field modes and a_i and $a_i^\dagger$ are the destruction and creation operators, respectively. An analogous representation is valid for the "ghost" field operators:

$$\hat{A}_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum e^i_\mu [\hat{a}_i(k) f_i^-(k,x) + \hat{a}_i^+(k) f_i^+(k,x)]. \tag{A2}$$

The essential point of thermo field dynamics [7] is the application of a rotated space of states. In contrast with the standard formalism, the ground state satisfies the conditions

$$\alpha_i(k,\beta)\rangle_\beta = 0, \qquad \hat{\alpha}_i(k,\beta)\rangle_\beta = 0, \tag{A3}$$

with destruction operators which are connected by the original creation and destruction operators by a Bogoliubov rotation

$$\begin{pmatrix} a_i(k) \\ \hat{a}_i^\dagger(k) \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \alpha_i(k,\beta) \\ \hat{\alpha}_i^\dagger(k,\beta) \end{pmatrix} \tag{A4}$$

and

$$\begin{pmatrix} a_i^\dagger(k) \\ \hat{a}_i(k) \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \alpha_i^\dagger(k,\beta) \\ \hat{\alpha}_i(k,\beta) \end{pmatrix}. \tag{A5}$$

Substituting the creation and destruction operators a_i and $a_i^\dagger$ according the relations (A4) and (A5) and forming the vacuum expectation values $\langle A_{\mu a} A_{\nu b}\rangle_\beta$, we obtain, for the Green's functions,

$$\langle A_\mu(x) A_\nu(x')\rangle_\beta = -\int d^3\vec{k} \sum e^i_\mu e^i_\nu g_{ij} \{f_i^-(k,x) f_j^+(k,x') + \sinh^2\theta [f_i^-(k,x) f_j^+(k,x') + f_i^+(k,x) f_j^-(k,x')]\}, \tag{A6}$$

$$\langle \hat{A}_\mu(x) \hat{A}_\nu(x')\rangle_\beta = -\int d^3\vec{k} \sum e^i_\mu e^j_\nu g_{ij} \{f_i^+(k,x) f_j^-(k,x') + \sinh^2\theta [f_i^+(k,x) f_j^-(k,x') + f_i^-(k,x) f_j^+(k,x')]\}, \tag{A7}$$

$$\langle \hat{A}_\mu(x) A_\nu(x')\rangle_\beta = \langle A_\mu(x) \hat{A}_\nu(x')\rangle_\beta = -\int d^3\vec{k} \sum e^i_\mu e^j_\nu g_{ij} \sinh\theta\cosh\theta [f_i^-(k,x) f_j^+(k,x') + f_i^+(k,x) f_j^-(k,x')]. \tag{A8}$$

Finally, by introducing standard notation, the mode summation leads to the well-known Green's functions for vanishing temperature:

$$D^-_{\mu\nu}(x,x') = i\int d^3\vec{k}\sum e^i_\mu e^j_\nu g_{ij} f^-_i(k,x) f^+_j(k,x'), \tag{A9}$$

$$D^1_{\mu\nu}(x,x') = i\int d^3\vec{k}\sum e^i_\mu e^j_\nu g_{ij}[f^-_i(k,x) f^+_j(k,x') + f^+_i(k,x) f^-_j(k,x')]. \tag{A10}$$

The modes $f^\pm_i$ have to satisfy the appropriate boundary conditions. Equation (A6) is in accordance with the corresponding Eq. (7). We have confirmed that both procedures agree.

APPENDIX B: PHOTON GREEN'S FUNCTIONS BETWEEN PLATES

To clarify the differences between Green's functions for the two-plate system with "thin" and "thick" plates, let us first consider the case of "thin" plates [4]:

$$^sD^-_{\mu\nu}(x,x') = \left(g_{\tilde\mu\tilde\nu} - \frac{\partial_{\tilde\mu}\partial_{\tilde\nu}}{\tilde\partial^2}\right) {}^sD^-_2(\tilde x - \tilde x', x_3, x'_3) + \begin{pmatrix} \frac{\tilde\partial_\mu\tilde\partial_\nu}{\tilde\partial^2} & 0 \\ 0 & -1 \end{pmatrix} D^-(x-x') \tag{B1}$$

$$= g_{\mu\nu}D^- + \left(g_{\tilde\mu\tilde\nu} - \frac{\partial_{\tilde\mu}\partial_{\tilde\nu}}{\tilde\partial^2}\right)\bar D^-(\tilde x - \tilde x', x_3, x'_3). \tag{B2}$$

This expression is valid for the inner and outer regions of the two-plate system. D^- and ${}^sD^-_2$ denote the standard Green's functions

$$D^-(x-x') = i\int \frac{d^4k}{(2\pi)^3} e^{ik(x-x')}\delta(k^2)\Theta(-k_0), \tag{B3}$$

$$^sD^-_2(x,x') = \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int \frac{d^2k_\perp}{2k_0} e^{-i\tilde k(\tilde x - \tilde x')}\sin\frac{n\pi}{d}x_3 \sin\frac{n\pi}{d}x'_3 \tag{B4}$$

$$= D^-(x-x') + \bar D^-(x,x'). \tag{B5}$$

Formula (B4) is valid between the two plates. Outside the two-plate system, the expressions valid for one plate is used.

"Thin" and "thick" plates differ by the treatment of the mode propagating parallel to the plates. For "thick" plates we have [4]

$$^sD^-_{d\mu\nu}(x,x') = \left(g_{\tilde\mu\tilde\nu} - \frac{\partial_{\tilde\mu}\partial_{\tilde\nu}}{\tilde\partial^2}\right) {}^sD^-_2(\tilde x - \tilde x', x_3, x'_3) + \begin{pmatrix} \frac{\tilde\partial_\mu\tilde\partial_\nu}{\tilde\partial^2} & 0 \\ 0 & -1 \end{pmatrix} D^-_d(x-x') + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\tilde D^-(x-x'). \tag{B6}$$

The additional contribution for "thick" plates $\tilde D^-(x-x')$ is built up from the discrete mode propagating parallel to the plates. An equivalent expression reads

$$^sD^-_{d\mu\nu}(x,x') = g_{\mu\nu}D^-_d + \left(g_{\tilde\mu\tilde\nu} - \frac{\partial_{\tilde\mu}\partial_{\tilde\nu}}{\tilde\partial^2}\right)\bar D^-(\tilde x - \tilde x', x_3, x'_3) - n_\mu n_\nu \tilde D^-(\tilde x - \tilde x'). \tag{B7}$$

The functions D^-_d and $\tilde D^-$ are given by

$$D^-_d(x-x') = i\int \frac{d^3\tilde k}{(2\pi)^2}\frac{1}{2d}\sum_{n=-\infty, n\neq 0}^{\infty} \exp\left(i\tilde k(\tilde x - \tilde x') - i\frac{2\pi n}{d}(x_3 - x'_3)\right)\delta(k^2)\Theta(-k_0) \tag{B8}$$

and

$$\tilde D^-(x-x') = i\int \frac{d^3\tilde k}{(2\pi)^2}\frac{1}{2d} e^{i\tilde k(\tilde x - \tilde x')}\delta(\tilde k^2)\Theta(-k_0). \tag{B9}$$

For explicit calculations, it is useful to apply the representation

$$^sD^-_2 = {}^sD^-_- + {}^sD^-_+, \tag{B10}$$

$$^sD^-_\pm = \frac{i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int \frac{d^2p_\perp}{2p_0}\exp[-i\tilde{p}(\tilde{x}-\tilde{x}')]\begin{pmatrix}\cos\frac{n\pi}{d}(x_3-x_3') \\ -\cos\frac{n\pi}{d}(x_3+x_3')\end{pmatrix}. \tag{B11}$$

Additional relations are $^sD^-_- = D^- + \bar{D}^-_-$, $^sD^-_+ = \bar{D}^-_+$. The transition from "thick" to "thin" plates is effected by setting $\tilde{D}^-$ to 0 and the substitution $D^-_d \to D^-$. The thermal Green's functions are obtained according to the rules given in Sec. II:

$$\begin{aligned}\langle A_\mu(x)A_\nu(x')\rangle_\beta &= i^sD^-_{\beta\;\mu\nu}(x,x') \\ &= i\left(g_{\tilde{\mu}\tilde{\nu}} - \frac{\partial_{\tilde{\mu}}\partial_{\tilde{\nu}}}{\tilde{\partial}^2}\right){}^sD^-_{2,\beta} + i\begin{pmatrix}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\partial}^2} & 0 \\ 0 & -1\end{pmatrix}D^-_\beta - in_\mu n_\nu \tilde{D}^-_\beta \\ &= ig_{\mu\nu}D^-_\beta + i\left(\tilde{g}_{\mu\nu} - \frac{\tilde{\partial}^x_\mu\tilde{\partial}^{x'}_\nu}{\tilde{\partial}^x\tilde{\partial}^{x'}}\right)\bar{D}^-_\beta - in_\mu n_\nu\tilde{D}^-.\end{aligned} \tag{B12}$$

The function $\bar{D}^-_\beta$ is defined as the difference between the Green's function $^sD^-_\beta$ satisfying the Dirichlet boundary condition and the free space function D^-_β, i.e., $\bar{D}^-_\beta = ^s D^-_\beta - D^-_\beta$. One further specification of the function $\bar{D}^-$ follows from the structure of the x_3, x'_3 dependence

$$\bar{D}^-_\beta(x,x') = \bar{D}^-_{\beta-}(\tilde{x}-\tilde{x}', x_3 - x'_3) + \bar{D}^-_{\beta+}(\tilde{x}-\tilde{x}', x_3 + x'_3). \tag{B13}$$

In fact the Green's functions splits according to

$$\begin{aligned}&^sD^-_{2,\beta}(x,x') = {}^sD^-_{\beta,-}(x,x') + {}^sD^-_{\beta,+}(x,x'), \\ &^sD^-_{\beta,-}(x,x') = D^-_\beta(x,x') + \bar{D}^-_{\beta,-}(x,x'), \quad {}^sD^-_{\beta,+}(x,x') = \bar{D}^-_{\beta,+}(x,x').\end{aligned} \tag{B14}$$

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