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Relativistic gauge conditions in quantum cosmology

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This paper studies the quantization of the electromagnetic field on a flat Euclidean background with boundaries. One-loop scaling factors are evaluated for the one-boundary and two-boundary backgrounds. The mode-by-mode analysis of Faddeev-Popov quantum amplitudes is performed by using ζ-function regularization and is compared with the space-time covariant evaluation of the same amplitudes. It is shown that a particular gauge condition exists for which the corresponding operator matrix acting on gauge modes is in diagonal form from the beginning. Moreover, various relativistic gauge conditions are studied in detail, to investigate the gauge invariance of the perturbative quantum theory.

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I. INTRODUCTION

In a quantum cosmological framework, the quantization of the electromagnetic field on flat Euclidean backgrounds with boundaries was first considered in a paper by Louko [1]. The one-loop correction to the Hartle-Hawking wave function of the Universe [2] was studied and the value of $\zeta(0)$ describing the scaling properties of the wave function was calculated by restricting the path-integral measure to the physical degrees of freedom (i.e., the transverse part of the potential). Later, it was found in Ref. [3] that the value of the scaling factor obtained by a space-time covariant method on using the formula for the A_2 Schwinger-DeWitt coefficient for arbitrary fields on manifolds with boundaries [4] disagrees with the result obtained in Ref. [1]. Analogous discrepancies were found for other higher spin fields on manifolds with boundaries [5–13], and for gravitons [14] and photons [15,16] on the Riemannian four-sphere representing the Wick-rotated version of de Sitter space-time.

Some attempts to understand the reasons of the discrepancies mentioned above were made in recent years. The first of these ideas suggests that the reason for discrepancies lies in the inappropriate implementation of a 3+1 split on the manifolds where this is ill definite [17]. The second idea is connected with the necessity to study the contribution of gauge and ghost modes to the quantum amplitudes [12,18]. The third approach stresses the necessity to pay attention to a correct definition of the measure in the corresponding path integrals [15,16,19]. The fourth one consists in the check of the covariant formulas for the A_2 Schwinger-DeWitt coefficient for arbitrary fields on manifolds with boundaries [20].

In our previous paper [21] we have investigated the correspondence between covariant and noncovariant formalisms for the Maxwell field on flat Euclidean four-space with boundaries by applying the first two approaches mentioned above. We were able to disentangle the eigenvalue equations for normal and longitudinal components of the electromagnetic potential A_μ in two relativistic gauges [18,21]:

$$\Phi_L \equiv {}^{(4)}\nabla^\mu A_\mu \ , \ \Phi_E \equiv {}^{(4)}\nabla^\mu A_\mu - A_0 \ \mathrm{Tr}\ K \ ,$$

where K is the extrinsic-curvature tensor of the boundary. Their contribution to $\zeta(0)$ on the manifold representing the part of flat Euclidean four-space bounded by two concentric three-spheres was then evaluated. It was shown that by taking into account the contributions of nonphysical modes and ghosts (which do not cancel each other in contrast with the usual experience on the manifolds without curvature or boundaries), one obtains results for the Faddeev-Popov amplitudes which agree with the space-time covariant calculation of the same amplitudes. An analogous result was obtained for gravitons in the de Donder gauge [22] and for photons in the Coulomb gauge [23]. Moreover, it was shown in Ref. [21] that, in the Lorentz gauge, the value of $\zeta(0)$ on flat Euclidean four-space bounded by only one three-sphere coincides with the value of the A_2 Schwinger-DeWitt coefficient [24] obtained by using the corrected formula derived in Ref. [20].

However, relativistic gauges different from the Lorentz gauge yield a different $\zeta(0)$ value when the boundary three-geometry consists of only one three-sphere [21]. A

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possible explanation of these results can be that the absence of a well-defined 3+1 decomposition of the electromagnetic four-vector potential on the one-boundary manifold makes the calculations in terms of physical degrees of freedom and normal and longitudinal components inconsistent. In the particular case of the Lorentz gauge, these calculations are still consistent because all the operators are in relativistically covariant form.

In this paper we continue the analysis of the electromagnetic field on manifolds with boundaries, by studying the problem of gauge invariance in Euclidean Maxwell theory. For this purpose, we study families of relativistic gauges for the manifolds with one and two boundaries. In particular, the gauge is found where the eigenvalue equations for normal and longitudinal components are decoupled without having to diagonalize operator matrices, and the calculations are especially simple. Since we study a model relevant for the quantization of closed cosmologies (although in the limiting case of a flat background [5]), the normal and tangential components of the electromagnetic potential are expanded on a family of three-spheres as [1,12,18,21]

$$A_0(x,\tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x) , \tag{1.1}$$

$$A_k(x,\tau) = \sum_{n=2}^{\infty} \Big[f_n(\tau) S_k^{(n)}(x) + g_n(\tau) P_k^{(n)}(x) \Big] \quad \text{for all } k=1,2,3 , \tag{1.2}$$

where $Q^{(n)}(x), S_k^{(n)}(x), P_k^{(n)}(x)$ are scalar, transverse, and longitudinal vector harmonics on S^3, respectively [25].

Section II shows that a gauge condition exists such that gauge modes for a spin-1 field can be decoupled without having to use the diagonalization method described in our previous paper [21]. The resulting $\zeta(0)$ value is obtained. Section III applies the same gauge condition of Sec. II to flat Euclidean four-space bounded by only one three-sphere. Section IV studies the most general family of relativistic gauge-averaging functionals depending linearly on gauge modes and their first derivatives. Results and open problems are presented in Sec. V.

II. DECOUPLING OF GAUGE MODES: TWO-BOUNDARY CASE, MAGNETIC AND ELECTRIC BOUNDARY CONDITIONS

Following Refs. [12,18,21], we study quantum amplitudes for Euclidean Maxwell theory within the framework of Faddeev-Popov formalism. Thus, the total Euclidean action is given by [12,18]

$$\widetilde{I}_E = I_{\rm gh} + \int_M \left[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\Big[\Phi(A)\Big]^2}{2\alpha} \right] \sqrt{\det g}\, d^4x , \tag{2.1}$$

where A_μ is the four-vector potential, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the electromagnetic-field tensor, g is the background four-metric, Φ is an arbitrary gauge-averaging functional defined on a space of connection one-forms, and α is a positive dimensionless parameter. $I_{\rm gh}$ is the corresponding ghost-field action.

A relevant class of choices for $\Phi(A)$ can be parametrized by a real number, say b, and it can be cast in the form [12,18]

$$\Phi^{(b)}(A) \equiv {}^{(4)}\nabla^\mu A_\mu - b\, A_0 \,{\rm Tr}\, K , \tag{2.2}$$

where K is the extrinsic-curvature tensor of the boundary. The two gauges studied in Ref. [21] are a particular case of (2.2), since $b=0$ leads to the Lorentz gauge, and $b=1$ yields the Esposito gauge [12,18,21]. If (2.2) is chosen as the gauge-averaging functional, the part $I_2(g,R)$ of the Euclidean action quadratic in gauge modes is (cf. Refs. [12,18])

$$I_2(g,R) = \frac{1}{2\alpha}\int_{\tau_-}^{\tau_+} \tau^3 \Big(\dot{R}_1 + \frac{3}{\tau}(1-b)R_1\Big)^2 d\tau + \sum_{n=2}^{\infty} \int_{\tau_-}^{\tau_+} \left[\frac{\tau}{2(n^2-1)}[\dot{g}_n - (n^2-1)R_n]^2 + \frac{\tau}{2\alpha}\Big(\tau \dot{R}_n + 3(1-b)R_n - \frac{g_n}{\tau}\Big)^2 \right] d\tau . \tag{2.3}$$

Of course, we need boundary conditions on the boundary surfaces. They can be magnetic, which implies setting to zero on the boundaries the magnetic field, the gauge-averaging functional and hence the Faddeev-Popov ghost field. They can also be electric, hence setting to zero on the boundaries the electric field, and leading to Neumann conditions on the ghost [12,18,21]. The former imply, in the gauge (2.2), Dirichlet boundary conditions for g_n and ghost modes, and Robin boundary conditions for R_n modes. The latter imply Neumann boundary conditions for g_n and ghost modes, and Dirichlet boundary conditions for R_n modes.

Integrating by parts in (2.3) and using the magnetic or electric boundary conditions described above one finds, for all $n \geq 2$,

$$I_2^{(n)}(g,R) = \frac{1}{2}\int_{\tau_-}^{\tau_+} \frac{\tau g_n}{(n^2-1)} (\widehat{A}_n g_n)\, d\tau + \frac{1}{2}\int_{\tau_-}^{\tau_+} \tau^3 R_n (\widehat{B}_n R_n)\, d\tau + \Big(1-\frac{1}{\alpha}\Big)\int_{\tau_-}^{\tau_+} g_n \frac{d}{d\tau}(\tau R_n)\, d\tau + \frac{3}{\alpha}\Big(b-\frac{2}{3}\Big)\int_{\tau_-}^{\tau_+} g_n R_n \, d\tau , \tag{2.4}$$

where the second-order elliptic differential operators $\widehat{A}_n$ and $\widehat{B}_n$ are

$$\widehat{A}_n(\tau) \equiv -\frac{d^2}{d\tau^2} - \frac{1}{\tau}\frac{d}{d\tau} + \frac{(n^2-1)}{\alpha\tau^2} , \tag{2.5}$$

$$\widehat{B}_n(\tau) \equiv -\frac{1}{\alpha}\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right) +\frac{1}{\tau^2}\left[n^2-1+\frac{3}{\alpha}\left[1+3b\left(b-\frac{4}{3}\right)\right]\right] . \quad (2.6)$$

Thus, if we choose the gauge-averaging functional as [cf. (2.2)]

$$\Phi_P(A) \equiv {}^{(4)}\nabla^\mu A_\mu - \frac{2}{3} A_0 \,\mathrm{Tr}\, K , \quad (2.7)$$

the action quadratic in the gauge modes becomes, for all $n \geq 2$,

$$I_2^{(n)}(g,R) = \frac{1}{2}\int_{\tau_-}^{\tau_+} -\frac{\tau g_n}{(n^2-1)}\left[\frac{d^2}{d\tau^2}+\frac{1}{\tau}\frac{d}{d\tau}-\frac{(n^2-1)}{\alpha\tau^2}\right] g_n \, d\tau +\frac{1}{2}\int_{\tau_-}^{\tau_+} -\tau^3 R_n\left[\frac{1}{\alpha}\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right)-\left(n^2-1-\frac{1}{\alpha}\right)\frac{1}{\tau^2}\right] R_n \, d\tau+\left(1-\frac{1}{\alpha}\right)\int_{\tau_-}^{\tau_+} g_n \frac{d}{d\tau}(\tau R_n)\, d\tau . \quad (2.8)$$

Remarkably, by setting to 1 the parameter α we get the decoupled eigenvalue equations for normal and longitudinal components of the electromagnetic potential

$$\frac{d^2 g_n}{d\tau^2}+\frac{1}{\tau}\frac{dg_n}{d\tau}-\frac{(n^2-1)}{\tau^2}g_n+\lambda_n g_n = 0 , \quad (2.9)$$

and

$$\frac{d^2 R_n}{d\tau^2}+\frac{3}{\tau}\frac{dR_n}{d\tau}-\frac{(n^2-2)}{\tau^2}R_n+\lambda_n R_n = 0 . \quad (2.10)$$

The regular solutions of Eqs. (2.9) and (2.10) are Bessel functions of noninteger order. However, to use the complex-contour technique of Refs. [8–11] it is convenient to set $\lambda_n = -M^2$ and then work with the corresponding modified Bessel functions. After making this change of variable, and defining $\nu \equiv \sqrt{n^2-1}$, we get the regular solutions for g_n and R_n:

$$g_n(\tau) = C_1\, I_\nu(M\tau) + C_2\, K_\nu(M\tau) , \quad (2.11)$$

$$R_n(\tau) = \frac{1}{\tau}\left(C_3\, I_\nu(M\tau) + C_4\, K_\nu(M\tau)\right) , \quad (2.12)$$

where C_i with $i = 1, \ldots, 4$ are constants. As in Ref. [21], both I and K functions contribute to regular gauge modes, since the singularity at the origin of flat Euclidean four-space is avoided in our elliptic boundary-value problem with two three-sphere boundaries.

Now, defining $I_\nu^- \equiv I_\nu(M\tau_-)$, $I_\nu^+ \equiv I_\nu(M\tau_+)$, $K_\nu^- \equiv K_\nu(M\tau_-)$, $K_\nu^+ \equiv K_\nu(M\tau_+)$, and imposing magnetic boundary conditions described above, one has the equations

$$C_1\, I_\nu^- + C_2\, K_\nu^- = 0 , \quad (2.13a)$$

$$C_1\, I_\nu^+ + C_2\, K_\nu^+ = 0 , \quad (2.13b)$$

$$C_3\, \left(I_{\nu-1}^- + I_{\nu+1}^-\right) - C_4\, \left(K_{\nu-1}^- + K_{\nu+1}^-\right) = 0 , \quad (2.13c)$$

$$C_3\, \left(I_{\nu-1}^+ + I_{\nu+1}^+\right) - C_4\, \left(K_{\nu-1}^+ + K_{\nu+1}^+\right) = 0 . \quad (2.13d)$$

The condition for the existence of nontrivial solutions for the system (2.13a)–(2.13d) is the vanishing of the determinants

$$\det \begin{pmatrix} I_\nu^- & K_\nu^- \\ I_\nu^+ & K_\nu^+ \end{pmatrix} = 0, \quad (2.14)$$

$$\det \begin{pmatrix} (I_{\nu-1}^- + I_{\nu+1}^-) & -(K_{\nu-1}^- + K_{\nu+1}^-) \\ (I_{\nu-1}^+ + I_{\nu+1}^+) & -(K_{\nu-1}^+ + K_{\nu+1}^+) \end{pmatrix} = 0. \quad (2.15)$$

Thus, we have found the condition on eigenvalues for normal and longitudinal components of the electromagnetic field and can evaluate their contribution to $\zeta(0)$ by using the algorithm of Refs. [8–11].

For this purpose, let us recall that $\zeta(0)$ can be expressed as

$$\zeta(0) = I_{\log} + I_{\mathrm{pole}}(\infty) - I_{\mathrm{pole}}(0) . \quad (2.16)$$

With our notation [8–11], one writes $f_n(M^2)$ for the function occurring in the equation obeyed by the eigenvalues by virtue of boundary conditions, and $d(n)$ for the degeneracy of the eigenvalues. One then defines the function [8–11]

$$I(M^2,s) \equiv \sum_{n=n_0}^{\infty} d(n)\, \frac{1}{n^{2s}}\, \ln f_n(M^2) . \quad (2.17a)$$

Such a function has a unique analytic continuation to the whole complex-s plane as a meromorphic function: i.e.,

$$\text{“}I(M^2,s)\text{”} = \frac{I_{\mathrm{pole}}(M^2)}{s} + I^R(M^2) + O(s) . \quad (2.17b)$$

Thus, $I_{\log} = I_{\log}^R$ is the coefficient of $\ln M$ from $I(M^2,s)$ as $M \to \infty$, and $I_{\mathrm{pole}}(M^2)$ is the residue at $s = 0$. Remarkably, $I_{\log}$ and $I_{\mathrm{pole}}(\infty)$ are obtained from uniform asymptotic expansions of modifies Bessel functions as their order tends to ∞ and $M \to \infty$, whereas $I_{\mathrm{pole}}(0)$ is obtained from the limiting behavior of such Bessel functions as $M \to 0$ [8–11]. The condition $\det \mathcal{I} = 0$ [see (2.14)–(2.15)] should be studied after eliminating fake roots $M = 0$. To obtain that, it is enough to divide $\det \mathcal{I}$ by the minimal power of M occurring in the determinant. It is easy to see by using the series expansion for modified Bessel functions that such a power is 0 for (2.14) and -2 for (2.15).

We begin with the calculation of $I_{\log}$ for g_n and R_n modes. Using uniform asymptotic expansions of modified Bessel functions, one can see, from (2.14), that the coefficient of ln M is -1, while (2.15), divided by M^{-2}, gives $+\ln M$. Hence

$$I_{\log} = I_{\log_{R_n}} + I_{\log_{g_n}} = 0 . \quad (2.18)$$

In a similar way one finds that $I_{\rm pole}(0) = 0$, whereas the contributions to $I_{\rm pole}(\infty)$ from g_n and R_n vanish separately.

The next problem is the calculation of the contribution to $\zeta(0)$ of the R_1 mode. In our gauge the eigenvalue equation for it is

$$\frac{d^2R_1}{d\tau^2} + \frac{3}{\tau}\frac{dR_1}{d\tau} + \frac{R_1}{\tau^2} - M^2R_1 = 0 , \quad (2.19)$$

whose solution is

$$R_1(\tau) = C_1\frac{1}{\tau}I_0(M\tau) + C_2\frac{1}{\tau}K_0(M\tau) . \quad (2.20)$$

Imposing Robin (i.e., magnetic) conditions on R_1,

$$\frac{dR_1}{d\tau} + \frac{R_1}{\tau} = 0 , \quad (2.21)$$

at the three-sphere boundaries, one gets the system of equations

$$C_1I_1^- - C_2K_1^- = 0 , \quad (2.22a)$$

$$C_1I_1^+ - C_2K_1^+ = 0 . \quad (2.22b)$$

The determinant of the system (2.22a) and (2.22b) should vanish and this gives the eigenvalue condition. Such a determinant has no fake roots. Thus, by using the uniform asymptotic expansions of Bessel functions one finds that the contribution owed to $I_{\log}$ is $-\frac{1}{2}$. As noted in Ref. [21], we have to add the number $N_D = 1$ of such decoupled modes to the full $\zeta(0)$ value. In fact, they are nontrivial since they involve zero-eigenvalues corresponding to nonvanishing eigenfunctions [21,26–28]. Hence one obtains

$$\zeta_{R_1}(0) = I_{\log} + N_D = -\frac{1}{2} + 1 = \frac{1}{2} . \quad (2.23)$$

Now we deal with the ghost operator. By studying the gauge transformation [12,18]

$$^{\epsilon}A_\mu \equiv A_\mu + {}^{(4)}\nabla_\mu\epsilon , \quad (2.24)$$

one gets, by virtue of (2.2),

$$\Phi^{(b)}(A) - \Phi^{(b)}({}^{\epsilon}A) = \sum_{n=1}^{\infty}\left[-\frac{d^2}{d\tau^2} - \frac{3}{\tau}(1-b)\frac{d}{d\tau} + \frac{(n^2-1)}{\tau^2}\right]\epsilon_n(\tau)Q^{(n)}(x) . \quad (2.25)$$

Hence the eigenfunctions of the ghost operator are related to [12,18]

$$\epsilon_n(\tau) = \tau^{(\frac{3b}{2}-1)}(B_1I_\nu(M\tau) + B_2K_\nu(M\tau)) , \quad (2.26)$$

where

$$\nu \equiv +\sqrt{n^2 - \frac{3b}{4}(4-3b)} . \quad (2.27)$$

In the case $b = \frac{2}{3}$, the order of the modified Bessel functions in (2.26) is $+\sqrt{n^2-1}$ as in (2.11) and (2.12). The contribution to $\zeta(0)$ of the ghost, in both cases (magnetic and electric) is zero. Bearing in mind that, from Ref. [21], the contribution to $\zeta(0)$ of transverse modes is $-\frac{1}{2}$ with magnetic boundary conditions, and $\frac{1}{2}$ when the boundary conditions are electric, one gets

$$\zeta(0) = \zeta_{\rm transversal\ photons}(0) + \zeta_{R_1}(0) = -\frac{1}{2} + \frac{1}{2} = 0 . \quad (2.28)$$

The calculation of $\zeta(0)$ in the electric case is immediate. In this case $\dot{g}_n = 0$ and $R_n = 0$ at the three-sphere boundaries, and only the decoupled mode contributes to the $\zeta(0)$ value and it yields $\zeta_{R_1}(0) = -\frac{1}{2}$. Thus, also in this case, one obtains $\zeta(0) = 0$.

Our results coincide with those obtained by a space-time covariant Schwinger-DeWitt method, where the vanishing of the A_2 coefficient results from the mutual cancellation of the contributions from the two boundaries, in the case of flat Euclidean four-space [24].

III. DECOUPLING OF GAUGE MODES IN THE ONE-BOUNDARY PROBLEM

Since the gauge condition studied in the previous section leads more easily to the decoupling of g_n and R_n modes, and it agrees with the results found in Ref. [21] in the two-boundary case, it appears necessary to study its properties in the one-boundary problem as well. Moreover, this analysis enables one to further check the gauge dependence of the one-loop quantum amplitudes [21]. Following the results of Sec. II, we can write the regular solution for g_n, R_n and ϵ_n as

$$g_n(\tau) = A\, I_\nu(M\tau) , \quad (3.1)$$

$$R_n(\tau) = B\, \frac{1}{\tau}I_\nu(M\tau) , \quad (3.2)$$

$$\epsilon_n(\tau) = C\, I_\nu(M\tau) , \quad (3.3)$$

where A, B, C are constants. Imposing magnetic boundary conditions at the three-sphere boundary of radius a, we get

$$I_\nu(Ma) = 0 \tag{3.4}$$

for g_n and ϵ_n, and

$$I'_\nu(Ma) = 0 \tag{3.5}$$

for R_n. Remarkably, the only possible form of the decoupled mode for normal photons is $R_1 \equiv 0$, since R_1 would be proportional to $I_0(M\tau)/\tau$ in our gauge, and hence cannot be regular at the origin (see also the end of Sec. IV).

First, we evaluate $I_{\log}$ for g_n and ϵ_n. Using, as usual, the uniform asymptotic expansion of modified Bessel functions, eliminating fake roots $M = 0$ and taking into account that the degeneracy of ghost modes is -2 times the degeneracy of g_n, we see that the coefficient of $\ln M$ is $\nu + \frac{1}{2}$, where M^ν is the power of fake roots. For R_n, after dividing by $M^{\nu-1}$, we find that the coefficient of $\ln M$ is $-(\nu - \frac{1}{2})$. Hence we obtain

$$I_{\log} = \sum_{n=2}^{\infty} \frac{n^2}{2} = -\frac{1}{2} \; . \tag{3.6}$$

For ϵ_1, which is proportional to $I_0(M\tau)$, we get, by a simple calculation,

$$I_{\log_{\epsilon_1}} = \frac{1}{2} \; . \tag{3.7}$$

It is easy to see that the contribution to $I_{\rm pole}(\infty)$ is equal to zero for g_n, R_n, and ϵ_n separately. Last, we have to evaluate $I_{\rm pole}(0)$. The contribution of g_n to $I_{\rm pole}(0)$ is obtained by taking the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \to \infty$ of

$$\frac{1}{2} n^2 \ln \frac{1}{\Gamma(\nu+1)} \; ,$$

while the structure of the term deriving from R_n which contributes to $I_{\rm pole}(0)$ is

$$\frac{1}{2} n^2 \ln \frac{1}{\Gamma(\nu)} \; .$$

Both terms contribute $-\frac{59}{720}$ to $I_{\rm pole}(0)$, and bearing in mind the different degeneracy between ghost and gauge modes one finds

$$I_{\rm pole}(0) = 0 \; . \tag{3.8}$$

Finally, taking into account the contribution to $\zeta(0)$ of the transverse part of the potential [1] we get the full $\zeta(0)$ as

$$\zeta(0) = -\frac{77}{180} \; . \tag{3.9}$$

Remarkably, this $\zeta(0)$ value agrees with the one obtained in Ref. [1], where ghost and gauge modes were not taken into account. The striking cancellation of $\zeta(0)$ contributions from ghost and gauge modes *in the particular gauge* (2.7) deserves further thinking.

If we choose electric boundary conditions at the three-sphere boundary, the roles of g_n and R_n are interchanged and ghost modes obey Neumann boundary conditions. Hence, a similar analysis leads to the full $\zeta(0)$ value [12,18]

$$\zeta(0) = \frac{13}{180} + \frac{1}{2} + \frac{3}{2} + 1 = \frac{553}{180} \; . \tag{3.10}$$

The results (3.9) and (3.10) show that, on choosing the gauge condition (2.7) in the one-boundary problem, the full $\zeta(0)$ value is different on imposing magnetic or electric boundary conditions. However, an analysis along the lines of Ref. [21] and of this section shows that, on imposing electric boundary conditions in the Lorentz gauge, one finds again $\zeta(0) = -\frac{31}{90}$ as in Ref. [21], where magnetic boundary conditions were studied in the one-boundary problem. The dependence of the $\zeta(0)$ value on the boundary conditions and on the gauge conditions seems to result from the ill-definite nature of the 3+1 split of our background with only one boundary (see Sec. V).

IV. THE MOST GENERAL GAUGE-AVERAGING FUNCTIONAL

In Refs. [12,18,21] and in Sec. II of this paper, gauge invariance of the Faddeev-Popov formalism in the presence of boundaries has been *assumed* to obtain a convenient set of eigenvalue equations leading to the full $\zeta(0)$ value for one-loop quantum amplitudes. To complete our analysis it is therefore necessary to study the most general gauge-averaging functional $\Phi(A)$. Of course, the family (2.2) of gauge functionals is only a *particular* case. Our $\Phi(A)$ should obey the following conditions.

(i) $\Phi(A)$ is linear in the gauge modes and their first derivatives, to ensure that the total Euclidean action is quadratic in the gauge modes and only involves second-order elliptic operators.

(ii) $\Phi(A)$ does not contain first derivatives of g_n modes. In fact, such derivatives only occur in the components of the electric field, but not in the Lorentz functional, or in the Coulomb functional, or in the A_0 Tr K term. Moreover, the variation of the total Euclidean action does not vanish if the contribution of $\dot{g}_n$ is added to $\Phi(A)$.

One is thus led to write $\Phi(A)$ in the form

$$\begin{aligned}\Phi(A) \equiv & \left(\gamma_1 \dot{R}_1 + \gamma_2 \frac{R_1}{\tau}\right) Q^{(1)}(x) \\ & + \sum_{n=2}^{\infty} \left(\gamma_1 \dot{R}_n + \gamma_2 \frac{R_n}{\tau} + \gamma_3 \frac{g_n}{\tau^2}\right) Q^{(n)}(x) \\ & = \gamma_1 {}^{(4)}\nabla^0 A_0 + \frac{1}{3}\gamma_2 \, A_0 \, {\rm Tr}\, K - \gamma_3 \, {}^{(3)}\nabla^i A_i \; ,\end{aligned} \tag{4.1}$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary dimensionless parameters independent of τ. Note that, if γ_1 does not vanish, it can be absorbed into the definition of α by setting $\frac{\gamma_1^2}{\alpha} \equiv \frac{1}{\tilde{\alpha}}$, whereas $\frac{\gamma_2}{\gamma_1} \equiv \tilde{\gamma}_2$, $\frac{\gamma_3}{\gamma_1} \equiv \tilde{\gamma}_3$. This implies that, if $\gamma_1 \neq 0$, one can always consider an equivalent quantum theory where $\gamma_1 = 1$, while γ_2 and γ_3 remain arbitrary. An equivalent classification is obtained by focusing on γ_2 or γ_3. With this understanding, the following (sub)families of nonvanishing gauge functionals may occur: (1) $\gamma_1 = 1$, $\gamma_2 \neq 0$, $\gamma_3 \neq 0$; (2) $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 \neq 0$; (3) $\gamma_1 = 1$, $\gamma_2 \neq 0$, $\gamma_3 = 0$; (4) $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = 0$; (5) $\gamma_1 = 0$, $\gamma_2 \neq 0$, $\gamma_3 \neq 0$; (6) $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$; (7) $\gamma_1 = 0$, $\gamma_2 \neq 0$, $\gamma_3 = 0$.

The cases (5)–(7) correspond to *degenerate* gauge functionals, in that they do not lead to second-order elliptic operators on R_n modes. They are not studied in this paper (cf. Ref. [23]). Hence we here focus on the cases (1)–(4), i.e., whenever γ_1 does not vanish (see above). The first problem we face is the attempt to decouple g_n and R_n modes by means of the operator matrix first applied in Ref. [21]. In our case, by virtue of (2.1) and (4.1), the coupled eigenvalue equations take the form (cf. Ref. [21])

$$\widehat{A}_n g_n(\tau) + \widehat{B}_n R_n(\tau) = 0 , \tag{4.2}$$

$$\widehat{C}_n g_n(\tau) + \widehat{D}_n R_n(\tau) = 0 , \tag{4.3}$$

where, on defining

$$\rho \equiv 1 + \frac{\gamma_3}{\alpha} , \tag{4.4}$$

$$\mu \equiv 1 + \frac{\gamma_2 \gamma_3}{\alpha} , \tag{4.5}$$

one has

$$\widehat{A}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau}\frac{d}{d\tau} - \frac{\gamma_3^2}{\alpha}\frac{(n^2-1)}{\tau^2} + \lambda_n , \tag{4.6}$$

$$\widehat{B}_n \equiv -\rho(n^2-1)\frac{d}{d\tau} - \frac{\mu(n^2-1)}{\tau} , \tag{4.7}$$

$$\widehat{C}_n \equiv \frac{\rho}{\tau^2}\frac{d}{d\tau} + \frac{\gamma_3}{\alpha}(1-\gamma_2)\frac{1}{\tau^3} , \tag{4.8}$$

$$\widehat{D}_n \equiv \frac{1}{\alpha}\frac{d^2}{d\tau^2} + \frac{3}{\alpha}\frac{1}{\tau}\frac{d}{d\tau} + \left[\frac{\gamma_2}{\alpha}(2-\gamma_2) - (n^2-1)\right]\frac{1}{\tau^2} + \lambda_n . \tag{4.9}$$

As we did in Ref. [21], we now look for a diagonalized matrix in the form

$$O_{ij}^{(n)} \equiv \begin{pmatrix} 1 & V_n(\tau) \\ W_n(\tau) & 1 \end{pmatrix} \begin{pmatrix} \widehat{A}_n & \widehat{B}_n \\ \widehat{C}_n & \widehat{D}_n \end{pmatrix} \times \begin{pmatrix} 1 & \alpha_n(\tau) \\ \beta_n(\tau) & 1 \end{pmatrix} . \tag{4.10}$$

Thus, on using the operator identities [21]

$$\left[\frac{d}{d\tau}, \alpha_n\right] = \frac{d\alpha_n}{d\tau} , \tag{4.11}$$

$$\left[\frac{d^2}{d\tau^2}, \alpha_n\right] = \frac{d^2\alpha_n}{d\tau^2} + 2\frac{d\alpha_n}{d\tau}\frac{d}{d\tau} , \tag{4.12}$$

and setting to zero the off-diagonal matrix element

$$O_{12}^{(n)} = \widehat{A}_n \alpha_n + \widehat{B}_n + V_n \widehat{C}_n \alpha_n + V_n \widehat{D}_n ,$$

one finds the system of equations (cf. Ref. [21])

$$V_n + \alpha_n = 0 , \tag{4.13}$$

$$2\frac{d\alpha_n}{d\tau} + 2\left(1 - \frac{1}{\alpha}\right)\frac{dV_n}{d\tau} + \frac{\left(\alpha_n + 3\frac{V_n}{\alpha}\right)}{\tau} - \rho(n^2-1) = 0 , \tag{4.14}$$

$$\frac{d^2\alpha_n}{d\tau^2} + \left(\frac{\rho V_n}{\tau^2} + \frac{1}{\tau}\right)\frac{d\alpha_n}{d\tau} - \frac{\gamma_3^2}{\alpha}\frac{(n^2-1)}{\tau^2}\alpha_n - (n^2-1)\frac{\mu}{\tau} + \frac{\gamma_3}{\alpha}\left(1-\gamma_2\right)V_n\alpha_n\frac{1}{\tau^3} + \left[\frac{\gamma_2}{\alpha}(2-\gamma_2) - (n^2-1)\right]\frac{V_n}{\tau^2} = 0 . \tag{4.15}$$

Equations (4.13) and (4.14) are solved by $V_n = -\alpha_n$, and

$$\alpha_n(\tau) = \frac{\alpha}{(\alpha-1)}\rho(n^2-1)\tau + \alpha_{0,n}\tau^{\frac{(3-\alpha)}{2}} , \tag{4.16}$$

where $\alpha_{0,n}$ is a constant. Since the insertion of (4.16) into (4.15) leads to an involved condition unless $\alpha = 1$, it is very interesting to study first the limiting case $\alpha \to \infty$. This does not affect the arbitrariness in the choice of the parameters $\gamma_1, \gamma_2, \gamma_3$ appearing in (4.1). One then finds the condition

$$\left(1 + \frac{\gamma_3}{\alpha}(2-\gamma_2)\right)\frac{\alpha^2}{(\alpha-1)^2}(n^2-1)^2\left(1+\frac{\gamma_3}{\alpha}\right)^2 + \left[\frac{\gamma_3^2}{\alpha}(n^2-1) + \frac{\gamma_2}{\alpha}(2-\gamma_2) - n^2\right]\frac{\alpha}{(\alpha-1)}(n^2-1) \times\left(1+\frac{\gamma_3}{\alpha}\right) = -(n^2-1)\left(1+\frac{\gamma_2\gamma_3}{\alpha}\right) , \tag{4.17}$$

which is identically satisfied for all $n \geq 2$, as $\alpha \to \infty$. This shows that the limiting form of $\alpha_n(\tau)$ as $\alpha \to \infty$, i.e.,

$$\alpha_n(\tau) \sim (n^2-1)\tau , \tag{4.18}$$

is indeed also a solution of Eq. (4.15).

One now has to set to zero the off-diagonal matrix element

$$O_{21}^{(n)} = W_n \widehat{A}_n + W_n \widehat{B}_n \beta_n + \widehat{C}_n + \widehat{D}_n \beta_n ,$$

in (4.10). By virtue of (4.6)–(4.9), and (4.11) and (4.12) applied to $\beta_n(\tau)$, one thus finds the system of equations

$$W_n + \beta_n = 0 \ , \tag{4.19}$$

$$2\frac{d\beta_n}{d\tau} + \frac{\left(W_n + \frac{3}{\alpha}\beta_n\right)}{\tau} + \frac{\rho}{\tau^2} = 0 \ , \tag{4.20}$$

$$\frac{1}{\alpha}\frac{d^2\beta_n}{d\tau^2} + \left(\frac{3}{\alpha}\frac{1}{\tau} - \rho(n^2-1)W_n\right)\frac{d\beta_n}{d\tau} - \mu(n^2-1)W_n\,\beta_n\,\frac{1}{\tau} + \left[\left(\frac{\gamma_2}{\alpha}(2-\gamma_2) - (n^2-1)\right)\beta_n - \frac{\gamma_3^2}{\alpha}(n^2-1)W_n\right]\frac{1}{\tau^2} + \frac{\gamma_3}{\alpha}(1-\gamma_2)\frac{1}{\tau^3} = 0 \ . \tag{4.21}$$

Equation (4.19) implies $W_n = -\beta_n$. Hence (4.20) is solved by

$$\beta_n(\tau) = \frac{\alpha}{(\alpha-1)}\frac{\rho}{3\tau} + \beta_{0,n}\,\tau^{\frac{1}{2}(1-\frac{3}{\alpha})} \ , \tag{4.22}$$

where $\beta_{0,n}$ is a constant. However, a direct calculation shows that the limiting form of $\beta_n(\tau)$ as $\alpha \to \infty$, i.e.,

$$\beta_n(\tau) \sim \frac{1}{3\tau} + \beta_{0,n}\sqrt{\tau} \ , \tag{4.23}$$

is not a solution of (4.21) as $\alpha \to \infty$. Moreover, if one studies finite values of α, the exact formulas (4.16) and (4.22), on insertion into (4.15) and (4.21) respectively, lead to equations which are not satisfied unless the parameters γ_2, γ_3 and α take very special values. For example, if $\gamma_2 = 1, \gamma_3 = -\alpha = -1$, $\alpha_n = \beta_n = V_n = W_n = 0$, the decoupling functional (2.7) is recovered.

Thus, our analysis shows that gauge modes cannot be decoupled for arbitrary gauge-averaging functionals, and one now faces the problem of evaluating their contribution to the full $\zeta(0)$ even though g_n and R_n are not expressed in terms of Bessel functions [12,18,21,29]. However, for the class of gauge conditions (2.2) involving the arbitrary dimensionless parameter b, the basis functions can be found by using the technique described in Ref. [21]. The resulting $\zeta(0)$ value in the two-boundary problem is again equal to zero for magnetic or electric boundary conditions, while in the one-boundary problem the $\zeta(0)$ value depends on b. In the case of magnetic boundary conditions one finds

$$\begin{aligned}\zeta_b(0) = &-\frac{8}{45} - \frac{1}{96}\left(3b-2\right)\left(27b^3 - 36b^2 - 12b - 8\right) \\ &+\left(\frac{|\,3b-2\,|}{2} - \frac{1}{4}\right)\left(1-\theta(b-1)\right) \\ &\times\left(1-\theta\left(\tfrac{1}{3}-b\right)\right) .\end{aligned} \tag{4.24}$$

Note that Eq. (4.24) reflects the absence of a regular decoupled mode R_1 for $b \in]\frac{1}{3}, 1[$, in agreement with what we found in the particular case of Sec. III. One can easily check that Eq. (4.24) agrees with the $\zeta(0)$ values obtained in Ref. [21] and in our Sec. III.

V. RESULTS AND OPEN PROBLEMS

In this paper we have obtained the following results. First, we have studied the class of gauge functionals for which the disentanglement of the eigenvalue equations for normal and longitudinal modes can be achieved, and we have pointed out one particular choice when such equations are decoupled from the beginning. Second, on using this particular gauge functional, the calculation of the full $\zeta(0)$ value, in the two-boundary problem, agrees with the evaluation performed in Ref. [21], where we have imposed other gauge conditions. Third, in the one-boundary problem, we have found that the one-loop quantum amplitudes are gauge-dependent and the computation of the full $\zeta(0)$ value is different on imposing magnetic or electric boundary conditions. These undesirable properties, as already noted in Refs. [17,21], seem to add evidence in favor of the 3+1 decomposition of the four-vector potential being ill defined on the manifolds bounded by only one three-surface. Fourth, we have studied the most general class of relativistic gauges and the corresponding eigenvalue equations have been obtained for the first time.

Interestingly, the recent work in the literature shows that the semiclassical amplitudes respect the properties of the underlying classical theory. For example, for a massless spin-$\frac{1}{2}$ field obeying the Weyl equation and subject to spectral or locally supersymmetric boundary conditions on a three-sphere, the regular modes turn out to obey the same boundary conditions [12,30]. In the one-loop quantum theory, the eigenvalue conditions are different, but the $\zeta(0)$ values turn out to coincide [6,7,9–12]. Moreover, Euclidean Maxwell theory in vacuum is invariant under duality transformations. Correspondingly, we have found that the one-loop amplitudes are independent of the choice of electric or magnetic boundary conditions, providing the Lorentz gauge is chosen in the one-boundary problem.

The main open problem in Euclidean Maxwell theory in the presence of boundaries seems to be the *explicit* proof of gauge invariance of one-loop amplitudes for relativistic gauges, in the case of flat Euclidean space bounded by two concentric three-spheres. For this purpose, one may have to show that, for coupled gauge modes, $I_{\log}$ and the difference $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ are not affected by a change in the gauge parameters $\gamma_1, \gamma_2, \gamma_3, \alpha$ of Sec. IV. Although this is what happens in the particular cases studied so far, at least three technical achievements are necessary to obtain a rigorous proof; i.e., (i) to relate the regularization at large x used in Refs. [12,18] to the regularization based on the Barvinsky-Kamenshchik-Karmazin-Mishakov (BKKM) function defined in (2.17a), (ii) to evaluate $I_{\log}$ from an asymptotic analysis of coupled eigenvalue equations, and (iii) to evaluate $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ by relating the analytic continuation to the whole complex-s plane of the difference $I(\infty, s) - I(0, s)$ [see (2.17a)] to the analytic continuation of the ζ function.

If this last step can be performed, it may involve an integral transform relating the BKKM function (2.17a) to the ζ function, and a non-trivial application of the Atiyah-Patodi-Singer theory of Riemannian four-manifolds with boundary [26,31]. In other words, one might have to prove that, *in the two-boundary problem only*, $I_{\rm pole}(\infty) - I_{\rm pole}(0)$ resulting from coupled gauge modes is the residue of a meromorphic function, invariant under a smooth variation in $\gamma_1, \gamma_2, \gamma_3, \alpha$ of the matrix of elliptic self-adjoint operators appearing in (4.6)–(4.9). Work is now in progress on this problem, and we hope to be able to solve it in a future publication.

There is also the problem of physical interpretation of the results obtained so far [18,21]. In the two-boundary case, where one has a well-defined 3+1 split of the electromagnetic potential, the contributions to $\zeta(0)$ which, jointly with transverse modes, enable one to obtain agreement with the space-time covariant calculation, result only from the decoupled gauge modes. Note that such decoupled modes should be treated separately, since they do not correspond to any Dirac constraint of the theory [19]. However, in the case of flat Euclidean space bounded by only one three-sphere, even on studying the Lorentz gauge which leads to agreement between mode-by-mode and space-time covariant calculations of Faddeev-Popov amplitudes, the nonvanishing contributions to $\zeta(0)$ are not due just to transverse modes and decoupled modes. By contrast, longitudinal, normal, and ghost modes play a role as well in obtaining the full $\zeta(0)$ value. Perhaps, the redefinition of the very notion of physical degrees of freedom is necessary in this case, and the problem deserves further consideration.

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