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Long wavelength iteration of Einstein's equations near a spacetime singularity

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We analyze the behavior of a very inhomogeneous spacetime near the singularity by using the recently developed long wavelength iteration scheme of Einstein's equations. Near the singularity, the local anisotropy cannot be neglected and we give the first order and third order solutions for any perfect fluid adiabatic index. We also clarify the links between a recently developed long wavelength iteration scheme of Einstein's equations, the Belinski-Khalatnikov-Lifschitz (BKL) general solution near a singularity, and the anti-Newtonian scheme of Tomita. We determine the regimes when the long wavelength or anti-Newtonian scheme is directly applicable and show how it can otherwise be implemented to yield the BKL oscillatory approach to a spacetime singularity.

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I. INTRODUCTION

Despite the fact that the universe is clearly inhomogeneous on the galactic scale and the possiblity, raised by some inflationary scenarios (see, e.g., [1]), that its geometry may be "chaotic" on scales larger than the Hubble radius, most works in cosmology are based on the homogeneous and isotropic models of Friedmann, Robertson, and Walker (FRW). Many convincing reasons, physical or philosophical, can be given to that state of affairs, but there is also a purely technical one: very few inhomogeneous solutions of cosmological interest are known (see, e.g., [2]).

Various approximate solutions, however, have been given in the past. A simple one is the "quasi-isotropic" solution of Lifschitz and Khalatnikov (see, e.g., [3]) the spatial sections of which (in a synchronous reference frame) are just uniformly stretched in the course of time [$ds^2 = -dt^2 + a^2(t)h_{ij}dx^i dx^k$, where the arbitrary "seed" metric $h_{ij}(x^k)$ depends on space only]. This metric is exact and reduces to the standard FRW metric if the spatial sections are maximally symmetric, and is a good approximation to an exact solution of Einstein's equations if, as we shall recall below, all spatial derivatives remain small, that is, if all "point to point" interactions are neglected.

A more general approximate solution when all gradients are neglected is the "anti-Newtonian" solution of Tomita [4], which, as we shall recall, depends on as many arbitrary functions as a generic solution of Einstein's equations. Finally, the "general oscillatory solution" studied by Belinski, Lifschitz, and Khalatnikov [5] is the most elaborate approximate description of a generic solution of Einstein's equations near the big bang.

There was recently a renewed interest in these approximation solutions first of all because the observations of the Cosmic Background Explorer (COBE) satellite urged a fresh view on the old problem of structure formation (see, e.g., [6]) but also because a new line of attack on Einstein's equations was pursued. Indeed, in a series of papers [7], Salopek and Stewart and collaborators developed a "long wavelength" iteration scheme not of Einstein's equations but of the Hamilton-Jacobi equation for general relativity. Their method, which consists at lowest order in neglecting all spatial gradients, leads back in most instances to the quasi-isotropic solution mentioned above, and yielded, for dust at least, the solution up to and including the third iteration (that is, accurate to order 6 in the gradients as will become clear below [8]). (The case of a more general perfect fluid is more awkward to handle in this Hamiltonian formalism.)

In another series of papers the present authors (together with Comer, Goldwirth, Tomita and Parry) [9–11] iterated in the same way the Einstein equations themselves. They noted that the zeroth order quasi-isotropic solution, although not generic, is an attractor at late cosmological times of the generic solution of Tomita. Concentrating then on this quasi-isotropic limit of the zeroth order solution they obtained the solution up to and including the second iteration (fifth order in the gradients) for matter being a perfect fluid with constant adiabatic index or a scalar field. In the particular case of dust their result is identical to that of [7] so that the link between the two methods could be clearly made.

The motivation for going beyond the zeroth order is the wish to describe inhomogeneities within the Hubble radius. Indeed the approximation at the root of these long wavelength iteration schemes is the following. Take a synchronous reference frame where the line element reads

$$ds^2 = -dt^2 + \gamma_{ij}(x^k,t)dx^i dx^k \qquad (i,j=1,2,3).$$

At each point define a local scale factor a and a Hubble time H^{-1} by

$$a^2 \equiv (\det\gamma_{ij})^{1/3}, \quad H \equiv \dot{a}/a,$$

where $\dot{a} \equiv \partial a/\partial t$. The Hubble time is the characteristic proper time on which the metric evolves. The characteristic comoving length on which it varies is denoted L: $\partial_i \gamma_{jk} \approx L^{-1}\gamma_{ij}$. The long wavelength approximation is the assumption that the characteristic scale of spatial variation is much bigger than the Hubble radius: that is,

$$\frac{1}{a}\partial_i\gamma_{jk} \ll \dot{\gamma}_{ij} \quad \Longleftrightarrow \quad aL \gg H^{-1}.$$

At lowest order then the long wavelength approximation is not suited to describe, e.g., the formation of structure within the present Hubble radius. One can hope, however, that the iteration scheme pushed at a sufficiently high order can give results valid within the Hubble radius. Some numerical investigation of this question has been undertaken by Deruelle and Goldwirth [10] (see also [7]) but further work is nevertheless required to assess the convergence properties of the approximation scheme.

In most of the previous works it was assumed that the local behavior was isotropic, i.e., that the expansion, when one goes forward in time, was similar in all directions. This assumption is justified since the local anisotropy then decays. But when one looks backwards in time near the singularity this local anisotropic effect can become very important. Thus one of the main purposes of this paper is to analyze in detail this phenomenon and to reinvestigate the validity of the long wavelength approximation scheme (LWAS), which naively seems to be justified near the singularity.

Another motivation to go beyond lowest order, which is the one for this paper, is to study, within a long wavelength approximation scheme, the behavior of a generic solution of Einstein's equations near a spacetime singularity and make the link between that scheme and the Belinski-Khalatnikov-Lifschitz (BKL) general solution referred to above.

The point of view is therefore very different from the one adopted in our previous papers since, instead of the late time quasi-isotropic solution, we consider here the solution near a singularity, thus going backward in time. In this case the quasi-isotropic behavior is no longer valid and the approximate solution becomes more complicated since it is no longer possible to separate the time dependence and the spatial dependence into a scale factor and a "seed" metric, respectively. The generic (i.e., without assuming quasi-isotropy) first order solution was given by Tomita [4] in the case of dust and radiation (it can also be found in [7]). Here we consider the more general case of a barytropic perfect fluid with an equation of state of the form $p/\epsilon = \Gamma - 1$, where Γ is a constant. Although an explicit solution for the first order solution cannot be given for Γ different from $\Gamma = 1$ (dust) and $\Gamma = 2$ (stiff matter), an explicit limit near the singularity can be given in all cases (Secs. II and III).

Once the first order solution has been given, we analyze the validity of the approximation near the singularity (Sec. IV). To do this we examine the time evolution of the terms which were neglected at first order. We find that they could not always be ignored and we give a condition of validity for the approximation scheme. In the cases when this condition is not satisfied, we are able to make the link with the work of Belinski, Lifschitz, and Kalatnikov. We believe that the way we recover the oscillatory behavior of the metric, which does not introduce intermediate Bianchi type-IX geometries, is more straightforward than the original approach of BKL, and will allow in particular an easier analysis of the genericity of the "spindle" singularities found by Bruni *et al.* [12].

We then give the generic third order solution (Sec. V). For the sake of simplicity we then apply in detail the approximation scheme to the case of spherical symmetry (Sec. VI). In particular we show that, in the case of dust, the third order solution corresponds to an expansion of the Tolman-Bondi solution in time. Finally, in Sec. VII, we give our conclusions and comment on the usefulness of the long wavelength approximation.

II. THE LONG WAVELENGTH ITERATION SCHEME

In this section we first rewrite Einstein's equations in a way convenient for our purposes and then describe the iteration procedure.

We place ourselves in a synchronous reference frame where the line element takes the form

$$ds^2 = -dt^2 + \gamma_{ij}(t,x^k)dx^i dx^j \qquad (i,j=1,2,3). \quad (2.1)$$

(Coordinate transformations involving four functions of space can still be performed without spoiling the synchronicity of the reference frame; see, e.g., [3].) Matter is taken to be a perfect fluid with pressure p, energy density ϵ, unit four-velocity u^μ ($\mu = 0,1,2,3$), and stress-energy tensor

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu} \quad (2.2)$$

with the further restriction that $p/\epsilon = \Gamma - 1$ where the index Γ is supposed to be constant, positive, and less than or equal to 2 (the limiting cases $\Gamma = 0$ and $\Gamma = 2$ correspond, respectively, to a cosmological constant and a "stiff" fluid whose speed of sound equals the speed of light; $\Gamma = 1$ is dust, $\Gamma = 4/3$ radiation; fluids with $0 < \Gamma < 2/3$ violate the strong energy condition and can be called "inflationary").

Einstein's equations are $R^\mu_\nu = S^\mu_\nu$ with $S^\mu_\nu \equiv \chi(T^\mu_\nu - \frac{1}{2}\delta^\mu_\nu T^\rho_\rho)$ and $\chi \equiv 8\pi G$, G being Newton's constant. In a synchronous reference frame the components of the Ricci tensor R^μ_ν are (see, e.g., [3])

$$R^0_0 = \tfrac{1}{2}\dot{\kappa} + \tfrac{1}{4}\kappa^i_j\kappa^j_i\ , \quad R^0_i = -\tfrac{1}{2}(\kappa^j_{i;j} - \kappa_{,i})\ ,$$
$$R^i_j = \tilde{R}^i_j + \tfrac{1}{2}\dot{\kappa}^i_j + \tfrac{1}{4}\kappa\kappa^i_j, \qquad (2.3)$$

where $\kappa_{ij} \equiv \dot{\gamma}_{ij}$ is the extrinsic curvature (an overdot denotes the derivative with respect to time t, a semicolon the covariant derivative with respect to γ_{ij}); all indices are raised with the inverse metric γ^{ij}; $\kappa \equiv \kappa^i_i$, and $\tilde{R}^i_j$ is the Ricci tensor associated with the metric γ_{ij}.

Now, one can always decompose $\kappa^i_j \equiv \gamma^{ik}\dot{\gamma}_{kj}$ into a trace and traceless part:

$$\kappa^i_j = 2H\delta^i_j + A^i_j/a^3 \qquad (2.4)$$

where the "anisotropy matrix" A^i_j is traceless ($A^i_i = 0$) and where we have introduced a local "scale factor" $a \equiv (\det\gamma_{ij})^{1/6}$, so that the local "Hubble parameter" H is $H \equiv \dot{a}/a$. When $a(t, x^k)$ and $A^i_j(t, x^k)$ are known the metric γ_{ij} is obtained by integrating the six linear equations

$$\dot{\gamma}_{ij} = 2H\gamma_{ij} + A^k_j\gamma_{ki}/a^3. \qquad (2.5)$$

(The matrix A^i_j is therefore such that $A^k_i\gamma_{kj} = A^k_j\gamma_{ki}$.)

Let us then rewrite Einstein's equations as equations for a and A^i_j. The traceless part of $R^i_j = S^i_j$ gives

$$\dot{A}^i_j = 2a^3(\bar{S}^i_j - \bar{R}^i_j) \qquad (2.6)$$

where $\bar{R}^i_j \equiv \tilde{R}^i_j - \frac{1}{3}\delta^i_j\tilde{R}$ and $\bar{S}^i_j \equiv \chi\epsilon\Gamma(u^iu_j - \frac{1}{3}\delta^i_ju^ku_k)$ are the traceless parts of $\tilde{R}^i_j$ and S^i_j.

The trace of $R^i_j = S^i_j$ together with the $\binom{0}{0}$ equation $R^0_0 = S^0_0$ give

$$2\dot{H} + 3\Gamma H^2 + \frac{||A||^2}{8a^6}(2-\Gamma) = \tfrac{1}{6}(2-3\Gamma)\tilde{R} - \tfrac{1}{3}\chi\epsilon(4-3\Gamma)\Gamma u^ku_k, \qquad (2.7)$$

$$\chi\epsilon(1+\Gamma u^ku_k) = 3H^2 - \frac{||A||^2}{8a^6} + \tfrac{1}{2}\tilde{R}, \qquad (2.8)$$

where $||A||^2 \equiv A^i_jA^j_i$.

Finally the $\binom{0}{i}$ equation $R^0_i = S^0_i$ reads

$$2\partial_iH - \frac{1}{2}\left(\frac{A^j_i}{a^3}\right)_{;j} = \chi\epsilon\Gamma u_i\sqrt{1+u^ku_k}. \qquad (2.9)$$

Equations (2.6)–(2.9) are strictly equivalent to Einstein's equations but are written in a form suitable for the implementation of the long wavelength iteration scheme. We shall also use the following consequence of Einstein's equations [obtained by differentiating (2.8) and using (2.6)–(2.9)]:

$$(1-\Gamma)\partial_i\epsilon = \Gamma\left\{D_j(\epsilon u^ju_i) + \frac{1}{a^3}\left[\epsilon u_ia^3\sqrt{1+u^ku_k}\right]^{\cdot}\right\}. \qquad (2.10)$$

The long wavelength approximation consists in neglecting all the terms quadratic in the gradients, that is, in the spatial derivatives, in Einstein's equations. Now, from Eq. (2.9) or (2.10) the three-velocity u_i is at least first order in the gradients so that the right-hand side of Eq. (2.6) is at least second order. At first order then it can be set equal to zero so that Eq. (2.6) gives that the anisotropy matrix A^i_j does not depend on time: $A^i_j \simeq {}^{(1)}A^i_j(x^k)$. Then Eq. (2.7), the right-hand side of which can be ignored at first order, is an equation which, when integrated, gives the scale factor ${}^{(1)}a(t, x^k)$. The anisotropy matrix and the scale factor being known, Eq. (2.5) yields the first order metric ${}^{(1)}\gamma_{ij}(t, x^k)$. Finally Eqs. (2.8) and (2.9) where at first order $\tilde{R}$ and u^ku_k can be ignored give the energy density ${}^{(1)}\epsilon(t, x^k)$ of the fluid as well as its three-velocity ${}^{(1)}u_i(t, x^k)$ [the three-velocity can equivalently be obtained from (2.10)]. This first order solution is given by Eardley, Liang, and Sachs [13], and Tomita [4] but only in the case $\Gamma = 0, 4/3$. It is reviewed in the next section.

III. THE GENERIC FIRST ORDER SOLUTION

In this section we give the general solution (see [13] and [4]) of the truncated Einstein equations (2.6)–(2.9) in which all terms of order greater than 1 in the spatial derivatives are neglected.

A. The anisotropy matrix

As already mentioned in Sec. II, Eq. (2.6) at lowest order reads $\dot{A}^i_j = 0$ and gives that the anisotropy matrix depends on space alone:

$$A^i_j = {}^{(1)}A^i_j(x^k) \quad \text{with} \quad {}^{(1)}A^i_i = 0. \qquad (3.1)$$

B. The scale factor

As for Eq. (2.7) for the scale factor a it reduces to

$$\dot{H} + 3\Gamma H^2 + \frac{||{}^{(1)}A||^2}{8a^6}(2-\Gamma) = 0, \qquad (3.2)$$

a first integral of which is readily obtained:

$$(a^3)^{\cdot} = \sqrt{\beta^4 + 4\tilde{a}^3a^{3(2-\Gamma)}} \qquad (3.3)$$

where $\tilde{a}(x^k)$ is an integration "constant" and where we have set $\beta^2 \equiv \left(3||{}^{(1)}A||^2/8\right)^{1/2}$. [We chose $\dot{a} > 0$ which will correspond to spacetimes emerging from a singularity. The collapsing situation $\dot{a} < 0$ is the time reversal of the solution presented here. As for $\tilde{a}$ it will have to be positive or zero: see Eq. (3.14) below.] Equation (3.3) can be explicitly integrated when the anisotropy matrix vanishes ($\beta^2 = 0$) or when matter is dust ($\Gamma = 1$), a radiation fluid ($\Gamma = 4/3$), or a stiff fluid ($\Gamma = 2$) (the particular case $\Gamma = 0$ is treated at the end of the section):

$$\Gamma = 1: \quad {}^{(1)}a^3 = u(\tilde{a}^3 u + \beta^2), \tag{3.4}$$

$$\Gamma = 4/3: \quad u = \frac{3\beta^4}{16\tilde{a}^{9/2}}\left[x\sqrt{x^2+1} - \ln\left(\sqrt{x^2+1}+x\right)\right]$$

$$\text{with} \quad x \equiv 2\tilde{a}^{\frac{3}{2}}({}^{(1)}a)/\beta^2, \tag{3.5}$$

$$\Gamma = 2: \quad {}^{(1)}a^3 = \bar{a}^3 u \quad \text{with} \quad \bar{a}^3 \equiv \sqrt{\beta^4 + 4\tilde{a}^3}, \tag{3.6}$$

where $u \equiv t - t_0(x^k)$ with $t_0(x^k)$ an integration "constant." For a general $0 < \Gamma < 2$ (and $\beta^2 \neq 0$) an approximate solution for small u is

$$0 < \Gamma < 2, \quad {}^{(1)}a^3 = \beta^2 u\left[1 + \tilde{c}u^{2-\Gamma} + O\left(u^{4-2\Gamma}\right)\right]$$

$$\text{with} \quad \tilde{c} \equiv \frac{2\tilde{a}^3}{3-\Gamma}\beta^{-2\Gamma}. \tag{3.7}$$

C. The metric

The anisotropy matrix and the scale factor being known, Eq (2.5) then gives the metric ${}^{(1)}\gamma_{ij}(t, x^k)$. The fact that $A^k_i\gamma_{jk} = A^k_j\gamma_{ik}$ implies that the matrix A^i_j is self-adjoint with respect to the metric γ_{ij}. Therefore ${}^{(1)}A^i_j$ is diagonalizable in an orthogonal basis (with respect to the metric γ_{ij}) Let us denote by $r_{(a)}(x^k)$ its three eigenvalues and by e_a three associated independent eigenvectors with components $e^i_a(x^k)$ that we shall normalize to unity. The triad e^a_i forms a basis of the tangent space. Defining the cotriad e^a_i by $e^a_i e^i_b = \delta^a_b$, the components of any tensor on this new basis are obtained by contracting its components in the coordinate basis with the triad or cotriad. In particular we define $\eta_{ab} \equiv \gamma_{ij}e^i_a e^j_b$ and $\kappa_{ab} \equiv \kappa_{ij}e^i_a e^j_b$. Since the triad is time independent the relation $\kappa_{ab} = \dot{\eta}_{ab}$ holds and Eq. (2.5) becomes $\dot{\gamma}_{ab} = r_{(a)}\gamma_{ab}/a^3$ with $\gamma_{ab} \equiv \eta_{ab}/a^2$, and where η_{ab} is diagonal. Hence we get the solution for the metric:

$${}^{(1)}\gamma_{ij} = e^a_i e^b_j {}^{(1)}\eta_{ab} \tag{3.8}$$

with

$${}^{(1)}\eta_{ab} = \delta_{ab}a^2 \exp\left(r_{(a)}\int dt a^{-3}\right).$$

The tracelessness of A^i_j implies that $\sum r_{(a)} = 0$; we also have $\sum r^2_{(a)} = ||{}^{(1)}A||^2 = 8\beta^4/3$. Explicit integration of Eq. (3.8) with a given by Eqs. (3.4), (3.6), and (3.7) gives

$$\Gamma = 1, \quad {}^{(1)}\eta_{ab} = \delta_{ab}C_{(a)}\beta^{2r_{(a)}/\beta^2}u^{2/3+r_{(a)}/\beta^2} \times(u\tilde{a}^3 + \beta^2)^{2/3-r_{(a)}/\beta^2}, \tag{3.9}$$

$$\Gamma = 2, \quad {}^{(1)}\eta_{ab} = \delta_{ab}\bar{a}^2 C_{(a)}u^{2/3+r_{(a)}/\bar{a}^3}, \tag{3.10}$$

$$0 < \Gamma < 2, \quad {}^{(1)}\eta_{ab} = \delta_{ab}\beta^{4/3}C_{(a)}u^{2/3+r_{(a)}/\beta^2} \times\left[1 + \tilde{c}\left(\frac{2}{3} - \frac{r_{(a)}}{\beta^2(2-\Gamma)}\right)u^{2-\Gamma} + O\left(u^{4-2\Gamma}\right)\right], \tag{3.11}$$

where $C_{(a)}(x^k)$ are three integration "constants." We have that

$$\det\gamma_{ij} = a^6 = \det{}^2(e^a_i)\det\eta_{ab} = \det{}^2(e^a_i)C_{(1)}C_{(2)}C_{(3)}a^6,$$

and therefore $\det^2(e^a_i)C_{(1)}C_{(2)}C_{(3)} = 1$. We note that when expanding the metric (3.9) in small u, one finds the metric (3.11) for $\Gamma = 1$. We also note that at leading order the metric (3.11), and therefore, because of the previous remark, also the metric (3.9), is Kasner-like. Indeed, setting $p_{(a)} = 1/3 + r_{(a)}/2\beta^2$, it reads ${}^{(1)}\eta_{ab} \propto u^{2p_{(a)}}$ with $\sum p_{(a)} = \sum p^2_{(a)} = 1$. However, it is important to note that the metric (3.10) for $\Gamma = 2$ is *not* Kasner-like, since the sum of the $p^2_{(a)}$ (with the definition $p_{(a)} = 1/3 + r_{(a)}/2\bar{a}^3$) is less than 1 but one has still $\sum p_{(a)} = 1$.

Finally, we should mention that the previous calculations are only valid for $\beta^2 \neq 0$. If $\beta^2 = 0$ then the anisotropy matrix vanishes, and the integration of Eq. (2.5) is obvious. The metric is quasi-isotropic and reads

$${}^{(1)}\gamma_{ij} = a^2(t)h_{ij}(x^k), \tag{3.12}$$

where h_{ij} is an arbitrary "seed" metric that depends only on space and the scale factor must be taken from Sec. III B depending on which type of matter one considers. This particular case of a quasi-isotropic metric was studied in detail in our previous paper [9].

D. The energy density

As for the energy density it is given by Eq. (2.8) which reduces to

$$\chi\epsilon = 3H^2 - ||{}^{(1)}A||^2/8a^6; \tag{3.13}$$

that is, using (3.3),

$$\chi\epsilon = \frac{4}{3}\tilde{a}^3 a^{-3\Gamma}. \tag{3.14}$$

(The positivity of ϵ implies that $\tilde{a}$ has to be positive.) For ${}^{(1)}a$ given by Eqs. (3.4), (3.6), and (3.7), (3.14) yields

$$\Gamma = 1, \quad \chi^{(1)}\epsilon = \frac{4}{3u(u + \beta^2/\tilde{a}^3)}, \tag{3.15}$$

$$\Gamma = 2, \quad \chi^{(1)}\epsilon = \frac{4\tilde{a}^3}{4\tilde{a}^3 + \beta^4}\frac{1}{3u^2}, \tag{3.16}$$

$0 < \Gamma < 2$,

$$\chi^{(1)}\epsilon = \frac{4\tilde{a}^3}{3}\beta^{-2\Gamma}u^{-\Gamma}\left[1 - \tilde{c}\Gamma u^{2-\Gamma} + O\left(u^{4-2\Gamma}\right)\right], \quad (3.17)$$

and one notices that the surface $u = 0 \Leftrightarrow t = t_0(x^k)$ is a singular surface of infinite density. [Note also that if, in the general case $0 < \Gamma < 2$, Eq. (3.14) yields (3.17), Eq. (3.13) only gives the leading part in $\chi\epsilon$.]

E. The three-velocity

As for the three-velocity at first order it follows from (2.9). However, Eq. (2.10), which at first order in the gradients reads

$$\tilde{a}^3 a^{(3-3\Gamma)}\Gamma u_i = (1-\Gamma)\int dt\, a^3\partial_i(\tilde{a}^3 a^{-3\Gamma}) + C_i, \quad (3.18)$$

gives us the behavior of u_i without having to resort to the full expression for the metric. In the case of dust it tells us, for example, that u_i is a function of space alone, and for $0 < \Gamma < 2$ it gives

$$\begin{aligned} {}^{(1)}u_i = \partial_i t_0 + \tilde{C}_i u^{\Gamma-1}[1 + (\Gamma-1)\tilde{c}u^{2-\Gamma}] \\ + \frac{1-\Gamma}{\Gamma}\frac{\partial_i\tilde{c}}{\tilde{c}}\frac{u}{2-\Gamma}, \end{aligned} \quad (3.19)$$

with $\tilde{C}_i = C_i\tilde{a}^{-3}\beta^{2(\Gamma-1)}/\Gamma$. To determine the three "constants" $\tilde{C}_i(x^j)$ as functions of the constants $C_{(a)}$ appearing in the metric, the more complete Eq. (2.9) must be used; their explicit expressions in the case of spherical symmetry will be given in Sec. VII.

F. Genericity

Let us now examine the genericity of the metric thus obtained. It depends on the following 12 arbitrary functions: ${}^{(1)}A^i_k(x^k)$ (eight functions) [or, equivalently, $r_{(a)}(x^k)$ (two functions) and $e^i_a(x^k)$ (six functions)]; $\tilde{a}(x^k)$ and $t_0(x^k)$; and the two functions $C_{(a)}(x^k)$.

Now four of these 12 functions can in principle be fixed by choosing a particular synchronous reference frame (see Sec. VII for an explicit implementation of such a gauge fixing in the case of spherical symmetry). One sees in particular that the reference frame can be chosen in such a way that the surface of infinite density is $t = 0$, that is, one can choose t_0 to be zero. Indeed, in an infinitesimal change of coordinates, $\tilde{t} = t + T$ and $\tilde{x}^i = x^i + X^i$ with $T = T(x^i)$ to preserve synchroniticity, the three-velocity transforms as $\tilde{u}_i = \frac{\partial t}{\partial \tilde{x}^i}u_0 + \frac{\partial x^j}{\partial \tilde{x}^i}u_j \simeq -\partial_i T u_0 + (\delta^j_i - \partial_i X^j)u_j \simeq u_i - \partial_i T$ if $|u_i| \ll 1$. Therefore the three-velocity can be set equal to zero locally by an appropriate choice of coordinates if it depends on space only, which is the case for dust. In all other cases this freedom of gauge can be used to set $\partial_i t_0 = 0$ as can be seen from Eq. (3.19).

The metric ${}^{(1)}\gamma_{ij}$ therefore depends on $12-4 = 8$ physically distinct arbitrary functions of space corresponding to the four degrees of freedom of gravity (the two gravitons) and the four degrees of freedom of a fluid (ϵ and u^i). It is therefore generic.

G. Late time limit

The scale factor being an increasing function of time, Eq. (3.3) tells us that when a is large the anisotropy β^2 becomes negligible (unless $\Gamma = 2$) and the scale factor tends to its Friedmann-Robertson-Walker value $a \propto u^{2/3\Gamma} \simeq t^{2/3\Gamma}$ (t_0 can be neglected for large t). Since then $\int dt a^{-3} \propto t^{(\Gamma-2)/\Gamma} \to 0$, Eq. (3.8) tells us that ${}^{(1)}\gamma_{ij}$ tends to a "quasi-isotropic" metric: ${}^{(1)}\gamma_{ij} \to t^{4/3\Gamma}h_{ij}(x^k)$ where $h_{ij}(x^k)$ is a "seed" metric depending on three physically distinct arbitrary functions. Five physical degrees of freedom are therefore diluted away: the traceless part of the intrinsic curvature and the epoch of the big bang (in the particular case $\Gamma = 2$, the metric does not become quasi-isotropic at late times and only the epoch of the big bang is lost). The quasi-isotropic scheme developed within a Hamilton-Jacobi framework by Salopek and Stewart and collaborators [7] and along the lines presented here in [9], which consists in iterating Einstein's equations starting from the restricted "seed" ${}^{(1)}\gamma_{ij} = t^{4/3\Gamma}h_{ij}(x^k)$, is therefore justified far away from a spacetime singularity. On the other hand, near a singularity, the full first order metric must be taken as starting point.

H. The case of vacuum

The Einstein equations for vacuum can be derived from the general equations for a perfect fluid by imposing in (2.6)–(2.9) $\epsilon = 0$ and $S^i_j = 0$. Note that in this case all the terms in (2.7) proportional to Γ cancel because of (2.8) with $\epsilon = 0$.

In the vacuum case the constraint (2.8), that is, (3.14), gives $\tilde{a} = 0$ and Eq. (3.3) gives ${}^{(1)}a^3 = \beta^2 u$ so that the metric (3.8) reads $\eta_{ab} = \delta_{ab}\beta^{4/3}C_{(a)}u^{2p_{(a)}}$ with $p_{(a)} \equiv \frac{1}{3} + r_{(a)}/2\beta^2$. It is a Kasner-like metric since $\sum p_{(a)} = \sum p^2_{(a)} = 1$. [This incidentally shows that matter becomes negligible near $u = 0$: see Eq. (3.11).] The metric depends on 11 functions but (2.9) gives three additional constraints. Therefore the solution depends on eight functions, that is, four physical degrees of freedom, those of the gravitons, as it should in vacuum.

I. The case of a cosmological constant

In the particular case of a cosmological constant ($\Gamma = 0$), the integration of (3.3) gives

$$\begin{aligned} \beta^2 \neq 0: \quad {}^{(1)}a^3 &= \frac{\beta^2}{\sqrt{3\Lambda}}\sinh(\sqrt{3\Lambda}\,u), \\ \beta^2 = 0: \quad {}^{(1)}a^3 &= \exp(\sqrt{3\Lambda}\,u) \end{aligned} \quad (3.20)$$

with $\Lambda \equiv 4\tilde{a}^3/3$ and the metric (3.8) reads

$\beta^2 \neq 0:$

$$^{(1)}\eta_{ab} = \delta_{ab}\left(\frac{4\beta^4}{3\Lambda}\right)^{1/3} C_{(a)}(\sinh\sqrt{3\Lambda/4}\,u)^{2/3+r_{(a)}/\beta^2} \times(\cosh\sqrt{3\Lambda/4}\,u)^{2/3-r_{(a)}/\beta^2},$$

$$\beta^2 = 0: \quad ^{(1)}\gamma_{ij} = e^{\frac{\sqrt{\Lambda}}{3}t}h_{ij}(x^k). \tag{3.21}$$

(In the case $\beta^2 = 0$ the integration constant t_0 can be absorbed in the seed metric h_{ij}.) Eq. (2.8) becomes a definition of the energy density:

$$\chi\, ^{(1)}\epsilon = \Lambda = \text{const} \tag{3.22}$$

and (2.10) says that Λ is a true constant, independent of space, and hence is not a true degree of freedom. The metric therefore depends on 11 functions, three of which disappear when the constraint (2.9) is imposed. In that case then, as in vacuum, the solution depends on four physical degrees of freedom, as it should.

IV. CONDITIONS OF VALIDITY

The purpose of this section is to establish in which situations the approximation scheme developed in the two previous sections is valid and which remedy to give in the cases where it is not. To check the validity of the approximation scheme, we simply compare the third order terms arising from the first iteration, which were ignored up to now, with the first order terms that we have just calculated. We consider the scheme to be valid if the third order terms remain small with respect to the first order ones. In the general case this of course does not guarantee the convergence of the whole series. However, in the particular case of spherical symmetry where an exact solution is known (cf. Sec. V) one can check that the difference between the LWAS and the corresponding exact solution tends to zero when u tends to zero. Far from a spacetime singularity when the first order metric reduces to its quasi-isotropic component this was already done in [7] and [9] with the conclusion that the next orders tend to zero as time increases if matter violates the strong energy condition, i.e., if the fluid is "inflationary." On the other hand, near a singularity where the anisotropy matrix cannot be ignored, the next order, as we shall see below, blows up generically as one approaches the singularity, whatever equation of state matter satisfies, and we shall recover the BKL oscillatory behavior for the metric.

Let us begin with the most tiresome part: the computation of the Ricci tensor built from the first order three-dimensional metric. We shall here compute the Ricci tensor for a general metric of the form

$$\gamma_{ij} = e^a_i e^b_j \eta_{ab}, \tag{4.1}$$

where the triad e^a_i depends only on the spatial coordinates whereas the metric η_{ab} depends on both spatial coordinates and time. The situation is therefore more complicated than if η_{ab} were only time dependent as is the case in the BKL analysis. However, we proceed along similar lines.

Let us first introduce the Ricci rotation coefficients (see [14]) defined by

$$\gamma_{abc} \equiv e_{(a)i;k}e^i_{(b)}e^k_{(c)}, \tag{4.2}$$

and their commutator

$$\lambda_{abc} \equiv \gamma_{abc} - \gamma_{acb}, \tag{4.3}$$

which have the property

$$\gamma_{abc} = -\gamma_{bac} + \partial_c\eta_{ab}, \tag{4.4}$$

thus enabling us to express the Ricci rotation coefficients in terms of their commutators:

$$\gamma_{abc} = \tfrac{1}{2}\left[\lambda_{abc} + \lambda_{bca} - \lambda_{cab} + \partial_c\eta_{ab} + \partial_b\eta_{ac} - \partial_a\eta_{bc}\right]. \tag{4.5}$$

It is important to rewrite the commutators λ_{abc} in the form

$$\lambda_{abc} = \eta_{ad}\mu^d{}_{bc} + \partial_c\eta_{ab} - \partial_b\eta_{ac}, \tag{4.6}$$

where

$$\mu^d{}_{bc} \equiv e^i_b e^k_c\left(\partial_k e^d_i - \partial_i e^d_k\right) \tag{4.7}$$

are *time independent.* In order to make the link with the terminology of BKL it is worth noticing that these coefficients can be rewritten in the form

$$\mu^a{}_{bc} = -\frac{1}{\det(e)}\epsilon_{dbc}\vec{e}^{\,d}\cdot\vec{\nabla}\times\vec{e}^{\,a}. \tag{4.8}$$

It is then straightforward to compute the components of the three-dimensional Riemann tensor in the nonholonomic basis. One finds

$$R_{abcd} = \eta_{ec}\eta_{fb}\partial_a\gamma_d{}^{ef} - \eta_{ec}\eta_{fa}\partial_b\gamma_d{}^{ef} + \gamma_{deb}\gamma^e{}_{ca} - \gamma_{dea}\gamma^e{}_{cb} + \gamma_{dce}\gamma^e{}_{ba} - \gamma_{dce}\gamma^e{}_{ab}. \tag{4.9}$$

Therefore the components of the Ricci tensor are

$$R_{ab} = \eta_{eb}\partial_a\gamma_c{}^{ec} - \eta^{cd}\eta_{eb}\eta_{fa}\partial_c\gamma_d{}^{ef} + \gamma^c{}_{ec}\gamma^e{}_{ba} - \gamma^c{}_{be}\gamma^e{}_{ac}. \tag{4.10}$$

In terms of the functions μ_{abc}, the Ricci rotation coefficients can be expressed as

$$\gamma_{abc} = \tfrac{1}{2}\left[\mu_{abc} + \mu_{bca} - \mu_{cab} + \partial_c\eta_{ab} + \partial_a\eta_{bc} - \partial_b\eta_{ac}\right] \tag{4.11}$$

so that the explicit expression of the Ricci tensor in terms of the triad and of the metric η_{ab} is given by

$$R_{ab} = \partial_a \mu^c{}_{bc} - \tfrac{1}{2}\partial_c \mu^c{}_{ba} - \tfrac{1}{2}\eta_{eb}\eta^{cd}\partial_c \mu^e{}_{ad} + \tfrac{1}{2}\eta_{ea}\eta^{cd}\partial_c\mu^e{}_{db}$$
$$+\tfrac{1}{2}\mu^c{}_{ec}\left(\mu^e{}_{ba} + \mu_{ba}{}^e - \mu_a{}^e{}_b\right) - \tfrac{1}{4}\left(\mu^c{}_{ae} + \mu_{ae}{}^c - \mu_e{}^c{}_a\right)\left(\mu^e{}_{bc} + \mu_{bc}{}^e - \mu_c{}^e{}_b\right)$$
$$+\tfrac{1}{2}\mu^c{}_{ec}\eta^{ed}\left(\partial_d\eta_{ab} - \partial_a\eta_{bd} - \partial_b\eta_{ad}\right) - \tfrac{1}{4}\mu^e{}_{ba}\eta^{cf}\partial_e\eta_{cf} + \tfrac{1}{4}\left(\mu_{ba}{}^e - \mu_a{}^e{}_b\right)\eta^{cf}\left(2\partial_f\eta_{ec} - \partial_e\eta_{fc}\right)$$
$$+\tfrac{1}{2}\mu^{cd}{}_a\left(2\partial_c\eta_{bd} + \partial_d\eta_{bc} - \partial_b\eta_{cd}\right) + \tfrac{1}{2}\mu_a{}^{cd}\partial_d\eta_{bc} + \tfrac{1}{2}\mu^{fe}{}_b\left(\partial_e\eta_{af} - \partial_a\eta_{ef}\right) - \tfrac{1}{2}\mu_b{}^{fe}\partial_f\eta_{ea}$$
$$+\tfrac{1}{4}\eta^{ed}\eta^{cf}\left(\partial_d\eta_{ab} - \partial_b\eta_{ad} - \partial_a\eta_{bd}\right)\left(2\partial_f\eta_{ec} - \partial_e\eta_{cf}\right) + \tfrac{1}{4}\eta^{ed}\eta^{cf}\partial_a\eta_{ef}\partial_b\eta_{cd}$$
$$+\tfrac{1}{2}\eta^{cd}\eta^{ef}\left(\partial_c\eta_{ae}\partial_d\eta_{bf} - \partial_c\eta_{be}\partial_f\eta_{da}\right) + \tfrac{1}{2}\eta^{cd}\left(\partial_a\partial_d\eta_{bc} - \partial_a\partial_b\eta_{dc} - \partial_c\partial_d\eta_{ab} + \partial_b\partial_c\eta_{ad}\right). \quad (4.12)$$

In addition to the curvature terms we have also ignored, in the first order approximation, the terms quadratic in the three-velocity. The approximate three-velocity that follows from the first order metric according to (2.9) is given in the new basis by

$$\chi\epsilon\Gamma u_a \simeq -\frac{1}{2}\left(\partial_b\kappa^b_a + \kappa^b_d\gamma^d{}_{ab} - \kappa^d_a\gamma^b{}_{db} - \partial_a\kappa\right), \quad (4.13)$$

where we have used the relation

$$\kappa^j_{i;k}e^i_a e^b_j e^k_c = \partial_c\kappa^b_a + \kappa^b_d\gamma^d{}_{ac} - \kappa^d_a\gamma^b{}_{dc}. \quad (4.14)$$

We have now to analyze the time dependence of these two expressions giving the Ricci tensor and the three-velocity and find the dominant term, i.e., that with the smallest power in u since we are heading towards the singularity. To do that we shall assume that (1) the new metric is diagonal, i.e., $\eta_{ab} = \eta_a\delta_{ab}$ (this amounts to supposing that the anisotropy matrix is diagonalizable) and (2) the spatial derivative of any component of the metric has the same time behavior as the component itself (this means in particular that we assume that t_0 is independent of space, which, as we saw, can be the case in an appropriate reference frame, and that we also ignore the logarithmic corrections that arise from the spatial derivative of the exponents). Inspection of the above expression then shows that all the terms in R_{ab} can be classified in one of the following categories, as far as their time behavior is concerned: constant (or logarithmic), η_a/η_c, η_b/η_c, η_c/η_d, and $\eta_a\eta_b/\eta_c\eta_d$, where a and b are fixed but c and d range from 1 to 3. We shall give here only the explicit expression for the three-velocity:

$$\chi\epsilon\Gamma^{(1)}u_a \simeq -\tfrac{1}{2}\left[\sum_{b\neq a}\frac{\dot\eta_b}{\eta_b}\gamma^b{}_{ab} - \frac{\dot\eta_a}{\eta_a}\gamma^b{}_{ab} - \sum_{b\neq a}\partial_a\left(\frac{\dot\eta_b}{\eta_b}\right)\right], \quad (4.15)$$

with

$$\gamma^b{}_{ab} = \mu^b{}_{ab} + \tfrac{1}{2}\left(\partial_b\eta_{ac} + \partial_c\eta_{ab} - \partial_a\eta_{bc}\right). \quad (4.16)$$

A. General case

Now, as was shown in Sec. III, the generic behavior of the first order metric near the singularity is of Kasner type, when $0 < \Gamma < 2$. Let us label the coordinates in such a way that $p_1 < p_2 < p_3$. We know that $-1/3 < p_1 < 0$, $0 < p_2 < 2/3$, and $2/3 < p_3 < 1$ (see [3]). In $\tilde R_{11}$, the dominant term then is

$$\tfrac{1}{2}\left(\mu^1{}_{23}\right)^2\eta_1^2/\eta_2\eta_3 \quad (4.17)$$

(there is no term of the form η_1^2/η_3^2 because of the antisymmetry of $\mu^a{}_{bc}$ in the two last indices). In $\tilde R_{22}$, the dominant term is

$$-\tfrac{1}{2}\left(\mu^1{}_{23}\right)^2\eta_1/\eta_3 \quad (4.18)$$

whereas the dominant term in $\tilde R_{33}$ is

$$-\tfrac{1}{2}\left(\mu^1{}_{23}\right)^2\eta_1/\eta_2. \quad (4.19)$$

As for the dominant contribution in the crossed terms, $\tilde R_{12}$, $\tilde R_{23}$, and $\tilde R_{31}$, it is more complicated since there are several terms involved. Therefore we quote only the time dependence:

$$\tilde R_{12} \sim \eta_1/\eta_3, \qquad \tilde R_{23} \sim \text{const}, \qquad \tilde R_{31} \sim \eta_1/\eta_2. \quad (4.20)$$

The dominant term in the scalar three-curvature $\tilde R$ is

$$-\tfrac{1}{2}\left(\mu^1{}_{23}\right)^2\frac{\eta_1}{\eta_2\eta_3}. \quad (4.21)$$

The three-curvature thus behaves as a power law:

$$\tilde R \sim u^{2(p_1-p_2-p_3)} \sim u^{4p_1}u^{-2}. \quad (4.22)$$

As we see the dominant terms always come from the cross product of the μ_{abc} so that the terms with the spatial derivatives of the metric η_{ab} do not play a role near the singularity. One can therefore expect that we will recover the results obtained by BKL who started their analysis on a Bianchi type-IX model where the metric η_{ab} is only time dependent.

The time behavior of the Ricci tensor being now known, let us see if we were allowed to neglect it. An analysis of the Einstein equations (2.6), (2.7), and (2.8) shows first that $u^2\tilde R^j_i$ and $\tilde R u^2$ must be convergent for the approximation to be valid. This is *not* the case if the dominant terms are those listed above because p_1 is negative. Indeed $u^2\tilde R$ is divergent as well as $u^2\tilde R^3_1$ and $u^2\tilde R^2_1$.

As for the term containing the three-velocity in Eq. (2.7), it varies as

$$u^2(\epsilon\Gamma u^a u_a) \sim u^{\Gamma-2p_3}. \quad (4.23)$$

Therefore this term is convergent only if $\Gamma > 2p_3$. Moreover, since $-1/3 < p_1 < 0$ and $2/3 < p_3 < 1$, one can see that when this term diverges it can be either more ($\Gamma < 4p_1+2p_3$) or less divergent than the curvature term.

The conclusion therefore is that the long wavelength approximation scheme breaks down in the general case when approaching the singularity.

B. Case $\mu^1{}_{23} = 0$

In the particular case where

$$\mu^1{}_{23} = 0, \tag{4.24}$$

all the dominant contributions listed above vanish and the validity of the scheme must be reconsidered. In that case the dominant time dependences are

$$\tilde{R}_{11} \sim \eta_1/\eta_3, \quad \tilde{R}_{22} \sim \eta_2/\eta_3, \quad \tilde{R}_{33} \sim \text{const}, \tag{4.25}$$

$$\tilde{R}_{12} \sim \eta_2/\eta_3, \quad \tilde{R}_{23} \sim \text{const}, \quad \tilde{R}_{31} \sim \text{const}. \tag{4.26}$$

Therefore

$$\tilde{R} \sim 1/\eta_3 \sim u^{-2p_3}. \tag{4.27}$$

Knowing that $2/3 < p_3 < 1$, one can conclude that $\tilde{R}u^2$ is then always convergent. In a similar manner one sees that all the quantities $u^2\tilde{R}_i^j$ are convergent.

Let us now look at the term quadratic in the velocity. The time dependence of this term remains the same as in the general case. Therefore the approximation of Eq. (2.7) is valid only (except the case $\Gamma = 4/3$) if $\Gamma > 2p_3$ or in the case where $u_3 = 0$ if $\Gamma > 2p_2$ or $u_2 = 0$, these conditions being rather restrictive.

This is, however, not yet the end of the story. The analysis of Eq. (2.8) imposes two further conditions.

(1) $u^a u_a$ must be small with respect to 1, which implies that $\Gamma > 1 + p_a$ for all a unless u_a vanishes.

(2) $\tilde{R}$ must be negligible with respect to the energy density ϵ: this is due to the fact, already mentioned, that the two first terms on the right-hand side compensate at leading order. The condition is therefore, in view of Eqs. (3.15)–(3.17), that $u^\Gamma \tilde{R}$ must be convergent. Therefore one must have $\Gamma > 2p_3$.

Note that the condition $\Gamma > 1+p_3$ implies all the other conditions, but this is very restrictive in general since p_3 is limited from above only by 1.

To summarize, we find that the first order solution given in Sec. III is a good approximation to a solution of Einstein's equations near a spacetime singularity, if the conditions $\mu^1{}_{23} = 0$, $\Gamma > 2p_3$, and $\Gamma > 1 + p_a$ (for all $a = 1, 2, 3$ unless $u_a = 0$) are satisfied. Diagonal anisotropy matrices form an important subclass of matrices satisfying the first condition and we shall see that in the context of spherical symmetry, studied in detail in the last section, the other conditions are also fulfilled for $\Gamma = 1$ or for $\Gamma > 4/3$. Now imposing (4.24) renders the first order solution nongeneric as it then depends on seven instead of eight physically distinct arbitrary functions. However, a qualitative analysis of what happens when $\mu^1{}_{23} \neq 0$ can be given, which follows closely the work of BKL.

C. BKL oscillatory behavior

We now consider the Einstein equations (2.6) and (2.7) where we do *not* neglect the curvature terms any longer. The time derivative of Eq. (2.4), after use of (2.7) and (2.8) and expressed in the new basis, gives

$$a^{-3}\left(a^3\kappa^a_b\right)^{\cdot} = (2-\Gamma)\chi\epsilon\delta^a_b + 2\chi\epsilon\Gamma u^a u_b - 2\tilde{R}^a_b. \tag{4.28}$$

As shown in Sec. III, the energy density ϵ varies as $u^{-\Gamma}$. We now assume that the curvature term evolving as u^{4p_1-2} is dominant over the velocity term, evolving as $u^{\Gamma-2-2p_3}$ (at worst). Assume moreover that the metric remains diagonal in the evolution and that at some time it is Kasner-like. In these conditions we are able to recover the behavior discovered by BKL, initially in the case of Bianchi type-IX and extended later to inhomogenous situations.

Following BKL, let us introduce a new time defined by

$$\tau = \ln u, \tag{4.29}$$

and let us write the metric in the form

$$[\eta_{ab}] = \text{Diag}[e^{2\alpha}, e^{2\beta}, e^{2\gamma}]. \tag{4.30}$$

Then keeping in (4.28) only the dominant contributions from the curvature term, we get the three following equations governing the coefficients of the diagonal metric:

$$\partial^2_\tau \alpha = -\tfrac{1}{2}\left(\mu^1{}_{23}\right)^2 e^{4\alpha}, \quad \partial^2_\tau \beta = \tfrac{1}{2}\left(\mu^1{}_{23}\right)^2 e^{4\alpha},$$

$$\partial^2_\tau \gamma = \tfrac{1}{2}\left(\mu^1{}_{23}\right)^2 e^{4\alpha}. \tag{4.31}$$

When the metric is Kasner-like, one has

$$\partial_\tau \alpha = p_1, \qquad \partial_\tau \beta = p_2, \qquad \partial_\tau \gamma = p_3. \tag{4.32}$$

The equation for α is similar to the equation for a particle with coordinate α moving in an exponential potential. Initially the particle moves with a constant velocity $\partial_\tau \alpha = p_1$. After reflection on the potential wall, the particle will move with the velocity $\partial_\tau \alpha = -p_1$. The two other equations then give the two other final velocities p_1: $\partial_\tau \beta = p_2 + 2p_1$ and $\partial_\tau \gamma = p_3 + 2p_1$. Therefore the initial Kasner-like metric evolves into another Kasner-like metric due to the influence of the curvature terms, given by

$$\eta_{ab} \sim \text{Diag}[u^{\frac{-p_1}{1+2p_1}}, u^{\frac{p_2+2p_1}{1+2p_1}}, u^{\frac{p_2+2p_1}{1+2p_1}}]. \tag{4.33}$$

We thus recover from our general analysis the oscillatory behavior between Kasner-like metrics, behavior which was studied in detail by BKL (see, e.g., [3]).

Let us conclude this section with the particular cases of the vacuum, a cosmological constant, and a stiff fluid. In the case of vacuum, one has still a Kasner-like metric as was shown in the Sec. III H. Therefore the above analysis applies without modification. The same conclusion arises from the cosmological constant case with anisotropy since the metric (see Sec. III I), when $u \to 0$ has the same

Kasner-like behavior as the metric (3.11). However, the case $\Gamma = 2$ gives a qualitatively different result. Indeed, there is more freedom for the coefficients $p_{(a)}$ and it is then possible that p_1, the smallest of the three powers, be positive, in which case all the terms $u^2\tilde{R}_i^j$ converge. Note that the energy equation is valid also only in the case where $p_1 > 0$. BLK showed that in the case of stiff matter (or scalar field) the oscillating behavior (if there exists one at the beginning, i.e., if p_1 is negative) will end after a few oscillations when one goes backwards in time. This means that, sufficiently near the singularity, the approximation scheme works.

V. THE GENERIC THIRD ORDER SOLUTION

Let us first rewrite the Einstein equations (2.6)–(2.9) in the new basis:

$$\dot{A}_b^a = 2a^3(\bar{S}_b^a - \bar{R}_b^a), \tag{5.1}$$

$$2\dot{H} + 3\Gamma H^2 + \frac{||A||^2}{8a^6}(2-\Gamma)$$
$$= \tfrac{1}{6}(2-3\Gamma)\tilde{R} - \tfrac{1}{3}\chi\epsilon(4-3\Gamma)\Gamma u^a u_a, \tag{5.2}$$

$$\chi\epsilon(1+\Gamma u^a u_a) = 3H^2 - \frac{||A||^2}{8a^6} + \tfrac{1}{2}\tilde{R}, \tag{5.3}$$

where $||A||^2 \equiv A_b^a A_a^b$ and

$$\chi\epsilon\Gamma u_a = -\frac{1}{2\sqrt{1+u^c u_c}}\left(\partial_b \kappa_a^b + \kappa_d^b \gamma^d{}_{ab} - \kappa_a^d \gamma^b{}_{db} - \partial_a \kappa\right). \tag{5.4}$$

In the previous sections we have only considered the first order approximation of the Einstein equations. We can now include third order corrections to the first order quantities:

$$a = {}^{(1)}a + {}^{(3)}a + \cdots, \quad A_b^a = {}^{(1)}A_b^a + {}^{(3)}A_b^a + \cdots. \tag{5.5}$$

These third order corrections follow from the approximate Einstein equations in which the terms that were ignored previously are now taken into account but are computed with the first order solution. Since the third order solution is supposed to be small with respect to the first order solution, one can linearize the Einstein equations and all the equations giving the third order terms will be linear ordinary differential equations. One sees that, in principle, one can repeat this procedure at any order and build iteratively the metric and the other quantities.

It is convenient to define the third order quantities

$$a_3 \equiv \frac{{}^{(3)}a}{{}^{(1)}a} \tag{5.6}$$

and

$$\mathbf{y}_3 \equiv {}^{(3)}\left(\frac{\mathbf{A}}{a^3}\right) = \frac{{}^{(3)}\mathbf{A}}{({}^{(1)}a)^3} - 3^{(1)}\left(\frac{\mathbf{A}}{a^3}\right)a_3, \tag{5.7}$$

where a boldfaced letter stands for a matrix. The third order correction to the extrinsic curvature, ${}^{(3)}\kappa_a^b$, can be decomposed, following (2.4), into

$${}^{(3)}\kappa_a^b = 2^{(3)}H\delta_a^b + (\mathbf{y}_3)_a^b, \tag{5.8}$$

where

$${}^{(3)}H = \dot{a}_3. \tag{5.9}$$

The expansion of the Einstein equations then gives

$${}^{(3)}A_b^a = 2\int dt\ {}^{(1)}a^3(\bar{S}_b^a - \bar{R}_b^a), \tag{5.10}$$

where the term under the integral is built from the first order solution and not from the exact solution as is the case in the exact Einstein equation (2.6). (Note also that the constant of integration that could arise from the above integral is supposed to be already included in the first order term ${}^{(1)}A_b^a$.) As for the equations for a and ϵ they become

$$2\ddot{a}_3 + 6\Gamma H\dot{a}_3 + \frac{2-\Gamma}{4}{}^{(1)}\left(\frac{\mathbf{A}}{a^3}\right)\cdot\mathbf{y}_3$$
$$= \frac{2-3\Gamma}{6}\tilde{R} - \frac{\Gamma(4-3\Gamma)}{3}\chi^{(1)}\epsilon u^a u_a. \tag{5.11}$$

The third order correction to the metric can be computed by expanding formula (2.4) (in the new basis). One finds

$${}^{(3)}\kappa_a^b = {}^{(3)}\dot{\eta}_{ac}\eta^{cb} - \dot{\eta}_{ac}\eta^{ce(3)}\eta_{de}\eta^{bd}. \tag{5.12}$$

Using the fact that the matrix $[\eta_{ab}]$ is diagonal,

$${}^{(1)}\eta_{ab} = \text{Diag}[\eta_c], \tag{5.13}$$

one finds

$${}^{(3)}\kappa_a^b = {}^{(3)}\dot{\eta}_{ab}\eta_b^{-1} - \dot{\eta}_a\eta_a^{-1(3)}\eta_{ab}\eta_b^{-1}, \tag{5.14}$$

where there is no summation on the indices. This equation can be integrated into

$${}^{(3)}\eta_{ab} = \eta_a \int dt \frac{\eta_b}{\eta_a}{}^{(3)}\kappa_a^b \tag{5.15}$$

and therefore

$${}^{(3)}\eta_{ab} = \eta_a \int dt \frac{\eta_b}{\eta_a}\left(2^{(3)}H\delta_a^b + (\mathbf{y}_3)_a^b\right). \tag{5.16}$$

One can then obtain the third order energy and velocity by inserting the metric ${}^{(1)}\eta_{ab} + {}^{(3)}\eta_{ab}$ in Eqs. (5.3) and (5.4) (taking for the quadratic term $u_a u^a$ the first order solution ${}^{(1)}u_a$). For instance, the third order energy density is given by

$$\chi^{(3)}\epsilon = 6^{(1)}H\dot{a}_3 - \frac{1}{4}{}^{(1)}\left(\frac{\mathbf{A}}{a^3}\right)\cdot \mathbf{y}_3 + \tfrac{1}{2}\tilde{R} \\ -\chi\Gamma^{(1)}\epsilon^{(1)}u^{a(1)}u_a. \quad (5.17)$$

VI. THE EXAMPLE OF SPHERICAL SYMMETRY

A. The equations

The line element of a spherically symmetric spacetime can be written in a suitable coordinate system (t, r, θ, ϕ) as

$$ds^2 = -dt^2 + \gamma_{rr}dr^2 + \gamma_{\theta\theta}d\Omega^2 \quad (6.1)$$

with

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2.$$

An infinitesimal change of coordinates $\tilde{t} = t + T, \tilde{r} = r + R$ preserves the synchronicity of the reference frame if $T = T(r)$ and $\dot{R} = T'/\gamma_{rr}$ (where a prime denotes a derivative with respect to r). It involves two arbitrary functions of space: $T(r)$ and the integration "constant" in the equation for R (see, e.g., [3]).

The extrinsic curvature $\kappa^i_j \equiv \gamma^{ik}\dot{\gamma}_{jk}$ is diagonal and therefore the matrix A^i_j in Eq. (4) is too, so that we are spared from the triad formalism of Secs. III–V. We shall give here the solution to third order in the gradients. This will illustrate the general discussion of the preceding sections, clarify the gauge issue, and allow a comparison with the known exact solution of Tolman and Bondi. This section is intended to be self-contained.

When the line element is (6.1) so that the traceless anisotropy matrix A^i_j in $\kappa^i_k \equiv 2H\delta^i_j + A^i_j/a^3$ (where $H \equiv \dot{a}/a$) is diagonal with eigenvalues $r_{(r)} = -A$, $r_{(\theta)} = r_{(\phi)} = A/2$, the Einstein's equations (2.6)–(2.9) for a perfect fluid yield

$$\gamma_{rr} = a^2\exp\left(-\int dt\,\frac{A}{a^3}\right), \quad \gamma_{\theta\theta} = a^2\exp\left[\tfrac{1}{2}\int dt\,\frac{A}{a^3}\right], \quad (6.2)$$

where A and a are given by

$$A = -2\int dt\, a^3(\bar{S} - \bar{R}) \quad (6.3)$$

with

$$\bar{S} = \frac{2\chi\epsilon}{3}\Gamma u^r u_r, \quad \bar{R} = \tilde{R}^r_r - \tfrac{1}{3}\tilde{R},$$

$$2\dot{H} + 3\Gamma H^2 + \frac{3}{16}\left(\frac{A}{a^3}\right)^2(2-\Gamma) \\ = \tfrac{1}{6}(2-3\Gamma)\tilde{R} - \tfrac{1}{2}(4-3\Gamma)\bar{S}. \quad (6.4)$$

As for the energy density and the radial velocity they are given by

$$\chi\epsilon(1+\Gamma u^r u_r) = 3H^2 - \frac{3}{16}\left(\frac{A}{a^3}\right)^2 + \tfrac{1}{2}\tilde{R}, \quad (6.5)$$

$$\chi\epsilon\Gamma u_r\sqrt{1+\Gamma u^r u_r} = 2H' + \tfrac{1}{2}\left(\frac{A}{a^3}\right)' + \tfrac{3}{4}\left(\frac{A}{a^3}\right)(\ln\gamma_{\theta\theta})'. \quad (6.6)$$

Simple counting gives that a generic metric solution of (6.2)–(6.4) depends on four arbitrary functions of r. The total number of physical degrees of freedom, however, is two (there are no gravitons in spherically symmetric spacetimes and the fluid is specified by its density and radial velocity). Two functions can therefore be eliminated by fixing the gauge, that is, choosing a particular synchronous reference frame, in agreement with the remark below Eq. (6.1). We can first give a geometrical meaning to the coordinate r by relating it to the surface of two-spheres: this will fix a function in $\gamma_{\theta\theta}$. To eliminate the remaining gauge freedom we note that, in an infinitesimal change of coordinates that preserves synchronicity, the radial velocity transforms as $\tilde{u}_r = \frac{\partial t}{\partial\tilde{r}}u_0 + \frac{\partial r}{\partial\tilde{r}}u_r \simeq -T'u_0 + \left(1-\frac{\partial R}{\partial r}\right)u_r \simeq u_r - T'$ if $u_r \ll 1$, so that an arbitrary function of space (T') can be subtracted from u_r.

B. The case of dust ($\Gamma = 1$)

At first order in the gradients Eq. (6.3) gives $A = A(r)$ and the solution of (6.4) is

$${}^{(1)}a^3 = \tilde{a}^3 u(u+\alpha) \quad (6.7)$$

with

$$u \equiv t - t_0 \quad \text{and} \quad \alpha \equiv \tfrac{3}{4}\frac{|A|}{\tilde{a}^3}$$

and where $\tilde{a}(r)$ and $t_0(r)$ are two integration "constants." The metric then follows from (6.2) [see Eq. (3.9)]:

$$\begin{aligned} {}^{(1)}\gamma_{rr} &= C_r u^{4/3}(1+\alpha/u)^{2/3(1+2\epsilon)}, \\ {}^{(1)}\gamma_{\theta\theta} &= C_\theta u^{4/3}(1+\alpha/u)^{2/3(1-\epsilon)}, \end{aligned} \quad (6.8)$$

where $\epsilon \equiv A/|A|$. From now on we shall consider only $\epsilon = 1$ since the metric, in the case $\epsilon = -1$, tends toward a metric of the Kasner type with the coefficients ($p_1 = 1, p_2 = 0, p_3 = 0$) which is nothing less than the flat metric as can be shown with a suitable change of coordinates (see [3]): therefore there is no singularity for $\epsilon = -1$. C_r and C_θ are two integration constants. The first order metric depends, as anticipated, on four functions of r: α, t_0, C_r, and C_θ, and two can be eliminated by fixing the gauge. To do that we first impose $C_\theta = r^2$, so that the radial velocity (6.6) becomes

$${}^{(1)}u_r = t_0' + \frac{3}{2r}\alpha, \quad (6.9)$$

which, as we already knew from (3.19) depends on space only and can be set equal to zero (indeed when matter is dust and hence follows geodesics there exists a synchronous reference frame where the particles remain at rest) by choosing $\alpha = -2t_0'r/3$.

The gauge being thus completely fixed, the first order spherical symmetric line element for dust finally reads

$$^{(1)}ds^2 = -dt^2 + \frac{\varrho'^2}{1-hr^2}dr^2 + \varrho^2 d\Omega^2 \qquad (6.10)$$

with the definitions $\varrho \equiv ru^{2/3}$, $u \equiv t - t_0$, and $C_r \equiv (1-hr^2)^{-1}$ (this form will be useful for a comparison with the exact solution of Tolman and Bondi). It depends on two arbitrary functions of r: h and t_0. The energy density follows from (6.5) [see Eq. (3.15)] and the radial velocity is zero:

$$\chi\, ^{(1)}\epsilon = \frac{4}{3u(u-2t_0'r/3)}, \quad ^{(1)}u_r = 0. \qquad (6.11)$$

Useful secondary quantities are

$$^{(1)}a^3 = \tilde{a}^3 u(u-2t_0'r/3), \quad ^{(1)}\left(\frac{A}{a^3}\right) = \frac{-8t_0'r}{9u(u-2t_0'r/3)}. \qquad (6.12)$$

To obtain the third order metric the easiest way is as follows. Writing $a = {}^{(1)}a(1+a_3)$ and $A/a^3 = {}^{(1)}(A/a^3) + y_3$, Eq. (6.2) gives

$$^{(1)}ds^2 + {}^{(3)}ds^2$$
$$= -dt^2 + \frac{\varrho'^2}{1-hr^2}(1+\gamma_r)dr^2 + \varrho^2(1+\gamma_\theta)d\Omega^2 \quad (6.13)$$

with $\gamma_r = 2a_3 - \int dt\, y_3$ and $\gamma_\theta = 2a_3 + \frac{1}{2}\int dt\, y_3$. Equation (6.4) for a_3 and y_3 is then transformed, by using the relation $\dot{y}_3 = -3Hy_3 - 3^{(1)}(A/a^3)\dot{a}_3 - 2(\bar{S} - \bar{R})$ which follows from (6.3), into an equation for γ_θ which eventually reads

$$\ddot{\gamma}_\theta + 3\dot{\gamma}_\theta\left[^{(1)}H + \tfrac{1}{4}{}^{(1)}(A/a^3)\right] - \bar{R} + \tfrac{1}{6}\tilde{R} = 0. \qquad (6.14)$$

With $^{(1)}H \equiv {}^{(1)}\dot{a}/a$ and $^{(1)}(A/a^3)$ given by (6.12), and the relevant components of the Ricci tensor for the line element (6.10) being

$$\bar{R} = \frac{(hr^2)'}{3\varrho\varrho'} - \frac{2hr^2}{3\varrho^2}, \quad \tilde{R} = \frac{2hr^2}{\varrho^2} + \frac{2(hr^2)'}{\varrho\varrho'}, \qquad (6.15)$$

Eq. (6.14) becomes $\ddot{\gamma}_\theta + 2\dot{\gamma}_\theta/u + hu^{-4/3} = 0$ which is readily integrated into

$$\gamma_\theta = -\frac{9}{10}hu^{2/3}. \qquad (6.16)$$

Turning then Eq. (6.4) into an equation for γ_r instead of γ_θ and using the relation $\dot{\gamma}_r = \dot{\gamma}_\theta - \frac{3}{2}y_3$ to eliminate y_3 one gets

$$\ddot{\gamma}_r + 3\dot{\gamma}_r\left[^{(1)}H - \tfrac{1}{4}{}^{(1)}(A/a^3)\right] + 2\bar{R}$$
$$+ \tfrac{1}{6}\tilde{R} - \tfrac{3}{4}\dot{\gamma}_\theta{}^{(1)}(A/a^3) = 0, \quad (6.17)$$

the integration of which yields

$$\gamma_r = -\frac{9}{10}(h+h'r)\frac{u^{5/3}}{u-2t_0'r/3} + \frac{6}{5}hrt_0'\frac{u^{2/3}}{u-2t_0'r/3}. \qquad (6.18)$$

At third order then, the spherically symmetric element for dust is (6.13) with γ_θ and γ_r given, respectively, by (6.16) and (6.18).

Now the exact solution for dust with spherical symmetry is known. It is the Tolman-Bondi solution, the line element of which can be written as (see, e.g., Ref. [3])

$$ds^2 = -dt^2 + \frac{\rho'^2}{1+f(r)}dr^2 + \rho^2 d\Omega, \qquad (6.19)$$

with

$$\rho = \frac{\mu}{2f}(\cosh\eta - 1), \quad t - t_0(r) = \frac{\mu}{2f^{3/2}}(\sinh\eta - \eta) \qquad (6.20)$$

for $f > 0$,

$$\rho = \frac{\mu}{-2f}(1-\cos\eta), \quad t - t_0(r) = \frac{\mu}{2(-f)^{3/2}}(\eta - \sin\eta) \qquad (6.21)$$

for $f < 0$, and

$$\rho = \frac{9\mu}{4}^{1/3}[t - t_0(r)]^{2/3} \qquad (6.22)$$

for $f = 0$. The line element is written in the comoving gauge where the three-velocity vanishes and the miscellaneous functions appearing in the metric have an easy physical interpretation: any particle is labeled by the coordinate r, the same at any time; $4\pi\rho^2(r,t)$ gives the area of the sphere containing this particle at time t; $\dot{\rho}(r,t)$ is the radial velocity of the particle; and $\mu(r)$ corresponds to the mass inside the sphere containing the particle.

We now consider the expansion of the Tolman-Bondi solution in the parameter $u = t - t_0(r)$, supposed to be small. We find, for ρ,

$$\rho = \left(\frac{3}{2}\right)^{2/3}\mu^{1/3}u^{2/3} + \frac{9}{20}\left(\frac{2}{3}\right)^{2/3}\mu^{-1/3}fu^{4/3}. \quad (6.23)$$

In the derivation of this expansion, we assume that u' is of the order of u. Going to the next to leading order in the expansion we thus find

$$\gamma_{rr} = \frac{1}{9}\left(\frac{3}{2}\right)^{4/3}\frac{(\mu u^2)^{2/3}}{1+f}\left[(\ln\mu u^2)'\right]^2$$
$$\times\left[1 + \frac{3fu^{2/3}}{5\mu^{2/3}}\left(\frac{2}{3}\right)^{1/3}(\ln\mu u^2)'\frac{(\ln f^3u^4/\mu)'}{(\ln\mu u^2)'}\right]. \qquad (6.24)$$

Identifying $f(r) = -hr^2$ and $\mu = 4r^3/9$, we recover the first and third orders (6.13), (6.16), and (6.18) given by the expansion scheme.

C. The general case ($0 < \Gamma < 2$)

The first order metric is obtained as before and, without going into details, it should be clear that it is given by (3.11), the index (a) being r and θ, with $r_{(r)} = -A = -4\epsilon\beta^2/3; r_{(\theta)} = r_{(\phi)} = A/2$ (again we shall consider only $\epsilon = +1$ since $\epsilon = -1$ does not describe a singularity). This metric depends on four functions of r: $C_r\beta^{4/3} = g(r)$, $C_\theta\beta^{4/3}$, t_0, and $\tilde{c}$ and two of those can be eliminated by particularizing the reference frame. As before we shall first choose $C_\theta\beta^{4/3} = r^2$. Then, from (3.19) and the discussion in Sec. III F, we know that t_0 can be chosen to be zero, and that fixes the gauge completely (note that when $\Gamma = 1$ it is not the comoving gauge chosen in the preceding paragraph). The first order line element therefore is (6.1) with

$$^{(1)}\gamma_{rr} = gt^{-2/3}\left[1 + \frac{2\tilde{c}}{3}\frac{(4-\Gamma)}{(2-\Gamma)}t^{2-\Gamma} + O\left(t^{4-2\Gamma}\right)\right], \tag{6.25}$$

$$^{(1)}\gamma_{\theta\theta} = r^2t^{4/3}\left[1 + \frac{2\tilde{c}}{3}\frac{(1-\Gamma)}{(2-\Gamma)}t^{2-\Gamma} + O\left(t^{4-2\Gamma}\right)\right], \tag{6.26}$$

which depends on two arbitrary functions: $g(r)$ and $\tilde{c}(r)$. As for the density and radial velocity they are given by (3.17) and (6.6):

$$\chi^{(1)}\epsilon = \tfrac{2}{3}(3-\Gamma)\tilde{c}t^{-\Gamma}\left[1 - \tilde{c}\Gamma t^{2-\Gamma} + O\left(t^{4-2\Gamma}\right)\right], \tag{6.27}$$

$$\begin{aligned}^{(1)}u_r \simeq{}& (\ln\tilde{c})'\frac{1-\Gamma}{\Gamma(2-\Gamma)}t + \frac{3}{\Gamma(3-\Gamma)}\frac{1}{r}\frac{t^{\Gamma-1}}{\tilde{c}}\\ &\times\left[1 + \tilde{c}(\Gamma-1)t^{2-\Gamma}\right].\end{aligned} \tag{6.28}$$

Let us now examine the conditions of validity of the approximation by determining the behavior of the third order terms. The analysis here is just a very particular case of the general discussion given in Sec. IV. Indeed we are in the case where $\mu^1{}_{23} = 0$, $u_2 = u_3 = 0$ and $p_1 = -1/3$, $p_2 = p_3 = 2/3$. Therefore we know that the curvature terms do not cause any trouble, that the condition $u_a u^a < 1$ implies $\Gamma > 2/3$, and finally that one must have $\Gamma > 4/3$ in order to ensure the validity of Eq. (6.5) giving the energy density. The reason that (6.27) fails to give the energy density when $\Gamma < 4/3$ is that the leading orders in (6.27) compensate (the Kasnerian metric is a vacuum solution), and that the subdominant term is superseded, when $\Gamma < 4/3$, by third order terms coming from the first iteration. Let us recover these results directly by determining the behavior of the third order correction.

At leading order the dominant term in the right-hand side (RHS) of (6.3) is $\bar{R} \simeq -\frac{2}{3r^2}t^{-4/3}$; as for $\bar{S}$ it remains negligible ($\bar{S} \propto t^{\Gamma-4/3}$). The first order scale factor being at leading order proportional to $t^{1/3}$ we have that the third order correction to A, built out of the leading part of the first order solution, is

$$^{(1)+(3)}\left(\frac{A}{a^3}\right) \simeq -\frac{4}{3t}\left(1 - \frac{3}{2r^2}t^{2/3}\right) \tag{6.29}$$

which has to be compared with $^{(1)}(A/a^3)$, built with the more accurate first order solution (6.25) and (6.26): that is,

$$^{(1)}\left(\frac{A}{a^3}\right) = -\frac{4}{3t}\left[1 - \tilde{c}t^{2-\Gamma} + O\left(t^{4-2\Gamma}\right)\right]. \tag{6.30}$$

We therefore see, in agreement with the general discussion of Sec. IV, that indeed the first order solution is a good approximation to a generic solution of Einstein's equations near a spacetime singularity up to *and including* terms in $t^{2-\Gamma}$ provided that $2/3 > 2 - \Gamma$, that is, $\Gamma > 4/3$. When $\Gamma < 4/3$ the metric (6.25) and (6.26) is still good at leading order near the singularity but the energy density cannot any longer be given by (6.27).

D. The case of stiff fluid ($\Gamma = 2$)

The first order metric is again obtained by particularizing the results of Sec. III to the case of spherical symmetry. The anisotropy matrix depends on r only: $^{(1)}A = A(r)$; the scale factor is given by (3.6): $^{(1)}a = \bar{a}u^{1/3}, u \equiv t - t_0$; and the metric (3.10) becomes

$$^{(1)}\gamma_{rr} = \bar{C}_r u^{2/3-4\alpha/3}, \quad ^{(1)}\gamma_{\theta\theta} = \bar{C}_\theta u^{2/3+2\alpha/3} \tag{6.31}$$

(with $\alpha \equiv 3A/4\bar{a}^3$). It depends on four functions: $\bar{C}_r, \bar{C}_\theta, t_0, \alpha$. We fix the gauge by choosing $\bar{C}_\theta = r^2$ and $t_0 = 0$ so that the generic first order line element, together with the energy density, given by (3.16), and the radial velocity derived from (6.6) are

$$^{(1)}ds^2 = -dt^2 + t^{2/3}\left[\bar{C}_r t^{-4\alpha/3} + r^2t^{2\alpha/3}d\Omega^2\right], \tag{6.32}$$

$$\chi^{(1)}\epsilon = \frac{1-\alpha^2}{3t^2}, \tag{6.33}$$

$$^{(1)}u_r = \frac{t}{1-\alpha^2}\left[\alpha\alpha'\ln t + \frac{3\alpha + r\alpha'}{r}\right]. \tag{6.34}$$

The solution depends on the two arbitrary functions $\bar{C}_r$ and α, and the positivity of ϵ imposes $\alpha^2 < 1$.

To obtain the metric at third order in the gradients we must first evaluate the RHS of Eq. (6.3), that is, compute $\bar{R}$ and $\bar{S}$ by means of the metric (6.31). These are sums of terms in $t^{\frac{-2}{3}(1-2\alpha)}\left[\text{const}, \ln t, (\ln t)^2\right]$ and in $t^{-\frac{2}{3}(1+\alpha)}$, that we shall denote collectively by Q. Integrating (6.3) and (6.4) will then yield $^{(3)}A$" = "$A + Qt^2$ and $^{(3)}a$" = "$^{(1)}a(1 + Qt^2)$, so that the third order metric will be of the form

$$^{(1)+(3)}\gamma_{rr} = {}^{(1)}\gamma_{rr}(1 + \gamma_r);$$

$$^{(1)+(3)}\gamma_{\theta\theta} = {}^{(1)}\gamma_{\theta\theta}(1 + \gamma_\theta) \tag{6.35}$$

where the time dependence of γ_r and γ_θ is Qt^2. Hence we see that the iteration scheme is valid if $Qt^2 < 1$. Since $\alpha^2 < 1$ this condition is satisfied near $t = 0$, in agreement with the general discussion of Sec. IV.

The detailed calculation gives, for $\alpha > 0$ and at leading order near the singularity,

$$\gamma_r \simeq \frac{9(1-2\alpha)}{r^2(7-2\alpha)(2-\alpha)^2} t^{\frac{2}{3}(2-\alpha)}, \tag{6.36}$$

$$\gamma_\theta \simeq \frac{9(4\alpha-5)}{2r^2(7-2\alpha)(2-\alpha)^2} t^{\frac{2}{3}(2-\alpha)}. \tag{6.37}$$

[For $\alpha < 0$ the dominant third order term near the singularity is in $(\ln t)^2 t^{\frac{4}{3}(1+\alpha)}$.] One can also check that the conditions of validity coming from the three-velocity are also satisfied since $\Gamma = 2$ and $p_a < 1$.

VII. CONCLUSIONS

In this work, we have studied the early time behavior of inhomogeneous spacetimes near the singularity. To do that we have used a long wavelength iteration scheme to approximate the Einstein equations. Our main concern was to test the validity of the approximation scheme, by comparing the terms we ignored with those we kept. The result of this investigation is that one should be very cautious with the use of the long wavelength approximation if one wishes to get *general* results. Indeed our analysis shows that, in the general case, there are very severe restrictions on the range of validity of this scheme. The troubles arise from two origins.

The curvature terms. The curvature terms, which are ignored in the first step, blow up in general near the singularity. Going beyond the long wavelength approximation by keeping them from the beginning enables us to recover the oscillatory behavior discovered by BKL. In the case where the curl of the vector field representing the axis of contraction (going forwards in time) is orthogonal to the vector field, then the curvature terms can be ignored and the approximation scheme is valid.

The velocity terms. The velocity terms may also blow up. They do not if the perfect fluid is sufficiently "stiff" to compensate for the dilatation ($\Gamma > 2p_a$) or if the component of the velocity along the dilatation (time going forwards) direction vanishes. Even if these conditions are satisfied it is not yet enough to get a valid energy equation, which demands that $\Gamma > 1 + p_a$ unless the component u_a vanishes.

In view of these results it should be interesting to reconsider the study of BKL when the velocity terms are dominant over the curvature terms. In this case the role of matter should become important and one should not be able to restrict oneself to the case of vacuum.

Finally we stress the fact that the problem of the velocity terms disappears in the case of a cosmological constant ($\Gamma = 0$), and in the case of *irrotational dust* where it is possible to choose a synchronous system of coordinates for which the three-velocity of dust is always zero. If we impose spherical symmetry the complete scheme works for $\Gamma > 4/3$ (including $\Gamma = 2$). However, the scheme works weakly, i.e., without the energy relation, for $\Gamma > 2/3$. The case $\Gamma = 1$ is special since one can choose a coordinate system so that the scheme works. The scheme also works in general for the stiff case as soon as there is local expansion along *all* the spatial directions.

All our conclusions should apply to a gravitational collapse, instead of a big bang, by just reversing the time.

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