Description of the Riemannian geometry in the presence of conical defects

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A consistent approach to the description of integral coordinate-invariant functionals of the metric on manifolds M_α with conical defects (or singularities) of the topology $C_\alpha \times \Sigma$ is developed. According to the proposed prescription M_α are considered as limits of the converging sequences of smooth spaces. This enables one to give a strict mathematical meaning to a number of invariant integral quantities on M_α and make use of them in applications. In particular, an explicit representation for the Euler numbers and Hirtzebruch signature in the presence of conical singularities is found. Also, a higher dimensional Lovelock gravity on M_α is shown to be well defined and the gravitational action in this theory is evaluated. Another series of applications is related to the computation of black hole entropy in the higher derivative gravity and in quantum two-dimensional models. This is based on its direct statistical-mechanical derivation in the Gibbons-Hawking approach, generalized to the singular manifolds M_α, and gives the same results as in the other methods.

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I. INTRODUCTION

Thin cosmic strings are known to give rise to remarkable gravitational effects. They do not affect immediately the local geometry of a space-time manifold but change instead its global properties. Placing the origin of the polar coordinate system on the string axis, one reveals a deficit $2\pi(1-\alpha)$ of the polar angle φ [1]. Thus, near the string world sheet Σ the space looks like the direct product $C_\alpha \times \Sigma$ where C_α is the conical space with the corresponding ranging of the angle $0 \leq \varphi \leq 2\pi\alpha$.

This peculiarity results in the interesting quantum effects which have been studied for both simple cones [2] and around cosmic strings [3]. Spaces with analogous features appear also in the other important physical applications. The well-known example is the orbifolds occurring in the string compactifications [4]. A similar set of spaces, called conifolds, has been proposed to generalize the histories included in Euclidean functional integrals in quantum gravity [5]. Finally, much attention has been paid recently to conical defects in connection with thermodynamics in the presence of black hole [6, 7], and cosmological [8] horizons, where the conical angle α is associated with the inverse temperature of a system.

One should also point out a number of mathematical results. For instance, the general theory of the Laplace and heat kernel operators on such a kind of cones has been developed in [9] and an explicit form of the DeWitt-Schwinger coefficients has been found in some cases [10]. In addition, many works were devoted to the functional determinants and the ζ function on the different types of orbifolds [11].

On the other hand, a consistent description of the geometrical quantities and invariant functionals of the metric on the conical defects seems to be absent. To elucidate this problem, let us recall that a cone is everywhere flat space (like the plane) except the tip where its curvature R is singular. Obviously, calculations by means of the standard formulas of the Riemannian geometry cannot reveal this δ-like singularity, and other methods must be used to get a correct result. One of these, based on topological arguments, was suggested many years ago by Sokolov and Starobinsky [12] for two-dimensional cones and used recently in higher dimensions in [13]. However, an approach like that does not seem to be quite satisfactory. It only concerns the computation of the scalar curvature, saying nothing about the components of the Riemann tensor, and faces difficulties under generalization to arbitrary invariant functionals.

In this paper, we consider a more natural recipe how to handle singularities with a particular topology $C_\alpha \times \Sigma$ and use it in the relevant examples. The corresponding manifolds will be denoted by $\mathcal{M}_\alpha$. This method is to replace the singular space by a sequence of regular manifolds. All the integral invariants are then well defined, and final results are obtained when the regularization is taken off. Some aspects of such a regularization have already been discussed in the literature (see, for instance, [5], [14], [15]) and we represent its further development. Thus, we show that although arbitrary curvature polynomials in such a procedure turn out to be divergent and depend on regularization, some specific integral quantities can be finite and have the strict mathematical meaning. We make use of this fact to derive a number of new results of both mathematical and physical interest.

The paper is organized as follows. The regularization method is described in Sec. II. Its features are discussed in detail for two-dimensional cones where the regularization ambiguity and the structure of the integral of R^2 and Polyakov-Liouville action are investigated. Then,

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the technique and results are extended for the higher dimensional cases. We evaluate the components of the Riemann tensor on $\mathcal{M}_\alpha$ and give examples of the functionals being quadratic in the curvature.

A number of consequences and applications is presented in the second part of the paper, in Sec. III, which starts with the discussion of the generalized variational principle on a class of spaces including $\mathcal{M}_\alpha$. Then, we analyze the higher order curvature polynomials that can be defined on $\mathcal{M}_\alpha$. An important example is the Euler characteristics and Hirtzebruch signature of $\mathcal{M}_\alpha$ for which the explicit integral representation is found. Also the Lovelock gravity turns out to be strictly defined on the manifolds with conical singularities and we give the corresponding generalization of the Lovelock action and equations. Finally, using our technique, we calculate the black hole entropy in the higher derivative gravity and in quantum two-dimensional models. This is based on a direct statistical-mechanical derivation of the entropy in the Gibbons-Hawking approach generalized to the singular manifolds and gives the same results as in the other methods. Some technical moments are clarified in the Appendix.

II. THE METHOD

A. Two-dimensional cones

1. Integral curvature

The method is worth illustrating when $\mathcal{M}_\alpha$ is a two-dimensional space with topology of the cone $\mathcal{C}_\alpha$. Then its metric reads

$$ds^2 = e^\sigma(d\rho^2 + \rho^2 d\phi^2) \equiv e^\sigma ds^2_C \ , \tag{2.1}$$

where ds^2_C is the line element on $\mathcal{C}_\alpha$, ϕ runs from 0 to $2\pi\alpha$, and the conformal factor σ is assumed to have the following expansion in the vicinity of $\rho = 0$:

$$\sigma = \sigma_1\rho^2 + \sigma_2\rho^4 + \cdots \ . \tag{2.2}$$

In general, σ_1 and σ_2 can be functions of the angle ϕ and a possible constant term in (2.2) can be absorbed by redefinition of ρ.

Because of asymptotics (2.2), the singularity comes out only from the conical metric ds^2_C. Hence, to understand how to introduce the regularization of $\mathcal{M}_\alpha$, consider an embedding of $\mathcal{C}_\alpha$ in three-dimensional (pseudo-)Euclidean space. It can be given by equations $x = \alpha\rho\cos(\phi/\alpha)$, $y = \alpha\rho\sin(\phi/\alpha)$, $z = \sqrt{|1-\alpha^2|}\rho$, that define the surface

$$z^2 - \frac{|1-\alpha^2|}{\alpha^2}(x^2+y^2) = 0, \quad z \geq 0 \quad . \tag{2.3}$$

Obviously, if $\alpha \neq 1$, there is a singularity at $z = 0$ where one cannot introduce the tangent space and calculate the curvature in the usual way.

It is easy to "roll off" the cone tip if going from $\mathcal{C}_\alpha$ to a surface $\tilde{\mathcal{C}}_\alpha$ with the equation $z = \sqrt{|1-\alpha^2|}f(\rho,a)$ where $f(\rho,a)$ is a smooth function and a is a regularization parameter such that $\lim_{a\to 0} f = \rho$. So far as for $\mathcal{C}_\alpha$ the function z has a minimum at $\rho = 0$, this should also be valid for f in the case of the regularized surface. Thus, the only additional condition on $f(\rho,a)$ is $\partial_\rho f|_{\rho=0} = 0$ and the line element on $\tilde{\mathcal{C}}_\alpha$ can be written as

$$ds^2_{\tilde{C}} = u d\rho^2 + \rho^2 d\phi^2, \quad u = \alpha^2 + (1-\alpha^2)(f'_\rho)^2 \ , \tag{2.4}$$

where the function u has the asymptotics

$$u|_{\rho=0} = \alpha^2, \quad u|_{\rho\gg a} = 1 \ . \tag{2.5}$$

The simplest example of the regularization is that corresponding to the change of $\mathcal{C}_\alpha$ to a hyperbolic space:

$$z^2 - \frac{|1-\alpha^2|}{\alpha^2}(x^2+y^2) = a^2, \quad z \geq 0 \ , \tag{2.6}$$

$$ds^2_H = \frac{\rho^2 + a^2\alpha^2}{\rho^2 + a^2}d\rho^2 + \rho^2 d\phi^2 \ . \tag{2.7}$$

Instead of the singular manifold (2.1) one can use now the regularized space $\tilde{\mathcal{M}}_\alpha$ with topology of $\tilde{\mathcal{C}}_\alpha$. To proceed, it is convenient to represent the scalar curvature on it in the form

$$R = e^{-\sigma}R_{\tilde{C}} - e^{-\sigma}\Box_{\tilde{C}}\sigma \ , \tag{2.8}$$

where $R_{\tilde{C}}$ and $\Box_{\tilde{C}}$ are the curvature and Laplace operator defined with respect to metric (2.4). Then, by taking into account the form of the regularized volume element $d\mu = e^\sigma\sqrt{u}\rho d\rho d\phi$ and asymptotics (2.5), one can evaluate the integral curvature on $\tilde{\mathcal{M}}_\alpha$:

$$\begin{aligned}\int_{\tilde{\mathcal{M}}_\alpha} R &= 2\pi\alpha\int_0^\infty d\rho u'_\rho u^{-\frac{3}{2}} - \int_0^\infty\int_0^{2\pi\alpha}\sqrt{u}\rho d\rho d\phi\ \Box_{\tilde{C}}\sigma \\ &= 4\pi(1-\alpha) - \int_0^\infty\int_0^{2\pi\alpha}\sqrt{u}\rho d\rho d\phi\ \Box_{\tilde{C}}\sigma \ . \end{aligned}\tag{2.9}$$

The conical metric results on the first term on the right hand side (RHS) of in (2.9) which does not depend on the regularization. The dependence on u appears only in the second "volume" term in (2.9), but the latter turns out to be finite in the limit when regularization is taken off and it coincides with integral curvature computed in the standard way on the smooth domain $\mathcal{M}_\alpha/\Sigma$ of $\mathcal{M}_\alpha$. (Let us recall that Σ denotes the singular set that is the point $\rho = 0$ in the given case.) Thus, in the limit $a \to 0$ one has, from (2.9),

$$\lim_{\tilde{\mathcal{M}}_\alpha\to\mathcal{M}_\alpha}\int_{\tilde{\mathcal{M}}_\alpha} R = 4\pi(1-\alpha) + \int_{\mathcal{M}_\alpha/\Sigma} R \ . \tag{2.10}$$

When in the region $\mathcal{M}_\alpha/\Sigma$ the curvature R equals to zero, Eq. (2.10) reproduces the formula of Sokolov and Starobinsky [12]. This result does not depend on the concrete behavior of the regularization function u, which can be shown to be a consequence of the Gauss-Bonnet theorem relating the integral curvature in two dimensions with the Euler characteristics. A more deep discussion of this point will be given in Sec. III.

So far as only a singular point $\rho = 0$ can give rise to the first term in (2.10), one can introduce a local representation for the curvature on $\mathcal{M}_\alpha$:

$$^{(\alpha)}R = \frac{2(1-\alpha)}{\alpha}\delta(\rho) + R \ , \tag{2.11}$$

where $\delta(\rho)$ is the δ-function normalized as

$$\int_0^\infty \delta(\rho)\rho d\rho = 1 \ .$$

In applications one also needs to handle with the higher order curvature polynomials or nonlocal functionals on conical defects. However, as distinct from the integral curvature (2.10), they include in general nonintegrable singularities such as $\delta^n(\rho)$. Let us consider the properties of such functionals on the simplest examples.

2. Integral of R^2

Without loss of generality, assume that metric (2.1) does not depend on the angle variable ϕ. In this case, introducing a new radial coordinate $x = \frac{\rho}{a}$, one can on $\tilde{\mathcal{M}}_\alpha$ write the curvature at the singular point as a decomposition:

$$R = (1 - a^2\sigma_1 x^2)\left(\frac{1}{a^2}\frac{u'}{xu^2} - \frac{4\sigma_1}{u} + \sigma_1\frac{xu'}{u^2} + O(a^2)\right) \ , \tag{2.12}$$

where $u' \equiv \frac{\partial u}{\partial x}$. Using this it can be shown that the equality

$$\int_{\tilde{\mathcal{M}}_\alpha} R^2 = \int_{\mathcal{M}_\alpha/\Sigma} R^2 + 2R(0)I_1(\alpha) + X(\alpha, a) \ , \tag{2.13}$$

$$X(\alpha, a) = -\frac{1}{4}R(0)I_2(\alpha) + \frac{1}{a^2}I_3(\alpha) \tag{2.14}$$

holds at small values of the regularization parameter. Here the relation $-4\sigma_1 = R(\rho = 0) \equiv R(0)$ has been used and $I_k(\alpha)$ denote the integrals:

$$I_1(\alpha) = 2\pi\alpha\int_0^\infty dx\left(\frac{u'_x}{u^{\frac{5}{2}}}\right) = \frac{4\pi}{3}\frac{(1-\alpha^3)}{\alpha^2} \ ,$$

$$I_2(\alpha) = 2\pi\alpha\int_0^\infty dx x\left(\frac{u'_x}{u^2}\right)^2 u^{\frac{1}{2}} \ ,$$

$$I_3(\alpha) = 2\pi\alpha\int_0^\infty \frac{dx}{x}\left(\frac{u'_x}{u^2}\right)^2 u^{\frac{1}{2}} \ . \tag{2.15}$$

The quantity $I_1(\alpha)$ that enters into the finite part of the integral R^2 does not depend on choice of the regularization function $u(\rho, a)$ and it is determined only by asymptotics (2.5). However, $I_2(\alpha)$ and $I_3(\alpha)$ change under variations of $u(\rho)$, including coordinate transformations. Taking this into account there is no reason to consider such terms in (2.13) separately and so they were gathered in a combination $X(\alpha, a)$. It follows from the form of (2.13), (2.14), and (2.15) that $X(\alpha, a)$ should be an invariant function, singular in the limit $a \to 0$. It is also important that at small conical deficits $X(\alpha, a)$ vanishes as fast as $(1-\alpha)^2$ and only finite terms in (2.13) dominate. This can be proved in general with the help of Eq. (2.4) but it is better to demonstrate for particular regularization (2.7):

$$I_2(\alpha) = \frac{16\pi(1-\alpha^2)^2}{15\alpha^2}\frac{(\alpha^2+3\alpha+1)}{(1+\alpha)^3} \ ,$$

$$I_3(\alpha) = \frac{8\pi(1-\alpha^2)^2}{15\alpha^4}\frac{(8\alpha^2+9\alpha+3)}{(1+\alpha)^3} \ . \tag{2.16}$$

The above consideration teaches that (2.13) and other similar invariant functionals cannot have a strict mathematical meaning in the presence of conical singularities. Nevertheless, the structure of singular terms in these integrals can be described and, as will be shown, in some important cases all of them cancel each other or do not contribute to the considered quantities leaving there finite terms not depending on the regularization.

3. Polyakov-Liouville action

This is an example of the nonlocal functional playing an important role in two-dimensional quantum gravity since it is result of integrating the conformal anomaly. It looks like [16]

$$W_{\rm PL} = \int R\psi \ , \tag{2.17}$$

where ψ is a solution of the equation

$$\Box\psi = R \ . \tag{2.18}$$

Consider (2.17) on the regularized space $\tilde{\mathcal{M}}_\alpha$ and make use of (2.8). Then (2.18) is solved as follows:

$$\psi = -\sigma + \psi_{\tilde{C}} \ ,$$

$$\psi_{\tilde{C}} = -2\ln\rho + C\int^\rho \frac{u^{\frac{1}{2}}}{\rho}d\rho + E \ , \tag{2.19}$$

where $\psi_{\tilde{C}}$ is a solution of Eq. (2.18) for the smoothed cone $\tilde{C}$ with the curvature $R_{\tilde{C}} = u'_\rho/(\rho u^2)$, C and E are constants. It should be noted that only for $C = \frac{2}{\alpha}$ can this be written in everywhere regular form,

$$\psi_{\tilde{C}} = \frac{2}{\alpha}\int_0^\rho \frac{u^{\frac{1}{2}} - \alpha}{\rho'}d\rho' + E \ , \tag{2.20}$$

and, moreover, the function $\psi_{\tilde{C}}$ coincides in the limit $a \to 0$ with the corresponding solution on the conical space C_α:

$$\psi_{\tilde{C}} \to \psi_C = 2\frac{1-\alpha}{\alpha}\ln\rho + E \ , \tag{2.21}$$

where the possible singular term $\sim \ln a$ absorbed in redefinition of constant E. The function ψ is important for analysis of quantum effects on gravitational background and enters into the formulas for the energy density of the Hawking radiation and black hole entropy [7].

The nonlocal action (2.17) on regularized two-dimensional manifold $\tilde{\mathcal{M}}_\alpha$ can be written in a more suitable form:

$$W_{\rm PL}[\tilde{\mathcal{M}}_\alpha] = \int_{\tilde{C}} (R_{\tilde{C}}\psi_{\tilde{C}} - 2\sigma R_{\tilde{C}} + \sigma\Box_{\tilde{C}}\sigma) \ . \qquad (2.22)$$

Obviously, the second and third terms on the RHS of (2.22) give a regular contribution when regularization is taken off. Taking into account Eq. (2.11), one gets, when $\tilde{C}_\alpha \to C_\alpha$,

$$\int_{\tilde{C}} \sigma R_{\tilde{C}} \to 4\pi(1-\alpha)\sigma(0) \ , \qquad (2.23)$$

where $\sigma(0)$ can be zero, if the asymptotics (2.2) is assumed. In addition, the limit

$$\int_{\tilde{C}} \sigma\Box_{\tilde{C}}\sigma \quad \to \quad \int_C \sigma\Box_C\sigma \equiv W_{\rm PL}[\mathcal{M}_\alpha/\Sigma] \qquad (2.24)$$

can be identified with the contribution to the Polyakov-Liouville action from the regular points of $\mathcal{M}_\alpha$. The remaining term in (2.22) for $E=0$ has a nonlocal form

$$\begin{aligned}\int_{\tilde{C}_\alpha} R_{\tilde{C}}\psi_{\tilde{C}} &= 2\pi\alpha \int_0^\infty \frac{u'_\rho}{u^{\frac{3}{2}}}\psi_{\tilde{C}}(\rho)d\rho \\ &= 2\pi\alpha \int_0^\infty \frac{u'_\rho}{u^{3/2}}d\rho \int_0^\rho \frac{u^{1/2}-\alpha}{u^{3/2}}d\rho' \\ &\equiv X(\alpha,a) \ . \end{aligned} \qquad (2.25)$$

As one can see, $X(\alpha,a)$ is regular in the limit $a \to 0$ [for the regularization (2.6) dependence on a is absent] but depends on the form of the regularization function u. From Eq. (2.4) other important property follows that when $\alpha \to 1$ the function $X(\alpha,a)$ vanishes as $(1-\alpha)^2$.

Finally, one obtains the action (2.17) on $\mathcal{M}_\alpha$ in the form

$$W_{\rm PL}[\mathcal{M}_\alpha] = W_{\rm PL}[\mathcal{M}_\alpha/\Sigma] + 8\pi(1-\alpha)\psi(\Sigma) + X(\alpha) \ , \qquad (2.26)$$

where $\psi = -\sigma$ is a solution of Eq. (2.18) when $\alpha = 1$, Σ is a singular point and $X(\alpha) \equiv X(\alpha, a=0)$. Thus the nonlocal action $W_{\rm PL}$ turns out to be finite (in the limit $a \to 0$) but regularization dependent. We will return to Eq. (2.26) in Sec. III.

B. Higher-dimensional case

The technique can be extended now to higher dimensions. Let us consider a two-dimensional cone C_α embedded in the Riemann d-dimensional manifold $\mathcal{M}_\alpha$ so that near the singularity ($\rho = 0$) the metric is represented as

$$\begin{aligned} ds^2 &= e^\sigma \Big(d\rho^2 + \rho^2 d\phi^2 \\ &\quad + \sum_{i,j=1}^{d-2} [\gamma_{ij}(\theta) + h_{ij}(\theta)\rho^2] d\theta^i d\theta^j + \cdots \Big) \\ &\equiv e^\sigma d\tilde{s}^2 \ , \end{aligned} \qquad (2.27)$$

where the ellipsis means terms of higher power in ρ^2 and ϕ runs from 0 to $2\pi\alpha$. For convenience we prefer to use the same parametrization as in two dimensions but, as distinct from this case, the singular set now is a $(d-2)$-dimensional surface Σ with coordinates $\{\theta^i\}$ and metric $\gamma_{ij}(\theta)$. Near it $\mathcal{M}_\alpha$ looks as a direct product $\mathcal{C}_\alpha \times \Sigma$. One can also consider $\mathcal{M}_\alpha$ having a number of singular sets Σ_i, each with the corresponding conical angle α_i. Hereafter the metric will be assumed not to depend on φ at least in the small region of Σ.

The metric (2.27) can be regularized with a parameter a as in two dimensions by changing the $g_{\rho\rho}$ component in the conical part

$$\begin{aligned} ds^2 &= e^\sigma \Big[u(\rho,a)d\rho^2 + \rho^2 d\phi^2 \\ &\quad + \sum_{i,j=1}^{d-2} [\gamma_{ij}(\theta) + h_{ij}(\theta)\rho^2] d\theta^i d\theta^j + \cdots \Big] \ . \end{aligned} \qquad (2.28)$$

The curvature tensors for a manifold $\tilde{\mathcal{M}}_\alpha$ with metric (2.28) and evaluation of the geometrical quantities in the limit $a \to 0$ are similar to that we considered in two dimensions. Leaving the details for Appendix A, it should be mentioned that only the two-dimensional conical part of (2.28) gives rise to the singular contributions.

We begin with formulas for components of the Riemann tensor that can be represented near Σ as

$$\begin{aligned} {}^{(\alpha)}R^{\mu\nu}{}_{\alpha\beta} &= R^{\mu\nu}{}_{\alpha\beta} + 2\pi(1-\alpha) \\ &\quad \times [(n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha)]\,\delta_\Sigma \ , \\ {}^{(\alpha)}R^\mu{}_\nu &= R^\mu{}_\nu + 2\pi(1-\alpha)(n^\mu n_\nu)\delta_\Sigma \ , \\ {}^{(\alpha)}R &= R + 4\pi(1-\alpha)\delta_\Sigma \ , \end{aligned} \qquad (2.29)$$

where δ_Σ is the δ function: $\int_{\mathcal{M}} f\delta_\Sigma = \int_\Sigma f$; $n^k = n^k_\mu dx^\mu$ are two orthonormal vectors orthogonal to Σ, $(n_\mu n_\nu) = \sum_{k=1}^2 n^k_\mu n^k_\nu$, and the quantities $R^{\mu\nu}{}_{\alpha\beta}$, $R^\mu{}_\nu$, and R are computed in the regular points $\mathcal{M}_\alpha/\Sigma$ by the standard method.

A consequence of (2.29) is the following important formula for the integral curvature of $\mathcal{M}_\alpha$

$$\int_{\mathcal{M}_\alpha} {}^{(\alpha)}R = 4\pi(1-\alpha)A_\Sigma + \int_{M_\alpha/\Sigma} R \ , \qquad (2.30)$$

where $A_\Sigma = \int_\Sigma$ is the area of Σ. Equation (2.30) already appeared in a number of recent publications for particular cases [6] and it was virtually implied in results of [13]. If $\mathcal{M}_\alpha$ has a number of singular surfaces Σ_i with different conical deficits $2\pi(1-\alpha_i)$ then the first term in (2.30) should be changed by the sum over all Σ_i.

As arbitrary functionals on $\mathcal{M}_\alpha$ are concerned, we give, as an example, the integrals of quadratic combinations in $R_{\mu\nu\lambda\rho}$. The chosen regularization leads to the results (for details see Appendix A)

$$\int_{\mathcal{M}_\alpha} R^2 = \int_{\mathcal{M}_\alpha/\Sigma} R^2 + \left(2I_1 - \frac{5d-8}{8(d-1)}I_2\right)\int_\Sigma (R - R_\Sigma) + 8\pi(1-\alpha)\int_\Sigma R_\Sigma + \frac{3}{4(d-1)}I_2 \int_\Sigma Q_\Sigma + Y(\alpha)A_\Sigma \ , \quad (2.31)$$

$$\int_{\mathcal{M}_\alpha} R^2_{\mu\nu} = \int_{\mathcal{M}_\alpha/\Sigma} R^2_{\mu\nu} + \frac{1}{d-1}\left(\frac{d}{2}I_1 - \frac{3d-4}{16}I_2\right)\int_\Sigma (R - R_\Sigma) + \frac{1}{d-1}\left(I_1 + \frac{1}{8}I_2\right)\int_\Sigma Q_\Sigma + \frac{1}{2}Y(\alpha)A_\Sigma \ , \quad (2.32)$$

$$\int_{\mathcal{M}_\alpha} R^2_{\mu\nu\alpha\beta} = \int_{\mathcal{M}_\alpha/\Sigma} R^2_{\mu\nu\alpha\beta} + \frac{1}{d-1}\left(2I_1 - \frac{d}{8}I_2\right)\int_\Sigma (R - R_\Sigma) + \frac{1}{d-1}\left(4I_1 - \frac{1}{4}I_2\right)\int_\Sigma Q_\Sigma + Y(\alpha)A_\Sigma \ , \quad (2.33)$$

where $Y(\alpha)$ is a quantity divergent in the limit $a \to 0$ and

$$Q_\Sigma = \frac{d}{2}R_{\mu\alpha\nu\beta}n^\mu_i n^\nu_i n^\alpha_j n^\beta_j - R_{\mu\nu}n^\mu_i n^\nu_j \ .$$

For $d = 2$ expression (2.31) coincides with that derived in the previous section. As in two dimensions, integrals (2.31)–(2.33) contain both divergent $[Y(\alpha)]$ and dependent on the regularization $[I_2(\alpha)]$ terms and can be brought into the same form as (2.13) by gathering these terms together. In this case the remaining part of (2.31)–(2.33) will be a sum of the integrals over the smooth domain of $\mathcal{M}_\alpha$ and regularization-independent additions in the form of surface integrals depending on either internal or external geometry of Σ. Obviously, one can proceed in this way and obtain similar expressions for functionals being higher order curvature polynomials on $\mathcal{M}_\alpha$. These examples follow below.

III. APPLICATIONS

A. Generalized variational principle

As the first straightforward application of Eq. (2.30), we consider the variational principle generalized on a class of manifolds admitting conical singularities. It can be used, for instance, in the description of gravitational effects caused by cosmic strings. The gravitational action including a cosmic string with the tension μ and two-dimensional world sheet Σ reads

$$W = -\frac{1}{16\pi G}\int R + \mu \int_\Sigma \equiv W_{\rm gr} + \mu \int_\Sigma \ . \quad (3.1)$$

Without loss of generality we assume that manifolds on which (3.1) is defined do not have the boundaries. Consider this functional on the spaces $\mathcal{M}_\alpha$ with conical singularities distributed over Σ and represent it, according to (2.30), as

$$W[\mathcal{M}_\alpha] = W_{\rm gr}[\mathcal{M}_\alpha/\Sigma] + \left(-\frac{1-\alpha}{4\pi G} + \mu\right)\int_\Sigma \ . \quad (3.2)$$

The form of (3.2) can be used now to find its variations on the given class of singular spaces, but without fixing the actual value of the deficit angle at the conical singularity. Thus, it is easy to see that, apart from the standard Einstein equations following from the first regular term on the RHS of (3.2), the independent change of the metric on Σ results in the additional condition

$$1 - \alpha = 4\pi G\mu \quad (3.3)$$

being the well-known relation between the string tension μ and the conical angle deficit [1]. Condition (3.3) is analogous to the "surface Einstein equations" in the presence of matter shells [17] that can also be obtained from variations of the gravitational functional [14]. As is seen, in the absence of strings (3.3) is satisfied only at the vanishing deficit angle, $\alpha = 1$. Therefore, even in the generalized variational principle the extrema of the Einstein action in vacuum are realized on the smooth manifolds. The same conclusion was previously derived in [14] for a minisuperspace model. On the other hand, spaces with a number of different conical defects cannot be extrema of the vacuum functional.

B. Topological characteristics of $\mathcal{M}_\alpha$

Let us turn to definition of the Euler numbers χ and the Hirtzebruch signature τ on manifolds with conical singularities. We are interested in these quantities so far as they are expressed through the integrals on powers of the Riemann tensor to which the regularization technique introduced above can be naturally applied. To be more specific, consider such a characteristic, say χ, on $\mathcal{M}_\alpha$ as a limit of this quantity taken on the converging sequence $\tilde{\mathcal{M}}_\alpha$:

$$\chi[\mathcal{M}_\alpha] \equiv \lim_{\tilde{\mathcal{M}}_\alpha \to \mathcal{M}_\alpha} \chi[\tilde{\mathcal{M}}_\alpha] = \chi \ . \quad (3.4)$$

By definition, the right-hand side of (3.4) is only determined by the topology of the smooth spaces and does not depend on the regularization parameter. Therefore, topological characteristics such as χ of a singular manifold $\mathcal{M}_\alpha$ simply coincide with those of $\tilde{\mathcal{M}}_\alpha$ and should be well-defined integral invariants. Our aim now is to find a concrete integral representation of χ and τ for $\mathcal{M}_\alpha$.

1. *Euler numbers*

To begin with, let us investigate the simplest example when $\mathcal{M}_\alpha$ is a closed four-dimensional space with one singular surface Σ. For its regularized analogue $\tilde{\mathcal{M}}_\alpha$ the Euler number reads

$$\chi = \frac{1}{32\pi^2}\int_{\tilde{\mathcal{M}}_\alpha} (R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta}) \ . \quad (3.5)$$

Using (2.31)–(2.33) and going from $\tilde{\mathcal{M}}_\alpha$ to $\mathcal{M}_\alpha$ one obtains from (3.5) a finite expression

$$\chi[\mathcal{M}_\alpha] = \frac{1}{32\pi^2}\int_{\mathcal{M}_\alpha/\Sigma}(R^2-4R^2_{\mu\nu}+R^2_{\mu\nu\alpha\beta})+(1-\alpha)\chi[\Sigma]\ , \tag{3.6}$$

where all the terms depending on the regularization are mutually canceled. Formula (3.6) gives the desired representation for $\chi[\mathcal{M}_\alpha]$ in which the first term on the RHS is the contribution to the integral from the regular points, and $\chi[\Sigma]=\frac{1}{4\pi}\int_\Sigma R_\Sigma$ is the Euler number of the surface Σ.

Equation (3.6) can be generalized to higher even dimensions $d=2p$. Without loss of generality, we confine ourselves to the compact spaces without boundaries. In this case, the Euler number of a $2p$-dimensional smooth manifold $\mathcal{M}$ is given by the integral [18]

$$\chi = c_p\int_{\mathcal{M}}\mathcal{L}_{2p}\sqrt{g}d^{2p}x\ , \tag{3.7}$$

where $\mathcal{L}_p$ is the quantity

$$\mathcal{L}_p = \epsilon_{\mu_1\mu_2\cdots\mu_{2p-1}\mu_{2p}}\epsilon^{\nu_1\nu_2\cdots\nu_{2p-1}\nu_{2p}}R^{\mu_1\mu_2}_{\ \ \ \nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}_{\ \ \ \nu_{2p-1}\nu_{2p}} \tag{3.8}$$

and the constant c_p is

$$c_p = \frac{1}{2^{2(p+1)}\pi^p p!}\ . \tag{3.9}$$

Now let the manifold in (3.7) be a smooth approximation $\tilde{\mathcal{M}}_\alpha$ of a $2p$-dimensional space $\mathcal{M}_\alpha$ with a singular $2(p-1)$-dimensional surface Σ. Then the Riemann tensor of $\tilde{\mathcal{M}}_\alpha$ can be represented as the sum

$$R^{\mu\nu}_{\ \ \alpha\beta} = R^{\mu\nu}_{(\text{reg})\ \alpha\beta} + R^{\mu\nu}_{(\text{con})\ \alpha\beta} \tag{3.10}$$

of a term remaining regular when the regularization is taken off and a term $R^{\mu\nu}_{(\text{con})\ \alpha\beta}$ provided by the conical singularity. The latter has only one nontrivial component [see (A3)]

$$R^{\phi\rho}_{(\text{con})\ \phi\rho} = \frac{1}{a^2}\frac{u'_x}{2xu^2}\ , \tag{3.11}$$

where $x=\frac{\rho}{a}$ and it is assumed that $\tilde{\mathcal{M}}_\alpha$ in the vicinity of Σ is covered by coordinates $\{\phi,\rho,\theta^i\}$ with metric (2.28). Inserting (3.10) into (3.8) one gets a polynomial with respect to $R^{\mu\nu}_{(\text{con})\ \alpha\beta}$. However, because of the antisymmetry of the ϵ tensor only the first order of this quantity survives and

$$\mathcal{L}_p = \mathcal{L}^{reg}_p + 4p\epsilon_{\phi\rho i_1\cdots i_{2p-2}}\epsilon^{\phi\rho j_1\cdots j_{2p-2}}R^{\phi\rho}_{(\text{con})\ \phi\rho}\times R^{i_1i_2}_{\ \ \ j_1j_2}\cdots R^{i_{2p-3}i_{2p-2}}_{\ \ \ j_{2p-3}j_{2p-2}}\ , \tag{3.12}$$

where indices i_k and j_k run from 1 to $2p-2$. This means that no singularities appear in (3.12) in the limit $a\to 0$ apart from an integrable δ function resulting in a surface addition on Σ. To evaluate it, choose the normal vectors n^k to Σ so that $n^1=\{n^1_\phi,0,...,0\}$ and $n^2=\{0,n^2_\rho,0,...,0\}$. Then the ϵ tensor reads

$$\epsilon_{\phi\rho i_1\cdots i_{2p-2}} = n^1_\phi n^2_\rho\epsilon_{i_1\cdots i_{2(p-1)}}\ ,$$

where $\epsilon_{i_1\cdots i_{2(p-1)}}$ is the rank $2(p-1)$ Levi-Civita tensor on Σ. Because of the orthonormality of the vectors n^k, the product of ϵ tensors in (3.12) becomes the product of their $2(p-1)$-dimensional analogues:

$$\epsilon_{\phi\rho i_1\cdots i_{2p-2}}\epsilon^{\phi\rho j_1\cdots j_{2p-2}} = \epsilon_{i_1\cdots i_{2p-2}}\epsilon^{j_1\cdots j_{2p-2}}\ .$$

In addition, so far as the extrinsic curvatures of the surface Σ vanish due to the isometry, the Gauss-Codacci equations [19] enable one to identify $R^{i_ki_n}_{\ \ \ j_lj_m}$ on Σ with the components of the Riemann tensor of this surface. Thus, in the limit $a\to 0$ one obtains the integral

$$\chi[\mathcal{M}_\alpha] = c_p\int_{\mathcal{M}_\alpha/\Sigma}\mathcal{L}_p + 8\pi p c_p(1-\alpha)\int_\Sigma \mathcal{L}_{(p-1)}\ , \tag{3.13}$$

where the first term on the RHS is evaluated in the regular points of $\mathcal{M}_\alpha$ and $\mathcal{L}_{(p-1)}$ takes the form (3.8) defined with respect to the metric on Σ. Finally, comparing this with (3.7) and using identity $c_{(p-1)}=8p\pi c_p$ one gets the desired formula for the Euler number (3.7). We will write this for the general case when $\mathcal{M}_\alpha$ has several singular surfaces Σ_i with the conical deficits $2\pi(1-\alpha_i)$:

$$\chi[\mathcal{M}_\alpha] = c_p\int_{\mathcal{M}_\alpha/\Sigma}\mathcal{L}_p + \sum_i(1-\alpha_i)\chi[\Sigma_i]\ . \tag{3.14}$$

As was expected the whole expression does not depend on the regularization and reproduces (3.6) as a particular case. This formula is also valid for a two-dimensional space when the Euler number is proportional to the integral curvature and the singular surfaces are the point sets. In this case (3.14) is a consequence of (2.10) if one takes into account that $\chi=1$ for a point. It is worth mentioning as well that (3.14) reminds one of a formula for the Euler characteristic of polygons where each vertex gives a contribution in χ determined by the corresponding angular defect [20].

The case is of special interest when $\mathcal{M}_\alpha$ possesses a continuous isometry rotation group in the polar coordinate ϕ [Eq. (2.27)] and all the singular surfaces in (3.14) have equal angles $\alpha_i=\alpha$. Then, if $\alpha=1$, the space is everywhere smooth. Otherwise, when $\alpha\neq 1$, $\mathcal{M}_\alpha$ can be obtained by the following chain of continuous topology preserving deformations: $\mathcal{M}_{\alpha=1}\to\tilde{\mathcal{M}}_{\alpha=1}\to\tilde{\mathcal{M}}_\alpha\to\mathcal{M}_\alpha$. Therefore, one can identify the Euler numbers $\chi[\mathcal{M}_\alpha]=\chi[\mathcal{M}_{\alpha=1}]$, which result, due to (3.14), in the interesting formula reducing the number χ of a manifold $\mathcal{M}_\alpha$ to that of the fixed points set of its Abelian isometry:

$$\chi[\mathcal{M}_{\alpha=1}] = \sum_i\chi[\Sigma_i]\ , \tag{3.15}$$

where we made use of the fact that for the given case the volume term in (3.14) equals $\alpha\chi[\mathcal{M}_{\alpha=1}]$. Equation (3.15) can be illustrated for the deformed hyperspheres S^d_α [8] with the conical deficits of the polar angle. Thus, the singular set of S^2_α consists of its "north" and "south" poles. Each of these points has $\chi=1$ and one gets, from (3.15), $\chi[S^2]=1+1=2$. On the other hand, the singular surface of S^d_α ($d \geq 3$) is S^{d-2} and from (3.15) the known identity $\chi[S^d] = \chi[S^{d-2}]$ follows. Note that Eq. (3.15) is valid only for spaces with continuous isometry in ϕ and it is violated for arbitrary kind orbifolds with conical singularities.

2. Hirtzebruch signature

We confine the analysis to the four-dimensional case that is of the most importance in applications. The Hirtzebruch signature τ on the smooth spaces without boundaries is represented by the integral [18]

$$\tau = \frac{1}{96\pi^2}\int_{\mathcal{M}} R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\gamma\sigma}\epsilon^{\alpha\beta\gamma\sigma}\sqrt{g}d^4x. \quad (3.16)$$

Consider this integral on the regularized space $\tilde{\mathcal{M}}_\alpha$ and use Eq. (3.10) to extract the term giving a singular contribution to the curvature tensor when regularization is removed. Because of the Levi-Civita tensor, the only additional surface term that can appear in (3.16) is defined by the quantity

$$R_{\rho\phi ij}R^{\rho\phi}_{(\text{con})}\ {}_{\rho\phi}\epsilon^{ij\rho\phi}\ ,$$

where $R_{\rho\phi ij}$ are regular components of the Riemann tensor taken on the singular surface Σ and ij indices are referred to its coordinates. However, taking into account the behavior (2.27) of the metric near Σ one can show that $R_{\rho\phi ij} = 0$ and the surface terms are absent. Therefore, the Hirtzebruch signature on $\mathcal{M}_\alpha$ has the same form as that on the smooth manifolds; it is given by the integral over the regular region

$$\tau[\mathcal{M}_\alpha] = \frac{1}{96\pi^2}\int_{\mathcal{M}_\alpha/\Sigma} R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\gamma\sigma}\epsilon^{\alpha\beta\gamma\sigma}\sqrt{g}d^4x. \quad (3.17)$$

One can also obtain $\tau[\mathcal{M}_\alpha]$ in higher dimensions and show that, similar to (3.17), it is represented by the integral over region $\mathcal{M}_\alpha/\Sigma$ without extra surface terms.

C. Lovelock gravity

Now a natural question arises: can one indicate higher order curvature polynomials not reducible to topological characteristics but still having strict meaning on the conical singularities? The answer is positive. To begin with, let us note that the integral of $(R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta})$ is the topological invariant only in four dimensions where it is reduced to a total derivative. Nevertheless, as one can show with the help of (2.31)–(2.33), this integral, having extended to higher dimensions, will be strictly defined as before and can be represented in the same form as its topological analog (3.6).

The given integral combination is a particular example of the so-called Lovelock gravity [21] and its property holds also for the general Lovelock gravitational action. This functional is introduced on a d-dimensional Riemannian manifold as the polynomial

$$W_L = \sum_{p=1}^{k_d}\lambda_p\int\frac{1}{2^{2p}p!}\delta^{[\nu_1\nu_2\cdots\nu_{2p-1}\nu_{2p}]}_{[\mu_1\mu_2\cdots\mu_{2p-1}\mu_{2p}]}R^{\mu_1\mu_2}{}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}{}_{\nu_{2p-1}\nu_{2p}} \equiv \sum_{p=1}^{k_d}\lambda_p W_p\ , \quad (3.18)$$

where $\delta^{[\cdots]}_{[\cdots]}$ is the totally antisymmetrized product of the Kronecker symbols and k_d is $(d-2)/2$ [or $(d-1)/2$] for even (odd) dimension d. In the four-dimensional case there is only one term $W_1 = \frac{1}{2}\int R$ in this functional and it is reduced to the Einstein action. It was argued [22] that the gravitational action similar to (3.18) arises in the low-energy expansion of string models. Moreover, because of antisymmetrization, no derivatives higher than second order appear in the equations in the Lovelock theory [21] and it turns out to be free of ghosts when expanding about flat space [22].

The fact that the Lovelock action is a finite and well-defined functional on manifolds with conical singularities can be proved along the lines given for the Euler characteristics. Indeed, each the integral W_p in W_L can be shown by using the properties of the Levi-Civita tensor to be a dimensional extension of the corresponding Euler number χ (3.7). Thus, the analysis showing that W_p is finite on $\mathcal{M}_\alpha$ and independent of the regularization is completely the same as that given for χ. The important things one should use for this are the antisymmetricity property and a helpful relation

$$\delta^{[\nu_1\cdots\nu_n]}_{[\mu_1\cdots\mu_n]} = \sum_{k=1}^{n}(-1)^{k+1}\delta^{\nu_k}_{\mu_1}\delta^{[\nu_2\cdots\nu_1\cdots\nu_n]}_{[\mu_2\cdots\mu_k\cdots\mu_n]}\ . \quad (3.19)$$

After a simple algebra the Lovelock action on $\mathcal{M}_\alpha$ can be represented as the sum of the volume and surface parts

$$W_L[\mathcal{M}_\alpha] = W_L[\mathcal{M}_\alpha/\Sigma] + 2\pi(1-\alpha)\sum_{p=0}^{k_d-1}\lambda_{p+1}W_p[\Sigma]\ , \quad (3.20)$$

where the first term is the action computed at the regular points and the second one is a Lovelock's action given on the singular surface. It should be stressed that integrals $W_p[\Sigma]$ are defined completely in terms of the Riemann tensor on Σ:

$$W_p[\Sigma] = \frac{1}{2^{2p}p!}\int_\Sigma \delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]}R^{i_1i_2}{}_{j_1j_2}\cdots R^{i_{2p-1}i_{2p}}{}_{j_{2p-1}j_{2p}} \quad (3.21)$$

and $W_0 \equiv \int_\Sigma$.

Formula (3.20) can be used to investigate the equations following from the extrema of $W_L(\mathcal{M}_\alpha)$. The variations of this functional at fixed α result in the normal Lovelock equations [21] and the surface ones:

$$2\pi(1-\alpha)\left(\sum_{p=1}^{k_d-1}\lambda_{p+1}\delta^{jj_1j_2\cdots j_{2p-1}j_{2p}}_{ii_1i_2\cdots i_{2p-1}i_{2p}}R^{i_1i_2}_{\ \ \ \ j_1j_2}\cdots R^{i_{2p-1}i_{2p}}_{\ \ \ \ \ \ \ \ j_{2p-1}j_{2p}}+\lambda_1\delta^j_i\right)=\mu\delta^j_i\ , \tag{3.22}$$

where μ is the density of a matter distributed over Σ. The latter equation generalizes relation (3.3) between string tension and polar angle deficit in the Einstein theory. Remarkably, in the higher dimensional case, an essential feature comes out: even if $\mu=0$, (3.22) may have nontrivial solutions different from these with $\alpha=1$. This means that singular manifolds $\mathcal{M}_\alpha$ can be extrema in the pure Lovelock gravity. However, further discussion of this point is outside the aim of this paper.

D. Calculus of black hole entropy

Manifolds with conical singularities naturally appear in the path integral approach to gravitational thermodynamics in the presence of the Killing horizons [6–8, 23]. Let the space-time possess a globally defined timelike Killing vector ∂_t and be static. Then the free energy of a field system at temperature $T=\beta^{-1}$ can be shown to coincide, up to multiplier β, with an effective action functional $W(\beta)$ given on an Euclidean section $\mathcal{M}_\beta$ of the corresponding background manifold. The time coordinate τ of this Euclidean space has to be periodical with the period β. In the case of the Killing horizon Σ, $\mathcal{M}_\beta$ acquires conical singularities on this surface and can be described near it by the metric (2.27) with $\alpha=\frac{\beta}{\beta_H}$. Here β_H^{-1} is the Hawking temperature at which conical singularities vanish and at which the black hole thermodynamics is considered. However, to get the entropy S from the partition function $Z(\beta)$ according to the standard definition

$$S=\left(-\beta\frac{\partial}{\partial\beta}+1\right)\ln Z(\beta)|_{\beta=\beta_H} \tag{3.23}$$

one should put β to be slightly different from β_H. In terms of the effective action Eq. (3.23) can be rewritten as

$$S=\left(\alpha\frac{\partial}{\partial\alpha}-1\right)W(\mathcal{M}_\alpha)\ , \tag{3.24}$$

where for the background manifold the previous notation $\mathcal{M}_\alpha$ has been introduced and $\alpha\equiv\beta\beta_H^{-1}$. Several examples how this formula can be used in the framework of the given regularization approach follow below.

1. Higher-derivative gravity

Consider the following gravitational action being quadratic in the curvature tensor:

$$W=\int\sqrt{g}d^dx\left(-\frac{1}{16\pi G}R+a_1R^2+a_2R^{\mu\nu}R_{\mu\nu}+a_3R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\right)\ . \tag{3.25}$$

The first term in (3.25) is the standard Einstein action, whereas the others are usually motivated by necessity to get rid off the one-loop ultraviolet divergences.

Obviously, a straightforward application of (3.24) to calculate the black hole entropy in such a theory faces a difficulty so far as the higher order terms are ill defined on the conical singularities and first one should change $\mathcal{M}_\alpha$ by its regular analog [7]. Then formulas (2.31)–(2.33) give the following expressions valid for any dimension d:

$$\int_{\mathcal{M}_\alpha}R=\alpha\int_{\mathcal{M}_{\alpha=1}}R+4\pi(1-\alpha)\int_\Sigma\ , \tag{3.26}$$

$$\int_{\mathcal{M}_\alpha}R^2=\alpha\int_{\mathcal{M}_{\alpha=1}}R^2+8\pi(1-\alpha)\int_\Sigma R+O((1-\alpha)^2)\ , \tag{3.27}$$

$$\int_{\mathcal{M}_\alpha}R^{\mu\nu}R_{\mu\nu}=\alpha\int_{\mathcal{M}_{\alpha=1}}R^{\mu\nu}R_{\mu\nu}+4\pi(1-\alpha)\times\int_\Sigma R_{\mu\nu}n^\mu_in^\nu_i+O((1-\alpha)^2)\ , \tag{3.28}$$

$$\int_{\mathcal{M}_\alpha}R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}=\alpha\int_{\mathcal{M}_{\alpha=1}}R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}+8\pi(1-\alpha)\times\int_\Sigma R_{\mu\nu\lambda\rho}n^\mu_in^\lambda_in^\nu_jn^\rho_j+O((1-\alpha)^2)\ , \tag{3.29}$$

where n^μ_i are two orthonormal vectors orthogonal to the horizon surface Σ. To get (3.26)–(3.29) we made use of the fact that $\mathcal{M}_\alpha$ is static and of the Gauss-Codacci identity on Σ:

$$R=R_\Sigma+2R_{\mu\nu}n^\mu_in^\nu_i-R_{\mu\rho\nu\sigma}n^\mu_in^\nu_in^\rho_jn^\sigma_j$$

in which the second fundamental forms are absent due to the symmetry. The first integrals in (3.26)–(3.29) are defined on the smooth space at $\alpha=1$, they are proportional to α and do not affect the entropy S. As for the terms $O((1-\alpha)^2)$ in (3.27)–(3.29), they depend on the regularization prescription and turn out to be singular in the limit $\tilde{\mathcal{M}}_\alpha\to\mathcal{M}_\alpha$, but they do not contribute to S and the energy of the system at the Hawking temperature ($\alpha=1$). Indeed, from (3.24) and (3.26)–(3.29) one obtains for S the following integral over the horizon Σ

$$S=\frac{1}{4G}A_\Sigma-\int_\Sigma\left(8\pi a_1R+4\pi a_2R_{\mu\nu}n^\mu_in^\nu_i+8\pi a_3R_{\mu\nu\lambda\rho}n^\mu_in^\lambda_in^\nu_jn^\rho_j\right)\ . \tag{3.30}$$

Remarkably, this expression differs from the Bekenstein-Hawking entropy $S=\frac{1}{4G}A_\Sigma$ in the Einstein gravity by the contributions depending both on internal and external geometry of the horizon due to higher order curvature terms in (3.25). It is easy to see that the effect of internal geometry of Σ is reduced to the integral curvature of this surface. In four dimensions ($d=4$) this, being a topological invariant, is an irrelevant addition to S.

It is worth noting that exactly the same expression can be derived by the Noether charge method suggested by Wald [24]. A difference between the two approaches is that Wald's method seems to be more general, but it is defined on the equations of motion, whereas the above derivation of (3.30) can be also applied off shell. A general proof of their equivalence when taken on shell has been given in [25].

2. Lovelock gravity

Expression (3.30) can be generalized to the theory with the gravitational action being an arbitrary polynomial in the Riemann tensor. A relevant example is again the Lovelock gravity, where the static black hole solutions do exist [26] and their thermodynamics can be treated along lines of thermodynamics in the Einstein gravity [27]. The entropy of a hole in this case can be inferred from the Lovelock action (3.20) associated with the free energy

$$S=\left(\alpha\frac{\partial}{\partial\alpha}-1\right)W_L(\mathcal{M}_\alpha)|_{\alpha=1}=-2\pi\sum_{p=0}^{k_d-1}\lambda_{p+1}W_p(\Sigma) \tag{3.31}$$

and it turns out to depend only on the internal geometry of Σ. Formula (3.31) has been previously derived in the Hamiltonian approach in [28], whereas arguments based on the dimensional continuation of the Euler characteristics have been used for its derivation in [13].

3. Two-dimensional quantum models

Two-dimensional models of quantum gravity represent a remarkable example when the one-loop effective action W can be found explicitly. Thus, in the two-dimensional (2d) dilaton gravity W is the combination

$$W=W_0-\frac{c}{96\pi}W_{\rm PL} \tag{3.32}$$

of the classical dilaton action

$$W_0=-\int d^2x\sqrt{g}[F(\Phi)R+G(\Phi)(\nabla\Phi)^2+U(\Phi)] \tag{3.33}$$

and the Polyakov-Liouville functional (2.17) generated by the quantum effects, c is a constant associated with the central charge.

The contribution of classical action W_0 (3.33) to the entropy can easily be found using Eq. (3.23):

$$S_0=4\pi F(\Phi_h), \tag{3.34}$$

where Φ_h is the value of the dilaton field Φ at the horizon which in two dimensions is a point x_h. This expression coincides with that previously obtained in [29, 30]. As for the quantum correction to S, it can be derived using formula (2.26) that defines $W_{\rm PL}$ on conical singularities and the fact that $X(\alpha)\simeq(1-\alpha)^2$. From (2.26) one immediately finds

$$S_1=\frac{c}{12}\psi(x_h)\quad . \tag{3.35}$$

The total entropy for the effective action (3.32) reads

$$S=S_0+S_1=4\pi F(\Phi_h)+\frac{c}{12}\psi_h\quad . \tag{3.36}$$

In the conformal gauge one puts $\psi_h=\sigma_h$. Again this result coincides with that previously obtained by means of the Wald method [31]. We see that the function $\psi(x)$ is not uniquely defined; one may add any solution of the homogeneous equation $w(x)$: $\Box w=0$. The concrete choice of $w(x)$ means the specification of the quantum state of the system and it can be found from appropriate boundary conditions [31]. Finally, it is worth noting that the last term in (3.36) determines a correction which comes from the conformal anomaly and for the dilaton holes it leads to a logarithmic dependence of the entropy on the mass of the hole [7, 32].

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APPENDIX

Here we present some technical details omitted in Sec. II. Thus, to compute the curvature on the regularized space $\tilde{\mathcal{M}}_\alpha$, one can take into account that the metric (2.28) is of the form $g_{\mu\nu}=e^{\sigma}\tilde{g}_{\mu\nu}$ and then make use of the formulas for the curvature tensors of two conformally related manifolds:

$$\begin{aligned}R[g]&=e^{-\sigma}\left(R[\tilde{g}]+\frac{1}{2}(d-1)\sigma^{\alpha}_{\ \alpha}\right)\ ,\\ R^{\mu}_{\ \nu}[g]&=e^{-\sigma}\left(R^{\mu}_{\ \nu}[\tilde{g}]+\frac{1}{4}[(d-2)\sigma^{\mu}_{\ \nu}+\delta^{\mu}_{\ \nu}\sigma^{\alpha}_{\ \alpha}]\right)\ ,\\ R^{\mu\nu}_{\ \ \alpha\beta}[g]&=e^{-\sigma}\left(R^{\mu\nu}_{\ \ \alpha\beta}[\tilde{g}]+\delta^{[\mu}_{\ [\alpha}\sigma^{\nu]}_{\ \beta]}\right)\ ,\end{aligned} \tag{A1}$$

where

$$\sigma_{\mu\nu}\equiv-2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma+\tilde{\nabla}_\mu\sigma\tilde{\nabla}_\nu\sigma-\frac{1}{2}\tilde{g}_{\mu\nu}(\tilde{\nabla}\sigma)^2\ . \tag{A2}$$

For metric $\tilde{g}_{\mu\nu}$ (2.28) we have, in the vicinity of $\rho=0$

$$\begin{aligned}\tilde{R}^{\phi}_{\ \rho\phi\rho}&=\frac{1}{a^2}\frac{u'_x}{2xu}+O(a^2)\ ,\\ \tilde{R}_{\rho\rho}&=\frac{1}{a^2}\frac{u'_x}{2xu}-h\left(1-\frac{xu'_x}{2u}\right)+O(a^2)\ ,\\ \tilde{R}^{\phi}_{\ \phi}&=\frac{1}{a^2}\frac{u'_x}{2xu^2}-\frac{h}{u}+O(a^2)\ ,\\ \tilde{R}&=\frac{1}{a^2}\frac{u'_x}{xu^2}+R_\Sigma-\frac{h}{u}\left(4-\frac{xu'_x}{u}\right)+O(a^2)\ ,\end{aligned} \tag{A3}$$

where $h=\gamma^{ij}h_{ij}$ and we introduced the variable $x=\frac{\rho}{a}$; R_Σ is the scalar curvature of Σ. Other components of the curvature tensors do not contain the terms divergent in the limit $a\to 0$. As for the tensor $\sigma_{\mu\nu}$ (A2), near the point $\rho = 0$ its components read

$$\sigma_{\rho\rho} = -8\sigma_1 + 4\sigma_1 \frac{xu'_x}{u} + O(a^2) \ ,$$

$$\sigma^{\phi}_{\ \phi} = -8\frac{\sigma_1}{u} + O(a^2) \ ,$$

$$\sigma^{\alpha}_{\ \alpha} = -16\sigma_1 \left(\frac{1}{u} - \frac{1}{4}\frac{xu'_x}{u^2}\right) + O(a^2) \ . \qquad \text{(A4)}$$

It is easy to see that in the limit $a \to 0$ only the two-dimensional conical part of the metric $\tilde{g}_{\mu\nu}$ gives singular contributions to the curvature tensors whereas the terms in (A3) result in regular additions. Finally, taking into account the form of the volume element

$$d\tilde{\mu} = a^2 e^{-d\sigma/2} u^{\frac{1}{2}} (1 + \tfrac{1}{2}a^2x^2h)\sqrt{\gamma} x dx d\phi d^{d-2}\theta \ ,$$

one obtains Eqs. (2.29).

Consider now the integrals of quadratic curvature combinations on $\mathcal{M}_\alpha$. By using (A2)–(A4) in the limit $a \to 0$ it can be shown that

$$\int_{\mathcal{M}_\alpha} R^2 = \alpha \int_{\mathcal{M}_{\alpha=1}} R^2 + 8\pi(1-\alpha)\int_\Sigma R_\Sigma - \left(8I_1 - \frac{5}{2}I_2\right) \times \int_\Sigma h + \left[(d-4)I_2 - 16(d-1)\left(I_1 - \frac{1}{4}I_2\right)\right] \times \int_\Sigma \sigma_1 + \frac{1}{a^2} I_3 A_\Sigma \ , \qquad \text{(A5)}$$

$$\int_{\mathcal{M}_\alpha} R^2_{\mu\nu} = \alpha \int_{\mathcal{M}_{\alpha=1}} R^2_{\mu\nu} + \left(\frac{3}{4}I_2 - 2I_1\right)\int_\Sigma h + \left[\frac{1}{2}(d-4)I_2 - 4d\left(I_1 - \frac{1}{4}I_2\right)\right]\int_\Sigma \sigma_1 + \frac{1}{2a^2} I_3 A_\Sigma \ , \qquad \text{(A6)}$$

$$\int_{\mathcal{M}_\alpha} R^2_{\mu\nu\alpha\beta} = \alpha \int_{\mathcal{M}_{\alpha=1}} R^2_{\mu\nu\alpha\beta} + \frac{1}{2}I_2 \int_\Sigma h + \left[(d-4) - 16\left(I_1 - \frac{1}{4}I_2\right)\right]\int_\Sigma \sigma_1 + \frac{1}{a^2} I_3 A_\Sigma \ , \qquad \text{(A7)}$$

where $A_\Sigma = \int_\Sigma \sqrt{\gamma} d^{d-2}\theta$ is the area of the singular surface Σ. By means of identities (A1) the integrals $\int_\Sigma h$ and $\int_\Sigma \sigma_1$ can be written in a coordinate invariant form in terms of the curvature tensors for the initial metric $g_{\mu\nu}$ (2.27):

$$\int_\Sigma h = \int_\Sigma \left(\frac{d}{4} R_{\mu\rho\nu\lambda} n^\mu_i n^\nu_i n^\lambda_j n^\rho_j - \frac{1}{2} R_{\mu\nu} n^\mu_i n^\nu_i\right) \ ,$$

$$8(d-1)\int_\Sigma \sigma_1 = \int_\Sigma (R_\Sigma - R - dR_{\mu\rho\nu\lambda} n^\mu_i n^\nu_i n^\lambda_j n^\rho_j + 2R_{\mu\nu} n^\mu_i n^\nu_i) \ . \qquad \text{(A8)}$$

Finally, when using (A8), expressions (A5)–(A7) take the invariant form of Eqs. (2.31)–(2.33).

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