Perturbations of an anisotropic spacetime: Formulation

Hyerim Noh
Korea Astronomy Observatory, San 36-1, Whaam-dong, Yusung-gu, Daejon, Korea

Jai-chan Hwang
Department of Astronomy and Atmospheric Science, Kyungpook National University, Taegu, Korea
(Received 10 February 1995)

We present a formulation for studying the evolution of perturbations in an anisotropic model universe. The formulation is designed to allow the systematic study of the evolution of perturbations in the spatially homogeneous and anisotropic Bianchi type-I spacetime. The spatial anisotropy in the background causes the couplings of the density perturbation mode with the gravitational wave and the rotation to the linear order. Our formulation includes the *imperfect fluid terms* in the background and the perturbations, and the *cosmological constant* in the background. We use the complete set of perturbed equations without fixing the gauge mode. After resolving the gauge issue we present the equations so that we can use the advantage of having the *gauge freedom* efficiently, i.e., in a gauge ready form. The formulation is extended so that it can treat the system of *multicomponents*.

PACS number(s): 98.80.Hw, 95.30.Sf

I. INTRODUCTION

The evolution of perturbations in cosmological models is important for describing the generation and the evolution of the observed large scale structure (see [1]). This study has been thoroughly made in a simple case where the background universe model is spatially homogeneous and isotropic. We call such a background cosmological model the Friedmann-Lemaître-Robertson-Walker (FLRW) model. In this work we will study the evolution of perturbations in a Bianchi type-I background model. The Bianchi type-I model is the simplest model in the class of homogeneous, but anisotropic universe models; for review see [2]. In the isotropic limit the Bianchi type-I model becomes the FLRW one with the vanishing spatial curvature. The anisotropic expansion stage is supported by the nonvanishing background shear in the metric. If we consider a fluid (or a field) in the energy-momentum tensor we can find models in which the early shear-dominated anisotropic expansion stage later becomes the matter-dominated isotropic FLRW universe. Thus, in such a scenario it is interesting to investigate the evolution of perturbations while the background model is isotropized. As an example, the anisotropic background metric can consistently include the magnetic field which is aligned in one of the principal axes. In the magnetized model we can investigate the evolution of a self-consistently excited (perturbed) magnetic field and the gravitational field.

There are only a few previous studies on the perturbations of the anisotropic background model; see [3–6]. The situation is particularly striking if we consider the considerable amount of literature concerning the perturbations in the isotropic background model. The isotropic background model can be considered as a (measure zero) special case of the anisotropic universe models. The authors of [3] considered the perturbations in the Newtonian context. In the Bianchi type-I model, the authors of [4] considered a dust medium in a particular gauge condition. The authors of [5] considered $w =$ const an ideal fluid using gauge invariant variables which correspond to the variables in certain gauge conditions. Related studies using the gauge invariant, covariant, or gauge specific treatments can be found in [6].

In [7] one can find a new formulation for studying the cosmological perturbations in a spatially homogeneous and isotropic background spacetime. The "gauge ready method" presented in [7] was originally suggested in [8]. In this paper we will present a companion part of [7] which is applicable to the perturbations in a spatially homogeneous but anisotropic spacetime. As in [7], our formulation is general and systematic. Using our formulation one can treat the general perturbations in various fluids and fields as long as the background model is supported by the Bianchi type-I metric.

The gauge ready method starts with the full perturbation equations without fixing any gauge. The gauge will be chosen depending on how much the gauge choice leads to the mathematical simplification of the problem at hand. In [7], thus in the FLRW model, we find six different temporal gauge conditions. Because of the spatial homogeneity of the background model the spatial gauge condition can be trivially treated. Except for the synchronous gauge condition which has been used popularly, the rest of gauge conditions fix the gauge mode completely. Any variable in such a gauge has a unique corresponding gauge invariant combination of variables. Thus, the variables in such a gauge condition can be considered as gauge invariant ones. Similar situations remain valid in the Bianchi type-I background. In the Bianchi type-I model, due to the spatial homogeneity of the background model, the spatial gauge transformation does not have an important role. We can find a unique set of gauge conditions which completely fix the spatial gauge transformation property. In order to fix the temporal gauge mode we can identify several different variables; compared with the isotropic case, due to the existence of the background shear in the anisotropic background, the zero-shear condition is replaced by a uniform-shear condition which fixes part of the perturbed shear tensor of the normal frame vector field (for details, see Sec. III B 1). Out of these variables, except for the synchronous gauge condition, the rest of the gauge condition fixes the temporal gauge transformation property completely. Thus, the variables in such a gauge condition have the unique corresponding gauge invariant combination of variables. Thus, even in an anisotropic model, the freedom in choosing the gauge depending on its convenience can be used as an advantage in treating each problem. In order to use the advantage, we present the equations without fixing the temporal gauge condition; the gauge condition will be chosen depending on the consequent physical or mathematical simplifications of the problem. We call this the *gauge ready method.*

Compared with the FLRW case, we have one new feature of the cosmological perturbations in a spatially anisotropic background. The nonvanishing background shear causes couplings between different modes. These are the scalar, vector, and tensor modes which evolve independently in the isotropic background. The nondecoupling of three modes was shown in [4]. In the set of linearly perturbed equations there appear the coupling terms between the background shear term and various perturbed order variables. Thus, these coupling terms disappear in the isotropic limit where we have the vanishing background shear. In the isotropic background the perturbations can be described by a second-order differential equation for the scalar mode, a second-order differential equation for the tensor mode, and a first-order differential equation for the vector mode. These three modes evolve independently. However, because of the couplings in anisotropic background, the equations are no longer decoupled. In general, we may have a higher order differential equation in which all three modes are combined. Thus, except for some particularly simple situations, we may be no longer able to analyze the evolution analytically. One expected feature from such couplings is that as the perturbations experience the anisotropic background evolution stage, the resulting amplitudes of the different modes of the perturbations in the subsequent isotropic stage will be related to each other. In a purely isotropic background, since all three modes are decoupled from each other, the initial condition for each mode should be given independently. It will be interesting to investigate the interaction between different modes while the background model evolves from anisotropic into isotropic stage.

As the inflation (accelerated expansion) model becomes popular as a part of the early universe scenario, the anisotropic models have lost some of their important merits as models for the early universe. The accelerated expansion of the model, if it occurs, will rapidly dilute (stretch) the preexisting classical structures (anisotropies or inhomogeneities) so that the observationally relevant local patch of the Universe becomes homogeneous and isotropic after the acceleration phase [9]. In the acceleration stage the currently observable scales become macroscopic during some e-folding times before the end of the acceleration era. If the acceleration phase lasts long enough, the seed fluctuations (for the scalar field, the gravitational wave, etc.) relevant for structures will be regenerated from quantum fluctuation during the isotropized stage near the end of the acceleration stage. The scalar- and the tensor-type fluctuations evolve separately in the isotropic background. However, since both fluctuations are generated from quantum fluctuations of each component in the same physical situation, the generated seed fluctuation of each component could be related. This relation between the scalar and the tensor modes in the isotropic model follows from the same seed generation mechanism [10]. Meanwhile, if the perturbations go through an anisotropic expansion stage, the couplings between the perturbations and the background shear cause the relation between the scalar and the tensor perturbations; thus, the relation follows from mixing due to the evolution of the background model.

In Sec. II we introduce the notation and equations for the background and the perturbations. Our method of the decomposition of variables into three different types is presented in Sec. II B. In Sec. III we discuss the gauge issue in detail. We identify a unique set of spatial (scalar and vector) gauge-fixing conditions. We identify several temporal gauge conditions, and each of which completely fixes the temporal gauge transformation property of every variable. Any variable under such gauge conditions corresponds to a unique gauge invariant combination of variables; this is discussed in Sec. III C. In Sec. IV we present a set of complete equations without fixing the temporal gauge condition. In Sec. V we present supplementary equations which can treat a system of multicomponent medium. In Sec. VI we present a discussion. For convenience, in the main text we present only the important ideas of our formulation. Details of the methods and the equations are presented in the Appendices. As a unit we set $c \equiv 1$.

II. NOTATIONS AND EQUATIONS

In order to derive a complete set of equations to the perturbed order we use the equations based on the ADM (Arnowitt-Deser-Misner) formulation. The ADM formulation of the Einstein's gravity theory is summarized in Appendix A; see also [11,12].

The metric to the perturbed order is introduced as

$$g_{00} \equiv -e^{2s}(1+2A), \quad g_{0\alpha} \equiv e^{2s}B_\alpha, \quad g_{\alpha\beta} \equiv e^{2s}(\gamma_{\alpha\beta}+C_{\alpha\beta}), \tag{1}$$

where

$$\gamma_{\alpha\beta} \equiv e^{2s_\alpha}\delta_{\alpha\beta}, \quad \sum_{\alpha=1}^{3} s_\alpha = 0, \quad s_\alpha = s_\alpha(t), \quad s = s(t). \tag{2}$$

$A(\mathbf{x},t)$, $B_\alpha(\mathbf{x},t)$, and $C_{\alpha\beta}(\mathbf{x},t)$ are the perturbed order metric variables. We consider B_α and $C_{\alpha\beta}$ as tensors based on $\gamma_{\alpha\beta}$. α, β, ... run from 1 to 3, and a, b, ... run from 0 to 3.

Comparing the metric in Eq. (1) with the ADM definitions in Eq. (A1) we can express the ADM metric notation in terms of the notation based on $\gamma_{\alpha\beta}$:

$$N = e^s(1+A), \quad N_\alpha = e^{2s}B_\alpha, \quad h_{\alpha\beta} = e^{2s}(\gamma_{\alpha\beta}+C_{\alpha\beta}). \tag{3}$$

We introduce fluid variables to the perturbed order directly through the ADM notation:

$$\begin{aligned} E &\equiv \mu + \varepsilon, \\ S &\equiv 3(p+\pi), \\ J_\alpha &\equiv e^s Q_\alpha, \\ \bar{S}_{\alpha\beta} &\equiv e^{2s}(\Pi_{\alpha\beta} + \delta\Pi_{\alpha\beta}), \end{aligned} \tag{4}$$

where Q_α, $\Pi_{\alpha\beta}$, and $\delta\Pi_{\alpha\beta}$ are based on $\gamma_{\alpha\beta}$. The background fluid variables $\mu(t)$, $p(t)$, and $\Pi_{\alpha\beta}(t)$ are the energy density, pressure, and anisotropic pressure, respectively. The perturbed fluid variables $\varepsilon(\mathbf{x},t)$, $\pi(\mathbf{x},t)$, $Q_\alpha(\mathbf{x},t)$, and $\delta\Pi_{\alpha\beta}(\mathbf{x},t)$ are the perturbed energy density, perturbed isotropic pressure, energy flux (or velocity), and perturbed anisotropic pressure, respectively.

Using Eqs. (3) and (4) ADM equations in Appendix A can be reexpressed in terms of our perturbation variables based on the Bianchi type-I spacetime and its perturbations. The perturbed set of equations is presented in Appendix B. In Appendix B we also present some useful quantities for derivation.

A. Bianchi type-I equations

The background order equations can be read from equations in Appendix B 2. We express the equations using s_α. From Eq. (2) we have

$$\dot{\gamma}_{\alpha\beta} = 2\dot{s}_\alpha\gamma_{\alpha\beta}, \tag{5}$$

where an overdot denotes the time derivative based on the background proper time, t; $dt = e^s dx^0$. Since the index in s_α is not a tensor index, we do not assume the summation convention on such index. We also introduce a convention such as

$$\dot{\gamma}_{\alpha\beta}\Pi^{\alpha\beta} = 2\sum_\alpha \dot{s}_\alpha\gamma_{\alpha\beta}\Pi^{\alpha\beta} \equiv 2\sum_\alpha \dot{s}_\alpha\Pi^\alpha_\alpha, \tag{6}$$

where in the second step the lowering index is done by the β index which is not affected by s_α. Thus, the background parts of Eqs. (B8), (B10), (B11), and (B13) become

$$\dot{\mu} + 3\dot{s}(\mu+p) = -\sum_\alpha \dot{s}_\alpha\Pi^\alpha_\alpha, \tag{7}$$

$$\ddot{s} + \dot{s}^2 = -\frac{4\pi G}{3}(\mu+3p) + \frac{\Lambda}{3} - \frac{1}{3}\sum_\alpha \dot{s}^2_\alpha, \tag{8}$$

$$\dot{s}^2 = \frac{8\pi G}{3}\mu + \frac{\Lambda}{3} + \frac{1}{6}\sum_\alpha \dot{s}^2_\alpha, \tag{9}$$

$$(\ddot{s}_\alpha + 3\dot{s}\dot{s}_\alpha)\,\delta^\alpha_\beta = 8\pi G\Pi^\alpha_\beta, \tag{10}$$

where Λ is the cosmological constant.

Equation (8) can be derived from Eqs. (7), (9), and (10). Combining Eqs. (8), and (9) we can derive

$$\ddot{s} = -4\pi G(\mu+p) - \frac{1}{2}\sum_\alpha \dot{s}^2_\alpha. \tag{11}$$

As we see in Eq. (B6), to the background order we have the nonvanishing trace-free part of the extrinsic curvature $\bar{K}^\alpha_\beta = -\dot{s}_\alpha\delta^\alpha_\beta$; this corresponds to the shear [see Eq. (E6)]. Thus, $\dot{s}_\alpha$ characterizes the background shear.

In the isotropic space limit of the background metric, the background becomes FLRW with zero spatial curvature. We have $s_\alpha = 0$, thus from Eq. (10) we have $\Pi_{\alpha\beta} = 0$. Introducing $a \equiv e^s$ and $H \equiv \dot{a}/a = \dot{s}$, our background equations [Eqs. (7)–(9)] reduce to the FLRW ones; compare with Eq. (21) of [7]. For a multicomponent system, in addition, we have Eqs. (44) and (E12).

B. Decomposition

We decompose the vector- and tensor-type perturbation variables as follows:

$$\begin{aligned} B_\alpha &\equiv B_{,\alpha} + B^{(v)}_\alpha, \\ C_{\alpha\beta} &\equiv C\gamma_{\alpha\beta} + \bar{C}_{,\alpha\beta} + 2C^{(v)}_{(\alpha,\beta)} + C^{(t)}_{\alpha\beta}, \\ Q_\alpha &\equiv Q_{,\alpha} + Q^{(v)}_\alpha, \\ \delta\Pi_{\alpha\beta} &\equiv \delta\Pi\gamma_{\alpha\beta} + \delta\bar{\Pi}_{,\alpha\beta} + 2\delta\Pi^{(v)}_{(\alpha,\beta)} + \delta\Pi^{(t)}_{\alpha\beta}, \end{aligned} \tag{12}$$

where the indices in Eq. (12) are based on $\gamma_{\alpha\beta}$; $C_{(\alpha,\beta)} \equiv \frac{1}{2}(C_{\alpha,\beta} + C_{\beta,\alpha})$. The superscripts (v) and (t) denote the vector-type and the tensor-type modes, respectively, defined as

$$\begin{aligned} &B^{(v)|\alpha}_\alpha \equiv 0, \quad C^{(v)|\alpha}_\alpha \equiv 0, \quad C^{(t)|\alpha}_{\alpha\beta} \equiv 0 \equiv C^{(t)\alpha}{}_\alpha, \\ &Q^{(v)|\alpha}_\alpha \equiv 0, \quad \delta\Pi^{(v)|\alpha}_\alpha \equiv 0, \quad \delta\Pi^{(t)|\alpha}_{\alpha\beta} \equiv 0 \equiv \delta\Pi^{(t)\alpha}{}_\alpha. \end{aligned} \tag{13}$$

(A derivative in superscript is denoted by a vertical bar; this is to indicate that the indices are raised by $\gamma_{\alpha\beta}$. Since the connection symbol based on $\gamma_{\alpha\beta}$ vanishes, the covariant derivative based on $\gamma_{\alpha\beta}$ is the same as ordinary spatial derivative.) Equations (12) and (13) can be considered as our *definition* of the scalar-, vector- and tensor-type modes. The vector-type mode is *defined* to preserve the trace-free condition. The tensor-type mode is *defined* to preserve the transverse and the trace-free conditions. In a decomposed set of perturbation equations we will see that the existence of the background anisotropy (shear $\sim \dot{s}_\alpha$) causes the couplings of all three types of modes to the linear order. From $\bar{S}^\alpha_\alpha = 0$ [Eq. (A3)] we have [Eqs. (4) and (12)]

$$\delta\Pi^\alpha_\alpha = 3\delta\Pi + \Delta\delta\bar{\Pi} = \Pi_{\alpha\beta}C^{\alpha\beta}. \tag{14}$$

We introduced $\Delta \equiv \gamma^{\alpha\beta}\partial_\alpha\partial_\beta \equiv \nabla^{(3)\alpha}\partial_\alpha$.

Using the decomposed variables in Eqs. (12) and (13) the perturbed set of equations in Appendix B 2 can be reexpressed. The vector- and tensor-type equations can be decomposed into two and three different types of equations, respectively. A method of such a decomposition can be found in Eq. (27). After a discussion of the gauge issue in Sec. III, we will present the decomposed equations with the spatial gauge mode fixed in a unique manner. The correspondence of our notation with the one used in the perturbed FLRW model is presented in Appendix G.

III. THE GAUGE ISSUE

A. The gauge transformation

We introduce a coordinate transformation

$$\hat{x}^a = x^a + \tilde{\xi}^a(x^e), \tag{15}$$

where $\tilde{\xi}^a$ is based on g_{ab}. Let us write the metric and the energy-momentum tensor to the perturbed order as

$$g_{ab} \equiv \bar{g}_{ab} + \delta g_{ab}, \quad T_{ab} \equiv \bar{T}_{ab} + \delta T_{ab}, \tag{16}$$

where an overbar and δ indicate the background order and the perturbed order, respectively. Using the transformation in Eq. (15) the metric and the energy-momentum tensor to the linear order transform as

$$\delta\hat{g}_{ab} = \delta g_{ab} - \bar{g}_{ab,c}\tilde{\xi}^c - \bar{g}_{cb}\tilde{\xi}^c{}_{,a} - \bar{g}_{ac}\tilde{\xi}^c{}_{,b}, \tag{17}$$

$$\delta\hat{T}_{ab} = \delta T_{ab} - \bar{T}_{ab,c}\tilde{\xi}^c - \bar{T}_{cb}\tilde{\xi}^c{}_{,a} - \bar{T}_{ac}\tilde{\xi}^c{}_{,b}. \tag{18}$$

We can impose four gauge conditions on δg_{ab} and δT_{ab} which can fix $\tilde{\xi}^a$.

In order to rewrite the gauge transformation properties in the perturbed anisotropic spacetime we introduce

$$\xi^0 \equiv \tilde{\xi}^0, \quad \xi^\alpha \equiv \tilde{\xi}^\alpha, \tag{19}$$

where ξ^α is based on $\gamma_{\alpha\beta}$. In order to derive the gauge transformation properties of the decomposed variables in Eq. (12) we introduce

$$\xi_\alpha \equiv \xi_{,\alpha} + \xi_\alpha^{(v)}, \quad \xi_\alpha^{(v)|\alpha} \equiv 0. \tag{20}$$

The gauge transformation properties of the decomposed metric and fluid variables [Eqs.(1),(4), and (12)] are presented in Appendix C.

B. The gauge conditions

We can impose three types of gauge conditions on perturbed variables. These will fix $\xi^t (\equiv e^s\xi^0)$, ξ, and $\xi_\alpha^{(v)}$. We call the gauge conditions on ξ^t, ξ, and $\xi_\alpha^{(v)}$ as the temporal, the spatial-scalar, and the rotational gauge conditions, respectively. The last two conditions belong to the spatial gauge conditions which fix ξ_α in Eq. (20).

Temporal gauge conditions. From Eqs. (C2)–(C17) we can identify the following temporal gauge conditions:

$$\begin{array}{lll} \text{synchronous gauge (SG)}: & A \equiv 0, & \xi^t = \xi^t(\mathbf{x}), \\ \text{comoving gauge (CG)}: & Q \equiv 0, & \xi^t = 0, \\ \text{uniform-curvature gauge (UCG)}: & C \equiv 0, & \xi^t = 0, \\ \text{uniform-expansion gauge (UEG)}: & \delta K \equiv 0, & \xi^t = 0, \\ \text{uniform-density gauge (UDG)}: & \varepsilon \equiv 0, & \xi^t = 0, \\ \text{uniform-pressure gauge (UPG)}: & \pi \equiv 0, & \xi^t = 0, \\ \text{uniform-shear gauge (USG)}: & \Sigma \equiv 0, & \xi^t = 0. \end{array} \tag{21}$$

Except for the synchronous gauge, other gauge conditions *completely fix* the temporal gauge mode. In an ideal fluid situation the uniform-pressure gauge is equivalent to the uniform-density gauge. For the uniform-shear gauge, see Sec. III B 1 below.

Spatial-scalar gauge conditions. We assume the temporal gauge mode ξ^t is completely removed by the temporal gauge condition. In such a case we can identify the following two spatial-scalar gauge fixing conditions:

$$\begin{array}{lll} \text{spatial-scalar } B \text{ gauge}: & B \equiv 0, & \dot{\xi} + (\dot{\Delta}/\Delta)\xi = 0, \\ \text{spatial-scalar } C \text{ gauge}: & \bar{C} \equiv 0, & \xi = 0. \end{array} \tag{22}$$

The spatial-scalar B gauge does not completely fix the spatial scalar mode, ξ. However, the spatial-scalar C gauge completely fixes ξ. For this reason, taking the spatial-scalar C gauge will be convenient. We have $\dot{\Delta} = \dot{\gamma}^{\alpha\beta}\partial_\alpha\partial_\beta = -\dot{\gamma}_{\alpha\beta}\nabla^{(3)\alpha}\nabla^{(3)\beta} = -2\sum_\alpha \dot{s}_\alpha \nabla^{(3)\alpha}\partial_\alpha$.

Rotational gauge conditions. We assume ξ^t and ξ are completely removed by the temporal and the spatial-scalar gauge conditions, respectively. In such a case, we can identify the following two rotational gauge-fixing conditions:

$$\begin{array}{lll} \text{rotational } B \text{ gauge}: & B_\alpha^{(v)} \equiv 0, & \xi_\alpha^{(v)} = \xi_\alpha^{(v)}(\mathbf{x}), \\ \text{rotational } C \text{ gauge}: & C_\alpha^{(v)} \equiv 0, & \xi_\alpha^{(v)} = 0. \end{array} \tag{23}$$

The rotational B gauge does not completely fix the rotational gauge mode, $\xi_\alpha^{(v)}$. However, the rotational C gauge completely fixes $\xi_\alpha^{(v)}$. On this regard, taking the rotational C gauge will be convenient.

Thus, if we take any temporal gauge condition mentioned above, except for the synchronous gauge, together with the spatial-scalar and rotational C-gauge conditions, all gauge modes will be completely fixed (removed).

We call the spatial-scalar and the rotational $C(B)$ gauge conditions simply as $C(B)$-gauge conditions. In the following we will concentrate on the gauge conditions which remove the gauge modes completely. Thus, for the spatial part we will take the C-gauge conditions. Together with the temporal gauge condition which fixes ξ^t completely, the C-gauge conditions fix both ξ and $\xi^{(v)}_\alpha$ completely. We do not choose the temporal gauge condition *a priori*. An appropriate temporal gauge condition will be chosen depending on how much the condition allows the problem to be simplified mathematically or on the correspondence of the result with Newtonian ones, etc.

1. The uniform-shear gauge

The trace-free part of the extrinsic curvature, $\bar{K}_{\alpha\beta}$ in Eq. (A4), contains information about the shear (with negative sign) of the normal frame vector; see Eq. (E6). From Eqs. (B6) and (3), using $\bar{K}_{\alpha\beta} \equiv h_{\alpha\gamma}\bar{K}^\gamma_\beta$, we can derive

$$\bar{K}_{\alpha\beta} = -e^{2s}\dot{s}_\alpha\gamma_{\alpha\beta} - \Sigma_{\alpha\beta},$$
$$\Sigma_{\alpha\beta} \equiv -e^{2s}\Bigg\{\dot{s}_\alpha\gamma_{\alpha\beta}(A-C) + e^{-s}\left(B_{,\alpha\beta} - \frac{1}{3}\gamma_{\alpha\beta}\Delta B\right) - \frac{1}{2}\left[\dot{\bar{C}}_{,\alpha\beta} - \frac{1}{3}\gamma_{\alpha\beta}\left(\Delta\bar{C}\right)^{\cdot}\right] + \frac{1}{2}e^{-s}\left(B^{(v)}_{\alpha,\beta} + B^{(v)}_{\beta,\alpha}\right) - \frac{1}{2}\left(\dot{C}^{(v)}_{\alpha,\beta} + \dot{C}^{(v)}_{\beta,\alpha} + \dot{C}^{(t)}_{\alpha\beta}\right)\Bigg\}, \tag{24}$$

where $\Sigma_{\alpha\beta}$ is based on $\gamma_{\alpha\beta}$. We note that $\bar{K}_{\alpha\beta}$ is trace-free, thus $\bar{K}^\alpha_\alpha = 0$, however $\Sigma^\alpha_\alpha \neq 0$.

In the FLRW limit, using the corresponding notation derived in Eq. (G13) we can show that $\Sigma_{\alpha\beta}$ becomes the shear tensor; thus,

$$-\bar{K}_{\alpha\beta} = \Sigma_{\alpha\beta} = \sigma_{\alpha\beta}, \tag{25}$$

which contains information about the scalar, vector, and tensor modes of the shear of the normal frame vector; see Eqs. (53), (64), and (81) of [13]. In the Bianchi type-I background, $\Sigma_{\alpha\beta}$ contains information about the perturbed part of the shear of the normal frame vector.

We can decompose $\Sigma_{\alpha\beta}$ into the scalar, vector, and tensor type modes as in Eqs. (12) and (13)

$$\Sigma_{\alpha\beta} \equiv \Sigma\gamma_{\alpha\beta} + \bar{\Sigma}_{,\alpha\beta} + 2\Sigma^{(v)}_{(\alpha,\beta)} + \Sigma^{(t)}_{\alpha\beta},$$
$$\Sigma^{(v)|\alpha}_\alpha \equiv 0, \quad \Sigma^{(t)|\alpha}_{\alpha\beta} \equiv 0 \equiv \Sigma^{(t)\alpha}{}_\alpha. \tag{26}$$

Each type of the decomposed variables can be derived from $\Sigma_{\alpha\beta}$ as

$$\Sigma = \frac{1}{2}\left(\Sigma^\alpha_\alpha - \frac{1}{\Delta}\Sigma_{\alpha\beta}{}^{|\alpha\beta}\right),$$
$$\Delta\bar{\Sigma} = \Sigma^\alpha_\alpha - 3\Sigma,$$
$$\Delta\Sigma^{(v)}_\alpha = \Sigma_{\alpha\beta}{}^{|\beta} - \frac{1}{\Delta}\Sigma_{\beta\gamma}{}^{|\beta\gamma}{}_\alpha,$$
$$\Sigma^{(t)}_{\alpha\beta} = \Sigma_{\alpha\beta} - \Sigma\gamma_{\alpha\beta} - \bar{\Sigma}_{,\alpha\beta} - 2\Sigma^{(v)}_{(\alpha,\beta)}. \tag{27}$$

From Eqs. (24) and (27) we can derive the decomposed parts of the perturbed shear using the metric variables. For Σ we have

$$\Sigma = -\frac{1}{2}e^{2s}\left[\frac{\dot{\Delta}}{2\Delta}(A-C) - \frac{2}{3}e^{-s}\Delta B + \frac{1}{3}\left(\Delta\bar{C}\right)^{\cdot} - \sum_\alpha \dot{s}_\alpha C^{(t)\alpha}{}_\alpha\right]. \tag{28}$$

The rest of the decomposed variables are presented in Appendix C1. Under the gauge transformation, using Eqs. (C2)–(C8) we can show that

$$\hat{\Sigma} = \Sigma - \frac{1}{2}e^{2s}\left(\frac{\ddot{\Delta}}{2\Delta} + \dot{s}\frac{\dot{\Delta}}{\Delta} - \frac{2}{3}e^{-2s}\Delta + 2\sum_\alpha \dot{s}^2_\alpha - \frac{4}{\Delta}\sum_\alpha \dot{s}^2_\alpha\nabla^{(3)\alpha}\partial_\alpha\right)\xi^t. \tag{29}$$

Thus, imposing $\Sigma = 0$ can be used as a temporal gauge-fixing condition which fixes the temporal gauge mode, ξ^t, completely. We call this condition the uniform-shear gauge condition as presented in Eq. (21). Although we named the condition $\Sigma \equiv 0$ the uniform-shear gauge, this does not mean that the total shear ($\Sigma_{\alpha\beta}$) is uniform in the hypersurface; see Eqs. (26) and (27). In the FLRW limit, using the notation in Eq. (G13), we can show that $\Sigma = -\frac{1}{3}\Delta\chi$, thus $\Sigma = 0$ leads to $\chi = 0$ which is the zero-shear gauge condition.

C. Gauge invariant combinations

From the gauge transformation properties of the variables presented in Appendix C we note the following. In an ideal fluid background, thus $\Pi_{\alpha\beta} = 0$, $Q^{(v)}_\alpha$, $\delta\Pi$ [for $\delta\bar{\Pi}$, see Eq. (14)], $\delta\Pi^{(v)}_\alpha$, and $\delta\Pi^{(t)}_{\alpha\beta}$ are gauge invariant; see Eqs. (C13)–(C17). *Assuming* $\Pi_{\alpha\beta} = 0$, we can construct the following set of gauge invariant combinations. Each combination becomes the first variable on the right-hand side in the comoving gauge condition:

$$A|_Q \equiv A + \left(\frac{e^s Q}{\mu + p}\right), \tag{30}$$

$$C|_Q \equiv C + \left(2\dot{s} + \frac{\dot{\Delta}}{2\Delta}\right)\frac{e^s Q}{\mu + p}, \tag{31}$$

$$C^{(t)}_{\alpha\beta}|_Q \equiv C^{(t)}_{\alpha\beta} + \gamma_{\alpha\beta}\left(2\dot{s}_\alpha - \frac{\dot{\Delta}}{2\Delta}\right)\frac{e^s Q}{\mu + p} - \frac{1}{\Delta}\left[\frac{\dot{\Delta}}{2\Delta} + 2\left(\dot{s}_\alpha + \dot{s}_\beta\right)\right]\frac{e^s Q_{,\alpha\beta}}{\mu + p}, \tag{32}$$

$$\varepsilon|_Q \equiv \varepsilon - 3\dot{s}e^s Q, \tag{33}$$

$$\pi|_Q \equiv \pi - 3\dot{s}c_s^2 e^s Q, \tag{34}$$

$$\delta K|_Q \equiv \delta K - \left(3\ddot{s} + e^{-2s}\Delta\right)\frac{e^s Q}{\mu + p}. \tag{35}$$

The gauge transformations of B and $B^{(v)}_\alpha$ depend on the spatial-scalar and rotational gauge modes, respectively. In order to construct the gauge invariant combinations we need the following steps. The following combinations are temporally gauge invariant:

$$\bar{C}|_Q \equiv \bar{C} - \frac{3\dot{\Delta}}{2\Delta^2}\frac{e^s Q}{\mu + p},$$

$$C^{(v)}_\alpha|_Q \equiv C^{(v)}_\alpha + \frac{1}{\Delta}\left(\frac{\dot{\Delta}}{\Delta} + 2\dot{s}_\alpha\right)\frac{e^s Q_{,\alpha}}{\mu + p}. \tag{36}$$

Thus,

$$\hat{\bar{C}}|_{\hat{Q}} = \bar{C}|_Q - 2\xi, \quad \hat{C}^{(v)}_\alpha|_{\hat{Q}} = C^{(v)}_\alpha|_Q - \xi^{(v)}_\alpha. \tag{37}$$

Using these combinations we can construct the following gauge invariant combinations which become B and $B^{(v)}_\alpha$, respectively, in the combination of the comoving gauge and the C gauge:

$$B|_{Q,\bar{C}|_Q} \equiv B|_Q - \frac{e^s}{2\Delta}\left(\Delta\bar{C}|_Q\right)^{\cdot} = B - \frac{Q}{\mu + p} - \frac{e^s}{2\Delta}\left(\Delta\bar{C} - \frac{3\dot{\Delta}}{2\Delta}\frac{e^s Q}{\mu + p}\right)^{\cdot}, \tag{38}$$

$$\begin{aligned} B^{(v)}_\alpha|_{Q,\bar{C}|_Q,C^{(v)}_\alpha|_Q} &\equiv B^{(v)}_\alpha + e^s\left[-\gamma_{\alpha\beta}\left(C^{(v)\alpha}|_Q\right)^{\cdot} + \left(\dot{s}_\alpha + \frac{\dot{\Delta}}{2\Delta}\right)\left(\bar{C}|_Q\right)_{,\alpha}\right] \\ &= B^{(v)}_\alpha + e^s\left[-\dot{C}^{(v)}_\alpha + 2\dot{s}_\alpha C^{(v)}_\alpha + \left(\dot{s}_\alpha + \frac{\dot{\Delta}}{2\Delta}\right)\bar{C}_{,\alpha}\right] \\ &\quad - e^s\left\{\left[\frac{1}{\Delta}\left(\frac{\dot{\Delta}}{\Delta} + 2\dot{s}_\alpha\right)\frac{e^s Q_{,\alpha}}{\mu + p}\right]^{\cdot} + \frac{1}{\Delta}\left(\frac{3}{4}\frac{\dot{\Delta}^2}{\Delta^2} - \frac{1}{2}\dot{s}_\alpha\frac{\dot{\Delta}}{\Delta} - 4\dot{s}^2_\alpha\right)\frac{e^s Q_{,\alpha}}{\mu + p}\right\}. \end{aligned} \tag{39}$$

One can similarly construct the gauge invariant combinations using C, δK, ε (or π), and Σ together with the C-gauge variables $\bar{C}$ and $C^{(v)}_\alpha$. In Eq. (21) we find some variables which can be used for imposing the temporal gauge conditions. Out of these variables, only the gauge transformation of Q depends on the background anisotropic pressure. Thus, similarly constructed gauge invariant combinations using C, δK, ε, and Σ with C-gauge variables are gauge invariant for general background with the nonvanishing anisotropic pressure.

Concerning the spatial gauge freedom, only the C-gauge conditions fix the corresponding gauge modes completely. Thus, in the following we will take the C gauge. In the C gauge, by a variable $B^{(v)}_\alpha|_{C\text{ gauge},Q}$ we imply $B^{(v)}_\alpha|_{Q,\bar{C}|_Q,C^{(v)}_\alpha|_Q}$ in Eq. (39). The C-gauge conditions impose $\bar{C} = 0 = C^{(v)}_\alpha$. From Eqs. (C6) and (C7) we have

$$\xi = \frac{3}{4}\frac{\dot{\Delta}}{\Delta^2}\xi^t, \quad \xi^{(v)}_\alpha = -\frac{1}{\Delta}\left(\frac{\dot{\Delta}}{\Delta} + 2\dot{s}_\alpha\right)\xi^t{}_{,\alpha}. \tag{40}$$

Thus, in the C-gauge condition, using Eq. (40), the gauge transformation properties of B and $B^{(v)}_\alpha$ in Eqs. (C3) and (C4) depend only on the temporal gauge transformation, ξ^t.

IV. EQUATIONS IN A GAUGE READY FORM

We will present a complete set of perturbation equations in a gauge ready form concerning the temporal gauge fixing condition. However, the spatial gauge transformation properties will be fixed using the C-gauge conditions which are the unique choice as explained in Sec. III B. Imposing the C-gauge conditions, the spatial gauge modes are completely fixed. This is true as long as we take a temporal gauge condition which also completely removes the temporal gauge mode; see Sec. III B. The C-gauge condition imposes

$$\bar{C} \equiv 0 \equiv C^{(v)}_\alpha. \tag{41}$$

The fundamental perturbation equations in this gauge condition are presented in Appendix D. We use the decomposed variables introduced in Sec. II B. The vector- and the tensor-type equations are also decomposed into three different modes; for a method, see Eq. (27). However, all three types of decomposed variables are coupled through the equations.

We have seven different temporal gauge conditions at our disposal; see Eq. (21). Except for the synchronous gauge, each of these gauge conditions fixes the temporal gauge freedom completely. As discussed in Sec. III C any variable using these gauge conditions has the unique gauge invariant counterpart. In this sense we can regard the variables in these gauge conditions as the gauge invariant ones.

V. MULTICOMPONENT SYSTEM

The formulation can be extended to the system including arbitrary number of different fluids and fields with general interactions between them. In a system of multicomponent sources, the energy-momentum tensor (T_{ab}) consists of all components. The total energy-momentum tensor in previous sections is the sum of the energy-momentum tensor of the individual component

$$T_{ab} = \sum_i T_{(i)ab}, \tag{42}$$

where a subindex (i) indicates the ith component of the source. The interaction between components can be characterized through the covariant conservation relation of each component as

$$T_{(i)a;b}^{\ \ \ \ \ b} \equiv \tilde{\mathbf{Q}}_{(i)a}, \quad \sum_i \tilde{\mathbf{Q}}_{(i)a} \equiv 0, \tag{43}$$

where $\tilde{\mathbf{Q}}_{(i)a}$ is based on g_{ab} and the second condition follows from the covariant conservation of the total energy-momentum tensor.

Since the fluid quantities are linearly related [see Eqs. (A3) and (E1)] to the energy-momentum tensor, the fluid quantities used in previous sections can be regarded as the sum of the individual fluid component as

$$\begin{aligned} &\mu = \sum_i \mu_{(i)}, \quad p = \sum_i p_{(i)}, \quad \Pi_{\alpha\beta} = \sum_i \Pi_{(i)\alpha\beta}, \\ &\varepsilon = \sum_i \varepsilon_{(i)}, \quad \pi = \sum_i \pi_{(i)}, \quad Q_\alpha = \sum_i Q_{(i)\alpha}, \\ &\delta\Pi_{\alpha\beta} = \sum_i \delta\Pi_{(i)\alpha\beta}. \end{aligned} \tag{44}$$

The decomposed fluid variables in Eq. (12) can be represented similarly.

The fundamental equations derived in this paper remain valid for the fluid quantities regarded as the total ones in Eq. (44). The metric quantities remain the same. The equations of motion of the individual fluid component can be derived from the covariant energy-momentum conservation equation [Eq. (43)]. For an easier derivation we introduce a covariant decomposition

$$\begin{aligned} &\tilde{\mathbf{Q}}_{(i)a} \equiv \tilde{\mathbf{Q}}_{(i)} n_a + \tilde{\mathbf{F}}_{(i)a}, \\ &n^a \tilde{\mathbf{F}}_{(i)a} \equiv 0, \\ &\sum_i \tilde{\mathbf{Q}}_{(i)} \equiv 0 \equiv \sum_i \tilde{\mathbf{F}}_{(i)a}, \end{aligned} \tag{45}$$

where $\tilde{\mathbf{F}}_{(i)a}$ is based on g_{ab} and n_a is a normal frame vector (see Appendix A). Using this notation Eq. (43) can be decomposed into

$$n^a T_{(i)a;b}^{\ \ \ \ \ b} = -\tilde{\mathbf{Q}}_{(i)}, \tag{46}$$

$$P_a^b T_{(i)b;c}^{\ \ \ \ \ c} = \tilde{\mathbf{F}}_{(i)a}, \tag{47}$$

where P_a^b is the projection tensor defined as $P_{ab} \equiv g_{ab} + n_a n_b$. We define

$$\mathbf{Q}_{(i)} \equiv \tilde{\mathbf{Q}}_{(i)}, \quad \mathbf{F}_{(i)\alpha} \equiv \tilde{\mathbf{F}}_{(i)\alpha}, \tag{48}$$

where $\mathbf{F}_{(i)\alpha}$ is based on $\gamma_{\alpha\beta}$. To the perturbed order, we write

$$\begin{aligned} &\mathbf{Q}_{(i)} \equiv \bar{\mathbf{Q}}_{(i)} + \delta\mathbf{Q}_{(i)}, \\ &\mathbf{F}_{(i)\alpha} \equiv \mathbf{F}_{(i),\alpha} + \mathbf{F}_{(i)\alpha}^{(v)}, \\ &\mathbf{F}_{(i)\alpha}^{(v)\ |\alpha} \equiv 0, \end{aligned} \tag{49}$$

where $\mathbf{F}_{(i)}$ and $\mathbf{F}_{(i)\alpha}^{(v)}$ are the perturbed order quantities.

Equations (46) and (47) lead to the energy and the momentum conservation equations of the ith component, respectively. The conservation equations for individual component are derived in Appendix E.

VI. DISCUSSION

Our formulation is very general. It can be applied to diverse physical situations as long as the background evolution is supported by the Bianchi type-I model. The equations are presented in a gauge ready form. In this way, we can *choose* a particular gauge condition which makes the mathematical manipulation most convenient; solutions in the other gauge can be expressed as the linear combinations of the known solutions in the particular gauge. Since our formulation includes the full imperfect fluid contribution both to the background and to the perturbations, it can be applied not only to various fluid systems, but also to various types of fields. The examples include the scalar field and the magnetic field. In fact, one of our main motivations for this study is to investigate the evolution of the perturbed magnetic field to the linear order. The energy-momentum tensor of the magnetic field contains magnetic field strength, $\mathbf{B}$, in $\mathbf{B}^2$ form. As we decompose $\mathbf{B}$ into $\bar{\mathbf{B}} + \delta\mathbf{B}$ we have $\mathbf{B}^2 = \bar{\mathbf{B}}^2 + 2\bar{\mathbf{B}}\cdot\delta\mathbf{B} + \delta\mathbf{B}^2$. The FLRW metric cannot accommodate the magnetic field as the energy-momentum tensor. Thus, in the FLRW background $\mathbf{B}^2 (= \delta\mathbf{B}^2)$ becomes the nonlinear order. However, in the anisotropic background, we can have nonvanishing $\bar{\mathbf{B}}$. To the linear order the term $\mathbf{B}^2 - \bar{\mathbf{B}}^2 = 2\bar{\mathbf{B}}\cdot\delta\mathbf{B}$ does not vanish. Thus, the study of the evolution of perturbations in a magnetized anisotropic background model will allow us to investigate the magnetohydrodynamic instabilities in general relativistic system self-consistently including the simultaneously excited gravitational field.

An ideal fluid is a case with

$$\Pi_{\alpha\beta} = 0 = \delta\Pi_{\alpha\beta}, \quad e \equiv \delta p - c_s^2 \delta\mu = 0, \tag{50}$$

where we introduced

$$w \equiv \frac{p}{\mu}, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}. \tag{51}$$

We call e as the entropic pressure. In cosmological study one often considers a case with $w =$ const. In this case we have $w = c_s^2$; see Eq. (7). Dust or radiation fluids correspond to $w = 0$ and $\frac{1}{3}$, respectively.

The energy-momentum tensor of a minimally coupled scalar field can be reinterpreted as an imperfect fluid with $e \neq 0$, but $\Pi_{\alpha\beta} = 0 = \delta\Pi_{\alpha\beta}$; see [7]. This is true even in a perturbed Bianchi type-I model. The energy-momentum tensor of a magnetic field can be reinterpreted as an imperfect fluid with $e \neq 0$ and $\Pi_{\alpha\beta} \neq 0 \neq \delta\Pi_{\alpha\beta}$. Applications to these fluid and field systems will be made in subsequent work.

Our imperfect fluid formulation can be also conveniently used when we use more generalized versions of gravity theory. For a class of generalized gravity theories, the contribution from such a gravity other than the Einstein tensor part can be reinterpreted as the new contribution to the energy-momentum tensor. Such reinterpretation is mathematically equivalent to the direct approach. In fact, such reinterpretation, in combination with the covariant formulation, allows a simple way of deriving the perturbation equations without heavy calculations. This approach was applied in the FLRW case in [7,14].

For a given equation of state, i.e., for a given energy-momentum tensor, the evolution of the Bianchi type-I model is characterized by the expansion rates s and s_α. For the Bianchi type-I model supported by the dust or radiation we can derive analytic solutions. These solutions show the transitions of the background models from the shear-dominated anisotropic stage to the matter- (dust- or radiation-) dominated isotropic FLRW stage. The evolution of perturbations in such background models will be considered in subsequent work.

Since the Bianchi type-I model is spatially flat we can decompose the spatial dependence of the perturbed variables using a wave vector $\mathbf{k}$ as $A(\mathbf{x}, t) \propto \int A_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k$. The $\mathbf{k}$ vector can be represented as $k_\alpha = (k_1, k_2, k_3)$ where $\hat{x}^1$, $\hat{x}^2$, and $\hat{x}^3$ represent three principal axes of the background anisotropy. From the transverse condition, a vector-type variable ($Q_\alpha^{(v)}$ for example) has two independent components. Whereas, from the transverse and trace-free conditions, a tensor-type variable ($C_{\alpha\beta}^{(t)}$ for example) has two independent components. We can classify perturbations according to the alignment of the wave propagation vector $\mathbf{k}$ with respect to the principal axes. These are presented in Appendix F.

In the Bianchi type-I model, the authors of [4] adopted the synchronous gauge in a dust medium for $k_\alpha = (0, 0, k_3)$. In a dust medium, the synchronous gauge is equivalent to the comoving gauge. The authors of [5] considered $w =$ const an ideal fluid using gauge invariant variables for $k_\alpha = (0, k_2, k_3)$. The gauge invariant variables used in [5] correspond to the variables in the comoving gauge and the uniform-curvature gauge together with the C gauge. Other studies can be found in [6].

The strength of our formulation can be summarized as follows. First, we derived the evolution equations without fixing the temporal gauge condition beforehand; the spatial gauge freedom is fixed in a unique way. We call this a gauge ready approach. This gives us the freedom to choose any gauge which turns out to be most suitable for the problem which we will deal with. We identified some gauge choices which remove the gauge freedom completely. The variables in such gauge are, in fact, equivalent to the gauge invariant ones. Using the gauge ready form of the equations we can easily relate a variable in one gauge to the other variable in other gauge. Since we are dealing with a linearized theory, a solution for a variable is linearly related to the solution for any other variable. Thus, from a known solution in a gauge choice we can derive all the other variables in the same gauge and also all the other variables in any other gauge choice. In this sense, it is convenient to work using some particular gauge choice where the mathematical manipulation becomes simplest. By comparing the behaviors of variables in different gauges we may be able to get a better perspective on the subject. For example, even in the FLRW analyses we found that, only in certain gauge(s) a variable shows the correct Newtonian limiting behavior.

Second, our formulation includes the full imperfect fluid contributions. As we discussed above, this implies that our formulation can be conveniently used for treating other fields (including the magnetic fluid) or other generalized gravity theories. Practically, such reinterpretation of the field or generalized gravity contributions as an imperfect fluid allows us to simplify a lot of the mathematical analyses involved; for the FLRW case, see [7,14]. We also include the cosmological constant in the equations.

Third, our formulation can be applied to the system which includes any number of different fluid or field (including the generalized gravity situations) components.

In essence, we present the formulation which can treat the evolution of general perturbations in the Bianchi type-I background. As long as we have the Bianchi type-I model as the background, the energy-momentum tensor can include various fluids, fields, and the combinations of those. As a gravity theory our formulation can treat a class of generalized gravity theories where the Einstein's gravity is a simple case. This formulation can be regarded as a companion of the similar formulation for the FLRW background case presented in [7]. Applications of our formulation to various fluid and field systems will be made in subsequent work.

ACKNOWLEDGMENTS

H.J. wishes to thank Dr. T. Futamase and Dr. M. Kasai for hospitality during the early part of the work in Hirosaki and for stimulating discussions.

APPENDIX A: ADM EQUATIONS

We summarize a set of equations based on the ADM formulation; see [11,12]. The spacetime is split into the spatial and the temporal parts based on a normal vector field. The metric is written as

$$g_{00} \equiv -N^2 + N^\alpha N_\alpha, \quad g_{0\alpha} \equiv N_\alpha, \quad g_{\alpha\beta} \equiv h_{\alpha\beta},$$
$$g^{00} = -N^{-2}, \quad g^{0\alpha} = N^{-2}N^\alpha,$$
$$g^{\alpha\beta} = h^{\alpha\beta} - N^{-2}N^\alpha N^\beta, \quad \text{(A1)}$$

where N_α is based on $h_{\alpha\beta}$. The normal vector n_a is introduced as

$$n_0 \equiv -N, \quad n_\alpha \equiv 0, \quad n^0 = N^{-1}, \quad n^\alpha = -N^{-1}N^\alpha. \quad \text{(A2)}$$

The fluid quantities are defined as

$$E \equiv n_a n_b T^{ab}, \quad J_\alpha \equiv -n_b T^b_\alpha,$$
$$S_{\alpha\beta} \equiv T_{\alpha\beta}, \quad S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \bar{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S, \quad \text{(A3)}$$

where J_α and $S_{\alpha\beta}$ are based on $h_{\alpha\beta}$. The extrinsic curvature is introduced as

$$K_{\alpha\beta} \equiv -\frac{1}{2N}\left(N_{\alpha|\beta} + N_{\beta|\alpha} - h_{\alpha\beta,0}\right),$$
$$K \equiv h^{\alpha\beta} K_{\alpha\beta},$$
$$\bar{K}_{\alpha\beta} \equiv K_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} K, \quad \text{(A4)}$$

where $K_{\alpha\beta}$ is based on $h_{\alpha\beta}$. A vertical bar denotes the covariant derivative based on $h_{\alpha\beta}$; this notation for a vertical bar applies only in this appendix and Appendix E 1. The intrinsic curvature is based on $h_{\alpha\beta}$ as the metric:

$$R^{(h)\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{(h)\alpha}{}_{\beta\delta,\gamma} - \Gamma^{(h)\alpha}{}_{\beta\gamma,\delta} + \Gamma^{(h)\epsilon}{}_{\beta\delta}\Gamma^{(h)\alpha}{}_{\gamma\epsilon} - \Gamma^{(h)\epsilon}{}_{\beta\gamma}\Gamma^{(h)\alpha}{}_{\delta\epsilon},$$
$$R^{(h)}_{\alpha\beta} \equiv R^{(h)\gamma}{}_{\alpha\gamma\beta},$$
$$R^{(h)} \equiv h^{\alpha\beta} R^{(h)}_{\alpha\beta},$$
$$\bar{R}^{(h)}_{\alpha\beta} \equiv R^{(h)}_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} R^{(h)},$$
$$\Gamma^{(h)\alpha}{}_{\beta\gamma} \equiv \frac{1}{2} h^{\alpha\delta}\left(h_{\beta\delta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}\right). \quad \text{(A5)}$$

A complete set of the ADM equations is the following. The energy constraint equation is

$$R^{(h)} = \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{2}{3}K^2 + 16\pi G E + 2\Lambda. \quad \text{(A6)}$$

The momentum constraint equation is

$$\bar{K}^\beta_{\alpha|\beta} - \frac{2}{3} K_{,\alpha} = 8\pi G J_\alpha. \quad \text{(A7)}$$

The energy conservation equation is

$$E_{,0}N^{-1} - E_{,\alpha}N^\alpha N^{-1} - K\left(E + \frac{1}{3}S\right) - \bar{S}^{\alpha\beta}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2 J^\alpha\right)_{|\alpha} = 0. \quad \text{(A8)}$$

The momentum conservation equation is

$$J_{\alpha,0}N^{-1} - J_{\alpha|\beta}N^\beta N^{-1} - J_\beta N^\beta{}_{|\alpha}N^{-1} - KJ_\alpha + EN_{,\alpha}N^{-1} + S^\beta_{\alpha|\beta} + S^\beta_\alpha N_{,\beta}N^{-1} = 0. \quad \text{(A9)}$$

The trace of ADM propagation equation is

$$K_{,0}N^{-1} - K_{,\alpha}N^\alpha N^{-1} + N^{|\alpha}{}_\alpha N^{-1} - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{1}{3}K^2 - 4\pi G(E + S) + \Lambda = 0. \quad \text{(A10)}$$

The trace-free ADM propagation equation is

$$\bar{K}^\alpha_{\beta,0}N^{-1} - \bar{K}^\alpha_{\beta|\gamma}N^\gamma N^{-1} + \bar{K}_{\beta\gamma}N^{\alpha|\gamma}N^{-1} - \bar{K}^\alpha_\gamma N^\gamma{}_{|\beta}N^{-1}$$
$$= K\bar{K}^\alpha_\beta - \left(N^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta N^{|\gamma}{}_\gamma\right)N^{-1} + \bar{R}^{(h)\alpha}{}_\beta - 8\pi G\bar{S}^\alpha_\beta. \quad \text{(A11)}$$

For a system of multicomponent medium, we have additional equations describing the evolution of individual component; see Eqs. (E7) and (E8) in the Appendix E 1.

APPENDIX B: A SET OF PERTURBATION EQUATIONS

1. Useful quantities

We can express the ADM notation in terms of the notation for the perturbed Bianchi type-I spacetime; see Eqs. (3) and (4). In the following we present some useful quantities.

The metric inverse is

$$g^{00} = -e^{-2s}(1 - 2A),$$
$$g^{0\alpha} = e^{-2s}B^\alpha,$$
$$g^{\alpha\beta} = e^{-2s}(\gamma^{\alpha\beta} - C^{\alpha\beta}). \quad \text{(B1)}$$

The connection is

$$\Gamma^{(h)\alpha}{}_{\beta\gamma} = \frac{1}{2}\left(C^\alpha_{\beta,\gamma} + C^\alpha_{\gamma,\beta} - C_{\beta\gamma}{}^{|\alpha}\right),$$
$$\Gamma^{(h)\beta}{}_{\alpha\beta} = \frac{1}{2}C^\beta_{\beta,\alpha}. \quad \text{(B2)}$$

A vertical bar indicates the covariant derivative based on $\gamma_{\alpha\beta}$. Since the spatial part of the Bianchi type-I background model is flat the connection based on $\gamma_{\alpha\beta}$ vanishes, and the covariant derivative based on $\gamma_{\alpha\beta}$ simply becomes an ordinary spatial derivative. However, for the derivatives in superscript, in order to indicate that the index is raised by $\gamma^{\alpha\beta}$ we denote it by a vertical bar; except in Appendices A and E 1 a superscript index "$|\alpha$" indicates $\nabla^{(3)\alpha} \equiv \gamma^{\alpha\beta}\partial_\beta$.

The intrinsic curvature is

$$R^{(h)\alpha}{}_{\beta\gamma\delta} = \frac{1}{2}\left(C^\alpha_{\delta,\beta\gamma} - C_{\beta\delta}{}^{|\alpha}{}_\gamma - C^\alpha_{\gamma,\beta\delta} + C_{\beta\gamma}{}^{|\alpha}{}_\delta\right),$$
$$R^{(h)\alpha}{}_\beta = \frac{1}{2}e^{-2s}\left(C^{\alpha\gamma}{}_{|\beta\gamma} - C^{\alpha|\gamma}_\beta{}_\gamma - C^\gamma_{\gamma|\beta}{}^\alpha + C_{\beta\gamma}{}^{|\gamma\alpha}\right),$$
$$R^{(h)} = e^{-2s}\left(C^{\alpha\beta}{}_{,\alpha\beta} - C^{\alpha|\beta}_\alpha{}_\beta\right). \quad \text{(B3)}$$

The extrinsic curvature is

$$K^{\alpha}_{\beta} = \left(\dot{s}\delta^{\alpha}_{\beta} + \frac{1}{2}\gamma^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}\right)(-1 + A) + \frac{1}{2}e^{-s}\left(B^{\alpha}{}_{,\beta} + B_{\beta}{}^{|\alpha}\right) - \frac{1}{2}\dot{C}^{\alpha}_{\beta} + \frac{1}{2}\dot{\gamma}^{\alpha\gamma}C_{\beta\gamma} + \frac{1}{2}C^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}, \quad \text{(B4)}$$

$$K = -3\dot{s} + 3\dot{s}A + e^{-s}B^{\alpha}{}_{,\alpha} - \frac{1}{2}\dot{C}^{\alpha}_{\alpha} \equiv -3\dot{s} + \delta K, \quad \text{(B5)}$$

$$\bar{K}^{\alpha}_{\beta} = \frac{1}{2}\gamma^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}(-1 + A) + \frac{1}{2}e^{-s}\left(B_{\beta}{}^{|\alpha} + B^{\alpha}{}_{,\beta}\right) - \frac{1}{2}\dot{C}^{\alpha}_{\beta} - \frac{1}{3}\left(e^{-s}B^{\gamma}{}_{,\gamma} - \frac{1}{2}\dot{C}^{\gamma}_{\gamma}\right)\delta^{\alpha}_{\beta} + \frac{1}{2}\dot{\gamma}^{\alpha\gamma}C_{\beta\gamma} + \frac{1}{2}\dot{\gamma}_{\beta\gamma}C^{\alpha\gamma}. \quad \text{(B6)}$$

An overdot denotes a time derivative based on the background proper time t, where $dt = e^{s}dx^{0}$.

2. Equations

The ADM equations can be expressed in a perturbed Bianchi type-I spacetime. We use the ADM metric and fluid variables expressed in terms of the perturbed Bianchi type-I variables [Eqs. (3) and (4)] and use some useful quantities presented in Appendix B 1. We introduce $\Delta \equiv \nabla^{(3)\alpha}\partial_{\alpha}$. The parts of the equations in curly brackets indicate the terms to the background order.

The definition of δK is

$$\delta K \equiv 3\dot{s}A + e^{-s}B^{\alpha}{}_{,\alpha} - \frac{1}{2}\dot{C}^{\alpha}_{\alpha}. \quad \text{(B7)}$$

The energy conservation equation is

$$\left\{\dot{\mu} + 3\dot{s}(\mu + p) + \frac{1}{2}\dot{\gamma}_{\alpha\beta}\Pi^{\alpha\beta}\right\} + \dot{\varepsilon} + 3\dot{s}\left[\varepsilon + \pi + (\mu + p)A\right] - (\mu + p)\delta K + e^{-s}Q^{\alpha}{}_{,\alpha} + \frac{1}{2}\dot{\gamma}_{\alpha\beta}\left(\delta\Pi^{\alpha\beta} - 2C^{\gamma\alpha}\Pi^{\beta}_{\gamma}\right) - \Pi^{\alpha\beta}\left(e^{-s}B_{\alpha,\beta} - \frac{1}{2}\dot{C}_{\alpha\beta}\right) = 0. \quad \text{(B8)}$$

The momentum conservation equation is

$$e^{s}\left(\dot{Q}_{\alpha} + 4\dot{s}Q_{\alpha}\right) + (\mu + p)A_{,\alpha} + \pi_{,\alpha} + \delta\Pi^{\beta}_{\alpha,\beta} + A_{,\beta}\Pi^{\beta}_{\alpha} - C^{\beta\gamma}{}_{,\beta}\Pi_{\alpha\gamma} + \frac{1}{2}C^{\beta}_{\beta,\gamma}\Pi^{\gamma}_{\alpha} - \frac{1}{2}C^{\gamma}_{\beta,\alpha}\Pi^{\beta}_{\gamma} = 0. \quad \text{(B9)}$$

The Raychaudhuri equation (trace of the ADM propagation) is

$$\left\{-3\left(\ddot{s} + \dot{s}^{2}\right) - 4\pi G(\mu + 3p) + \Lambda + \frac{1}{4}\dot{\gamma}_{\alpha\beta}\dot{\gamma}^{\alpha\beta}\right\} + \delta\dot{K} + 2\dot{s}\delta K - 4\pi G(\varepsilon + 3\pi) + \left[e^{-2s}\Delta - 3\dot{s}^{2} - 4\pi G(\mu + 3p) + \Lambda - \frac{1}{4}\dot{\gamma}_{\alpha\beta}\dot{\gamma}^{\alpha\beta}\right]A + \dot{\gamma}^{\alpha\beta}\left(\frac{1}{2}\gamma_{\beta\gamma}\dot{C}^{\gamma}_{\alpha} - e^{-s}B_{\alpha,\beta}\right) = 0. \quad \text{(B10)}$$

The energy constraint equation is

$$\left\{-6\dot{s}^{2} - \frac{1}{4}\dot{\gamma}_{\alpha\beta}\dot{\gamma}^{\alpha\beta} + 16\pi G\mu + 2\Lambda\right\} + 4\dot{s}\delta K + 16\pi G\varepsilon - e^{-2s}\left(C^{\alpha\beta}{}_{,\alpha\beta} - \Delta C^{\alpha}_{\alpha}\right) + \frac{1}{2}\dot{\gamma}_{\alpha\beta}\dot{\gamma}^{\alpha\beta}A - e^{-s}\dot{\gamma}_{\alpha\beta}B^{\alpha|\beta} - \frac{1}{2}\dot{\gamma}^{\alpha\beta}\gamma_{\beta\gamma}\dot{C}^{\gamma}_{\alpha} = 0. \quad \text{(B11)}$$

The momentum constraint equation is

$$-\frac{2}{3}\delta K_{,\alpha} - 8\pi G e^{s}Q_{\alpha} + \frac{1}{2}e^{-s}\left(\Delta B_{\alpha} + \frac{1}{3}B^{\beta}{}_{,\alpha\beta}\right) - \frac{1}{2}\left(\dot{C}^{\beta}_{\alpha,\beta} - \frac{1}{3}\dot{C}^{\beta}_{\beta,\alpha}\right) + \frac{1}{2}\dot{\gamma}_{\alpha\beta}A^{|\beta} + \frac{1}{2}\dot{\gamma}^{\beta\gamma}C_{\alpha\gamma,\beta} + \frac{1}{2}\dot{\gamma}_{\alpha\gamma}C^{\beta\gamma}{}_{,\beta} - \frac{1}{4}C^{\beta|\gamma}_{\beta}\dot{\gamma}_{\alpha\gamma} + \frac{1}{4}C^{\gamma\beta}{}_{,\alpha}\dot{\gamma}_{\beta\gamma} = 0. \quad \text{(B12)}$$

The trace-free ADM propagation equation is

$$\left\{e^{-3s}\left(e^{3s}\gamma^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}\right)^{\cdot}-16\pi G\Pi^{\alpha}_{\beta}\right\}+\ddot{C}^{\alpha}_{\beta}+3\dot{s}\dot{C}^{\alpha}_{\beta}-\frac{1}{3}\delta^{\alpha}_{\beta}\left(\ddot{C}^{\gamma}_{\gamma}+3\dot{s}\dot{C}^{\gamma}_{\gamma}\right)-e^{-3s}\left[e^{3s}\left(\dot{\gamma}^{\alpha\gamma}C_{\beta\gamma}+\dot{\gamma}_{\beta\gamma}C^{\alpha\gamma}\right)\right]$$
$$+e^{-2s}\left[C^{\alpha\gamma}{}_{,\beta\gamma}+C_{\beta\gamma}{}^{|\gamma\alpha}-\Delta C^{\alpha}_{\beta}-C^{\gamma|\alpha}_{\gamma\ \ \beta}-\frac{2}{3}\delta^{\alpha}_{\beta}\left(C^{\gamma\delta}{}_{,\gamma\delta}-\Delta C^{\gamma}_{\gamma}\right)\right]$$
$$-\left(\gamma^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}A\right)^{\cdot}-2e^{-2s}\left(A^{|\alpha}{}_{\beta}-\frac{1}{3}\delta^{\alpha}_{\beta}\Delta A\right)-\gamma^{\alpha\gamma}\dot{\gamma}_{\beta\gamma}\delta K$$
$$-e^{-s}\left(B^{\alpha}{}_{,\beta}+B_{\beta}{}^{|\alpha}\right)^{\cdot}+\frac{2}{3}\delta^{\alpha}_{\beta}e^{-s}\dot{B}^{\gamma}{}_{,\gamma}+2\dot{s}e^{-s}\left[-\left(B^{\alpha}{}_{,\beta}+B_{\beta}{}^{|\alpha}\right)+\frac{2}{3}\delta^{\alpha}_{\beta}B^{\gamma}{}_{,\gamma}\right]$$
$$+e^{-s}\left(B^{\alpha|\gamma}\dot{\gamma}_{\beta\gamma}+\dot{\gamma}^{\alpha\gamma}B_{\gamma,\beta}\right)-16\pi G\left(\delta\Pi^{\alpha}_{\beta}+A\Pi^{\alpha}_{\beta}-C^{\alpha\gamma}\Pi_{\beta\gamma}\right)=0. \quad \text{(B13)}$$

For a system of multicomponent we supplement Eqs. (44), (E10), and (E11).

APPENDIX C: GAUGE TRANSFORMATION

The gauge transformation properties for our metric variables in Eqs. (1) and (12) can be derived from Eq.(17). For fluid variables we can use Eqs. (4), (12), (A3), and (18). The energy-momentum tensor can be constructed from our perturbation variables through the ADM notation in Eq. (A3). We have

$$\tilde{T}_{00}=e^{2s}(1+2A)E,$$
$$\tilde{T}_{0\alpha}=-e^{s}J_{\alpha}+B^{\beta}S_{\alpha\beta},$$
$$\tilde{T}_{\alpha\beta}=S_{\alpha\beta}. \quad \text{(C1)}$$

We express the equations in terms of t, $\xi^t\equiv e^s\xi^0$, and s_α [Eqs. (2), (5), (19), and (20)]:

$$\hat{A}=A-\dot{\xi}^t, \quad \text{(C2)}$$
$$\hat{B}=B+e^{-s}\xi^t-e^{s}\left(\dot{\xi}+\frac{\dot{\Delta}}{\Delta}\xi\right), \quad \text{(C3)}$$
$$\hat{B}^{(v)}_{\alpha}=B^{(v)}_{\alpha}+e^{s}\left[-\gamma_{\alpha\beta}\dot{\xi}^{(v)\beta}+\left(2\dot{s}_{\alpha}+\frac{\dot{\Delta}}{\Delta}\right)\xi_{,\alpha}\right], \quad \text{(C4)}$$
$$\hat{C}=C-\left(2\dot{s}+\frac{\dot{\Delta}}{2\Delta}\right)\xi^t, \quad \text{(C5)}$$
$$\hat{\bar{C}}=\bar{C}+\frac{3\dot{\Delta}}{2\Delta^2}\xi^t-2\xi, \quad \text{(C6)}$$
$$\hat{C}^{(v)}_{\alpha}=C^{(v)}_{\alpha}-\xi^{(v)}_{\alpha}-\frac{1}{\Delta}\left(\frac{\dot{\Delta}}{\Delta}+2\dot{s}_{\alpha}\right)\xi^{t}{}_{,\alpha}, \quad \text{(C7)}$$
$$\hat{C}^{(t)}_{\alpha\beta}=C^{(t)}_{\alpha\beta}-\gamma_{\alpha\beta}\left(2\dot{s}_{\alpha}-\frac{\dot{\Delta}}{2\Delta}\right)\xi^t+\frac{1}{\Delta}\left[\frac{\dot{\Delta}}{2\Delta}+2\left(\dot{s}_{\alpha}+\dot{s}_{\beta}\right)\right]\xi^{t}{}_{,\alpha\beta}, \quad \text{(C8)}$$
$$\delta\hat{K}=\delta K+\left(3\ddot{s}+e^{-2s}\Delta\right)\xi^t, \quad \text{(C9)}$$
$$\hat{\varepsilon}=\varepsilon-\dot{\mu}\xi^t, \quad \text{(C10)}$$
$$\hat{\pi}=\pi-\dot{p}\xi^t, \quad \text{(C11)}$$
$$\hat{Q}=Q+e^{-s}\left[(\mu+p)\xi^t+\Pi_{\alpha\beta}\frac{1}{\Delta}\xi^{t|\alpha\beta}\right], \quad \text{(C12)}$$
$$\hat{Q}^{(v)}_{\alpha}=Q^{(v)}_{\alpha}+e^{-s}\left(\Pi_{\alpha\beta}\xi^{t|\beta}-\Pi_{\beta\gamma}\frac{1}{\Delta}\xi^{t|\beta\gamma}{}_{\alpha}\right), \quad \text{(C13)}$$
$$\delta\hat{\Pi}=\delta\Pi-\sum_{\alpha}\dot{s}_{\alpha}\Pi^{\alpha}_{\alpha}\xi^t+\frac{1}{2}\left(\dot{\Pi}_{\alpha\beta}+2\dot{s}\Pi_{\alpha\beta}\right)\frac{1}{\Delta}\xi^{t|\alpha\beta}, \quad \text{(C14)}$$
$$\delta\hat{\bar{\Pi}}=\delta\bar{\Pi}+\sum_{\alpha}\dot{s}_{\alpha}\Pi^{\alpha}_{\alpha}\frac{1}{\Delta}\xi^t-\frac{3}{2}\left(\dot{\Pi}_{\alpha\beta}+2\dot{s}\Pi_{\alpha\beta}\right)\frac{1}{\Delta^2}\xi^{t|\alpha\beta}-2\Pi_{\alpha\beta}\frac{1}{\Delta}\left(\xi^{|\alpha\beta}+\xi^{(v)\alpha|\beta}\right), \quad \text{(C15)}$$

$$\delta\hat{\Pi}^{(v)}_{\alpha} = \delta\Pi^{(v)}_{\alpha} - \left(\dot{\Pi}_{\alpha\beta} + 2\dot{s}\Pi_{\alpha\beta}\right)\frac{1}{\Delta}\xi^{t|\beta} + \left(\dot{\Pi}_{\beta\gamma} + 2\dot{s}\Pi_{\beta\gamma}\right)\frac{1}{\Delta^2}\xi^{t|\beta\gamma}{}_{\alpha}$$
$$-\Pi_{\alpha\beta}\left(\xi^{|\beta} + \xi^{(v)\beta}\right) + \Pi_{\beta\gamma}\frac{1}{\Delta}\left(\xi^{|\gamma\beta}{}_{\alpha} + \xi^{(v)\gamma|\beta}{}_{\alpha}\right), \quad \text{(C16)}$$

$$\delta\hat{\Pi}^{(t)}_{\alpha\beta} = \delta\Pi^{(t)}_{\alpha\beta} - \left(\dot{\Pi}_{\alpha\beta} + 2\dot{s}\Pi_{\alpha\beta}\right)\xi^{t} + \gamma_{\alpha\beta}\left[\sum_{\gamma}\dot{s}_{\gamma}\Pi^{\gamma}_{\gamma}\xi^{t} - \frac{1}{2}\left(\dot{\Pi}_{\gamma\delta} + 2\dot{s}\Pi_{\gamma\delta}\right)\frac{1}{\Delta}\xi^{t|\gamma\delta}\right]$$
$$-\sum_{\gamma}\dot{s}_{\gamma}\Pi^{\gamma}_{\gamma}\frac{1}{\Delta}\xi^{t}{}_{,\alpha\beta} - \frac{1}{2}\left(\dot{\Pi}_{\gamma\delta} + 2\dot{s}\Pi_{\gamma\delta}\right)\frac{1}{\Delta^2}\xi^{t|\gamma\delta}{}_{\alpha\beta} + 2\frac{1}{\Delta}\xi^{t|\gamma}{}_{(\alpha}\left(\dot{\Pi}_{\beta)\gamma} + 2\dot{s}\Pi_{\beta)\gamma}\right). \quad \text{(C17)}$$

1. Decomposed parts of the perturbed shear variable

In the following we present the decomposed parts of the perturbed shear variable in terms of the metric variables. From Eqs. (24), and (27) we can derive [Σ is presented in Eq. (28)]

$$\Delta\bar{\Sigma} = -e^{2s}\left[-\frac{3}{4}\frac{\dot{\Delta}}{\Delta}(A - C) + \Delta\left(e^{-s}B - \frac{1}{2}\dot{\bar{C}}\right) + \sum_{\alpha}\dot{s}_{\alpha}\left(-2C^{(v)|\alpha}_{\alpha} + \frac{1}{2}C^{(t)\alpha}{}_{\alpha}\right)\right], \quad \text{(C18)}$$

$$\Delta\Sigma^{(v)}_{\alpha} = -e^{2s}\left[\left(\dot{s}_{\alpha} + \frac{\dot{\Delta}}{2\Delta}\right)(A - C)_{,\alpha} + \frac{1}{2}\Delta\left(e^{-s}B^{(v)}_{\alpha} - \dot{C}^{(v)}_{\alpha}\right) + \sum_{\beta}\dot{s}_{\beta}\left(C^{(v)|\beta}_{\beta}{}_{\alpha} - C^{(t)|\beta}_{\alpha\beta}\right)\right], \quad \text{(C19)}$$

$$\Sigma^{(t)}_{\alpha\beta} = -e^{2s}\left[\gamma_{\alpha\beta}\left(\dot{s}_{\alpha} - \frac{\dot{\Delta}}{4\Delta}\right)(A - C) - \frac{1}{\Delta}\left(\dot{s}_{\alpha} + \dot{s}_{\beta} + \frac{\dot{\Delta}}{4\Delta}\right)(A - C)_{,\alpha\beta}\right.$$
$$\left.-\frac{1}{2}\dot{C}^{(t)}_{\alpha\beta} + \frac{1}{2}\gamma_{\alpha\beta}\sum_{\gamma}\dot{s}_{\gamma}C^{(t)\gamma}{}_{\gamma} + \frac{1}{\Delta}\sum_{\gamma}\dot{s}_{\gamma}\left(-\frac{1}{2}C^{(t)\gamma}{}_{\gamma,\alpha\beta} + C^{(t)|\gamma}_{\alpha\gamma}{}_{\beta} + C^{(t)|\gamma}_{\beta\gamma}{}_{\alpha}\right)\right]. \quad \text{(C20)}$$

APPENDIX D: EQUATIONS IN THE C GAUGE

We present a complete set of perturbation equations in decomposed forms. We take the spatial C gauge without losing any generality; thus we let $\bar{C} \equiv 0 \equiv C^{(v)}_{\alpha}$.

The definition of δK is

$$\delta K = 3\dot{s}A + e^{-s}\Delta B - \frac{3}{2}\dot{C}. \quad \text{(D1)}$$

The energy conservation equation is

$$\dot{\varepsilon} + 3\dot{s}(\varepsilon + \pi) + (\mu + p)(3\dot{s}A - \delta K) + e^{-s}\Delta Q = \frac{1}{2}\dot{\Delta}\delta\bar{\Pi} - \sum_{\alpha}\dot{s}_{\alpha}\left(2\delta\Pi^{(v)\alpha}{}_{,\alpha} + \delta\Pi^{(t)\alpha}{}_{\alpha}\right) + \left[e^{-s}\left(B^{|\alpha}{}_{\beta} + B^{(v)\alpha}{}_{,\beta}\right)\right.$$
$$\left.+C\sum_{\alpha}\dot{s}_{\alpha}\delta^{\alpha}_{\beta} + \sum_{\beta}\dot{s}_{\beta}C^{(t)\alpha}{}_{\beta} - \frac{1}{2}\dot{C}^{(t)\alpha}{}_{\beta}\right]\Pi^{\beta}_{\alpha}. \quad \text{(D2)}$$

The scalar part of the momentum conservation equation is

$$\dot{Q} + 4\dot{s}Q + e^{-s}\left[\pi + (\mu + p)A\right] = -\frac{2}{\Delta}\sum_{\alpha}\dot{s}_{\alpha}Q^{(v)\alpha}{}_{,\alpha} - e^{-s}\left(\delta\Pi + \Delta\delta\bar{\Pi}\right) - e^{-s}\Pi^{\beta}_{\alpha}\left[\frac{1}{\Delta}\left(A + \frac{1}{2}C\right)^{|\alpha}{}_{\beta} - \frac{1}{2}C^{(t)\alpha}{}_{\beta}\right]. \quad \text{(D3)}$$

The vector part of the momentum conservation equation is

$$\dot{Q}^{(v)}_{\alpha} + 4\dot{s}Q^{(v)}_{\alpha} - \frac{2}{\Delta}\sum_{\beta}\dot{s}_{\beta}Q^{(v)\beta}{}_{,\beta\alpha} = -e^{-s}\Delta\delta\Pi^{(v)}_{\alpha} + e^{-s}\left[-\Pi^{\beta}_{\alpha}\left(A + \frac{1}{2}C\right)_{,\beta} + \frac{1}{\Delta}\Pi^{\gamma}_{\beta}\left(A + \frac{1}{2}C\right)^{|\beta}{}_{\gamma\alpha}\right]. \quad \text{(D4)}$$

The Raychaudhuri equation is

$$\delta\dot{K} + 2\dot{s}\delta K + \left[e^{-2s}\Delta - 3\dot{s}^2 - 4\pi G(\mu + 3p) + \Lambda + \sum_\alpha \dot{s}_\alpha^2\right] A - e^{-s}\dot{\Delta}B - 4\pi G(\varepsilon + 3\pi)$$

$$= -2e^{-s}\sum_\alpha \dot{s}_\alpha B^{(v)\alpha}{}_{,\alpha} + \sum_\alpha \dot{s}_\alpha \dot{C}^{(t)\alpha}{}_\alpha. \quad \text{(D5)}$$

The energy constraint equation is

$$16\pi G\varepsilon + 4\dot{s}\delta K - 2\sum_\alpha \dot{s}_\alpha^2 A + e^{-s}\dot{\Delta}B + 2e^{-2s}\Delta C = 2e^{-s}\sum_\alpha \dot{s}_\alpha B^{(v)\alpha}{}_{,\alpha} - \sum_\alpha \dot{s}_\alpha \dot{C}^{(t)\alpha}{}_\alpha. \quad \text{(D6)}$$

The scalar part of the momentum constraint equation is

$$8\pi G e^s Q + \frac{2}{3}\left(\delta K - e^{-s}\Delta B\right) + \frac{\dot{\Delta}}{2\Delta}\left(A - \frac{3}{2}C\right) = \frac{1}{2}\sum_\alpha \dot{s}_\alpha C^{(t)\alpha}{}_\alpha. \quad \text{(D7)}$$

The vector part of the momentum constraint equation is

$$8\pi G e^s Q_\alpha^{(v)} - \frac{1}{2}e^{-s}\Delta B_\alpha^{(v)} = \left(\dot{s}_\alpha + \frac{\dot{\Delta}}{2\Delta}\right)\left(A - \frac{3}{2}C\right)_{,\alpha} - \sum_\beta \dot{s}_\beta C^{(t)\beta}{}_{\alpha,\beta}. \quad \text{(D8)}$$

The trace-free ADM propagation equation is

$$\ddot{C}^{(t)\alpha}{}_\beta + \left[3\dot{s} + 2(\dot{s}_\alpha - \dot{s}_\beta)\right]\dot{C}^{(t)\alpha}{}_\beta - e^{-2s}\Delta C^{(t)\alpha}{}_\beta$$
$$= 2\dot{s}_\beta\delta^\alpha_\beta\left(\delta K + \dot{A} - 3\dot{s}A\right) + e^{-2s}\left[2\left(A^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta A\right) + C^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta C\right]$$
$$+ e^{-s}\left[2\left(\dot{B}^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta\dot{B}\right) + 4\dot{s}\left(B^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta B\right) - \frac{2}{3}\delta^\alpha_\beta\dot{\Delta}B - 2(\dot{s}_\alpha + \dot{s}_\beta)B^{|\alpha}{}_\beta\right]$$
$$+ e^{-s}\left[\dot{B}^{(v)}_{\gamma,\beta}\gamma^{\alpha\gamma} + \dot{B}^{(v)|\alpha}_\beta + 2\dot{s}\left(B^{(v)|\alpha}_\beta + B^{(v)\alpha}{}_{,\beta}\right) - 2\left(\dot{s}_\alpha B^{(v)|\alpha}_\beta + \dot{s}_\beta B^{(v)\alpha}{}_{,\beta}\right)\right]$$
$$+ 16\pi G\left[\delta^\alpha_\beta\delta\Pi + \delta\bar{\Pi}^{|\alpha}{}_\beta + \delta\Pi^{(v)\alpha}{}_{,\beta} + \delta\Pi^{(v)|\alpha}_\beta + \delta\Pi^{(t)\alpha}{}_\beta\right.$$
$$\left. - (C - 2A)\Pi^\alpha_\beta - \Pi^\alpha_\gamma C^{(t)\gamma}{}_\beta\right]. \quad \text{(D9)}$$

The scalar part of the trace-free ADM propagation equation is

$$\dot{B} + \left(2\dot{s} + \frac{\dot{\Delta}}{\Delta}\right)B + e^{-s}\left(A + \frac{1}{2}C\right) - \frac{3}{4}e^s\frac{\dot{\Delta}}{\Delta^2}\left(\delta K + \dot{A} - 3\dot{s}A\right) = \frac{3}{\Delta}\left(-\sum_\alpha \dot{s}_\alpha B^{(v)\alpha}{}_{,\alpha} + \frac{e^s}{\Delta}\sum_{\alpha,\beta}\dot{s}_\alpha\dot{s}_\beta C^{(t)\alpha|\beta}{}_{\beta\ \alpha}\right)$$
$$-12\pi G\frac{e^s}{\Delta}\left[\delta\Pi + \Delta\delta\bar{\Pi} - \frac{1}{\Delta}\Pi^\beta_\alpha(C - 2A)^{|\alpha}{}_\beta\right]. \quad \text{(D10)}$$

The vector part of the trace-free ADM propagation equation is

$$\dot{B}^{(v)}_\alpha + \left(2\dot{s} + \frac{\dot{\Delta}}{\Delta}\right)B^{(v)}_\alpha - \frac{2}{\Delta}\sum_\beta \dot{s}_\beta B^{(v)\beta}{}_{,\beta\alpha} = -\frac{e^s}{\Delta}\left\{\left(2\dot{s}_\alpha + \frac{\dot{\Delta}}{\Delta}\right)\left(\delta K + \dot{A} - 3\dot{s}A - e^{-s}\Delta B\right)_{,\alpha}\right.$$
$$-2\sum_\beta \dot{s}_\beta\dot{C}^{(t)\beta}{}_{\alpha,\beta} + \frac{4}{\Delta}\sum_{\beta,\gamma}\dot{s}_\beta\dot{s}_\gamma C^{(t)\beta|\gamma}{}_{\gamma\ \beta\alpha}$$
$$+16\pi G\left[\Delta\delta\Pi^{(v)}_\alpha - \Pi^\beta_\alpha(C - 2A)_{,\beta}\right.$$
$$\left.\left. + \frac{1}{\Delta}\Pi^\beta_\gamma(C - 2A)^{|\gamma}{}_{\beta\alpha} - \Pi^\beta_\gamma C^{(t)\gamma}{}_{\alpha,\beta}\right]\right\}. \quad \text{(D11)}$$

The tensor part of the trace-free ADM propagation equation is

$$\ddot{C}^{(t)\alpha}{}_{\beta} + \left[3\dot{s} + 2\left(\dot{s}_\alpha - \dot{s}_\beta\right)\right]\dot{C}^{(t)\alpha}{}_{\beta} - \frac{2}{\Delta}\sum_{\gamma}\dot{s}_\gamma\left(\dot{C}^{(t)\gamma}{}_{\beta|\gamma}{}^{\alpha} + \dot{C}^{(t)\alpha|\gamma}{}_{\gamma\ \beta}\right) - e^{-2s}\Delta C^{(t)\alpha}{}_{\beta}$$

$$+\frac{2}{\Delta}\sum_{\gamma,\delta}\dot{s}_\gamma\dot{s}_\delta\left(\frac{1}{\Delta}C^{(t)\gamma|\delta}{}_{\delta\ \gamma}{}^{\alpha}{}_{\beta} + \delta^\alpha_\beta C^{(t)\gamma|\delta}{}_{\delta\ \gamma}\right) - \frac{4}{\Delta}\sum_{\gamma}\dot{s}_\gamma\left(\dot{s}_\alpha - \dot{s}_\gamma\right)C^{(t)\alpha|\gamma}{}_{\gamma\ \beta}$$

$$= \left(2\dot{s}_\beta - \frac{\dot{\Delta}}{2\Delta}\right)\delta^\alpha_\beta\left(\delta K + \dot{A} - 3\dot{s}A\right) - \frac{1}{\Delta}\left[2\left(\dot{s}_\alpha + \dot{s}_\beta\right) + \frac{\dot{\Delta}}{2\Delta}\right]\left(\delta K + \dot{A} - 3\dot{s}A\right)^{|\alpha}{}_{\beta}$$

$$-e^{-s}\left[2\left(\dot{s}_\beta B^{(v)\alpha}{}_{,\beta} + \dot{s}_\alpha B^{(v)|\alpha}_{\beta}\right) + \frac{\dot{\Delta}}{\Delta}\left(B^{(v)|\alpha}_{\beta} + B^{(v)\alpha}{}_{,\beta}\right) + 2\sum_{\gamma}\dot{s}_\gamma\left(\frac{1}{\Delta}B^{(v)\gamma}{}_{|\gamma}{}^{\alpha}{}_{\beta} - \delta^\alpha_\beta B^{(v)\gamma}{}_{,\gamma}\right)\right]$$

$$+16\pi G\left\{\delta\Pi^{(t)\alpha}{}_{\beta} - \frac{1}{2\Delta}\left(3\delta\Pi + \Delta\delta\bar{\Pi}\right)^{|\alpha}{}_{\beta} + \frac{1}{2}\delta^\alpha_\beta\left(3\delta\Pi + \Delta\delta\bar{\Pi}\right)\right.$$

$$-\left(C - 2A\right)\Pi^\alpha_\beta + \frac{1}{\Delta}\left[\left(C - 2A\right)_{|\beta}{}^{\gamma}\Pi^\alpha_\gamma + \left(C - 2A\right)^{|\alpha}{}_{\gamma}\Pi^\gamma_\beta\right]$$

$$\left.-\frac{1}{2\Delta}\Pi^\delta_\gamma\left[\frac{1}{\Delta}\left(C - 2A\right)^{|\gamma}{}_{\delta}{}^{\alpha}{}_{\beta} + \delta^\alpha_\beta\left(C - 2A\right)^{|\gamma}{}_{\delta}\right] - \Pi^\alpha_\gamma C^{(t)\gamma}{}_{\beta} + \frac{1}{\Delta}\Pi^\gamma_\delta\left(C^{(t)\alpha|\delta}{}_{\gamma\ \beta} + C^{(t)|\alpha\delta}_{\beta\gamma}\right)\right\}. \tag{D12}$$

For a system of a multicomponent we supplement Eqs. (44) and (E13)–(E15).

APPENDIX E: MULTICOMPONENT SYSTEM

1. ADM equations

In the following we *derive* the energy and the momentum conservation equations for the individual component using the ADM notation (see Appendix A). We can derive the equations directly from Eq. (43). An easier way is to derive the conservation equation for the individual component using the covariant formulation (based on the normal frame vector), then to derive the equations using the ADM notation from the covariant equations. For the covariant formulation of the Einstein gravity, see [15] and the Appendix of [13]. The ith component of the energy-momentum tensor for a general imperfect fluid in the normal frame (n_a) is

$$T_{(i)ab} = \mu_{(i)}n_a n_b + \left(p_{(i)} + \pi_{(i)}\right)P_{ab} + q_{(i)a}n_b + q_{(i)b}n_a + \pi_{(i)ab}, \tag{E1}$$

where $q_{(i)a}$ and $\pi_{(i)ab}$ are the energy flux vector and the anisotropic pressure of the ith component; $\pi_{(i)}$ is an entropic part of the isotropic pressure and differs from the definitions in Eqs. (4) and (44). From Eqs. (46) and (47) we can derive

$$\tilde{\mathbf{Q}}_{(i)} = \dot{\mu}_{(i)} + \left(\mu_{(i)} + p_{(i)}\right)\theta + q_{(i)}{}^{a}{}_{;a} + q_{(i)a}a^a + \pi^{ab}_{(i)}\sigma_{ab}, \tag{E2}$$

$$\tilde{\mathbf{F}}_{(i)a} = \left(\mu_{(i)} + p_{(i)}\right)a_a + P^b_a\left(p_{(i),b} + \pi_{(i)b;c}^{\ c} + \dot{q}_{(i)b}\right) + \left(\sigma_{ab} + \frac{4}{3}\theta P_{ab}\right)q^b_{(i)}. \tag{E3}$$

We used the kinematic quantities

$$\theta \equiv n^a{}_{;a},$$
$$a_a \equiv \dot{n}_a \equiv n_{a;b}n^b,$$
$$\sigma_{ab} \equiv P^c_a P^d_b n_{c;d} - \frac{1}{3}\theta P_{ab} = n_{a;b} + a_a n_b, \tag{E4}$$

where θ, a_a, and σ_{ab} are the volume expansion scalar (trace of the expansion tensor $\theta_{ab} \equiv P^c_{(a}P^d_{b)}n_{c;d}$), acceleration vector, and shear tensor (trace-free part of the expansion tensor) of the normal frame vector, respectively. The rotation of the normal frame vector vanishes, $\omega_{ab}(n_c) \equiv P^c_{[a}P^d_{b]}n_{c;d} = 0$. In Eqs. (E2)–(E4) an overdot denotes the covariant derivative following the frame vector; $\dot{\mu} \equiv \mu_{;a}n^a$, etc.

In terms of the ADM notation, the fluid quantities become

$$E = \mu,$$
$$J_\alpha = q_\alpha,$$
$$S = 3p,$$
$$\bar{S}_{\alpha\beta} = \pi_{\alpha\beta}, \tag{E5}$$

and are similar for the individual component. The kinematic quantities can be expressed using the ADM notation

$$\theta = -K,$$
$$\sigma^{\alpha\beta} = -\bar{K}^{\alpha\beta},$$
$$a^\alpha = \frac{N^{|\alpha}}{N}, \tag{E6}$$

and $\sigma^{0a} = 0 = a^0$. Thus we can derive

$$E_{(i),0}\frac{1}{N} - E_{(i),\alpha}\frac{N^\alpha}{N} - \left(E_{(i)} + \frac{1}{3}S_{(i)}\right)K$$
$$+J^\alpha_{(i)|\alpha} + 2\frac{N_{,\alpha}}{N}J^\alpha_{(i)} - \bar{S}^{\alpha\beta}_{(i)}\bar{K}_{\alpha\beta} = \mathbf{Q}_{(i)}, \tag{E7}$$

$$J_{(i)\alpha,0}\frac{1}{N} - J_{(i)\alpha|\beta}\frac{N^\beta}{N} - J_{(i)\beta}\frac{N^\beta{}_{|\alpha}}{N} - KJ_{(i)\alpha}$$
$$+\left(E_{(i)} + \frac{1}{3}S_{(i)}\right)\frac{N_{,\alpha}}{N} + \frac{1}{3}S_{(i),\alpha}$$
$$+\bar{S}_{(i)\alpha|\beta}^{\ \ \beta} + \bar{S}_{(i)\alpha}^{\ \ \beta}\frac{N_{,\beta}}{N} = \bar{\mathbf{F}}_{(i)\alpha}. \tag{E8}$$

The vertical bar in Eqs. (E6)–(E8) indicates the covariant derivative based on $h_{\alpha\beta}$. We introduced

$$\tilde{\mathbf{Q}}_{(i)} \equiv \mathbf{Q}_{(i)}, \quad \tilde{\mathbf{F}}_{(i)\alpha} \equiv \bar{\mathbf{F}}_{(i)\alpha}, \tag{E9}$$

where $\bar{\mathbf{F}}_{(i)\alpha}$ is based on $h_{\alpha\beta}$.

2. Perturbed equations

In the following we present the perturbed set of equations which can be used to treat a system of many components. From Eqs. (E7) and (E8) we can derive the energy and the momentum conservation equations in a perturbed Bianchi type-I model:

$$\left\{\dot{\mu}_{(i)} + 3\dot{s}\left(\mu_{(i)} + p_{(i)}\right) + \frac{1}{2}\dot{\gamma}_{\alpha\beta}\Pi^{\alpha\beta}_{(i)} - \bar{\mathbf{Q}}_{(i)}\right\} + \dot{\varepsilon}_{(i)} + \left(\mu_{(i)} + p_{(i)}\right)(3\dot{s}A - \delta K) + 3\dot{s}\left(\varepsilon_{(i)} + \pi_{(i)}\right) + e^{-s}Q_{(i)}{}^\alpha{}_{,\alpha}$$
$$+\frac{1}{2}\dot{\gamma}_{\alpha\beta}\left(\delta\Pi^{\alpha\beta}_{(i)} - 2C^{\gamma\alpha}\Pi_{(i)\gamma}^{\ \ \beta}\right) - \Pi^{\alpha\beta}_{(i)}\left(e^{-s}B_{\alpha,\beta} - \frac{1}{2}\dot{C}_{\alpha\beta}\right) - \delta\mathbf{Q}_{(i)} - \bar{\mathbf{Q}}_{(i)}A = 0, \tag{E10}$$

$$e^s\left(\dot{Q}_{(i)\alpha} + 4\dot{s}Q_{(i)\alpha}\right) + \pi_{(i),\alpha} + \left(\mu_{(i)} + p_{(i)}\right)A_{,\alpha} + \delta\Pi_{(i)\alpha,\beta}^{\ \ \beta} + \Pi_{(i)\alpha}^{\ \ \beta}\left(A_{,\beta} + \frac{1}{2}C^\gamma_{\gamma,\beta} - C^\gamma_{\beta,\gamma}\right) - \frac{1}{2}C^\gamma_{\beta,\alpha}\Pi_{(i)\gamma}^{\ \ \beta} - \mathbf{F}_{(i)\alpha} = 0, \tag{E11}$$

where for $\bar{\mathbf{Q}}_{(i)}$, $\delta\mathbf{Q}_{(i)}$, and $\mathbf{F}_{(i)\alpha}$, see Eqs. (48) and (49); $\mathbf{F}_{(i)\alpha}$ is based on $\gamma_{\alpha\beta}$. Equations (E10) and (E11) can be supplemented to equations in Appendix B 2.

To the background order, from Eq. (E10) we have

$$\dot{\mu}_{(i)} + 3\dot{s}\left(\mu_{(i)} + p_{(i)}\right) = -\sum_\alpha \dot{s}_\alpha\Pi_{(i)\alpha}^{\ \ \alpha} + \bar{\mathbf{Q}}_{(i)}. \tag{E12}$$

The perturbed parts of Eqs. (E10) and (E11) can be decomposed using Eqs. (44) and (12). In the C gauge, using s_α, we have

$$\dot{\varepsilon}_{(i)} + \left(\mu_{(i)} + p_{(i)}\right)(3\dot{s}A - \delta K) + 3\dot{s}\left(\varepsilon_{(i)} + \pi_{(i)}\right) + e^{-s}\Delta Q_{(i)}$$
$$= \frac{1}{2}\dot{\Delta}\delta\bar{\Pi}_{(i)} - \sum_\alpha \dot{s}_\alpha\left(2\delta\Pi^{(v)\alpha}_{(i)}{}_{|\alpha} + \delta\Pi^{(t)\alpha}_{(i)\alpha}\right) + C\sum_\alpha \dot{s}_\alpha\Pi_{(i)\alpha}^{\ \ \alpha}$$
$$+\Pi^{\alpha\beta}_{(i)}\left[e^{-s}\left(B_{,\alpha\beta} + B^{(v)}_{\alpha,\beta}\right) + 2\sum_\beta \dot{s}_\beta C^{(t)}_{\alpha\beta} - \frac{1}{2}\dot{C}^{(t)}_{\alpha\beta}\right] + \delta\mathbf{Q}_{(i)} + \bar{\mathbf{Q}}_{(i)}A, \tag{E13}$$

$$e^s\left(\dot{Q}_{(i)}+4\dot{s}Q_{(i)}\right)+\left(\mu_{(i)}+p_{(i)}\right)A+\pi_{(i)}$$

$$=-2\frac{e^s}{\Delta}\sum_\alpha \dot{s}_\alpha Q^{(v)\alpha}_{(i)\ ,\alpha}-\delta\Pi_{(i)}-\Delta\delta\bar{\Pi}_{(i)}-\Pi^{\alpha\beta}_{(i)}\left[\frac{1}{\Delta}\left(A+\frac{1}{2}C\right)_{,\alpha\beta}-\frac{1}{2}C^{(t)}_{\alpha\beta}\right]+\mathbf{F}_{(i)}, \quad \text{(E14)}$$

$$e^s\left(\dot{Q}^{(v)}_{(i)\alpha}+4\dot{s}Q^{(v)}_{(i)\alpha}-\frac{2}{\Delta}\sum_\beta \dot{s}_\beta Q^{(v)\beta}_{(i)\ ,\beta\alpha}\right)=-\Delta\delta\Pi^{(v)}_{(i)\alpha}-\left(A+\frac{1}{2}C\right)_{,\beta}\Pi_{(i)\alpha}^{\ \ \beta}+\frac{1}{\Delta}\left(A+\frac{1}{2}C\right)_{,\beta\gamma\alpha}\Pi^{\beta\gamma}_{(i)}+\mathbf{F}^{(v)}_{(i)\alpha}, \quad \text{(E15)}$$

where for $\mathbf{F}_{(i)}$ and $\mathbf{F}^{(v)}_{(i)\alpha}$ see Eq. (49). Equations (E13)–(E15) can be supplemented to equations in Appendix D.

APPENDIX F: CASES OF THE WAVE PROPAGATION

We take three principal axes of the background anisotropy as $\hat{x}^1$, $\hat{x}^2$, and $\hat{x}^3$. The wave propagation vector $\mathbf{k}$ can be in arbitrary direction. Depending on the alignment of $\mathbf{k}$ relative to the principal axes we can consider three situations: (i) $\mathbf{k}$ is directed along one principal axis as $k_\alpha=(0,0,k_3)$; (ii) $\mathbf{k}$ is in a plane of two principal axes as $k_\alpha=(0,k_2,k_3)$; (iii) $\mathbf{k}$ is in arbitrary direction as $k_\alpha=(k_1,k_2,k_3)$. The transverse condition for the vector-type variable, and the transverse and trace-free conditions for a tensor-type variable leave two independent components for each of the vector mode and the tensor mode; see Eq.(13). We can take $Q^{(v)}_1$ and $Q^{(v)}_2$, and $C^{(t)1}_{\ \ \ 1}$ and $C^{(t)1}_{\ \ \ 2}$ as characterizing the vector mode and the tensor mode, respectively. The two polarization states of the gravitational wave are characterized by $C^{(t)1}_{\ \ \ 1}$ and $C^{(t)1}_{\ \ \ 2}$. According to the direction of the wave propagation the other components can be expressed in terms of these components.

(i) In the $k_\alpha\equiv(0,0,k_3)$ case,

$$Q^{(v)}_3=0, \quad \text{(F1)}$$

$$C^{(t)2}_{\ \ \ 2}=-C^{(t)1}_{\ \ \ 1},$$
$$C^{(t)2}_{\ \ \ 1}=\gamma^{22}\gamma_{11}C^{(t)1}_{\ \ \ 2}=e^{-2(s_2-s_1)}C^{(t)1}_{\ \ \ 2},$$
$$\text{and } 0, \text{ otherwise.} \quad \text{(F2)}$$

(ii) In the $k_\alpha\equiv(0,k_2,k_3)$ case,

$$Q^{(v)}_3=-\frac{k^2}{k^3}Q^{(v)}_2, \quad Q^{(v)3}=-\frac{k^2}{k_3}Q^{(v)}_2, \quad \text{(F3)}$$

$$C^{(t)2}_{\ \ \ 2}=-\frac{k^3k_3}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}_{\ \ \ 1}, \quad C^{(t)3}_{\ \ \ 3}=-\frac{k^2k_2}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}_{\ \ \ 1},$$
$$C^{(t)3}_{\ \ \ 2}=\frac{k^3k_2}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}_{\ \ \ 1}, \quad C^{(t)2}_{\ \ \ 3}=\frac{k^2k_3}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}_{\ \ \ 1},$$
$$C^{(t)1}_{\ \ \ 3}=-\frac{k^2}{k^3}C^{(t)1}_{\ \ \ 2},$$
$$C^{(t)3}_{\ \ \ 1}=-\gamma_{11}\frac{k^2}{k_3}C^{(t)1}_{\ \ \ 2},$$
$$C^{(t)2}_{\ \ \ 1}=\gamma_{11}\gamma^{22}C^{(t)1}_{\ \ \ 2}. \quad \text{(F4)}$$

(iii) In the $k_\alpha\equiv(k_1,k_2,k_3)$ case,

$$Q^{(v)}_3=-\frac{1}{k^3}\left(k^1Q^{(v)}_1+k^2Q^{(v)}_2\right),$$
$$Q^{(v)3}=-\frac{1}{k_3}\left(k^1Q^{(v)}_1+k^2Q^{(v)}_2\right), \quad \text{(F5)}$$

$$C^{(t)3}_{\ \ \ 1}=\gamma^{33}\gamma_{11}C^{(t)1}_{\ \ \ 3},$$
$$C^{(t)3}_{\ \ \ 2}=\gamma^{33}\gamma_{22}C^{(t)2}_{\ \ \ 3},$$
$$C^{(t)2}_{\ \ \ 1}=\gamma^{22}\gamma_{11}C^{(t)1}_{\ \ \ 2},$$
$$C^{(t)3}_{\ \ \ 3}=-\left(C^{(t)1}_{\ \ \ 1}+C^{(t)2}_{\ \ \ 2}\right),$$
$$C^{(t)1}_{\ \ \ 3}=-\frac{1}{k^3}\left(C^{(t)1}_{\ \ \ 1}k^1+C^{(t)1}_{\ \ \ 2}k^2\right),$$
$$C^{(t)2}_{\ \ \ 2}=-\frac{1}{k_2k^2+k_3k^3}[\left(k_1k^1+k_3k^3\right)C^{(t)1}_{\ \ \ 1}+2k_1k^2C^{(t)1}_{\ \ \ 2}],$$
$$C^{(t)2}_{\ \ \ 3}=\frac{1}{k^3\left(k_2k^2+k_3k^3\right)}\left[k^2\left(k_1k^1+k_3k^3\right)C^{(t)1}_{\ \ \ 1}+\frac{k_1}{\gamma_{22}}\left(k_2k^2-k_3k^3\right)C^{(t)1}_{\ \ \ 2}\right]. \quad \text{(F6)}$$

APPENDIX G: FLRW LIMIT

We consider a limit where the shear terms are negligible. From Eqs. (10) and (14), we have

$$\dot{s}_\alpha\to 0, \quad \dot{\Delta}\to 0, \quad \Pi_{\alpha\beta}=0, \quad \Delta\delta\bar{\Pi}=-3\delta\Pi. \quad \text{(G1)}$$

Equations in Appendix D become

$$\delta K=3\dot{s}A+e^{-s}\Delta B-\frac{3}{2}\dot{C}, \quad \text{(G2)}$$

$$\dot{\varepsilon} + 3\dot{s}\,(\varepsilon + \pi) + (\mu + p)\,(3\dot{s}A - \delta K) + e^{-s}\Delta Q = 0, \quad \text{(G3)}$$

$$\dot{Q} + 4\dot{s}Q + e^{-s}\Big[\pi + (\mu + p)\,A\Big] = 2e^{-s}\delta\Pi, \quad \text{(G4)}$$

$$\dot{Q}^{(v)}_{\alpha} + 4\dot{s}Q^{(v)}_{\alpha} = -e^{-s}\Delta\delta\Pi^{(v)}_{\alpha}, \quad \text{(G5)}$$

$$\delta\dot{K} + 2\dot{s}\delta K + \left[e^{-2s}\Delta - 3\dot{s}^2 - 4\pi G\,(\mu + 3p) + \Lambda\right] A - 4\pi G\,(\varepsilon + 3\pi) = 0, \quad \text{(G6)}$$

$$16\pi G\varepsilon + 4\dot{s}\delta K + 2e^{-2s}\Delta C = 0, \quad \text{(G7)}$$

$$8\pi G e^{s} Q + \frac{2}{3}\left(\delta K - e^{-s}\Delta B\right) = 0, \quad \text{(G8)}$$

$$8\pi G e^{s} Q^{(v)}_{\alpha} - \frac{1}{2}e^{-s}\Delta B^{(v)}_{\alpha} = 0, \quad \text{(G9)}$$

$$\dot{B} + 2\dot{s}B + e^{-s}\left(A + \frac{1}{2}C\right) = 24\pi G\frac{e^{s}}{\Delta}\delta\Pi, \quad \text{(G10)}$$

$$\dot{B}^{(v)}_{\alpha} + 2\dot{s}B^{(v)}_{\alpha} = -16\pi G e^{s}\delta\Pi^{(v)}_{\alpha}, \quad \text{(G11)}$$

$$\ddot{C}^{(t)\alpha}{}_{\beta} + 3\dot{s}\dot{C}^{(t)\alpha}{}_{\beta} - e^{-2s}\Delta C^{(t)\alpha}{}_{\beta} = 16\pi G\delta\Pi^{(t)\alpha}{}_{\beta}. \quad \text{(G12)}$$

1. Correspondence with FLRW notation

In the following we show the relation between our perturbation variables in the FLRW limit and the ones used in [12,7]. For metric variables we have

$$\begin{aligned} &A = \alpha, \quad B = -\beta, \quad B^{(v)}_{\alpha} = -B^{(v)}Y^{(v)}_{\alpha}, \\ &C = 2\varphi, \\ &\bar{C} = 2\gamma, \\ &C^{(v)}_{\alpha} = -\frac{1}{k}H^{(v)}_{T}Y^{(v)}_{\alpha}, \\ &C^{(t)}_{\alpha\beta} = 2H^{(t)}_{T}Y^{(t)}_{\alpha\beta}, \\ &\delta K = \kappa, \quad \chi \equiv a\,(\beta + a\dot{\gamma})\,. \end{aligned} \quad \text{(G13)}$$

For fluid variables we have

$$\begin{aligned} &\varepsilon = \delta\mu, \quad \pi = \delta p, \\ &Q = \frac{1}{a}\Psi = -\frac{1}{ak}pf^{(s)}Y^{(s)}, \quad Q^{(v)}_{\alpha} = \frac{1}{a}pf^{(v)}Y^{(v)}_{\alpha}, \\ &\delta\Pi = -\frac{1}{3}\frac{\Delta}{a^2}\sigma = \frac{1}{3}p\pi^{(s)}_{T}Y^{(s)}, \\ &\delta\Pi^{(v)}_{\alpha} = -\frac{1}{2k}p\pi^{(v)}_{T}Y^{(v)}_{\alpha}, \quad \delta\Pi^{(t)}_{\alpha\beta} = p\pi^{(t)}_{T}Y^{(t)}_{\alpha\beta}. \end{aligned} \quad \text{(G14)}$$

Y's are the harmonic functions introduced in Secs. 2.1.1, 5.1, and 5.2 of [7]. From Eq. (14) we have $\Delta\delta\bar{\Pi} = -3\delta\Pi$. Since we take the normal frame ($u_a = n_a$) where $n_\alpha \equiv 0$, we have $v + k\beta = 0$ and $v^{(v)} - B^{(v)} = 0$ (see above Eq. (8) of [7]). Thus, $\psi = \Psi$ and $v_c = 0$ (see Eq. (8) and below Eq. (98) of [7]).

The C gauge condition corresponds to taking $\gamma = 0 = H^{(v)}_{T}$. In this case we have $\beta = \chi/a$, thus $B = -\chi/a$. Using the above FLRW notation, Eqs. (G2)–(G12) reduce to the FLRW ones in Eqs. (22)–(28), (98), and (102) of [7].

2. Background evolution

We are considering a flat FLRW model with constant w and $\Lambda = 0$. From Eqs. (7) and (9) we have

$$\dot{\mu} = -3\,(1 + w)\,\dot{s}\mu, \quad \dot{s}^2 = \frac{8\pi G}{3}\mu. \quad \text{(G15)}$$

The solution is

$$a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}}. \quad \text{(G16)}$$

In perturbation analysis we need a ratio between the scale and the horizon scale, $k/(a\dot{s}) = H^{-1}/(a/k) \sim l_H/l_\lambda$. It is convenient to introduce

$$\bar{\Delta} \equiv \frac{\Delta}{(a\dot{s})^2} = -\left(\frac{k}{a\dot{s}}\right)^2 = -\left(\frac{t}{t_H}\right)^{\frac{2(1+3w)}{3(1+w)}} = -\left(\frac{a}{a_H}\right)^{1+3w}, \quad \text{(G17)}$$

where we define $\bar{\Delta} = -1$ at $t = t_H$; this relation determines t_H. Using η we have

$$1 = \frac{k}{a\dot{s}}\bigg|_{s_H} = \frac{1+3w}{2}k\eta_H. \quad \text{(G18)}$$

3. Scalar perturbations in the comoving gauge

In the comoving gauge we let $Q \equiv 0$. We consider an ideal fluid system; thus, $\delta\Pi = 0$. From Eqs. (G3–G6) we can derive

$$\ddot{\delta} + (2 - 3w)\dot{s}\dot{\delta} - \left[4\pi G\mu(1 - w)(1 + 3w) + we^{-2s}\Delta\right]\delta = 0, \quad \text{(G19)}$$

where $\delta \equiv \delta\mu/\mu$. This equation can be transformed into a spherical Bessel equation. The solution is

$$\delta = x^{\frac{6w}{1+3w}}\left[d_1 j_{\frac{2}{1+3w}}(x) + d_2 n_{\frac{2}{1+3w}}(x)\right], \quad \text{(G20)}$$

where [using Eqs. [G16) and (G18)]

$$x \equiv \sqrt{w}k\eta = \sqrt{w}\frac{2}{1+3w}\frac{\eta}{\eta_H} = \sqrt{w}\frac{2}{1+3w}e^{\frac{1+3w}{2}(s-s_H)}. \quad \text{(G21)}$$

$d_1(k)$ and $d_2(k)$ are the integration constants. In the large scale limit ($x \ll 1$), we have

$$\delta \propto a^{1+3w}, a^{-\frac{3}{2}(1-w)} \propto t^{\frac{2(1+3w)}{3(1+w)}}, t^{-\frac{1-w}{1+w}}. \quad \text{(G22)}$$

In the radiation dominated era ($w = \frac{1}{3}$) we have

$$\delta = x\left[d_1\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) + d_2\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)\right]. \quad \text{(G23)}$$

4. Gravitational wave

The evolution of the gravitational wave is described by Eq. (G12). In an ideal fluid case, using $G \propto C^{(t)\alpha}_{\ \ \ \ \beta}$ we have

$$\ddot{G} + 3\dot{s}\dot{G} - e^{-2s}\Delta G = 0. \quad \text{(G24)}$$

This equation can be transformed into a spherical Bessel equation. The solution is

$$G = x^{-\frac{1-3w}{1+3w}}\left[g_1 j_{\frac{1-3w}{1+3w}}(x) + g_2 n_{\frac{1-3w}{1+3w}}(x)\right], \quad \text{(G25)}$$

where [using Eqs. (G16) and (G18)]

$$x \equiv k\eta = \frac{2}{1+3w}\frac{\eta}{\eta_{\rm H}} = \frac{2}{1+3w}e^{\frac{1+3w}{2}(s-s_{\rm H})}. \quad \text{(G26)}$$

$g_1(k)$ and $g_2(k)$ are the integration constants. In the large scale limit we have

$$G \propto \text{const},\ a^{-\frac{3}{2}(1-w)}. \quad \text{(G27)}$$

In the radiation-dominated era ($w = \frac{1}{3}$) we have

$$G = g_1\frac{\sin x}{x} - g_2\frac{\cos x}{x}. \quad \text{(G28)}$$

---

[1] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1973); P. J. E. Peebles, *Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980); Ya. B. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics, Volume 2: The Structure and Evolution of the Universe* (University of Chicago Press, Chicago, 1983); T. Padmanabhan, *Structure Formation in the Universe* (Cambridge University Press, Cambridge, England, 1993); P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993).

[2] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, 1975); M. A. H. MacCallum, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979), p. 533.

[3] A. G. Doroshkevich, Astrophys. **2**, 15 (1966); A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Sov. Phys. JETP **33**, 1 (1971).

[4] T. E. Perko, R. A. Matzner, and L. C. Shepley, Phys. Rev. D **6**, 969 (1972).

[5] K. Tomita and M. Den, Phys. Rev. D **34**, 3570 (1986); M. Den, Prog. Theor. Phys. **77**, 653 (1987); **79**, 1110 (1988).

[6] B. L. Hu, Phys. Rev. D **18**, 969 (1978); P. K. S. Dunsby, *ibid.* **48**, 3562 (1993); P. G. Miedema and W. A. van Leeuwen, *ibid.* **47**, 3151 (1993).

[7] J. Hwang, Astrophys. J. **375**, 443 (1991).

[8] J. M. Bardeen, in *Particle Physics and Cosmology*, edited by A. Zee (Gordon and Breach, London, 1988), p. 1.

[9] D. S. Goldwirth and T. Piran, Phys. Rep. **214**, 223 (1992).

[10] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992); A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).

[11] J. M. York, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, 1979), p. 83.

[12] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[13] J. Hwang and E. T. Vishniac, Astrophys. J. **353**, 1 (1990).

[14] J. Hwang, Class. Quantum Grav. **7**, 1613 (1990).

[15] J. Ehlers, Gen. Relativ. Gravit. **25**, 1225 (1993); G. F. R. Ellis, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic Press, New York, 1971), p. 104; in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gordon and Breach, New York, 1973), p. 1.