PHYSICAL REVIEW D VOLUME 52, NUMBER 4 15 AUGUST 1995

Estimation of the post-Newtonian parameters in the gravitational-wave emission of a coalescing binary

Andrzej Królak*
Max-Planck-Society, Research Unit "Theory of Gravitation" at the Friedrich-Schiller-University, D-07743 Jena, Germany

Kostas D. Kokkotas
Section Astrophysics, Astronomy and Mechanics, Department of Physics, Aristotle University of Thessaloniki, 540 06 Thessaloniki, Macedonia, Greece

Gerhard Schäfer
Max-Planck-Society, Research Unit "Theory of Gravitation" at the Friedrich-Schiller-University, D-07743 Jena, Germany
(Received 7 March 1995)

The effect of the recently calculated second post-Newtonian correction on the accuracy of the estimation of the parameters of the gravitational-wave signal from a coalescing binary is investigated. It is shown that the addition of this correction degrades considerably the accuracy of the determination of the individual masses of the members of the binary. However the chirp mass and the time parameter in the signal are still determined to a very good accuracy. The possibility of estimating the effects of other theories of gravity is investigated. The performance of the Newtonian filter is investigated and it is compared with the performance of post-Newtonian search templates introduced recently. It is shown that both search templates can extract accurately useful information about the binary.

PACS number(s): 04.30.Db, 04.80.Nn, 95.85.Sz, 97.80.Af

I. INTRODUCTION

It is currently believed that the gravitational waves that come from the final stages of the evolution of compact binaries just before their coalescence are very likely signals to be detected by long-arm laser interferometers [1]. The reason is that in the case of binary systems we can predict the gravitational waveform very well; and the amplitudes are reasonably high for sources at distances out to 200 Mpc. An estimate based on the number of compact binaries known in our galaxy and extrapolated to the rest of the Universe shows that there should be one neutron star compact binary coalescence per year out to the distance of 200 Mpc [2,3]. This estimate is a safe lower bound on the rate of binary coalescence. Arguments based on progenitor evolution scenarios suggest that there should be 100 of two-neutron-star coalescences, 5 neutron-star–black-hole coalescences, and 0.5 two-black-hole coalescences out to 200 Mpc [4]. The waveform derived using the quadrupole formula has been known for quite some time [5]. A standard optimal method to detect the signal from a coalescing binary in a noisy data set and to estimate its parameters is to correlate the data with the filter matched to the signal and vary the parameters of the filter until the correlation is maximal. The parameters of the filter that maximize the correlation are estimators for the parameters of the signal. The detailed algorithms and the performance of the matched-filtering method in application to a coalescing binary gravitational-wave signal has been investigated by several authors, e.g., [6–10]. It has recently been realized [11] that the correlation is very sensitive even to very small variations of the phase of the filter because of the large number of cycles in the signal. Consequently the addition of small corrections to the phase of the signal due to the post-Newtonian effects decreases the correlation considerably. Thus the post-Newtonian effects in the coalescing binary waveform can be detected and estimated to a much higher accuracy than it was thought before [12]. This opens up new prospects but also considerable data analysis challenges for the Laser Interferometric Gravitational Wave Observatory (LIGO), VIRGO, and GEO600 projects which are rapidly progressing. It was also found [11] that the post-Newtonian series is not converging rapidly for a binary near coalescence. Hence higher post-Newtonian corrections will affect the correlation. Currently three post-Newtonian corrections to the quadrupole formula are already known [13] and the calculation of further ones is in progress. In this work we analyze the estimation of parameters of the second post-Newtonian signal. This part of the work complements a recent detailed analysis of the 3/2 post-Newtonian signal performed in Ref. [10]. We also examine the detectability of the post-Newtonian signal and estimation of its parameters using the Newtonian waveform as a filter. This filter can be used as the simplest search template. We compare the Newtonian search templates with the post-Newtonian search templates recently investigated in Ref. [14].

The paper is organized as follows. In the first part of Sec. II we present the gravitational-wave signal from a binary system to the currently known second post-Newtonian order. In this work we analyze the signal

*Permanent address: Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-950 Warsaw, Poland.

in the "restricted" post-Newtonian approximation (i.e., only the phase of the signal is given to the second post-Newtonian accuracy whereas the amplitude of the signal is calculated from the quadrupole formula), we assume circularized orbits, and we assume the spin parameters to be constant. In the second part we briefly describe the optimal method of detection of such a signal in noise and the maximum likelihood (ML) method to estimate the parameters of the signal. We derive a number of properties of the ML estimators of the parameters of our signal and we examine the bounds on their variances. Our analysis is based on the Cramer-Rao bound. In the third part we give the approximate rms errors of the estimators for the signal at various post-Newtonian orders. In the fourth part of Sec. II we consider the effects of other theories of gravity and their detectability from gravitational-wave measurements. We consider Jordan-Fierz-Brans-Dicke theory and Damour-Esposito-Farèse biscalar tensor theory. In Sec. III we consider the so-called "search templates" introduced in [11]. These are simple filters containing as few parameters as possible to effectively detect the multiparameter signals. In the first part of Sec. III we analyze the simplest search template—the Newtonian filter which is the waveform of the gravitational signal from a binary in the quadrupole approximation. We examine the Newtonian filter as a tool both to detect the signal and also to determine its nature. In the second part of Sec. III we compare the Newtonian filter with the other search template analyzed recently [14] based on the full post-Newtonian signal. In Sec. IV we summarize conclusions from our results. A number of results is left to appendices. In Appendix A we examine the first-order effects on the phase of the signal due to eccentricity. In Appendix B we give numerical values of the covariance matrices at various post-Newtonian orders. In Appendix C we give certain detailed formulas for the Damour-Esposito-Farèse theory. In Appendix D we briefly review the theory of optimal detection of known signal in noise and we generalize it to nonoptimal detection. In Appendix E we give a useful analytic approximation to the correlation integral of the optimal filter with the signal from a binary. The units are chosen such that $G = c = 1$.

II. POST-NEWTONIAN EFFECTS

A. Gravitational wave signal from a coalescing binary

Let us first give the formula for the gravitational waveform of a binary with the three currently known post-Newtonian corrections. We make the following approximations. We work within the so-called "restricted" post-Newtonian approximation; i.e., we only include the post-Newtonian corrections to the phase of the signal keeping the amplitude in its Newtonian form; this is because the effect of the phase on the correlation is dominant. The inclusion of post-Newtonian effects in amplitudes will not qualitatively change our results. Because of the effect of rapid circularization of the orbit by radiation reaction one can assume that the orbit is quasicircular. For example in the case of the gravitational-wave signal from the Hulse-Taylor binary pulsar PSR1913+16 at the characteristic frequency of the detector for such a signal of around 47 Hz the eccentricity e would be $\sim 10^{-6}$. Moreover the first-order contribution to the phase of the signal due to eccentricity goes like e^2. Nevertheless for completeness we include the first-order correction to the signal due to eccentricity in our formulas. We give a detailed derivation of this correction in Appendix A. We neglect the tidal effects. All tidal contributions to the gravitational wave signal from a coalescing binary were estimated to be small [12,15]. There is also a small additional contribution to the phase due to tail effects detectability of which has been considered in detail [16] and was found to be small. This correction is formally of the fourth post-Newtonian order and consequently we neglect it in the present analysis.

With these approximations the waveform, as a function of time, is given by the expression

$$h(t) = A\, f(t)^{2/3} \cos\left(2\pi \int_{t_a}^{t} f(t')dt' - \phi\right), \tag{1}$$

where

$$A = \frac{8}{5}\pi^{2/3}\frac{\mu m^{2/3}}{R} \tag{2}$$

and where ϕ is an arbitrary phase; μ and m are the reduced and the total mass of the binary, respectively; t_a is a time parameter; and R is the distance to the source. A is the rms average amplitude over all Euler angles determining the position of the binary in the sky and the inclination angle between the plane of the orbit of the binary and the line of sight. The rms amplitude A is 2/5 of the maximum possible amplitude. The characteristic time for the evolution of the binary to the currently known second post-Newtonian order is given by

$$\tau_{2\mathrm{PN}} := \frac{f}{df/dt} = \frac{5}{96}\frac{1}{\mu m^{2/3}}\frac{1}{(\pi f)^{8/3}}\left\{1 - \frac{157}{24}\frac{I_e}{f^{19/9}} + \left(\frac{743}{336} + \frac{11}{4}\frac{\mu}{m}\right)(\pi m f)^{2/3} - (4\pi - s_o)(\pi m f) + \left[\frac{3\,058\,673}{1\,016\,064} + \frac{5429}{1008}\frac{\mu}{m} + \frac{617}{144}\left(\frac{\mu}{m}\right)^2 + s_s\frac{m}{\mu}\right](\pi m f)^{4/3}\right\}, \tag{3}$$

where I_e is the asymptotic eccentricity invariant,

$$I_e = e_0^2 f_0^{19/9}, \tag{4}$$

and e_0 is the eccentricity of the binary at gravitational frequency f_0 (see Appendix A for derivation and explanations). The quantities s_o and s_s are spin-orbit and spin-spin parameters, respectively. They are given by the formulas

$$s_o = \frac{113}{12}(s_1 + s_2) + \frac{25}{4}\left(s_1\frac{m_2}{m_1} + s_2\frac{m_1}{m_2}\right), \tag{5}$$

$$s_s = \frac{247}{48}\mathbf{s}_1 \cdot \mathbf{s}_2 - \frac{721}{48}s_1 s_2, \tag{6}$$

$$\mathbf{s}_1 = \frac{\mathbf{S}_1}{m^2}, \quad \mathbf{s}_2 = \frac{\mathbf{S}_2}{m^2}, \tag{7}$$

$$s_1 = \mathbf{L} \cdot \mathbf{s}_1, \quad s_2 = \mathbf{L} \cdot \mathbf{s}_2, \tag{8}$$

where $\mathbf{L}$ is the total orbital angular momentum and $\mathbf{S}_1$, $\mathbf{S}_2$ are the spin angular momenta of the two bodies. The terms in curly brackets in Eq. (3) are, respectively, at lowest order, the Newtonian (quadrupole); at order $f^{-19/9}$, the lowest order contribution due to eccentricity (see Appendix A); at order $f^{2/3}$, the 1PN [17]; at order f, the nonlinear effect of "tails" of the wave (4π term) [18–21] and spin-orbit effects [22]; and at order $f^{4/3}$, 2PN [13] and spin-spin effects [22].

In general the spin parameters vary with time. It was shown [10] that s_o is nearly conserved, it never deviates from its average value by more than ~ 0.25. Moreover the time-dependent part of the spin parameter is oscillatory which reduces considerably its influence on the phase of the signal [10]. In this work we shall assume that both spin-orbit and spin-spin parameters are constant. We also neglect the effect of the precession of the orbital plane due to spin on the waveform. The effects of the spin on the waveform of the signal from an inspiralling binary have been investigated in detail in [23]. If we take the available estimate of the moment of inertia for the pulsar in the Hulse-Taylor binary and assume masses of the neutron stars in the binary of 1.4 solar masses then $s_o \simeq 4.8 \times 10^{-2}$ and $s_s \simeq 2.4 \times 10^{-5}$. If such values are typical then spin effects will make negligible contributions to the phase of the signal. However this may not be the case for binaries containing black holes. Moreover if cosmic censorship is violated and black holes rotate at a higher rate than allowed by maximally rotating Kerr black holes the spin effects will significantly affect the gravitational waveform.

In the analysis of the detection of the above signal and estimation of its parameters it is convenient to work in the Fourier domain. The expression for the Fourier transform of our signal in the stationary phase approximation is given by (cf. [1,24,6,10])

$$\tilde{h} = \tilde{A} f^{-7/6} \exp\left(i\{2\pi f t_a - \phi - \pi/4 + \tfrac{5}{48}[a(f;f_a)k + a_e(f;f_a)k_e + a_1(f;f_a)k_1 + a_{3/2}(f;f_a)k_{3/2} + a_2(f;f_a)k_2]\}\right) \tag{9}$$

(for $f > 0$ and by the complex conjugate of the above expression for $f < 0$), where

$$\tilde{A} = \frac{1}{(30)^{1/2}}\frac{1}{\pi^{2/3}}\frac{\mu^{1/2}m^{1/3}}{R}, \tag{10}$$

$$k = \frac{1}{\mu m^{2/3}}, \tag{11}$$

$$k_e = \frac{1}{\mu m^{2/3}} e_0^2 (\pi f_0)^{19/9}, \tag{12}$$

$$k_1 = \frac{1}{\mu}\left(\frac{743}{336} + \frac{11}{4}\nu\right), \tag{13}$$

$$k_{3/2} = \frac{m^{1/3}}{\mu}(4\pi - s_o), \tag{14}$$

$$k_2 = \frac{m^{4/3}}{\mu}\left(\frac{3\,058\,673}{1\,016\,064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2 + \frac{s_s}{\nu}\right), \tag{15}$$

$$a(f;f_a) = \frac{9}{40}\frac{1}{(\pi f)^{5/3}} + \frac{3}{8}\frac{\pi f}{(\pi f_a)^{8/3}} - \frac{3}{5}\frac{1}{(\pi f_a)^{5/3}}, \tag{16}$$

$$a_e(f;f_a) = -\frac{157}{24}\left(\frac{81}{1462}\frac{1}{(\pi f)^{34/9}} + \frac{9}{43}\frac{\pi f}{(\pi f_a)^{43/9}} - \frac{9}{34}\frac{1}{(\pi f_a)^{34/9}}\right), \tag{17}$$

$$a_1(f;f_a) = \tfrac{1}{2}\frac{1}{\pi f} + \tfrac{1}{2}\frac{\pi f}{(\pi f_a)^2} - \frac{1}{\pi f_a}, \tag{18}$$

$$a_{3/2}(f;f_a) = -\left(\frac{9}{10}\frac{1}{(\pi f)^{2/3}} + \frac{3}{5}\frac{\pi f}{(\pi f_a)^{5/3}} - \frac{3}{2}\frac{1}{(\pi f_a)^{2/3}}\right), \tag{19}$$

$$a_2(f;f_a) = \frac{9}{4}\frac{1}{(\pi f)^{1/3}} + \frac{3}{4}\frac{\pi f}{(\pi f_a)^{4/3}} - \frac{3}{(\pi f_a)^{1/3}} \tag{20}$$

hold and where $\nu = \frac{\mu}{m}$ and $f_a = f(t_a)$.

The stationary phase approximation, Eq. (10), is an excellent approximation of the Fourier transform of the signal for frequencies which are not influenced by the finite time window of the measurement. In the above expressions for the gravitational-wave signal from a binary we can make an arbitrary choice of the time parameter and the phase of the signal.

We also point out that going from the time to the frequency domain we have made yet another approximation. Namely we have taken the modulus $|\tilde{h}|$ of the Fourier transform to be the Newtonian one, i.e., $|\tilde{h}| \sim f^{-7/6}$. In the stationary phase approximation $|\tilde{h}|$ goes like $1/\sqrt{f}$ and consequently by Eq. (4) there would be other powers of frequency due to the post-Newtonian effects. We neglect those additional terms since the post-Newtonian corrections to the phase have the dominant effect. The inclusion of the post-Newtonian amplitudes to the signal will not qualitatively change the results of this work.

A convenient parameter is the *chirp mass* defined as $\mathcal{M} := k^{-3/5}$. In the quadrupole approximation the gravitational-wave signal from a binary is entirely determined by the chirp mass.

We shall consider three models of binaries: neutron-star–neutron-star (NS-NS), neutron-star–black-hole (NS-BH), and black-hole–black-hole (BH-BH) binaries with parameters summarized in Table I where $M_\odot$ means solar mass.

For neutron stars we calculated the spin using the available estimate of the moment of inertia for the neutron star in the Hulse-Taylor binary pulsar. We have taken black holes to be spinning at half the maximum rate (i.e., $s_i = 0.5m_i^2/m^2$). The orbital momenta vectors were assumed to be parallel to the spin vectors.

To have an idea of the size of the post-Newtonian corrections in the gravitational-wave signal from a binary when it enters the observation window of the laser interferometer we have evaluated the characteristic time $\tau_{2\mathrm{PN}}$ for the above three models at the frequency $f_0' = 47$ Hz which is the characteristic frequency of the detector for this signal (see below). We have made explicit the contributions to the characteristic time from the three post-Newtonian corrections:

$$\tau^1_{2\mathrm{PN}} = 44(1 + 0.046[from\ 1\mathrm{PN}] - 0.025[from\ 3/2\mathrm{PN}] + 0.0012[from\ 2\mathrm{PN}])\ \mathrm{sec}, \quad (21)$$

$$\tau^2_{2\mathrm{PN}} = 9.9(1 + 0.10[from\ 1\mathrm{PN}] - 0.071[from\ 3/2\mathrm{PN}] + 0.0060[from\ 2\mathrm{PN}])\ \mathrm{sec}, \quad (22)$$

$$\tau^3_{2\mathrm{PN}} = 1.7(1 + 0.17[from\ 1\mathrm{PN}] - 0.12[from\ 3/2\mathrm{PN}] + 0.018[from\ 2\mathrm{PN}])\ \mathrm{sec}. \quad (23)$$

One concludes from the above numbers that for the earth-based laser interferometers post-Newtonian corrections are significant. Moreover several things are apparent. The quadrupole term is dominant for all the three models. This indicates a very good accuracy of the quadrupole formula even in the regime of strongly gravitating bodies. This has been noticed in other studies, for example, in the numerical investigation of the gravitational-wave emission from the two black hole collisions [25]. The difference in size between the first post-Newtonian correction and the 3/2 post-Newtonian correction (tail term) is rather small. They differ by a factor of 2 for NS-NS binary and only by a factor of around 1.5 for binaries with a black hole. The second post-Newtonian correction is noticably smaller then the 3/2 post-Newtonian correction. The difference varies form a factor of 20 for a NS-NS binary to a factor of 7 for a BH-BH binary. The convergence of the post-Newtonian series appears to be worse for BH-BH binaries and in this case it would be desirable to have accurate numerical waveforms and not only the ones based on the post-Newtonian approximation. Such waveforms should be available as a result of the numerical projects such as the Grand Challenge project currently under way in the United States.

B. Detection of the signal and estimation of its parameters

For the purpose of this investigation we shall use a fit to the total spectral density $S_h(f)$ of the noise in the advanced LIGO detectors, devised in [10]. This fit comprises thermal, shot, and quantum noises in the detector:

$$S_h(f) = S_0\{(f_0/f)^4 + 2[1 + (f/f_0)^2]\}/5, \quad (24)$$

where $f_0 = 70$ Hz and $S_0 = 3 \times 10^{-48}\ \mathrm{Hz}^{-1}$. It is an excellent approximation to the detailed formulas for various noises given in [6] over the range of frequencies from 10 to 1000 Hz. For frequencies below 10 Hz the seismic noise dominates. The *sensitivity function* sen(f) of the detector is defined as $1/S_h(f)$. The sensitivity function

TABLE I. Numerical values of the parameters of the three fiducial binary systems. Black holes are of 10 solar masses and neutron stars are of 1.4 solar masses. Spin parameters are assumed to be constant. Spin for neutron stars was calculated from the typical estimate of the moment of inertia I for a neutron star of $I = 10^{38}\ \mathrm{kg\,m^2}$.

Binary	$m_1[M_\odot]$	$m_2[M_\odot]$	$\mathcal{M}[M_\odot]$	s_1	s_2	s_o	s_s
1. NS-NS	1.4	1.4	1.2	1.5×10^{-3}	1.5×10^{-3}	4.8×10^{-2}	2.4×10^{-5}
2. NS-BH	1.4	10	3.0	3.0×10^{-5}	0.38	4.0	3.5×10^{-4}
3. BH-BH	10	10	8.7	0.13	0.13	3.9	0.15

Binary	$k[M_\odot^{-5/3}]$	$k_1[M_\odot^{-1}]$	$k_{3/2}[M_\odot^{-2/3}]$	$k_2[M_\odot^{-1/3}]$
1. NS-NS	0.72	4.1	25	13
2. NS-BH	0.16	2.0	16	15
3. BH-BH	2.7×10^{-2}	0.58	4.7	7.7

has the maximum at frequency f_0 given above and its *half width half magnitude* (HWHM) σ_0 is around 48 Hz.

To determine whether or not there is a signal in a noisy data set we use the Neyman-Pearson test (see Appendix D). When the noise in the detector is Gaussian the Neyman-Pearson test is the *correlator test.* It consists of linear filtering the data with the filter which Fourier transform is the Fourier transform of the signal divided by the spectral density of the noise [26]. The signal-to-noise ratio d that can be achieved by optimal filtering with the filter bandwith form frequency f_i to f_u is given by $d = (h|h)^{1/2}$ where, following [10], the scalar product $(h_1|h_2)$ is defined by

$$(h_1|h_2) = 4\mathrm{Re}\int_{f_i}^{f_u} \frac{\tilde{h}_1^*\tilde{h}_2}{S_h(f)}df. \tag{25}$$

Thus we have

$$d^2 = 4\tilde{A}^2 \int_{f_i}^{f_u} \frac{df}{S_h(f)f^{7/3}}. \tag{26}$$

By introducing a lower limit of integration we take into account the seismic noise. We shall call the integrand of the above signal-to-noise integral *signal sensitivity function* and we denote it by ind(f). This function has the maximum at the frequency f_0' where $f_0' = 47$ Hz and its HWHM σ_0' is $\simeq 26$ Hz, around half of that of the sensitivity function. This is the signal-to-noise ratio after filtering of the data. We see that linear filtering introduces an effective narrowing of the detector bandwidth [27].

In the case of our chirp signal the linear filtering increases the signal-to-noise ratio by an amount given roughly by the square root of the number $n(f)$ of cycles spent near the frequency f_0' where $n(f)$ is defined by [1]

$$n(f) := f\tau \simeq \frac{5}{96\pi}\frac{1}{\mathcal{M}^{5/3}}\frac{1}{(\pi f)^{5/3}}. \tag{27}$$

Consequently the effectiveness of matched filtering falls with the chirp mass. On the other hand the amplitude A of the signal increases with the chirp mass such as $\mathcal{M}^{5/3}$ and the overall factor in the signal-to-noise ratio increases as $\mathcal{M}^{5/6}$. This is born out by the amplitude $\tilde{A}$ of the Fourier transform. Thus the probability of detection of binaries with the same rate of occurrence increases with the chirp mass.

To estimate the parameters of the signal it is proposed to use the *maximum likelihood* (ML) estimation [28]. It is by no means guaranteed that this is the best or the ultimate method. It may sometimes fail to give an estimate and other methods may lead to more accurate estimates. The ML method consists of maximizing the likelihood function with respect to the parameters of the filter. In the case of the Gaussian noise the logarithm of the likelihood function Λ is given by [28]

$$\ln\Lambda = (x|h_F) - \tfrac{1}{2}(h_F|h_F), \tag{28}$$

where h_F which we call the filter has the form of the signal but with arbitrary parameters and x are the data. We assume that the noise n in the detector is additive, i.e., $x = h+n$. The maximum likelihood (ML) estimators of the parameters of the signal are given by the following set of differential equations providing that one can differentiate under the integration sign of the scalar product defined above:

$$(x - h_F|h_{F,i}) = 0, \tag{29}$$

where $h_{F,i}$ is the derivative of h_F with respect to the ith parameter. Rarely these equations can be solved analytically. It was shown [7] that in the case of the signal from a binary within the stationary phase approximation analytic expressions can be obtained for the maximum likelihood estimators of the amplitude and the phase.

The ML estimators are random variables since they depend on the noise. It is important to know the statistical properties of these estimators and their probability distributions so that we can determine how well they estimate the true values of the parameters. The most important quantities are the *expectation value* of the estimator and its *variance.* We would like to have the expectation value of the estimator to be as close as possible to the true value of the parameter and we would like the variance of the estimator to be as small as possible. The difference between the expectation value of an estimator of a parameter and the true value of the parameter is called the *bias* of the estimator. The ML estimator is not guaranteed to be either unbiased or minimum variance.

We have the following useful general inequality called the Cramer-Rao (CR) inequality [29] that gives the lower bound of the variance of estimators. Let (θ_i) be a set of n parameters and let θ_I be one of the parameters then the variance of its estimator $\hat{\theta}_I$ satisfies the inequality

$$\mathrm{Var}[\hat{\theta}_I] \geq (\Gamma^{-1})_{ij}\alpha^i\alpha^j, \tag{30}$$

where α^i and Γ^{ij} are given by

$$\alpha^i = \frac{\partial E[\hat{\theta}_I]}{\partial\theta_i}, \tag{31}$$

$$\Gamma^{ij} = E\left[\frac{\partial\ln\Lambda}{\partial\theta_i}\frac{\partial\ln\Lambda}{\partial\theta_j}\right], \tag{32}$$

where E is the expectation value. The matrix $\mathbf{\Gamma}$ is called the Fisher information matrix and its inverse is called the covariance matrix. One easily sees from the above inequality that when an estimator $\hat{\theta}_I$ is unbiased then the lower bound on its variance is given by the (II) component of the covariance matrix. For this inequality to hold certain mathematical assumptions must be satisfied [29]: (1) The likelihood function must be a differentiable function with respect to all the parameters θ_i; (2) the order of differentiation with respect to parameters and the integration in the expectation value integral must be interchangeable; (3) the variances of the estimators must be bounded; (4) the Fisher information matrix must be positive definite.

The Cramer-Rao inequality is very general. It holds no matter what the probability distribution of the data is and it applies to any estimator providing the regularity conditions mentioned above are satisfied. The above inequality guarantees only that the variance of an estimator is greater then a certain amount. It is important for us to know how well the right-hand side of the Cramer-Rao inequality approximates the actual variance of an estimator. It was shown [26,6] that in the case of Gaussian noise and in the limit of high signal-to-noise ratio d to the first order the maximum likelihood estimators are Gaussian random variables and moreover they are unbiased and their covariances are given by the covariance matrix defined above. In statistical literature there also exists a series of refined Cramer-Rao bounds called Battacharyya bounds [29]. However in our case a useful approach to have an idea of the accuracy of the Cramer-Rao lower bound is given in [10] where the maximum likelihood equations were solved iteratively and a formula for the covariance matrix of the ML estimators was derived to one higher order then given by the inverse of the Fisher information matrix. This formula can be treated as an approximation to the variances of the ML estimators by a series in $1/d$ where d is the signal-to-noise ratio. The first-order terms given by the inverse of Fisher matrix go as $1/d^2$ and the correction terms go like $1/d^4$. Consequently one can expect that for signal-to-noise ratios of 10 or so the diagonal elements of the inverse of the Fisher matrix give variances of the ML estimators to an accuracy of a few percent.

We shall show that the set of parameters that we have chosen for our chirp signal has particularly useful properties. Note that the phase of the Fourier transform is linear in the phase, the time parameter, and the mass parameters k_i. We shall call these parameters *phase parameters*. Moreover the Fourier transform is linear in the amplitude parameter $\tilde{A}$. The maximum likelihood estimators are those values of the parameters that maximize the likelihood function. The expectation value of the log likelihood is given by

$$E[\ln \Lambda] = (h|h_F) - \tfrac{1}{2}(h_F|h_F), \tag{33}$$

where $(h|h_F)$ is called the correlation function and is denoted by H. Using the stationary phase approximation to the Fourier transform of the signal H is given by the integral

$$H(\Delta t, \Delta\phi, \Delta k, \Delta k_e, \Delta k_1, \Delta k_{3/2}, \Delta k_2) = 4\tilde{A}\tilde{A}_F \int_{f_i}^{\infty} \frac{df}{S_h(f) f^{7/3}} \cos\{2\pi f \Delta t - \Delta\phi + \tfrac{5}{48}[a(f;f_a)\Delta k + a_e(f;f_a)\Delta k_e \times a_1(f;f_a)\Delta k_1 + a_{3/2}(f;f_a)\Delta k_{3/2} + a_2(f;f_a)\Delta k_2]\}, \tag{34}$$

where Δt means the difference in time parameters of the signal and the filter. The expectation of the log likelihood function depends on the phase parameters only through the correlation integral since $(h_F|h_F) = H(0,0,0,0,0,0,0) = d^2$ where d is the signal-to-noise ratio. We see that the correlation function depends only on the differences between the values of the phase parameters in the signal and the filter and it has the maximum when the differences are zero. Moreover the value of the correlation is the same if we move by the same amount from the maximum in any direction for a given parameter, i.e., $H(-\Delta t,0,0,0,0,0,0) = H(\Delta t,0,0,0,0,0,0)$ and so on for all phase parameters.[1] This property means that the probability distribution of any estimator of the phase parameter will be an even function of the difference between the estimator and its true value. In other words the probability distributions of the estimators of the phase parameters are symmetric about their true values. Consequently we have

$$m_l := E[(\hat{\theta}_I - \theta_I)^l] = 0 \quad \text{for } l \text{ odd} \tag{35}$$

and moreover for l even the moments m_l are independent of the true values of the phase parameters. Thus the ML estimators of the phase parameters are *unbiased* [this is immediate from Eq. (35) for $l = 1$] and the covariance matrix of the estimators of the phase parameters is independent of their values. The probability distributions of the phase parameters will depend on the signal-to-noise ratio. We know that for large signal-to-noise ratio they will tend to Gaussian probability distributions. The estimator of the amplitude parameter *is biased* nevertheless by the symmetry property of the probability distributions of the phase parameters its bias is independent of the values of the phase parameters. These properties of the parameters can be also seen explicitly from the first two terms of the series solution of the ML equations [Eq. (29)] given in [10]. The properties of our chosen set of parameters greatly simplify calculation of the Cramer-Rao bounds. In our case the Fisher information matrix Γ is given by

$$\Gamma^{ij} = \frac{\partial H}{\partial\theta_{iS}\partial\theta_{jF}}_{|\theta_{kS}=\theta_{kF}}, \tag{36}$$

where S refers to the parameters of the signal and F to the parameters of the filter. The inverse of the $\mathbf{\Gamma}$ matrix is called the covariance matrix and is denoted by $\mathbf{C}$. It is easily seen that Γ^{Ai} components are all equal to zero when $i \neq A$. Thus the amplitude parameter decouples from the phase parameters. Because the phase parameters are unbiased the lower bounds of their variances are given just by the appropriate diagonal elements of the co-

[1] We are indebted to J.A. Lobo for this observation, see also [26], p. 276.

variance matrix **C**. In the case of the amplitude parameter the Cramer-Rao bound is given by $\mathrm{Var}A \geq b'(A)/\Gamma^{AA}$ where $b'(A)$ is the derivative of the bias of amplitude parameter with respect to amplitude and $\Gamma^{AA} = d^2/\tilde{A}^2$. Note that Γ^{AA} is independent of $\tilde{A}$. This is a consequence of the linearity of the signal in the amplitude.

It is clear from the linearity of the function H in the differences $\Delta\theta$ that the $\mathbf{\Gamma}$ matrix is independent of the values of the phase parameters. Thus the Cramer-Rao bound on these parameters is also independent of the values of the parameters. From the argument above we know that this holds not only for the bounds on the variances but also for the variances themselves.

To obtain the maximum of the correlation each phase parameter of the filter has to match a corresponding parameter in the signal [see Eq. (34)]. Thus by linear filtering we shall get estimates of the time parameter t_a, phase, and the mass parameters k_i. In the filter one can always make an arbitrary choice of the time parameter t_a. For example, instead of choosing t_a as the time at which frequency is f_a one can choose time t'_a as the time at which the frequency is equal to f'_a. This new choice is equivalent to the transformation

$$t'_a = t_a + \bar{\delta}k + \bar{\delta}_e k_e + \bar{\delta}_1 k_1 + \bar{\delta}_{3/2} k_{3/2} + \bar{\delta}_2 k_2, \tag{37}$$

$$\phi' = \phi + \delta k + \delta_e k_e + \delta_1 k_1 + \delta_{3/2} k_{3/2} + \delta_2 k_2, \tag{38}$$

where

$$\bar{\delta} = \frac{5}{256}\left(\frac{1}{(\pi f_a)^{8/3}} - \frac{1}{(\pi f'_a)^{8/3}}\right), \tag{39}$$

$$\delta = \frac{1}{16}\left(\frac{1}{(\pi f_a)^{5/3}} - \frac{1}{(\pi f'_a)^{5/3}}\right), \tag{40}$$

$$\bar{\delta}_e = -\frac{785}{11\,008}\left(\frac{1}{(\pi f_a)^{43/9}} - \frac{1}{(\pi f'_a)^{43/9}}\right), \tag{41}$$

$$\delta_e = -\frac{785}{4352}\left(\frac{1}{(\pi f_a)^{34/9}} - \frac{1}{(\pi f'_a)^{34/9}}\right), \tag{42}$$

$$\bar{\delta}_1 = \frac{5}{192}\left(\frac{1}{(\pi f_a)^{2}} - \frac{1}{(\pi f'_a)^{2}}\right), \tag{43}$$

$$\delta_1 = \frac{5}{48}\left(\frac{1}{\pi f_a} - \frac{1}{(\pi f'_a)}\right), \tag{44}$$

$$\bar{\delta}_{3/2} = -\frac{1}{32}\left(\frac{1}{(\pi f_a)^{5/3}} - \frac{1}{(\pi f'_a)^{5/3}}\right), \tag{45}$$

$$\delta_{3/2} = -\frac{5}{32}\left(\frac{1}{(\pi f_a)^{2/3}} - \frac{1}{(\pi f'_a)^{2/3}}\right), \tag{46}$$

$$\bar{\delta}_2 = \frac{5}{128}\left(\frac{1}{(\pi f_a)^{4/3}} - \frac{1}{(\pi f'_a)^{4/3}}\right), \tag{47}$$

$$\delta_2 = \frac{5}{16}\left(\frac{1}{(\pi f_a)^{1/3}} - \frac{1}{(\pi f'_a)^{1/3}}\right). \tag{48}$$

The mass parameter frequency functions $a_i(f; f_a)$ ($i = 0, 1, 3/2, 2$) in Eq. (10) are then transformed to $a_i(f; f'_a)$. The mass parameters remain invariant under the above transformations. By linear filtering with the template parametrized by the new time parameter and the new phase given by the above transformation we estimate the new time parameter t'_a and the new phase ϕ' but the same mass parameters k_i.

There is also a particularly simple parametrization of the signal. Let us rewrite the Fourier transform of the gravitational-wave signal from a binary in the form

$$\tilde{h} = \tilde{A} f^{-7/6} \exp\Bigg[i \Bigg(2\pi f t_c - \phi_c - \pi/4 + \frac{3}{128}\frac{k}{(\pi f)^{5/3}} - \frac{4239}{11696}\frac{k_e}{(\pi f)^{34/9}} + \frac{5}{96}\frac{k_1}{\pi f} - \frac{3}{32}\frac{k_{3/2}}{(\pi f)^{2/3}} + \frac{15}{64}\frac{k_2}{(\pi f)^{1/3}} \Bigg) \Bigg] \tag{49}$$

(for $f > 0$ and by the complex conjugate of the above expression for $f < 0$), where t_c and ϕ_c are coalescence time and phase, respectively, and they are given by

$$t_c = t_a + \frac{5}{256}\frac{k}{(\pi f_a)^{8/3}} - \frac{785}{110\,008}\frac{k_e}{(\pi f)^{43/9}} + \frac{5}{192}\frac{k_1}{(\pi f_a)^2} - \frac{1}{32}\frac{k_{3/2}}{(\pi f_a)^{5/3}} + \frac{5}{128}\frac{k_2}{(\pi f_a)^{4/3}}, \tag{50}$$

$$\phi_c = \phi_a + \frac{1}{16}\frac{k}{(\pi f_a)^{5/3}} - \frac{785}{4352}\frac{k_e}{(\pi f)^{34/9}} + \frac{5}{48}\frac{k_1}{\pi f_a} - \frac{5}{32}\frac{k_{3/2}}{(\pi f_a)^{2/3}} + \frac{5}{16}\frac{k_2}{(\pi f_a)^{1/3}}. \tag{51}$$

Coalescence time amd coalescence phase are obtained when the time parameter t'_a is such that the corresponding frequency f'_a is infinite which occurs when the two point masses coalesce. We can estimate the coalescence time and the coalescence phase of the template if we filter for combinations of the time and phase parameters with the mass parameter given precisely by the right-hand sides of Eqs. (50) and (51). There is also a transformation of the phase that we shall find useful (see the following section):

$$\phi'' = \phi - 2\pi f_p t_a, \tag{52}$$

where f_p is some arbitrary constant frequency. Using the new phase parameter in the filter given by the above transformation we shall estimate a new value of the phase shifted by the amount $2\pi f_p t_a$. It is not difficult to show that all the above transformations do not change the CR bound on the mass parameters however the transformation, Eq. (48), changes the bound for time and phase parameters whereas the transformation, Eq. (52), changes the bound on the phase. We can use the freedom of these transformations in the filter to obtain better accuracies of estimation of the time parameter and the phase.

C. Numerical analysis of the rms errors of the estimators

First of all we investigate the influence of the increasing number of post-Newtonian parameters on the accuracy of their estimation. To this end we have calculated the covariance matrices for the signal containing only the quadrupole term, then covariance matrices for first post-Newtonian, 3/2 post-Newtonian, and second post-Newtonian signal, and finally for the second post-Newtonian signal with first-order contribution due to eccentricity. The results are summarized in Table II where we have given the square roots of the diagonal elements of the inverse of the Fisher matrix as the rms errors of the phase parameters. We have given the rms errors for the time and phase of coalescence t_c and ϕ_c, respectively. We have also determined the frequency f_m for which the error in the time parameter is minimum and we have given the minimum error Δt_m in the time parameter and the corresponding error $\Delta\phi$ in phase. As we have explained in the previous subsection the rms errors of the phase parameters are independent of their numerical values. They depend only inversely proportionally on the amplitude parameter $\tilde{A}$ in front of the Fourier transform of the signal. Thus the rms errors scale with the chirp mass precisely as $\mathcal{M}^{-5/6}$. To get the numerical values of Table II we have adopted the chirp mass $\mathcal{M} = 1M_\odot$ and the distance of 100 Mpc. We have taken the range of integration in the Fisher matrix integrals from 10 Hz to infinity. The signal-to-noise ratio in such a case is around 25.

From Table II we see that increasing the number of post-Newtonian corrections and parameters we filter for decreases the accuracy of estimation of the parameters independently of the size of the post-Newtonian correction. Thus searching for a negligible correction due to eccentricity increases the rms error in other parameters by over 100%.

For completeness in Appendix B we give the numerical values of covariance matrices for the phase parameters at various post-Newtonian orders and the corresponding values of the frequency $f_{\rm min}$.

As we have indicated above the estimator of the amplitude parameter is biased however if one takes the expansion of the variance of the estimator in the inverse powers of the signal-to-noise ratio (see [10] for a general formula) then the leading term for the variance of the amplitude is just $1/\Gamma^{AA}$ where Γ^{AA} is independent of $\tilde{A}$. The higher order corrections to the CR bounds of the amplitude go like $1/d^4$ and they do depend on the value of the amplitude. As an amplitude parameter we find convenient to choose $A_\oplus$ given by

$$A_\oplus = \frac{\mathcal{M}_\odot^{5/6}}{r_{100\,\rm Mpc}}, \tag{53}$$

where $\mathcal{M}_\odot$ is the chirp mass in the units of solar masses and $r_{100\,\rm Mpc}$ is the distance in units of 100 Mpc. For our reference binary the amplitude $A_\oplus = 1$ and thus the approximate rms error in its ML estimator is $A_\oplus/d \simeq 1/25 = 0.04 \left[\frac{\mathcal{M}_\odot^{5/6}}{100\rm Mpc}\right]$ and as explained above this last number is independent of the true value of the amplitude.

It is important to assess the accuracy of the estimation of the physical parameters of the binary, i.e., the two masses of its members and the spin parameters s_o and s_s. This means that we have to make a transformation to a different parameter set. An important property of the ML estimators is the following. Let $\hat{\theta}_i$ be the maximum likelihood estimators of the set of parameters θ_i. Let $f(\theta_i)$ be a function of the parameters then $f(\hat{\theta}_i)$ is the maximum likelihood estimator of the function f (see [28]). However it is not true in general that if estimators of the parameters θ_i are unbiased then the estimator $f(\hat{\theta}_i)$ is an unbiased estimator of $f(\theta_i)$. Consequently by just transforming the Γ matrix to new variables one will not get the Cramer-Rao bound on the new set of parameters. However we know that Cramer-Rao bounds are approximately equal to the true variances in the limit of high signal-to-noise ratio d, correction terms being of the order of $1/d^2$. Hence by transforming the CR bounds one gets the rms errors of the estimators accurate to the order $1/d$. Another important point is that the transformation to the new parameter set may be singular. Then the determinant of the $\mathbf{\Gamma}'$ matrix for the new set of parameters is zero and thus $\mathbf{\Gamma}'$ is not positive definite, consequently the Cramer-Rao inequality does not hold. A way to get errors of estimators of the new parameters in such a case could be to attempt to calculate the bias and the variance directly from some approximate probability distributions for the estimators (see [10] for such treatment to determine the accuracy of the distance to the binary). It may happen however that the probability density function is such that the expectation value and the variance do not exist (an example is Cauchy probability distribution) and then one may have to use another measure of bias and error, e.g., median and interquartile distance. The other method proposed in [10] is to use confidence intervals. We shall return to this problem in the future work [30,31].

The transformation from the four mass parameters k_I to new parameters, total mass (m), reduced mass (μ), and the spin parameters s_o and s_s, is regular. Thus we can obtain approximate values of the errors of the esti-

TABLE II. The rms errors for the phase parameters at various post-Newtonian orders for the reference binary of chirp mass of 1 solar mass at the distance of 100 Mpc. Expected advanced LIGO noise spectral density is assumed and the integration range from 10 Hz to infinity is taken giving signal-to-noise ratio of around 25. The rms errors scale with the chirp mass as $\mathcal{M}^{-5/6}$.

Δt_m (msec)	$\Delta\phi$	Δt_c (msec)	$\Delta\phi_c$	$\Delta k[M_\odot^{-5/3}]$	$\Delta k_1[M_\odot^{-1}]$	$\Delta k_{3/2}[M_\odot^{-2/3}]$	$\Delta k_2[M_\odot^{-1/3}]$	$\Delta k_e[M_\odot^{-5/3}100\text{Hz}^{19/9}]$
0.14	0.073	0.17	0.10	8.3×10^{-6}	-	-	-	-
0.15	0.087	0.27	0.33	4.0×10^{-5}	5.8×10^{-3}	-	-	-
0.18	0.14	0.54	1.9	1.7×10^{-4}	0.70×10^{-1}	0.52	-	-
0.24	0.14	1.6	24	6.6×10^{-4}	0.50	7.2	28	-
0.25	0.17	2.3	45	2.3×10^{-3}	1.3	17	59	1.2×10^{-6}

TABLE III. Degradation of the accuracy of estimation of the chirp mass, the reduced mass, and the total mass for the fiducial neutron-star–neutron-star binary with increasing number of parameters in the signal. 2PN means that the phase of signal is taken to second post-Newtonian order with spin parameters included and we maximize the correlation of the signal with a template matched to the signal for all the phase parameters.

PN order	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\mu/\mu$	$\Delta m/m$
1PN	0.0054%	0.57%	0.86%
3/2PN	0.023%	6.5%	9.8%
2PN	0.080%	42%	64%

mators of the reduced mass, the total mass, and the spin parameters. However the transformation from m and μ to individual masses m_1 and m_2 is singular (determinant of the Jacobian of the transformation is zero when masses are equal, see [10]). Consequently the errors in the determination of the masses cannot be obtained from the CR bounds calculated above.

In Table III we show the degradation of the accuracy of estimation of the chirp mass, the reduced mass, and the total mass with the increasing number of parameters in the signal for the NS-NS binary at a distance of 200 Mpc.

For the calculation of the numbers in the table above and all other tables in the remaining part of this section we have taken the range of integration in the Fisher matrix integrals to be from 10 Hz to frequency $f = (6^{3/2}\pi m)^{-1}$ corresponding to the last stable orbit of the test particle in Schwarzschild space-time. This may very roughly correspond to the last stable orbit in a binary [32,33].

In Table IV we give the signal-to-noise ratios and the Cramer-Rao bounds for the mass and the spin parameters in percents of their true values for the second post-Newtonian signal for our three representative binary systems at the distance of 200 Mpc. We have also given the improvement factors $\sqrt{n}$ in the S/N due to filtering.

We see that only the rms error in the chirp mass is small and also the accuracy of the determination of the spin-orbit parameter for NS-BH binary is satisfactory. The errors in reduced and total masses are large.

One can derive simple general formulas for the accuracy of determination of the chirp mass, the reduced mass, and the total mass in terms of rms errors of the mass parameters k. From the definition of the chirp mass one immediately obtains the following formula for the relative rms error in terms of the rms error in the mass parameter k:

$$\Delta\mathcal{M}/\mathcal{M} = \frac{3}{5} r_{100\,\mathrm{Mpc}} \Delta k \mathcal{M}_\odot^{5/3}. \tag{54}$$

For the errors in the reduced and the total mass we obtain the following general formulas using the standard law of propagation of errors:

$$\Delta\mu = \frac{\sqrt{(\frac{\partial k_1}{\partial m})^2(\Delta k)^2 - 2\frac{\partial k}{\partial m}\frac{\partial k_1}{\partial m}C_{kk_1} + (\frac{\partial k}{\partial m})^2(\Delta k_1)^2}}{\det}, \tag{55}$$

$$\Delta m = \frac{\sqrt{(\frac{\partial k_1}{\partial \mu})^2(\Delta k)^2 - 2\frac{\partial k}{\partial m}\frac{\partial k_1}{\partial m}C_{kk_1} + (\frac{\partial k}{\partial \mu})^2(\Delta k_1)^2}}{\det}, \tag{56}$$

where

$$\det = \frac{\partial k}{\partial \mu}\frac{\partial k_1}{\partial m} - \frac{\partial k}{\partial m}\frac{\partial k_1}{\partial \mu}, \tag{57}$$

Δk, Δk_1 are rms error in mass parameters k and k_1, respectively, and C_{kk_1} is the correlation coefficient between the k and k_1 mass parameters. The formula above is the same when the first post-Newtonian, the 3/2 post-Newtonian, and the second post-Newtonian corrections are included. Thus we see that independently of the post-Newtonian order the errors in μ and m depend only on the masses and the rms errors in the parameters k and k_1. The other mass parameters influence the errors in μ and m only through their correlations with the mass parameters k and k_1 and only through the functional form of the corrections as the rms error in the mass parameters are independent of their values. The errors in μ and m are independent of the numerical values of the parameters $k_{3/2}$ and k_2. Since in general the rms error Δk is considerably smaller than Δk_1 and also the correlation coefficient C_{kk_1} is much smaller then $(\Delta k_1)^2$ (see Appendix B) we get the following simplified expressions for the relative errors in the reduced and the total mass:

$$\Delta\mu/\mu \simeq \frac{1}{a} r_{100\,\mathrm{Mpc}} \mu_\odot \Delta k_1, \tag{58}$$

$$\Delta m/m \simeq \frac{3}{2a} r_{100\,\mathrm{Mpc}} \mu_\odot \Delta k_1, \tag{59}$$

where $a = 743/336 - 33/8\mu/m$. We see that the error in the determination of the reduced mass and the total mass is determined by error in the first post-Newtonian mass parameter k_1. Since the ratio μ/m is $\leq 1/4$ to a fairly good approximation we can take the value of a roughly equal to 1.

If the spin effects did not exist we would only have two parameters less to estimate with the reduced mass

TABLE IV. Accuracy of estimation of the parameters of the second post-Newtonian signal for the three fiducial binaries.

Binary	S/N	$\sqrt{n}$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\mu/\mu$	$\Delta m/m$	$\Delta s_o/s_o$	$\Delta s_s/s_s$
NS-NS	15	32	0.080%	42%	64%	62×10^2%	12×10^6%
NS-BH	32	15	0.26%	40%	60%	15%	19×10^4%
BH-BH	77	6	0.92%	160%	240%	250%	860%

and the total mass as unknown in the mass parameters k_i then we could achieve the accuracies in the parameters of the signal summarized in Table V. We considered three fiducial binary systems and second post-Newtonian signal but with spin-orbit and spin-spin parameters removed. We see that if spin parameters could be neglected we would have an excellent accuracy of estimation of the reduced and the total mass of the binary.

D. The effects of other theories of gravity

We shall consider two alternative theories. One is the Jordan-Fierz-Brans-Dicke (JFBD) theory (see [35] for a detailed discussion) and the other is a multi-scalar field theory recently proposed in [37].

In the JFBD theory in addition to the tensor gravitational field there is also a scalar field. The theory can be characterized by a coupling constant that we denote by ω. General relativity is obtained when ω goes to infinity. The JFBD theory has two effects on gravitational-wave emission. It admits dipole gravitational radiation and secondly there is a modification of the quadrupole emission due to the interaction of the scalar field with gravitating bodies. In the case of a binary system the effects of the JFBD theory has been studied in great detail [34] and a general formula for the change of orbital period was derived [[35], Eq. (14.22)]. From that formula we get the following expression for the characteristic time τ of the evolution of the binary due to radiation reaction in the case of circularized orbits and assuming that the contribution due to the dipole term is small:

$$\tau = \frac{5}{96}\frac{1}{\mu m^{2/3}}\frac{\mathcal{G}^{4/3}}{\kappa}\frac{1}{(\pi f)^{8/3}} \times\left(1-\frac{5}{192}k_B\frac{\mathcal{G}^{4/3}}{\kappa}\frac{\Sigma^2}{(\pi m f)^{2/3}}\right), \tag{60}$$

where

$$k_B = \frac{1}{2+\omega}, \tag{61}$$

$$\mathcal{G} = 1-\frac{k_B}{2}(\mathcal{C}_1+\mathcal{C}_2-\mathcal{C}_1\mathcal{C}_2), \tag{62}$$

$$\kappa = \mathcal{G}^2\left(1-\frac{k_B}{2}+\frac{k_B}{12}\gamma^2\right), \tag{63}$$

$$\gamma = 1-\frac{m_1\mathcal{C}_2+m_2\mathcal{C}_1}{m_1+m_2}, \tag{64}$$

$$\Sigma = \mathcal{C}_1-\mathcal{C}_2. \tag{65}$$

TABLE V. Accuracy of estimation of the parameters for three fiducial binary systems and the second post-Newtonian signal but with spin parameters removed. Thus the number of parameters estimated is 2 less than for the signal considered in Table IV.

Binary	S/N	Δt_c (msec)	$\Delta\phi_c$	$\Delta\mu/\mu$	$\Delta m/m$
NS-NS	15	0.47	0.82	0.29%	0.43%
NS-BH	32	0.32	0.47	0.19%	0.28%
BH-BH	77	0.18	0.24	0.27%	0.37%

$\mathcal{C}_1$ and $\mathcal{C}_2$ are "sensitivities" of the two bodies to changes of the scalar field. For a black hole the sensitivity $\mathcal{C}$ is always equal to 1. For a neutron star $\mathcal{C}$ depends on the equation of state. For neutron stars the sensitivity has been studied in [37] for a number of equations of state and it was found for a wide range of such equations that it is proportional to the mass of the neutron star with proportionality constant varying from 0.17 to 0.31. Here we shall assume that $\mathcal{C}_i = 0.21 m_{i\odot}$ for a neutron star of $m_{i\odot}$ solar masses. From the above formulas one sees that the dipole radiation will vanish if the binary system consists of two black holes or the neutron stars in the binary are the same.

The Fourier transform of the signal in the stationary phase approximation including contributions due to JFBD theory is given by (we neglect any contributions due to eccentricity)

$$\tilde{h} = \tilde{A}f^{-7/6}\exp\big(i\{2\pi f t_a-\phi-\pi/4 +\tfrac{5}{48}[a(f;f_a)k'+a_1(f;f_a)k_1+a_{3/2}(f;f_a)k_{3/2} +a_2(f;f_a)k_2+a_d(f;f_a)k_d]\}\big) \tag{66}$$

(for $f>0$ and by the complex conjugate of the above expression for $f<0$), where the function $a_d(f;f_a)$ due to dipole radiation has the form

$$a_d(f;f_a) = -\frac{5}{192}\left(\frac{9}{70}\frac{1}{(\pi f)^{7/3}}+\frac{3}{10}\frac{f}{(\pi f_a)^{10/3}} -\frac{3}{7}\frac{1}{(\pi f_a)^{7/3}}\right), \tag{67}$$

where

$$k' = \frac{1}{\mu m^{2/3}}\frac{\mathcal{G}^{4/3}}{\kappa}, \tag{68}$$

$$k_d = \frac{1}{\mu m^{4/3}}k_B\frac{\mathcal{G}^{8/3}}{\kappa^2}\Sigma^2 \tag{69}$$

and where the functions a are given by formulas (16), (18), (19), and (20). Current observational tests constrain ω to be greater than 600 and from the timing of binary pulsar a lower limit on ω of 200 can be set. Thus it is sufficient to keep only the terms of first order in $1/\omega$. Then the two parameters above are approximately given by

$$k' = \frac{1}{\mu m^{2/3}}(1-dk_d), \tag{70}$$

$$k_d = \frac{1}{\mu m^{4/3}}k_B\Sigma^2, \tag{71}$$

where

$$dk_d = k_B\left[\frac{1}{3}(\mathcal{C}_1+\mathcal{C}_2-\mathcal{C}_1\mathcal{C}_2)+\tfrac{1}{2}-\frac{\gamma^2}{12}\right]. \tag{72}$$

We thus see the JFBD theory introduces a new parameter k_d due to the dipole radiation and modifies the standard chirp mass parameter k by fraction dk_d.

We have investigated the potential accuracy of estimation of the parameter k_d assuming that the spin effects are negligible. We have taken neutron-star–black-hole binary with parameters given in Table I at the distance of 200 Mpc. The result is summarized in Table VI.

The potential accuracy of determination of the dipole radiation parameter k_d is high. Current observational constraints indicate however that this parameter is small. We have the numerical values

$$k_d = 3.2 \times 10^{-5} \left(\frac{500}{\omega}\right) \left(\frac{\Sigma^2}{0.5}\right) \left(\frac{32}{\mu_\odot m_\odot^{4/3}}\right), \qquad (73)$$

$$\frac{\Delta k_d}{k_d} = 0.7 \left(\frac{\omega}{500}\right) \left(\frac{0.5}{\Sigma^2}\right) \left(\frac{\mu_\odot m_\odot^{4/3}}{32}\right). \qquad (74)$$

We conclude that the gravitational-wave measurement by planned long-arm laser interferometers have the potential of testing the JFBD theory to the accuracy comparable to tests in the solar system and measurements from the binary pulsars [36].

From the general class of tensor-multiscalar theories studied recently [37] we shall consider a two-parameter subclass of tensor-biscalar theories denoted by T(β',β''). Theories in this subclass have two scalar fields and they tend smoothly to general theory of relativity when both parameters β' and β'' tend to zero. The subclass is defined in such a way that the dipole radiation vanishes. From the general formulas [37] one can calculate the characteristic time τ. For circularized orbits the only modification is an effective change of the chirp mass parameter k given by the formula

$$k'' = k - d_{\rm DF}, \qquad (75)$$

$$d_{\rm DF} = \frac{5}{144}\kappa_0(m_1, \mathcal{C}_1, m_2, \mathcal{C}_2) + \frac{1}{6}[\kappa_q(m_1, \mathcal{C}_1, m_2, \mathcal{C}_2) + \kappa_{d1}(m_1, \mathcal{C}_1, m_2, \mathcal{C}_2)] + \frac{5}{48}\kappa_{d2}(m_1, \mathcal{C}_1, m_2, \mathcal{C}_2), \qquad (76)$$

where coefficients $\kappa_0, \kappa_q, \kappa_{d1}, \kappa_{d2}$ are due to contributions from quadrupole helicity zero, corrections to quadrupole helicity two, and dipole radiation respectively. They are complicated functions of the masses and sensitivities. We give the detailed formulas in Appendix C. In all the tensor-multiscalar field theories whenever one of the components is a black hole corrections to the radiation reaction vanish. We have also found that for a simple model where sensitivities are proportional to masses of neutron stars and the proportionality constant is the same the correction $d_{\rm DF}$ does not depend on the parameter β''. For a system of two identical neutron stars the correction $d_{\rm DF}$ takes a simple form

$$d_{\rm DF} = 0.21\beta\mathcal{C}^2, \qquad (77)$$

where $\mathcal{C}$ is the sensitivity of the neutron star to changes of the scalar field introduced above. Current observations constrain parameter β to be less than 1. For circularized orbits (the case considered above) the biscalar theory does not introduce a new mass parameter in the phase of the signal but only a shift in the "Newtonian" mass parameter k. We shall consider the possibility of estimating this shift in the next section.

TABLE VI. The rms error for signal parameters in JFBD theory assuming spins are negligible for the binary of 1.4 solar mass neutron star and 10 solar mass black hole.

S/N	Δt_c (msec)	$\Delta\phi_c$	$\Delta\mu/\mu$	$\Delta m/m$	$\Delta k_d[M_\odot^{-5/3}]$
32	0.47	0.97	0.57%	0.73%	2.3×10^{-5}

III. SEARCH TEMPLATES

A. The Newtonian filter

We have seen in the previous section that the accuracy of estimation of the parameters is significantly degraded with increasing number of corrections even though a correction may be small. If we include the second post-Newtonian correction and filter for all unknown parameters then the accuracy of determination of the masses of the binary becomes undesirably low. Moreover we cannot entirely exclude unpredicted small effects in the gravitational-wave emission (e.g., corrections to general theory of gravity) that we at present cannot model. Thus there is a need for simple filters or *search templates* that will enable us to scan the data effectively and isolate stretches of data where the signal is most likely to be [11]. The simplest such filter is just a Newtonian waveform h_N which Fourier transform in stationary phase approximation is given by

$$\tilde{h}_N = \frac{1}{30^{1/2}}\frac{1}{\pi^{2/3}}\frac{\mu^{1/2}m^{1/3}}{R} f^{-7/6} \exp\{i[2\pi f t_c - \phi_c - \pi/4 + k\tfrac{3}{128}(\pi f)^{-5/3}]\}. \qquad (78)$$

We shall call the *Newtonian filter* the filter which Fourier transform is given by the above formula and we shall denote it by NF. This filter has been investigated by the present authors [9,39,40] and also by other researchers [41–44,14]. A different search template based on the post-Newtonian signal has recently been introduced in [14]. We discuss this alternative search template in the next subsection.

In this subsection we examine the performance of the Newtonian filter. We demonstrate that such a template will perform well in detecting the signal from a binary and it also gives a reasonable idea of the nature of the binary. We shall investigate the performance of the Newtonian filter both analytically and numerically.

Let us consider the correlation of the post-Newtonian signal with the Newtonian filter. Such an integral has the same form as the correlation integral given by Eq. (34) in Section II A except that all post-Newtonian mass parameters will be unmatched by the parameters of the filter. The correlation will be high if we can reduce the oscillations due to the cosine function as much as possible. Since the integrand of the correlation integral is fairly sharply peaked (HWHM $\simeq$ 26 Hz) around its maximum at the frequency $f_0' \simeq 47$ Hz we can achieve this by making the phase as small as possible around the peak frequency f_0'. The argument Φ of the cosine in the integrand of the correlation of the post-Newtonian signal with the Newtonian filter including the effects due to eccentricity and dipole radiation takes the form

$$\begin{aligned}\Phi(f) &= 2\pi f \Delta t + \Delta\phi \\ &+\frac{5}{48}[a(f;f_a)\Delta k + a_e(f;f_a)k_e + a_1(f;f_a)k_1 \\ &+a_{3/2}(f;f_a)k_{3/2} + a_2(f;f_a)k_2 + a_d(f;f_a)k_d].\end{aligned} \tag{79}$$

First we note that for all the mass parameter frequency functions $a_i(f;f_a)$ the functions and their first derivatives vanish at the frequency f_a. We shall therefore choose $f_a = f_0'$. Let us also transform the phase parameter according to transformation given by Eq.(52) with $f_p = f_0'$. In the new parametrization the phase Φ takes the form

$$\begin{aligned}\Phi(f) &= 2\pi(f - f_0')\Delta t' + \Delta\phi'' \\ &+\frac{5}{48}[a(f;f_0')\Delta k + a_e(f;f_0')k_e + a_1(f;f_0')k_1 \\ &+a_{3/2}(f;f_0')k_{3/2} + a_2(f;f_0')k_2 + a_d(f;f_0')k_d].\end{aligned} \tag{80}$$

Let us examine the functional behavior of $\Phi(f)$ around the frequency f_0'. We find

$$\begin{aligned}\Phi(f) &\simeq 2\pi(f - f_0')\Delta t' + \Delta\phi'' \\ &+\frac{5}{96}(f/f_0' - 1)^2\left(\frac{\Delta k}{(\pi f_0')^{5/3}} - \frac{157}{24}\frac{k_e}{(\pi f_0')^{19/9}} + \frac{k_1}{\pi f_0'} - \frac{k_{3/2}}{(\pi f_0')^{2/3}} + \frac{k_2}{(\pi f_0')^{1/3}} - \frac{5}{192}\frac{k_d}{(\pi f_0')^{7/3}}\right) \\ &+ O[(f/f_0' - 1)^3].\end{aligned} \tag{81}$$

We see that in the above approximation we can make the phase Φ vanish to the order $(f/f_0' - 1)^3$ when the following conditions hold:

$$\Delta t_{\max} = t'_{F\max} - t' = 0, \tag{82}$$

$$\Delta\phi'' = \phi''_{F\max} - \phi'' = 0, \tag{83}$$

$$\begin{aligned}\Delta k_{\max} &= k_{F\max} - k \\ &= -\frac{157}{24}\frac{k_e}{(\pi f_0')^{19/9}} + k_1(\pi f_0')^{2/3} - k_{3/2}(\pi f_0') \\ &\quad +k_2(\pi f_0')^{4/3} - \frac{5}{192}\frac{k_d}{(\pi f_0')^{2/3}},\end{aligned} \tag{84}$$

where the subscript $F_{\max}$ means the value of the parameter of the Newtonian filter that maximizes the correlation. Hence we can expect to match the Newtonian template to the post-Newtonian signal with the Newtonian mass parameter k shifted from the true value by a certain well-defined amount. The shift depends both on the parameters of the two-body system and the noise in the detector through the frequency f_0'. However the value of the shift in the k parameter is independent of the choice of the time parameter and the phase in the Newtonian filter.

In Table VII we have given the numerical values of the shift in the parameter k calculated from Eq. (85) for the three binary systems considered in the previous section. We have given three values of the shifts including one (δk_1), two ($\delta k_{3/2}$), and finally three (δk_2) post-Newtonian corrections.

We have also investigated the problem numerically and we have found the maxima to be located at the values of the shifts in the phase, the time, and the mass parameter k given in Table VIII, top (first post-Newtonian shift), middle (3/2 post-Newtonian shift), bottom (second post-Newtonian shift) below. We have also given the factor l which is defined as

$$l = \sqrt{\frac{(h|h_N)}{(h|h)}}. \tag{85}$$

In a previous work by these authors ([39,40]) we have claimed the factor l to be the drop in the signal-to-noise ratio as a result of using nonoptimal (Newtonian) filter. However the signal-to-noise ratio falls as *square* of the factor l (see Appendix D).[2] We also give the range of integration over which we calculated the correlation. We have found that we gain very little by extending the integration beyond that range. For the case of a neutron star binary increasing the range of integration up to 800 Hz increases the factor l by less than 1%. The reason for this is the effective narrowing of the band of the detector by the chirp signal discussed in the previous section.

We see that the agreement between the predicted values of the shifts in the parameters and the numerical values given above is very good. In particular the difference between the predicted values and the values of the shifts for the k parameter obtained numerically differ by less then 5%.

The results of the detailed analysis carried out in [14] show that when the amplitude and phase modulations

[2] We are grateful to T. A. Apostolatos for pointing this to us.

TABLE VII. Numerical values of the shifts in the mass parameter of the Newtonian filter calculated from the analytic formula [Eq. (85)].

Binary	δk_1	$\delta k_{3/2}$	δk_2
NS-NS	0.033 28	0.015 12	0.015 97
NS-BH	0.016 41	0.005 052	0.006 023
BH-BH	0.004 660	0.001 276	0.001 775

due to the time dependence of the spin parameters are taken into account then in the worst case $l = 0.63$ for the correlation of the Newtonian filter with the 3/2PN signal.

We have also performed the correlation using the signal in the time domain and evaluating the correlation using the fast Fourier transform. We kept the amplitude Newtonian. As we have remarked earlier the restricted post-Newtonian approximations are not equivalent in the frequency and the time domain. So the results are not the same. (See Table IX.) We therefore conclude that the Newtonian filter will perform reasonably well in *detecting* the post-Newtonian signal.

Using the Newtonian filter we would not like to lose any signals. We can achieve this by suitably lowering the detection threshold when filtering the data with the Newtonian filter. By this procedure we would isolate stretches of data where correlation has crossed the lowered threshold. The reduced data would contain all the signals that would be detected with the optimal filter but would also contain false alarms the number of which would be increased comparing to the number of false alarms with the optimal filter. This is the effect of lowering the threshold. Some numerical examples are given in Appendix D. The next step would be to analyze the reduced set of data with more accurate templates and the initial threshold to make the final detection. The theory of filtering with a suboptimal filter is outlined in Appendix D.

TABLE VIII. Numerical values of the factor l and shifts in the parameters of the Newtonian filter with respect to the true values for various post-Newtonian orders calculated numerically by maximizing the correlation function.

Binary	l_1	δk_1	$\delta t'$	$\delta\phi''$	Range (Hz)
NS-NS	0.68	0.037 21	3.0×10^{-3}	0.61	30–200
NS-BH	0.76	0.018 67	1.7×10^{-3}	-0.53	30–100
BH-BH	0.85	0.004 931	-4.1×10^{-3}	-0.20	30–100
Binary	$l_{3/2}$	$\delta k_{3/2}$	$\delta t'$	$\delta\phi''$	Range (Hz)
NS-NS	0.90	0.015 64	-3.5×10^{-3}	-0.40	30–200
NS-BH	0.87	0.004 905	0.61×10^{-3}	0.068	30–100
BH-BH	0.87	-0.001 219	0.39×10^{-3}	0.030	30–100
Binary	l_2	δk_2	$\delta t'$	$\delta\phi''$	Range (Hz)
NS-NS	0.85	0.016 58	-5.5×10^{-3}	-0.44	30–200
NS-BH	0.87	0.006 014	-1.1×10^{-3}	-0.024	30–100
BH-BH	0.87	0.001 789	-0.51×10^{-3}	-0.018	30–100

We have also calculated the covariance matrix for the parameters estimated with the Newtonian filter. Calculating the second derivatives of the correlation function at the maximum given by the numerical values of the parameters in Table VIII one gets the Γ matrix. The inverse gives the covariance matrix. The square roots of a diagonal components of the covariance matrix give lower bounds on the accuracy of determination of parameters with the Newtonian filter and they are approximate rms error for high signal-to-noise ratio as explained in Sec. II. The results are summarized in Table X for our three binary systems located at a distance of 200 Mpc. The numbers are given for signals with the currently known post-Newtonian corrections but without the eccentricity and the dipole terms.

One can easily calculate from Table II that the accuracy of determination of the mass parameter k with the Newtonian filter lies between the accuracy of determination of k for 1 and 3/2 post-Newtonian signal. In Appendix E we have derived a useful formula for the correlation function based on the approximation to the phase Φ considered above.

We shall next show that the Newtonian filter can also give a useful estimator characterizing the binary system. From the analytic investigation of the Newtonian filter given above it is clear that we can obtain an estimator of an *effective mass parameter* k_E of the binary system given approximately by [cf. Eq. (85)]

$$k_E = k - \frac{157}{24}\frac{k_e}{(\pi f_0')^{19/9}} + k_1(\pi f_0')^{2/3} - k_{3/2}(\pi f_0') + k_2(\pi f_0')^{4/3} - \frac{5}{192}\frac{k_d}{(\pi f_0')^{2/3}} \quad (86)$$

and the numerical investigation above shows that the Newtonian filter will determine the effective mass parameter the value of which is accurately given by the above analytic formula (cf. Tables VII and VIII). The k_E parameter can be used to give an estimate of the chirp mass of the binary system. We define *generalized chirp mass* $\mathcal{M}_g$ as

$$\mathcal{M}_g = 1/k_E^{3/5}. \quad (87)$$

We have calculated numerically the generalized chirp mass using the analytic formula (86) and we have found that it deviates from the true value by less than 4% for the range of masses from 1.4 to 10 solar mass. For the range of masses from 1.01 to 1.64 which is the expected range of neutron star masses given present observations of binary pulsars [45] the generalized chirp mass is always less than the true one by around 4% but with a very small range of 0.5% around the average value.

Because of the inequality $m \geq 2^{6/5}\mathcal{M}$ and the closeness of the generalized chirp mass to the true chirp mass from the generalized chirp mass $\mathcal{M}_g$ we get a lower bound on the total mass of the system. Thus from its estimate we can determine what binary system we observe. Also the right-hand side (RHS) of the above inequality gives a poor man's estimate of the total mass. For the range of

TABLE IX. Numerical values of the l factor and the shifts obtained from the correlation of the Newtonian template with the signal in the time domain at various post-Newtonian orders.

Binary	l_1	δk_1	$l_{3/2}$	$\delta k_{3/2}$	l_2	δk_2	Range (Hz)
NS-NS	0.67	0.040 97	0.97	0.015 76	0.88	0.018 99	30–200
NS-BH	0.87	0.019 16	1.00	0.004 889	0.93	0.010 97	30–100
BH-BH	0.97	0.005 130	1.00	0.001 896	0.94	0.005 874	30–100

masses of $(1M_\odot, 10M_\odot)$ it deviates by 50% from the true value of the total mass but for the range of $(1.01M_\odot, 1.64M_\odot)$ acceptable for neutron star binaries it is only 5% smaller than the true mass.

Another application of this estimate is that it can be used as an additional check on whether we are observing the real signal. If our estimate would fall out considerably from the expected range of $\mathcal{M}_g$ we could veto the detection.

An interesting application of the Newtonian filter would be to determine unexpected effects in the binary interaction that we would not be able to model and introduce into multiparameter numerical templates because we do not know their form. The idea is to use the estimates of the effective mass parameter k_E. Particularly useful would be estimates of k_E in the case of neutron star binaries. Since the range of the neutron star masses in a binary system is rather narrow the range of the allowable values for the generalized chirp mass is also narrow. From the analysis in [45] the range from the least lower bound and to the greatest upper bound is $(1.01M_\odot, 1.64M_\odot)$ and the range from greatest lower bound to least upper bound is as narrow as $(1.34M_\odot, 1.43M_\odot)$. This implies the respective ranges in k_E to be (0.57, 1.26) and (0.71, 0.79). From the population of estimates of the parameter k_E we can determine its probability distribution and also the mean, the variance or the range of the observed values of k_E. One can then compare the observed distribution of k_E and its characteristics with the ones obtained from observations of the neutron star binaries in our Galaxy or from the theoretical analysis and search for differences. As an example we consider Damour-Esposito-Farèse biscalar tensor theory described at the end of Sec. II D. The shift in the Newtonian mass parameter k due to effects of this theory is given by formula (77). We have calculated this shift numerically and we have found that for the range of neutron star masses $(1.01M_\odot, 1.64M_\odot)$ and the parameter $\beta = 1$ (current observational bound) the shift is in the range of (0.018, 0.022). This shift is much larger than rms error in estimation of k_E of 0.000 37 (see Table X). Consequently the effects of the biscalar theory could be determined to an accuracy depending on how well we would know the probability distribution of the neutron star masses and the number of available detections of gravitational waves from binaries.

TABLE X. Accuracy of determination of parameters of the Newtonian filter for the three fiducial binaries located at the distance of 200 Mpc.

Binary	Δt_{aN}[msec]	$\Delta k_N [M_\odot^{-5/3}]$
NS-NS	2.9	0.37×10^{-3}
NS-BH	0.53	0.051×10^{-3}
BH-BH	0.22	0.021×10^{-3}

B. Post-Newtonian search templates

In a recent work [14] different search templates than the Newtonian filter were recommended and extensively analyzed. The proposed templates are the post-Newtonian waveforms with all the spin effects and parameters removed. They have four phase parameters: time parameter, phase, reduced mass, total mass. We shall denote such search templates by 1PNF, 3/2PNF, 2PNF where the number in front refers to the order of post-Newtonian effects included. In [14] the fitting factor (FF) (FF $= l^2$ see Appendix D) of the 3/2PNF search template was calculated and it was concluded that this template family works quite well even for signals with both spin-modulational and the nonmodulated 3/2 post-Newtonian effects combined. In this subsection we investigate the performance of the 2PNF search template for the case of the second post-Newtonian signal in the approximation considered in Sec. II. This means that we ignore all post-Newtonian effects in the amplitudes of both the signal and the template and we assume that the spin-orbit and the spin-spin parameters s_o and s_s in the signal are constant. In Table XI we give the factor l and the shift in the time parameter, phase, reduced mass, and total mass for the three representative binary systems described in Sec. II. We have also given the shifts in the reduced and the total mass parameters in percentages of their true values.

We see that the 2PNF search template fits the signal better than the Newtonian search template NF investigated in Sec. III A. There are two reasons for this. The 2PNF template has one more parameter than the NF template and the phase of the 2PNF template has all post-Newtonian frequency evolution terms whereas the phase of the NF template has only Newtonian frequency evolution $f^{-5/3}$. Also, in the case of the NS-NS binary which has small spin parameters, the expectation values of the estimates of the reduced and the total masses are close to their true values.

The advantage of the Newtonian search template might be its simple analytic form: It has the least possible number of parameters and hence the least computational time is needed to implement such a template in data analysis algorithms. Before the detailed data analysis schemes are developed for the real detectors it is use-

TABLE XI. Performance of the second post-Newtonian search template for the three fiducial binaries located at the distance of 200 Mpc.

Binary	l	$\delta\mu$	$\frac{\delta\mu}{\mu}$	δm	$\frac{\delta m}{m}$	δt (msec)	$\delta\phi$
NS-NS	0.98	0.002 8	0.5%	-0.017	0.61%	-9.5×10^{-3}	0.000 27
NS-BH	0.95	0.52	42%	-4.8	42%	-3.0	-0.28
BH-BH	0.98	1.9	38%	-7.8	39%	2.3×10^{-3}	-0.000 53

ful to investigate theoretically a wide range of possible search templates.

We have also calculated the covariance matrix for the 2PNF template. The results are summarized in Table XII where we have given the rms errors in the time, reduced mass, and the total mass parameters of this search template for the three binary systems. We have also given the errors in the reduced and the total mass in percentage of their true values. We see that the rms errors of the parameters of the post-Newtonian search template are comparable to rms errors obtained with optimal filtering of the signal with spin parameters removed.

IV. CONCLUSIONS

The analysis of the accuracy of estimation of parameters of the second post-Newtonian signal (Sec. II C) has shown that main characteristics of this signal: chirp mass and the time parameter can be estimated to a very good accuracy: chirp mass to 0.1–1.0 % and time parameter to a quarter of a millisecond for typical binaries. A typical binary consists of compact objects of 1.4 to 10 solar masses and is located at the distance of 200 Mpc from Earth and the amplitude of its gravitational wave signal is averaged over all directions and orientations. The signal-to-noise ratio of typical binaries varies from 15 to 77 for the planned advanced LIGO interferometers. However the accuracy of determination of post-Newtonian effects is considerably degraded due to large number parameters: six parameters in the phase of the second post-Newtonian signal (Table II). Consequently the errors in determination of the reduced mass and the total mass are large and range from 50 to 200 % for typical systems (Table IV). If spin effects could be neglected thereby reducing the number of parameters by 2 the rms errors of estimation of reduced and total masses would have a very impressive value of a fraction of a percent (Table V). Analysis of the accuracy of estimation of the effects of the dipole radiation in the Jordan-Fierz-Brans-Dicke theory of gravity has shown that the planned laser interferometric gravitational wave detectors should have ability of testing alternative theories of gravity comparable to that of current observations in the solar system and our Galaxy.

The numerical analysis of Sec. II supports the need for the search templates emphasized in [11]. The results of Sec. III show that the Newtonian filter (a search template with only one mass parameter) will perform reasonably well at least for the case of of constant spin parameters. Such a filter can be used to perform an on-line scan of the data to search for the candidates for real signals. The measurement of the mass parameter of the Newtonian signal provides an accurate estimate of an effective mass parameter k_E of the binary [see Eq. (86)]. The value of this parameter gives the information about the binary analogous to the chirp mass in the analysis of the signal in the quadrapole approximation. Moreover this parameter contains information about the post-Newtonian effects and it can contain information about the effects that we cannot at present model for example about the effects due to unknown corrections to general relativity in the strong field regime. Such information can be extracted if we built a probability distribution of k_E from its estimators by the Newtonian filter. The post-Newtonian search templates analyzed in [14] perform better than Newtonian filters and considering increasing computational capability they can also be used in the on line analysis of the data. In the case of large spin parameters it would be useful to obtain relations of the two mass parameters in such templates to the true masses and spins similar to relation of the effective mass parameter of the Newtonian filter to the other parameters of the binary [Eq. (86)]. For the case of the observed binary systems, binaries consisting of two neutron stars with small spin parameters the Newtonian filter will provide an accurate estimate of the chirp mass whereas the post-Newtonian search templates will provide accurate estimates of the reduced and the total masses.

After this work was completed we learned about a parallel analysis of parameter estimation using second post-Newtonian waveforms by Poisson and Will [48]. The main difference with the results presented here is that Poisson and Will take into account *a priori* information

TABLE XII. The rms errors in the estimators of the parameters of the second post-Newtonian search template for the three fiducial binary systems located at the distance of 200 Mpc.

Binary	Δt_a (msec)	$\Delta\mu_{\rm PN}[M_\odot]$	$\frac{\Delta\mu_{\rm PN}}{\mu}$	$\Delta m_{\rm PN}[M_\odot]$	$\frac{\Delta m_{\rm PN}}{m}$
NS-NS	0.80	0.0078	1.1%	0.011	0.39%
NS-BH	0.40	0.012	1.0%	0.0068	0.06%
BH-BH	0.16	0.0090	0.2%	0.0050	0.03%

about the size of the spin parameters. This allows to reduce the rms error in the estimation of the mass parameters of the binary.

ACKNOWLEDGMENTS

One of us (A. Królak) would like to thank the Max-Planck-Gesellschaft for support and the Arbeitsgruppe Gravitationstheorie an der Friedrich- Schiller-Universität in Jena for hospitality during the time this work was done. We would like to thank T. A. Apostolatos and K. S. Thorne for helpful discussions. We are grateful to G. Tsegas for help in numerical work. This work was supported in part by KBN Grant No. 2 P302 076 04.

APPENDIX A: THE EFFECTS OF ECCENTRICITY

In this appendix we derive the first-order correction due to eccentricity in the phase of the gravitational-wave signal from a binary system. The derivation is due to Wex [38].

Let a and e be, respectively, the semimajor axis and the eccentricity of the Keplerian orbit of a binary. From the quadrupole formula one obtains the following expressions for the secular changes of a and e averaged over an orbit [46]:

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{\beta}{a^3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}, \quad \left\langle \frac{de}{dt} \right\rangle = -\frac{304}{15}\frac{\beta}{a^4} \frac{e\left(1 + \frac{121}{304}e^2\right)}{(1-e^2)^{5/2}}, \tag{A1}$$

where $\beta = \frac{64}{5} m^2 \mu$. From these equations we get da/de which can be integrated with respect to e. The result is

$$a(e) = a_0 \frac{\xi(e)}{\xi(e_0)}, \quad \xi(e) \equiv e^{12/19} \frac{\left(1 + \frac{121}{304}e^2\right)^{870/2299}}{1-e^2}, \tag{A2}$$

where e_0 is an arbitrary initial eccentricity and $a_0 = a(e_0)$. From Kepler's third law $\pi f = m^{1/2} a^{3/2}$, where f is the gravitational wave frequency, we get an analytic expression for f as a function of e:

$$f(e) = f_0 \frac{\eta(e_0)}{\eta(e)}, \quad \eta(e) \equiv e^{18/19} \frac{\left(1 + \frac{121}{304}e^2\right)^{1305/2299}}{(1-e^2)^{3/2}}, \tag{A3}$$

where $f_0 = f(e_0)$. For small eccentricities we find

$$e = e_0 \left(\frac{f}{f_0}\right)^{-19/18} \left[1 + O(e_0^2)\right]. \tag{A4}$$

Thus to first order in e the quantity $I_e = e_0^2 f_0^{19/9}$ is a constant. We call I_e the *asymptotic eccentricity invariant.* The *characteristic time* for the evolution of the binary system is given by

$$\tau_e := \frac{f}{df/dt} = f\left(\frac{df}{da}\frac{da}{dt}\right)^{-1}. \tag{A5}$$

From Kepler's third law we find

$$\tau_e = \frac{5}{96}\frac{1}{\mu m^{2/3}}\frac{1}{(\pi f)^{8/3}}\frac{(1-e^2)^{7/2}}{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}. \tag{A6}$$

For small eccentricities e we get

$$\tau_e = \frac{5}{96}\frac{1}{\mu m^{2/3}}\frac{1}{(\pi f)^{8/3}}\left[1 - \frac{157}{24}e^2 + O(e^4)\right]. \tag{A7}$$

Therefore, using Eq. (A4) we can express the characteristic time with first-order correction due to eccentricity as

$$\tau_e = \frac{5}{96}\frac{1}{\mu m^{2/3}}\frac{1}{(\pi f)^{8/3}}\left[1 - \frac{157}{24}e_0^2\left(\frac{f}{f_0}\right)^{-19/9}\right]. \tag{A8}$$

The phase of the Fourier transform of the signal in the stationary phase approximation is given by

$$\begin{aligned}\varphi[f] &= 2\pi f t_i - \varphi_i - \pi/4 - 2\pi\int_{f_i}^{f} \tau_e(f')(1 - f/f')\,df \\ &= 2\pi f t_a - \varphi + \frac{1}{128\mu m^{2/3}}\left[\left(\frac{3}{(\pi f)^{5/3}} + \frac{5\pi f}{(\pi f_a)^{8/3}} - \frac{8}{(\pi f_a)^{5/3}}\right)\right. \\ &\left.\quad - \frac{785}{1462}e_0^2(\pi f_0)^{19/9}\left(\frac{9}{(\pi f)^{34/9}} + \frac{34\pi f}{(\pi f_a)^{43/9}} - \frac{43}{(\pi f_a)^{34/9}}\right)\right]\end{aligned} \tag{A9}$$

TABLE XIII. The rms errors of the parameters of the signal with a first-order contribution due to eccentricity for a binary of two neutron stars of 1.4 solar mass each at the distance of 200 Mpc.

S/N	Δt_c (msec)	$\Delta\phi_c$	$\Delta\mu/\mu$	$\Delta m/m$	$\Delta k_e[M_\odot^{-5/3}(100\text{ Hz})^{19/9}]$
15	0.56	1.2	0.50%	0.74%	3.6×10^{-7}

and consequently the Fourier transform of our signal in the stationary phase approximation has the form

$$\tilde{h}(f) = \mathcal{A}f^{-7/6}\exp\left(i\{2\pi f t_a - \varphi - \pi/4 + \tfrac{5}{48}[a(f;f_a)k + a_e(f;f_a)k_e)]\}\right) \text{ for } f>0 \quad \text{(A10)}$$

(and by the complex conjugate of the above expression for $f<0$), where

$$\tilde{A} = \frac{1}{30^{1/2}}\frac{1}{\pi^{2/3}}\frac{\mu^{1/2}m^{1/3}}{R}, \quad \text{(A11)}$$

$$k = \frac{1}{\mu m^{2/3}}, \quad k_e = \frac{1}{\mu m^{2/3}}e_0^2(\pi f_0)^{19/9}, \quad \text{(A12)}$$

$$a(f;f_a) = \frac{9}{40}\frac{1}{(\pi f)^{5/3}} + \frac{3}{8}\frac{\pi f}{(\pi f_a)^{8/3}} - \frac{3}{5}\frac{1}{(\pi f_a)^{5/3}}, \quad \text{(A13)}$$

$$a_e(f;f_a) = -\frac{157}{24}\left(\frac{81}{1462}\frac{1}{(\pi f)^{34/9}} + \frac{9}{43}\frac{\pi f}{(\pi f_a)^{43/9}} - \frac{9}{34}\frac{1}{(\pi f_a)^{34/9}}\right). \quad \text{(A14)}$$

We have investigated the accuracy of measurements of parameters of the above signal with first-order eccentricity contribution. We have considered neutron-star–neutron-star binary. The results are summarized in Table XIII.

The potential accuracy of estimation of the eccentricity parameter k_e is very good however for the currently observed binaries the eccentricity invariant I_e is extremely small. For the Hulse-Taylor pulsar $I_e = 1.8\times 10^{-13}[M_\odot^{-5/3}100\text{ Hz}^{19/9}]$. We have the numerical values

$$k_e = 1.3\times 10^{-13}\left(\frac{I_e}{1.8\times 10^{-13}}\right)\left(\frac{1.2}{\mathcal{M}_\odot}\right)^{5/3}[M_\odot^{-5/3}100\ \text{ Hz}^{19/9}], \quad \text{(A15)}$$

$$\frac{\Delta k_e}{k_e} = 2.8\times 10^6 r_{200\,\text{Mpc}}\left(\frac{I_e}{1.8\times 10^{-13}}\right)\left(\frac{\mathcal{M}_\odot}{1.2}\right)^{5/3}, \quad \text{(A16)}$$

where $r_{200\,\text{Mpc}}$ is the distance in 200 Mpc. Thus for eccentricity effects to be measured one would need extremely short period binaries of high eccentricity. Such binaries could perhaps occur in the center of a galaxy or be created as a result of some supernova explosions.

APPENDIX B: COVARIANCE MATRICES AT VARIOUS POST-NEWTONIAN ORDERS

In this appendix we give the numerical values of the covariance matrices at various post-Newtonian orders for the reference binary. The reference binary has the chirp mass $\mathcal{M}$ of 1 solar mass and is located at the distance of 100 Mpc. We only give reduced covariance matrices, i.e., covariance matrices for the phase parameters. As indicated in Sec. II the estimator of the amplitude parameter is uncorrelated with phase parameters. The integration range in the Fisher matrix integrals was taken to be from 10 Hz to infinity and the spectral density of advanced LIGO detectors was assumed [Eq. (24)]. The frequency f_a was chosen such that the rms error in the time parameter is minimum. The minimum frequency is denoted by f_m and its numerical value is given for each covariance matrix. The subscripts N, 1PN, 3/2PN, 2PN, 2PNe refer to signal including quadrupole radiation, first post-Newtonian correction, 3/2 post-Newtonian correction, second post-Newtonian correction, and first-order effect due to eccentricity, respectively. The order of parameters in the matrices is t_m, ϕ, k, k_1, $k_{3/2}$, k_2, k_e:

$f_m^{\text{N}} = 70$ Hz,

$$C_{\text{N}} = \begin{pmatrix} 1.96\times 10^{-8} & 8.2\times 10^{-6} & 1.21\times 10^{-10} \\ 8.2\times 10^{-6} & 0.005\,37 & 2.08\times 10^{-7} \\ 1.21\times 10^{-10} & 2.08\times 10^{-7} & 6.65\times 10^{-11} \end{pmatrix}, \quad \text{(B1)}$$

$f_m^{\text{1PN}} = 100$ Hz,

$$C_{\text{1PN}} = \begin{pmatrix} 2.24\times 10^{-8} & 9.93\times 10^{-6} & 8.87\times 10^{-10} & -6.61\times 10^{-8} \\ 9.93\times 10^{-6} & 0.007\,54 & -6.28\times 10^{-7} & 0.000\,147 \\ 8.87\times 10^{-10} & -6.28\times 10^{-7} & 1.4\times 10^{-9} & -1.97\times 10^{-7} \\ -6.61\times 10^{-8} & 0.000\,147 & -1.97\times 10^{-7} & 0.000\,028\,9 \end{pmatrix}, \quad \text{(B2)}$$

$f_m^{3/2\mathrm{PN}} = 160$ Hz,

$$C_{3/2\mathrm{PN}} = \begin{pmatrix} 3.41\times10^{-8} & 0.000\,02 & -9.65\times10^{-9} & 3.36\times10^{-6} & 0.000\,021\,2 \\ 0.000\,02 & 0.0191 & -9.34\times10^{-7} & -0.000\,403 & -0.007 \\ -9.65\times10^{-9} & -9.34\times10^{-7} & 2.23\times10^{-8} & -8.71\times10^{-6} & -0.0000622 \\ 3.36\times10^{-6} & -0.000\,403 & -8.71\times10^{-6} & 0.003\,49 & 0.0253 \\ 0.000\,021\,2 & -0.007 & -0.000\,062\,2 & 0.0253 & 0.185 \end{pmatrix}, \quad \text{(B3)}$$

$f_m^{2\mathrm{PN}} = 100$ Hz,

$$C_{2\mathrm{PN}} = \begin{pmatrix} 5.62\times10^{-8} & 0.000\,028\,6 & 2.39\times10^{-9} & -9.48\times10^{-6} & -0.000\,199 & -0.001\,02 \\ 0.000\,028\,6 & 0.0189 & 0.000\,016 & -0.0162 & -0.26 & -1.11 \\ 2.39\times10^{-9} & 0.000\,016 & 2.3\times10^{-7} & -0.000\,162 & -0.002\,19 & -0.007\,96 \\ -9.48\times10^{-6} & -0.0162 & -0.000\,162 & 0.116 & 1.59 & 5.86 \\ -0.000\,199 & -0.26 & -0.002\,19 & 1.59 & 22. & 81.5 \\ -0.001\,02 & -1.11 & -0.007\,96 & 5.86 & 81.5 & 305 \end{pmatrix}, \quad \text{(B4)}$$

$f_m^{2\mathrm{PNe}} = 120$ Hz,

$$C_{2\mathrm{PNe}} = \begin{pmatrix} 6.36\times10^{-8} & 0.000\,036\,9 & -3.56\times10^{-8} & 4.9\times10^{-7} & -0.000\,135 & -0.000\,977 & -4.86\times10^{-11} \\ 0.000\,036\,9 & 0.0279 & 0.000\,035 & -0.0315 & -0.474 & -1.89 & -5.21\times10^{-9} \\ -3.56\times10^{-8} & 0.000\,035 & 2.36\times10^{-6} & -0.001\,21 & -0.0143 & -0.0464 & 1.33\times10^{-9} \\ 4.9\times10^{-7} & -0.0315 & -0.001\,21 & 0.63 & 7.57 & 24.7 & -6.52\times10^{-7} \\ -0.000\,135 & -0.474 & -0.0143 & 7.57 & 91.5 & 301 & -7.59\times10^{-6} \\ -0.000\,977 & -1.89 & -0.0464 & 24.7 & 301 & 999 & -0.000\,024 \\ -4.86\times10^{-11} & -5.21\times10^{-9} & 1.33\times10^{-9} & -6.52\times10^{-7} & -7.59\times10^{-6} & -0.000\,024 & 8.28\times10^{-13} \end{pmatrix}. \quad \text{(B5)}$$

APPENDIX C: COEFFICIENTS IN THE DAMOUR-ESPOSITO-FARÈSE BISCALAR $T(\beta',\beta'')$ THEORY

The coefficients $\kappa_0, \kappa_q, \kappa_{d1}, \kappa_{d2}$ in the shift of the Newtonian mass parameter k due to the biscalar $\mathrm{T}(\beta',\beta'')$ theory [Eq. (77) in Sec. III D] are given by

$$\kappa_0 = \tfrac{1}{2}\beta' B(\mathcal{C}_1^2 + \mathcal{C}_2^2), \quad \text{(C1)}$$

$$\kappa_q = \beta' B(\mathcal{C}_1^2 x_2 + \mathcal{C}_2^2 x_1), \quad \text{(C2)}$$

$$\kappa_{d1} = \tfrac{1}{2}\beta' B(\mathcal{C}_1^2 x_1 - \mathcal{C}_2^2 x_2)(x_1 - x_2), \quad \text{(C3)}$$

$$\kappa_{d2} = (ab_{121} - ab_{221})x_1 + (ab_{212} - ab_{112})x_2, \quad \text{(C4)}$$

where

$$x_1 = \frac{m_1}{m}, \quad \text{(C5)}$$

$$x_2 = \frac{m_2}{m}, \quad \text{(C6)}$$

and the constants A and B have the values

$$A = 2.156\,917\,6, \quad B = 1.026\,152\,9. \quad \text{(C7)}$$

$\mathcal{C}_1$ and $\mathcal{C}_2$ are sensitivities of the two bodies to changes of the scalar field. The functions ab are given by

$$\begin{aligned} ab_{121} = {} & \beta'[-\mathcal{C}_2 - B\mathcal{C}_1^2 + (A-3B)\mathcal{C}_2^2 - (A-B)2\mathcal{C}_2\mathcal{C}_1^2 + (2A^2 - 7AB + 5B^2)\mathcal{C}_2^2\mathcal{C}_1^2] \\ & +\beta'^2 B^2(-3\mathcal{C}_1^3 + 2\mathcal{C}_1^2\mathcal{C}_2^2 + \mathcal{C}_1^4 + \tfrac{1}{2}\mathcal{C}_2\mathcal{C}_1^4 + A\mathcal{C}_2^2\mathcal{C}_1^4) + \tfrac{1}{2}\beta'' B\mathcal{C}_2^2, \end{aligned} \quad \text{(C8)}$$

$$\begin{aligned} ab_{212} = {} & \beta'[-\mathcal{C}_1 - B\mathcal{C}_2^2 + (A-3B)\mathcal{C}_1^2 - (A-B)2\mathcal{C}_1\mathcal{C}_2^2 + (2A^2 - 7AB + 5B^2)\mathcal{C}_1^2\mathcal{C}_2^2] \\ & +\beta'^2 B^2(-3\mathcal{C}_1^3 + 2\mathcal{C}_1^2\mathcal{C}_2^2 + \mathcal{C}_1^4 + \tfrac{1}{2}\mathcal{C}_1\mathcal{C}_2^4 + A\mathcal{C}_1^2\mathcal{C}_2^4) + \tfrac{1}{2}\beta'' B\mathcal{C}_1^2, \end{aligned} \quad \text{(C9)}$$

$$\begin{aligned} ab_{221} = {} & \beta'[-\mathcal{C}_2 - \tfrac{1}{2}B(\mathcal{C}_1^2 + \mathcal{C}_2^2) + (A-3B)\mathcal{C}_2^2 - (A-B)\mathcal{C}_2(\mathcal{C}_1^2 + \mathcal{C}_2^2) \\ & +\tfrac{1}{2}(2A^2 - 7AB + 5B^2)\mathcal{C}_2^2(\mathcal{C}_1^2 + \mathcal{C}_2^2)] + \beta'^2 B^2[-3\mathcal{C}_2^3 + \mathcal{C}_2^2(\mathcal{C}_1^2 + \mathcal{C}_2^2) \\ & +\mathcal{C}_2^4 + \tfrac{1}{2}\mathcal{C}_2^3\mathcal{C}_1^2 + A\mathcal{C}_1^2\mathcal{C}_2^4] + \tfrac{1}{2}\beta'' B\mathcal{C}_2^2, \end{aligned} \quad \text{(C10)}$$

$$\begin{aligned} ab_{112} = {} & \beta'[-\mathcal{C}_2 - \tfrac{1}{2}B(\mathcal{C}_1^2 + \mathcal{C}_2^2) + (A-3B)\mathcal{C}_1^2 - (A-B)\mathcal{C}_1(\mathcal{C}_1^2 + \mathcal{C}_2^2) \\ & +\tfrac{1}{2}(2A^2 - 7AB + 5B^2)\mathcal{C}_1^2(\mathcal{C}_1^2 + \mathcal{C}_2^2)] + \beta'^2 B^2[-3\mathcal{C}_1^3 + \mathcal{C}_1^2(\mathcal{C}_1^2 + \mathcal{C}_2^2) \\ & +\mathcal{C}_1^4 + \tfrac{1}{2}\mathcal{C}_2^2\mathcal{C}_1^3 + A\mathcal{C}_2^2\mathcal{C}_1^4] + \tfrac{1}{2}\beta'' B\mathcal{C}_1^2. \end{aligned} \quad \text{(C11)}$$

For a detailed exposition of the theory the reader should consult [37].

APPENDIX D: DETECTION OF THE KNOWN SIGNAL WITH A NONOPTIMAL FILTER

Suppose that we would like to know whether or not in a given data set x there is present a signal h. We assume that the noise n in the data is additive. There are two alternatives:

$$\begin{aligned} \text{no signal:} \quad & x = n, \\ \text{signal:} \quad & x = h + n. \end{aligned} \tag{D1}$$

A standard method to determine which of the two alternatives holds is to perform the Neyman-Pearson test [28]. This test consists of comparing the *likelihood ratio* Λ, the ratio of probability density distributions of the data x when the signal is present and when the signal is absent, with a threshold. The threshold is determined by the false alarm probability that we can tolerate (the false alarm probability is the probability of saying that the signal is present when there is no signal). The test is optimal in the sense that it maximizes the probability of detection of the signal. In the case of Gaussian noise and deterministic signal h the logarithm of Λ is given by

$$\ln \Lambda = (x|h) - \tfrac{1}{2}(h|h). \tag{D2}$$

Thus in this case the optimal test consists of correlating the data with the expected signal and it is equivalent to comparing the correlation $G := (x|h)$ with a threshold. The probability distributions p_0 and p_1 of G when, respectively, the signal is absent and present are given by

$$p_0(G;d) = \frac{1}{\sqrt{2\pi d^2}} \exp\left[-\frac{G^2}{d^2}\right], \tag{D3}$$

$$p_1(G;d) = \frac{1}{\sqrt{2\pi d^2}} \exp\left[-\frac{(G-d^2)^2}{d^2}\right], \tag{D4}$$

where d is the optimal signal-to-noise ratio, $d^2 = (h|h)$ and we assumed that the noise is a zero mean Gaussian process.

Let T be a given threshold. This means that we say that the signal is present in a given data set if $G > T$. The probabilities P_F and P_D of false alarm and detection, respectively, are given by

$$P_F(T,d) = \int_T^\infty p_0(G;d)\, dG, \tag{D5}$$

$$P_D(T,d) = \int_T^\infty p_1(G;d)\, dG. \tag{D6}$$

In the Gaussian case they can be expressed in terms of the error functions.

$$P_F(d_T,d) = \tfrac{1}{2}\mathrm{erfc}\left(\frac{d_T^2}{\sqrt{2}d}\right), \tag{D7}$$

$$P_D(d_T,d) = \tfrac{1}{2}\left[1 + \mathrm{erf}\left(\frac{d^2 - d_T^2}{\sqrt{2}d}\right)\right], \tag{D8}$$

where erf and erfc are the error and the complementary error functions, respectively [47], and we have introduced for convenience the quantity $d_T := \sqrt{T}$ that we call the *threshold signal-to-noise ratio.* In practice we adopt a certain value of the false alarm probability that we can accept and from formula (D7) we calculate the detection threshold T.

Let F be a linear filter and let n be the additive noise in data x; then

$$(x|F) = (h|F) + (n|F). \tag{D9}$$

The signal-to-noise (S/N) ratio is defined by

$$(S/N)^2 := \frac{E_1[(h|F)^2]}{E_1[(n|F)^2]} = \frac{(h|F)^2}{(F|F)}, \tag{D10}$$

where E_1 means expectation value when the signal is present. By the Schwarz inequality we immediately see that (S/N) is maximal and equal to d when the linear filter is matched to the signal, i.e., $F = h$. This is another interpretation of the matched filter—it maximizes the signal-to-noise ratio over all linear filters [28]. However when the noise is not Gaussian the matched filter is not the optimal filter; it does not maximize the probability of detection of the signal. We see that in the case of Gaussian noise the problem of detecting a known signal by optimal filter is determined by one parameter—the optimal signal-to-noise ratio d.

Suppose that because of certain restrictions of practical nature we cannot afford to use the optimal filter h and we use a suboptimal one h_N, which is not perfectly matched to the signal. Thus $(h|h_N) < (h|h)$. We denote $\sqrt{(h|h_N)}$ by d_0 and we assume that $(h_N|h_N) = (h|h) = d^2$. Our suboptimal correlation function is given by $G_N = (x|h_N)$ and its probability distributions p_{N0} and p_{N1} when, respectively, the signal is absent and present are given by

$$p_{N0}(G_N;d) = \frac{1}{\sqrt{2\pi d^2}} \exp\left[-\frac{G_N^2}{d^2}\right], \tag{D11}$$

$$p_{N1}(G_N;d,d_0) = \frac{1}{\sqrt{2\pi d^2}} \exp\left[-\frac{(G_N-d_0^2)^2}{d^2}\right]. \tag{D12}$$

We see that the suboptimal detection problem is determined by two parameters: d and d_0, square roots of the expectation values of the optimal and suboptimal correlations when the signal is present. The false alarm and detection probabilities as in the optimal case can be expressed in terms of the error functions:

$$P_F(d_T,d) = \tfrac{1}{2}\mathrm{erfc}\left(\frac{d_T^2}{\sqrt{2}d}\right), \tag{D13}$$

$$P_D(d_T,d,d_0) = \tfrac{1}{2}\left[1 + \mathrm{erf}\left(\frac{d_0^2 - d_T^2}{\sqrt{2}d}\right)\right]. \tag{D14}$$

We see that the probability of false alarm for the suboptimal case is the same as in the optimal case however the probability of detection in the suboptimal case is always less than the probability of detection in the optimal case since $d_0 < d$ and the error function erf(x) is an increasing function of the argument x. The signal-to-noise ratio in the case of suboptimal linear filter h_N is given by

$$(S/N)^2 = \frac{(h|h_N)^2}{(h_N|h_N)} = d^2 \left(\frac{d_0}{d}\right)^4 . \tag{D15}$$

Let us denote the ratio d_0/d by l. The ratio l measures the drop in the expectation value of the correlation function as a result of nonoptimal filtering. We see that due to suboptimal filtering the signal-to-noise ratio decreases by *square* of the factor l. We denote l^2 by FF and following [14] call it the *fitting factor*.

In our considerations we need to calculate the number of events that will be detected by linear filtering. We shall make a number of simplifying assumptions. We shall assume a Euclidean universe where in the sphere of radius r_0 we have one source and that at the distance r_0 the optimal signal-to-noise ratio is d. Moreover we shall assume that the magnitudes of the signal h and the suboptimal filter h_N are inversely proportional to the distance r from the source. Then the square roots d_r and d_{0r} of the expectation values of the optimal and suboptimal correlations at the distance r are given by

$$d_r = \frac{r_0}{r} d, \tag{D16}$$

$$d_{0r} = \frac{r_0}{r} d_0. \tag{D17}$$

We assume that the sources are uniformly distributed in space. Then the expected number of detected real events N and N_N in the optimal and the suboptimal case, respectively, is given by

$$N(d_T, d) = \frac{4\pi \int_0^\infty r^2 P_D(d_T, d_r)\, dr}{\frac{4\pi}{3} r_0^3} = 3 \int_0^\infty x^2 P_D(d_T, d/x)\, dx, \tag{D18}$$

$$N_N(d_T, d, d_0) = \frac{4\pi \int_{r_0}^\infty r^2 P_{\rm ND}(d_T, d, d_0/r)\, dr}{\frac{4\pi}{3} r_0^3} = 3 \int_0^\infty x^2 P_{\rm ND}(d_T, d, d_0/x)\, dr. \tag{D19}$$

TABLE XIV. Comparison of number of true events and false alarms obtained with the optimal filter and the Newtonian filter. We assume the signal-to-noise ratio threshold $d_T = 5$ and we assume we have one signal for the optimal signal-to-noise ratio d. N is the expected number of detected signals with the optimal filter, N_F is the number of false alarms, N_N is the number of detected signals with the Newtonian filter, T_N is the lowered threshold, N_L is the number of signals with the lowered threshold, and $N_{\rm fl}$ is the number of false alarms with the lowered threshold.

d	FF	N	N_F	N_N	T_N	N_L	$N_{\rm fl}$
15	0.81	27	0.055	20	4.5	28	0.16
15	0.36	27	0.055	5.6	3.225	28	2.1
30	0.81	225	1.1	165	4.5	230	2.2
30	0.25	225	1.1	31	2.875	229	32

The assumptions that led to the above formulas mean that we neglect general relativistic, cosmological, and evolutionary effects. Because of the noise even if there is no signal there is always a nonzero probability that the correlation function crosses the threshold. Thus there will be a certain number N_F of false events. For a given optimal signal-to-noise ratio and a threshold d_T this number is the same for both the optimal and suboptimal filter and it is given by

$$N_F = \frac{4\pi \int_{r_0}^\infty r^2 P_F(d_T, d/r)\, dr}{\frac{4\pi}{3} r_0^3} = 3 \int_o^\infty x^2 P_F(d_T, d/x)\, dx. \tag{D20}$$

We observe that in the Gaussian case the integrals in formulas (D18)–(D20) are convergent even though we integrate over the all infinite Euclidean volume. In Table XIV we have given numerical examples of the effect of lowering the threshold.

APPENDIX E: AN APPROXIMATE FORMULA FOR THE CORRELATION FUNCTION

In this appendix we shall derive an approximate formula for the correlation integral. Let us consider the expression for the correlation function given by (34). The integrand of the correlation integral is the product of the integrand of the signal-to-noise integral ind(f) considered in Sec. II and oscillating factor. We know that the ind(f) is a fairly sharply peaked around a certain frequency f_0' consequently to obtain a reasonable approximation we expand the phase around the frequency f_0'. Keeping only the terms to second order we get

$$\Phi(f) \simeq 2\pi(f - f_0')\Delta t' - \Delta\phi'' + \frac{5}{96}\frac{(f/f_0' - 1)^2}{(\pi f_0')^{5/3}}\Delta k_E + O[(f/f_0' - 1)^3], \tag{E1}$$

where

$$\phi_0'' = \phi_0 - 2\pi f_0' t_0' \tag{E2}$$

and

$$k_E = k - \frac{157}{24}\frac{k_e}{(\pi f_0')^{19/9}} + k_1(\pi f_0')^{2/3} - k_{3/2}(\pi f_0') + k_2(\pi f_0')^{4/3} - \frac{5}{192}\frac{k_d}{(\pi f_0')^{2/3}}. \tag{E3}$$

We shall call k_E the *effective mass parameter.* Δk_E is the difference in the effective mass parameter of the signal and the filter. Thus in the above approximation the post-Newtonian signal can be parametrized by one mass parameter—the parameter k_E. In other words the dimension of the parameter space of the filters is effectively reduced. This last interpretation has been emphasized in [44] where first post-Newtonian corrections to the phase were considered. The mass parameter estimated by Newtonian filter considered in Sec. III is just the effective mass parameter. We stress that the parameter k_E depends not only on the parameters of the two-body system but also on the characteristic frequency f_0' of the noise in the detector.

The next step is to obtain a manageable approximation to the function ind(f). We approximate it by a Gaussian function with the mean equal to the frequency f_0' and the standard deviation equal to the HWHM σ_0' of the function ind(f). We extend the range of integration from $-\infty$ to $+\infty$. We introduce a normalization factor such that the integral of the approximate integrand is equal to the optimal signal-to-noise ratio d. It is then useful to introduce a reduced correlation integral $H' = H/d^2$ where d is the S/N ratio. Thus our approximate formula for the reduced correlation integral takes the form

$$H_a' = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \exp[-(f-f_0')/(2{\sigma_0'}^2)] \cos\left(2\pi(f-f_0')\Delta t' - \Delta\phi'' + \frac{5}{96}\frac{(f/f_0'-1)^2}{(\pi f_0')^{5/3}}\Delta k_E\right). \tag{E4}$$

The above integration can be done analytically. It is convenient to introduce the new variables and new parameters

$$y = \frac{f-f_0'}{\sqrt{2{\sigma_0'}^2}}, \tag{E5}$$

$$\vartheta'' = \Delta\phi'', \tag{E6}$$

$$\tau = 2\pi\Delta t'\sqrt{2{\sigma_0'}^2}, \tag{E7}$$

$$\kappa = \frac{5}{96}\frac{\Delta k_E}{{f_0'}^2(\pi f_0')^{5/3}} 2{\sigma_0'}^2 \tag{E8}$$

then our integral takes a simple form

$$H_a'(\vartheta,\tau,\kappa) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp[-y^2]\cos[-\vartheta'' + \tau y + \kappa y^2]dy. \tag{E9}$$

We see that in the new variables introduced above the reduced correlation integral is independent of the characteristics of the integrand ind(f), i.e., f_0' and σ_0'. The analytic formula for the function $H_a'(\vartheta'',\tau,\kappa)$ is given by

$$H_a'(\vartheta'',\tau,\kappa) = \frac{1}{(1+\kappa^2)^{1/4}} \exp\left(-\frac{\tau^2}{4(1+\kappa^2)}\right) \cos\left[\tfrac{1}{2}\left(\arctan\kappa - \frac{\kappa\tau^2}{2(1+\kappa^2)}\right) - \vartheta''\right]. \tag{E10}$$

By appropriate transformations given in Sec. II we can obtain approximate formulas to the correlation integral for an arbitrary choice of the time and the phase parameters. Let us first consider the transformation given by Eq. (52) for $f_m = f_0'$ and $t_a = t'$. In the coordinates introduced above it takes the form

$$\vartheta'' = \vartheta - \frac{1}{\sqrt{2}}\tau\rho, \tag{E11}$$

where

$$\rho = f_0'/\sigma_0'. \tag{E12}$$

Then the approximate formula for the correlation function is given by

$$H_a'(\vartheta,\tau,\kappa) = \frac{1}{(1+\kappa^2)^{1/4}} \exp\left(-\frac{\tau^2}{4(1+\kappa^2)}\right) \cos\left[\tfrac{1}{2}\left(\arctan\kappa - \frac{\kappa\tau^2}{2(1+\kappa^2)}\right) + \frac{1}{\sqrt{2}}\tau\rho - \vartheta\right]. \tag{E13}$$

We see that in these new coordinates for $\kappa = 0$ the correlation function oscillates with the maxima at the discrete values of τ coordinate given by

$$\tau_{\max} = \frac{2\sqrt{2}\pi}{\rho}n, \tag{E14}$$

where n is an integer. In the original coordinates Eq. (E14) takes the form $\Delta t = 1/f_0'$. Thus the correlation integral oscillates with the period determined by the characteristic frequency which depends both on properties of the noise and the signal.

The expressions for the correlation function for a different choice of the time and the phase parameters can be obtained by the following transformations. These are transformations given by Eqs. (37) and expressed in our dimensionless coordinates:

$$\vartheta = \vartheta' - \frac{3}{5}\kappa\rho^2(1-\delta^{5/3}), \tag{E15}$$

$$\tau = \tau' - \frac{3}{4\sqrt{2}}\kappa\rho(1-\delta^{8/3}), \tag{E16}$$

where

$$\delta = f_0'/f_a. \tag{E17}$$

The above transformations are obtained from general transformations given by Eqs. (37) and (38) with $t_a' = t', k = k_E, f_a' = f_0'$ and with all the post-Newtonian mass parameters removed. From the approximate formula for the correlation function obtained above we see that the correlation is given by the product of an oscillating cosine function and an envelope. In the cosine function there are oscillations with the period of $1/f_0'$. The envelope function is exponentially damped if we move away from the maximum at the center except for the direction given by $\tau = 0$ along which the damping is least. The equation of the ridge $\tau = 0$ in the primed coordinates is given by

$$\tau' = \frac{3}{4\sqrt{2}}\kappa\rho(1 - \delta^{8/3}), \tag{E18}$$

and in the original coordinates it takes the form

$$\Delta t = \frac{5}{256}\frac{1}{(\pi f_0')^{8/3}}\left[1 - \left(\frac{f_0'}{f_a}\right)^{8/3}\right]\Delta k_E. \tag{E19}$$

Consequently we conclude that the general appearance of the correlation function in coordinates $\Delta t'$ and Δk is a series of peaks aligned along a straight line given by Eq. (E19) above and occurring with the period $1/f_0'$ in the time coordinate. Numerical investigation shows that the correlation integral exhibits these properties and that our analytic formula reproduces qualitatively its behavior. The approximate formula obtained above may be a useful tool for developing algorithms to recognize the chirp signal in a noisy data set.

---

[1] K. S. Thorne, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987), pp. 330–458.

[2] R. Narayan, T. Piran, and A. Shemi, Astrophys. J. **379**, L17 (1991).

[3] E. S. Phinney, Astrophys. J. **380**, L17 (1991).

[4] A. V. Tutukov and L. R. Yungelson, Mon. Not. R. Astron. Soc. **260**, 675 (1993).

[5] P. C. Peters, Phys. Rev. **136**, B1224 (1964).

[6] L. S. Finn and D. F. Chernoff, Phys. Rev. D **47**, 2198 (1993).

[7] A. Królak, J. A. Lobo, and B. J. Meers, Phys. Rev. D **48**, 3451 (1993).

[8] B. S. Sathyaprakash and S. V. Dhurandhar, Phys. Rev. D **44**, 3819 (1993); **49**, 1707 (1994).

[9] K. D. Kokkotas, A. Królak, and G. Tsegas, Class. Quantum Grav. **11**, 1901 (1994).

[10] C. Cutler and E. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).

[11] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, E. E. Flanagan, D. Kennefick, D. M. Markovic, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).

[12] A. Królak, in *Gravitational Wave Data Analysis*, edited by B. F. Schutz (Kluwer, Dordrecht, 1989), pp. 59–69.

[13] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).

[14] T. A. Apostolatos, Ph.D. thesis, California Institute of Technology, 1994 (unpublished).

[15] C. W. Lincoln and C. M. Will, Phys. Rev. D **42**, 1123 (1990).

[16] L. Blanchet and B. S. Sathyaprakash, Phys. Rev. Lett. **74**, 1067 (1995).

[17] R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976); **215**, 984 (1977).

[18] E. Poisson, Phys. Rev. D **47**, 1497 (1993).

[19] L. Blanchet and T. Damour, Phys. Rev. D **46**, 4301 (1992).

[20] A. G. Wiseman, Phys. Rev. D **48**, 4757 (1993).

[21] L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993).

[22] L. E. Kidder, C. M. Will, and A. G. Wiseman, Phys. Rev. D **47**, R4183 (1993).

[23] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D **49**, 6274 (1994).

[24] S. Dhurandhar, A. Królak, B. F. Schutz, and J. Watkins (unpublished).

[25] L. Smarr, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, 1979).

[26] C. W. Helström, *Statistical Theory of Signal Detection*, 2nd ed. (Pergamon Press, London, 1968).

[27] S. V. Dhurandhar and B. F. Schutz, Phys. Rev. D **50**, 2390 (1994).

[28] M. H. A. Davies, in *Gravitational Wave Data Analysis*, edited by B. F. Schutz (Kluwer, Dordrecht, 1989), pp. 73–94.

[29] E. L. Lehmann, *Theory of Point Estimation* (Wiley, New York, 1983).

[30] P. Jaranowski, K. D. Kokkotas, A. Królak, and G. Tsegas (unpublished).

[31] A. Królak and J. A. Lobo (unpublished).

[32] N. Wex and G. Schäfer, Class. Quantum Grav. **10**, 2729 (1993).

[33] L. E. Kidder, C. M. Will, and A. G. Wiseman, Class. Quantum Grav. **9**, L125 (1992); Phys. Rev. D **47**, 3281 (1993).

[34] C. M. Will and H. W. Zaglauer, Astrophys. J. **346**, 366 (1989).

[35] C. M. Will, *Theory and Experiment in Gravitational Physics*, revised ed. (Cambridge University Press, Cambridge, 1993).

[36] When this work was completed we learned that accuracy of determination of the ω parameter in JFBD theory by gravitational-wave observations from coalescing binaries has already been considered in detail by Clifford Will [C. M. Will, Phys. Rev. D **50**, 6058 (1994)] and a conclusion similar to ours was reached. Moreover detection of scalar-type gravitational wave emission from gravitational collapse in JFBD theory by laser interferometers

was investigated in a paper: M. Shibata, K. Nakao, and T. Nakamura, Phys. Rev. D **50**, 7304 (1994).

[37] T. Damour and G. Esposito-Farèse, Class. Quantum Grav. **9**, 2093 (1992).

[38] N. Wex (unpublished).

[39] A. Królak, in *Proceedings of the Cornelius Lanczos International Centenary Conference*, edited by J. D. Brown, M. T. Chu, D. C. Ellison, and R. J. Plemmons (SIAM, Philadelphia, 1994), p. 482.

[40] K. D. Kokkotas, A. Królak, and G. Schäfer, in *Particle Astrophysics, Atomic Physics and Gravitation*, Proceedings of the 29th Rencontres de Moriond, Villars sur Ollon, Switzerland, 1994, edited by J. Tran Thanh Van, G. Fontaine, and E. Hinds (Editions Frontières, Gif-sur-Yvette, France, 1994), p. 485.

[41] E.E. Flanagan (private communication).

[42] S. L. Finn (private communication).

[43] R. Balasubramanian and S. V. Dhurandhar, Phys. Rev. D **50**, 6080 (1994).

[44] B. S. Sathyaprakash, Phys. Rev. D **50**, R7111 (1994).

[45] S. L. Finn, Phys. Rev. Lett. **73**, 1878 (1994).

[46] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

[47] The error function erf(x) is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt \tag{E20}$$

and the complementary error function erfc(x) is defined by

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x). \tag{E21}$$

[48] E. Poisson and C. M. Will, Phys. Rev. D **52**, 848 (1995).