Extrinsic time in quantum cosmology

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Extrinsic time is identified in most isotropic and homogeneous cosmological models by matching it with the *ideal clock*—a parametrized system whose only "degree of freedom" is time. Once this matching is established, the cosmological models are quantized in the same way as the ideal clock. The space of solutions of the Wheeler-DeWitt equation is turned into a Hilbert space by inserting a time-dependent operator in the inner product, yielding a unitary theory equivalent to the phase-space-reduced theory.

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I. INTRODUCTION

Much effort has been made in the last two decades to construct a quantum theory of gravitation. The problems arising when one intends to join, in the same theory, the principles of quantum mechanics and general relativity show that quantum gravity is a very complicated and not completely understood discipline.

One of the most difficult features is the problem of time [1–3]. In quantum mechanics, time is an absolute parameter, differently treated from the other coordinates, which turn out to be operators and observables. Instead, in general relativity "time" is merely an arbitrary label of a spatial hypersurface, and physically significant quantities are independent of those labels: they are invariant under diffeomorfisms.

General relativity is an example of a parametrized system, i.e., a system whose action is invariant under change of the integrating parameter ("reparametrization"). One can obtain such a type of system by starting from an action which does not possess invariance (this means that the integrating parameter t *is* time), and raising time to the rank of a dynamic variable. So the original degrees of freedom and time are left as functions of some physically irrelevant parameter τ. Time t can be varied independently of the other degrees of freedom when a constraint with a Lagrange multiplier is added.

One of the proposals to understand quantum gravity, the so-called "internal Schrödinger interpretation" [1], states that time is hidden among the dynamic variables, and must be picked up before quantization. Once time is identified, the system is quantized by means of a Schrödinger equation. Thus, the space of wave functions can be turned out into a Hilbert space by defining the usual inner product, so providing a satisfactory statistical interpretation.

However, not all the actions which are invariant under reparametrization, have the time hidden among the dynamic variables: Jacobi's principle [4,5] allows one to get the trajectory of a conservative system in the phase space, without information about time evolution, by varying a parametrized action that does not contain time among the dynamic variables [6].

Therefore, a criterion is needed to establish whether time is hidden in a parametrized system, joined with a method to pick it up. The action for a parametrized system has the general form

$$S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - N\mathcal{H}(q^i, p_i) \right) d\tau, \qquad (1.1)$$

where $\mathcal{H}$ is the constraint and N is the Lagrange multiplier.

This action is invariant under reparametrization,

$$\delta q^i = \epsilon(\tau)\frac{dq^i}{d\tau}(\tau), \quad \delta p_i(\tau) = \epsilon(\tau)\frac{dp_i}{d\tau}(\tau),$$
$$\delta N = \frac{d}{d\tau}(N\epsilon), \quad \epsilon(\tau_1) = 0 = \epsilon(\tau_2), \qquad (1.2)$$

which is equivalent to changing $\tau \to \tau + \epsilon(\tau)$ on the path $(q(\tau), p(\tau))$, $\int_{\tau_1}^{\tau_2} N\, d\tau$ remaining invariant. In addition, the action (1.1) is invariant under a gauge transformation:

$$\delta q^i = \epsilon(\tau)\{q^i, \mathcal{H}\}, \quad \delta p_i(\tau) = \epsilon(\tau)\{p_i, \mathcal{H}\},$$
$$\delta N = \frac{d\epsilon}{d\tau}, \quad \epsilon(\tau_1) = 0 = \epsilon(\tau_2). \qquad (1.3)$$

Both transformations are not independent but differ by an "equation-of-motion symmetry" [7]. On the classical path, the reparametrization is equal to a gauge transformation with parameter $N\epsilon$.

The solutions of the Hamilton equations associated with (1.1) have the form

$$q^i = q^i\left(\int N\,d\tau\right), \quad p_i = p_i\left(\int N\,d\tau\right). \tag{1.4}$$

So $\int N\,d\tau$, instead of τ, plays the role of physically meaningful time parameter.

It is always possible to solve (1.4) locally for $\int N\,d\tau$; i.e., there are locally well-defined functions $t = t(q^i, p_i)$ in the phase space coinciding with $\int N\,d\tau$ when evaluated on the functions (1.4). However, a global solution may not exist; if it does exist, one calls it a "global phase time" [8–11]. To find a global phase time amounts to getting a globally well-defined function $t = t(q^i, p_i)$ such that

$$\{t, \mathcal{H}\}|_{\mathcal{H}=0} = 1. \tag{1.5}$$

In fact, $dt/d\tau = N(\tau)\{t,\mathcal{H}\}|_{\mathcal{H}=0} = N(\tau)$ on any trajectory. Furthermore, let us suppose that we know a globally well defined function $\bar{t} = \bar{t}(q^i, p_i)$ such that

$$\{\bar{t}, \mathcal{H}\}|_{\mathcal{H}=0} = F(q,p) > 0. \tag{1.6}$$

Then $\bar{t}$ is a global phase time associated with the constraint

$$\bar{\mathcal{H}} \equiv F(q,p)\mathcal{H}. \tag{1.7}$$

But $\bar{\mathcal{H}}$ and $\mathcal{H}$ are entirely equivalent; they describe the same parametrized system because their respective Hamiltonian vectors, $\mathbf{H} = (H^q, H^p) = (\partial\mathcal{H}/\partial p, -\partial\mathcal{H}/\partial q)$, whose field lines coincide with the classical trajectories, are proportional on the constraint surface (remember that the τ evolution is physically irrelevant). Therefore, $\bar{t}$ should be also considered as a global phase time for the system described by $\mathcal{H}$. This teaches us that global phase time should be rather defined by means of [8]

$$\{t, \mathcal{H}\}|_{\mathcal{H}=0} > 0. \tag{1.8}$$

Equation (1.8) tells us that

$$H^A \frac{\partial t}{\partial x^A} > 0,$$

where $x^A = (q,p)$; i.e., $t(q,p)$ monotonically increases along any dynamic trajectory; each surface $t = \text{const}$ is crossed by the dynamic trajectories only once (so the $\mathbf{H}$ field lines are necessarily open).

A global phase time can play the role of time in an internal Schrödinger interpretation. The quantization should be performed by solving the constraint for p_t, the momentum associated with time, to obtain the Hamiltonian entering the Schrödinger equation for the "reduced" system [1–4].

Unfortunately the solution of Eq. (1.8), whenever a global one exists, is not unique. This lack of uniqueness is called the "multiple choice problem" [1] because it can lead to different quantum theories. Even if the job of solving p_t was successful for some chosen time, the resulting Hamiltonian of the reduced system may be so intricate as to desist from quantizing the system.

Another proposal to understand quantum gravity consists in solving the Wheeler-DeWitt equation, which comes from constraining the wave function according to the Dirac method,

$$\hat{\mathcal{H}}\varphi = 0, \tag{1.9}$$

where $\hat{\mathcal{H}}$ is an operator associated with the constraint. This is a second-order hyperbolic differential equation in the usual variables of gravity, which resembles the Klein-Gordon equation. As in the latter case the Wheeler-DeWitt equation allows for a conserved inner product, defined on a "spacelike" hypersurface in the "superspace"; however this product fails to be positive definite, leaving this approach without a clear interpretation [12].

In this paper we shall search for global phase time in cosmological models. We are going to deal with "extrinsic time" [1,13], i.e., one that is intended to be associated not only with the coordinates but also with the momenta. Extrinsic times deserve special attention in quantum gravity because it is known that there is no chance of reducing the system by identifying a time among the coordinates, as in the case of the relativistic particle, due to the fact that the potential in the super-Hamiltonian constraint is not definite positive [1,12].

A simple model of a parametrized system, the "ideal clock," will suggest how the space of solutions of the Wheeler-DeWitt equation can be turned out into a Hilbert space, with an inner product matching the expectation values of the Schrödinger approach associated with extrinsic time.

In Sec. II the ideal clock, a parametrized system having no genuine degrees of freedom, is introduced. It is quantized by following the Dirac recipe, where a singular operator must be inserted in the inner product, in order to recover the physical expectation values of the corresponding reduced system. In Sec. III the ideal clock is suitably generalized, in order to allow a comparison with the cosmological models. In Sec. IV isotropic and homogeneous cosmological models are studied. Extrinsic time is picked up in most of these minisuperspaces. Finally, in Sec. V the problem of quantizing a parametrized system with genuine degrees of freedom, namely, a homogeneous scalar field in a Robertson-Walker metric, is glanced at in light of the experience acquired with the ideal clock.

II. THE IDEAL CLOCK

Let us start by considering a system without degrees of freedom. Then, its "action" is not a functional of dynamic variables but merely an arbitrary function of time t:

$$S = \int f(t)dt.$$

In order to parametrize the system, one changes the integration variable t to τ:

$$S = \int f(t) \frac{dt}{d\tau} d\tau.$$

Thus, S could be regarded as a Hamiltonian functional action by identifying $f(t)$ with p_t. However, in order that the function S does not change, this identification should reenter through a constraint,

$$\mathcal{H} = p_t - f(t) = 0, \tag{2.1}$$

and the action is

$$S[t, p_t, N] = \int d\tau \left[p_t \frac{dt}{d\tau} - N\mathcal{H} \right]. \tag{2.2}$$

Now (t, p_t) is a pair of dynamic conjugated variables. We remark that the constraint is linear in p_t; this will play a significant role in the quantization of the system.

By varying (2.2) with respect to t, p_t, and N, one obtains the dynamic equations and the constraint

$$\frac{dt}{d\tau} = N,$$
$$\frac{dp_t}{d\tau} = N\frac{df}{dt},$$
$$\mathcal{H} = p_t - f(t) = 0.$$

From the first equation

$$\frac{dt}{d\tau} = N \Rightarrow t = \int N \, d\tau,$$

so N relates the variations of τ with the variations of t, and it is known as *lapse function.* Then the second equation reads

$$\frac{d}{d\tau}[p_t - f(t)] = 0,$$

so $p_t - f(t)$ is a constant of motion. Finally, the third equation fixes the conserved quantity. As the only generalized coordinate of this system is the time t, we call it *ideal clock.*

We can proceed with the quantization of this system. The wave function satisfies the Schrödinger equation

$$i\frac{d\psi}{d\tau} = N\hat{\mathcal{H}}\psi.$$

But, following the Dirac method, the constraint must be imposed on the wave function,

$$\hat{\mathcal{H}}\psi = [\hat{p}_t - f(t)]\psi = 0,$$

to get the "physical states." So the physical states do not depend on τ. Since $\hat{p}_t = -i\partial/\partial t$, the constraint equation turns out to be

$$i\frac{\partial\psi}{\partial t} = -f(t)\psi, \tag{2.3}$$

which is the Schrödinger equation in the time t for the *reduced* system described by the Hamiltonian $h = -f(t)$; its only solution is

$$\psi(t) = \exp\left(i \int f(t) dt \right). \tag{2.4}$$

This wave function is nothing but a phase, which can be understood by remembering that the clock has no degrees of freedom.

Let us define the inner product as

$$\langle \psi, \psi \rangle = \int_a^b dt \, \psi^*(t)\psi(t);$$

$\hat{p}_t = -i\partial/\partial t$ is a Hermitian operator on the space of physical states (2.4) whatever (a, b) is (in fact, $\psi^*\psi|_a^b = 0$). Since time is an unbounded variable, the interval should be $(-\infty, \infty)$ or (a, ∞). So the inner product $\langle \psi, \psi \rangle$ between physical states diverges, as is typical for constrained systems. In order to get a physically meaningful result (ψ, ψ), a Hermitian singular operator must be inserted [7]:

$$(\psi, \psi) = \langle \psi, \hat{\mu}_{t_0}\psi \rangle,$$
$$\hat{\mu}_{t_0} = \delta(\chi_{t_0}(t))|\{\chi_{t_0}, \mathcal{H}\}|, \tag{2.5}$$

where χ_{t_0} is any function such that $\chi_{t_0}(t) = 0$ is a gauge condition ($\{\chi, \mathcal{H}\} \neq 0$) fixing t in t_0 (for instance $\chi_{t_0} = t - t_0$), and $|\{\chi, \mathcal{H}\}|$ is the Faddeev-Popov "determinant". In this way, the physical state (2.4) becomes normalized:

$$(\psi, \psi) = 1.$$

In Eq. (2.5) t_0 must be regarded as the time at which the physical inner product is evaluated. Since (ψ, ψ) does not depend on t_0, the time evolution is unitary. In particular,

$$\hat{t}(\hat{\mu}_{t_0}\psi) = t_0\hat{\mu}_{t_0}\psi.$$

We remark on the importance of having in $\mathcal{H}$ a term linear in momentum, in order to reach the Schrödinger equation (2.3) in time t. However, that is not the case in parametrized systems of physical interest such as general relativity, where the constraint is quadratic in the momenta. In order to change the constraint of the ideal clock to one that is quadratic in momentum, let us choose the (arbitrary) function $f(t)$ to be

$$f(t) = t^2,$$

and perform the canonical transformation

$$Q = p_t, \quad P = -t.$$

Thus, the constraint of the ideal clock results in

$$\mathcal{H} = -P^2 + Q, \tag{2.6}$$

and the "dynamics" of the ideal clock can be gotten by varying the action

$$S = \int \left[P\frac{dQ}{d\tau} - N(-P^2 + Q) \right] d\tau,$$

which differs from the original action in a surface term.

The global phase time is

$$t(Q, P) = -P,$$

which, of course, satisfies Eq. (1.5). The constraint surface $\mathcal{H} = 0$ is a parabola in variables (Q, P), which intersects once and only once each surface $t =$ const. The Hamiltonian vector

$$\mathbf{H} = (-2P, -1)$$

crosses the $t =$ const surfaces in the direction of increasing time.

The physical quantum states for the system with constraint (2.6) come from the solution of the equation

$$\frac{d^2\varphi}{dQ^2} + Q\varphi = 0. \tag{2.7}$$

Differently from (2.3), this is a second-order equation, which has two linearly independent solutions: the Airy functions $\mathrm{Ai}(-Q)$, $\mathrm{Bi}(-Q)$. However, there are boundary conditions to be satisfied. If $Q \in (-\infty, \infty)$, $\mathrm{Bi}(-Q)$ should be discarded because it diverges when $Q \to -\infty$ [14]. Then,

$$\varphi(Q) = \sqrt{2\pi}\mathrm{Ai}(-Q), \tag{2.8}$$

which goes to zero in both infinities. The physical state (2.8) is the Fourier transform of the wave function (2.4). One can guess what the singular operator $\hat{\mu}_{t_0}$ should be in the Q representation by inserting the identity twice in (2.5):

$$\hat{\mu}_{t_0}\varphi(Q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dQ' e^{-it_0(Q-Q')}\varphi(Q'). \tag{2.9}$$

Then,

$$(\varphi, \varphi) = \langle \varphi, \hat{\mu}_{t_0}\varphi\rangle = 1,$$

since

$$\left|\int_{-\infty}^{\infty} dQ\, e^{-it_0 Q}\mathrm{Ai}(Q)\right| = 1.$$

In addition,

$$-\hat{P}(\hat{\mu}_{t_0}\varphi) = t_0(\hat{\mu}_{t_0}\varphi),$$

in agreement with the fact that $-P$ is the time.

III. GENERALIZATION OF THE QUADRATIC CONSTRAINT

The constraint (2.6) is still too simple to fit those of the cosmological models. One can get a more general form for the constraint of the ideal clock, although keeping it quadratic in the momentum, by performing another canonical transformation, a coordinate change, to the variables (Ω, π_Ω) defined as

$$Q = V(\Omega), \quad P = \frac{d\Omega}{dQ}\pi_\Omega = \left(\frac{dV}{d\Omega}\right)^{-1}\pi_\Omega, \tag{3.1}$$

where $V(\Omega)$ is a monotonic function unbounded from above [14]. In these variables, the constraint turns out to be

$$\mathcal{H}(\Omega, \pi_\Omega) = -\left(\frac{dV}{d\Omega}\right)^{-2}\pi_\Omega^2 + V(\Omega), \tag{3.2}$$

and the global phase time is

$$t(\Omega, \pi_\Omega) = -P(\Omega, \pi_\Omega) = -\left(\frac{dV}{d\Omega}\right)^{-1}\pi_\Omega. \tag{3.3}$$

Since the coordinate change (3.1) should not modify the wave function (whenever the wave function is regarded as a scalar), the kinetic term of the constraint operator should be associated with the invariant D'Alembertian operator.

In general, the constraints appearing in problems of interest have the form

$$\bar{\mathcal{H}}(\Omega, \pi_\Omega) = g(\Omega)\pi_\Omega^2 + v(\Omega), \tag{3.4}$$

with $g(\Omega) < 0$. So we are faced with the issue of knowing whether or not a constraint such as (3.4) hides an ideal clock.

The constraint (3.4) describes an ideal clock if it has the form (1.7), where $\mathcal{H}$ should be the one of Eq. (3.2). Then

$$VV'^2 = -\frac{v}{g}. \tag{3.5}$$

Since $V(\Omega)$ in Eq. (3.1) is a monotonic function, $V(\Omega)$ has no more than one zero. Because of Eq. (3.5), the same must be accomplished by the function $v(\Omega)$ (in the other case, $\bar{\mathcal{H}}$ does not correspond with an ideal clock). Let Ω_0 be such that $v(\Omega_0) = 0$; then the solution of Eq. (3.5) is

$$V(\Omega) = \mathrm{sgn}(v)\left[\frac{3}{2}\int_{\Omega_0}^{\Omega}\left(\frac{|v|}{-g}\right)^{1/2} d\Omega\right]^{2/3}, \tag{3.6}$$

and

$$F = \frac{v}{V} = -gV'^2. \tag{3.7}$$

It might happen that $F(\Omega)$ is zero or ill defined in Ω_0. Since $\bar{\mathbf{H}}|_{\mathcal{H}=0} = F\mathbf{H}$, in such a case the system would stop or its evolution would be ill defined, when $Q_0 \equiv V(\Omega_0) = 0$ is reached, i.e., at $t(Q_0) = -P(Q_0) = 0$. So, in order that $t = -P$ can evolve to infinity, the time should be regarded as a positive variable, and only values $t_0 > 0$ should be considered in the gauge condition.

Since the constraints $\mathcal{H}$ and $\bar{\mathcal{H}} = F\mathcal{H}$ are equivalent, they should lead to the same quantization. In fact, the relation between both Dirac quantizations is

$$\begin{aligned} \hat{\bar{\mathcal{H}}} &= \hat{F}^{1/2}\hat{\mathcal{H}}\hat{F}^{1/2}, \\ \bar{\varphi} &= \hat{F}^{-1/2}\varphi, \\ \hat{\bar{\mu}} &= \hat{F}^{1/2}\hat{\mu}\hat{F}^{1/2}. \end{aligned} \tag{3.8}$$

Then,

$$\hat{\bar{\mathcal{H}}}\bar{\varphi} = 0,$$
$$(\bar{\varphi}, \bar{\varphi}) = (\varphi, \varphi).$$

The transformation (3.8) can be regarded as induced by a finite unitary transformation $U = \exp[(i/2)(\eta \ln F\, \mathcal{P} - \mathcal{P} \ln F\, \eta)]$, in the Hilbert space of the Becchi-Rouet-Stora-Tyutin (BRST) quantization where $(\eta, \mathcal{P})$ are the ghosts associated with the constraint (cf. Refs. [7,15]). In the next sections we shall examine in what extent the Hamiltonian constraint of the isotropic and homogeneous cosmological models correspond with an ideal clock.

IV. COSMOLOGICAL MODELS

The action of general relativity with cosmological constant $\tilde{\Lambda}$ is

$$S = -\frac{1}{16\pi G}\int dt\, d^3x \sqrt{-g}(\mathcal{R} + 2\tilde{\Lambda}),$$

where $\mathcal{R}$ is the curvature scalar. We shall restrict ourselves to spatially isotropic and homogeneous geometries, so we shall deal with Robertson-Walker metrics:

$$ds^2 = \tilde{N}^2(\tau)d\tau^2 - R(\tau)^2\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right),$$

with $K = \pm 1, 0$. In this way the original system is left with a finite number of degrees of freedom (*minisuperspace models*).

The Robertson-Walker metrics satisfy the requirement for the consistency of the minisuperspace models, viz. that the same dynamics must result either by replacing the minisuperspace metric in the Einstein equations or by varying the action with respect to the remaining degrees of freedom:

$$S = -\frac{3\pi}{4G}\int \tilde{N}\, d\tau \left(\frac{R\dot{R}^2}{\tilde{N}^2} - KR + \frac{\tilde{\Lambda}}{3}R^3\right).$$

The Hamiltonian form of the action is

$$S = \int d\tau(\dot{\Omega}\pi_\Omega - N\bar{\mathcal{H}}),$$

where

$$\bar{\mathcal{H}} = -\tfrac{1}{4}e^{-3\Omega}\pi_\Omega^2 - Ke^{\Omega} + \Lambda e^{3\Omega}, \qquad (4.1)$$

and

$$\Omega = \ln\left[\left(\frac{3\pi}{4G}\right)^{1/2} R\right], \quad N = \left(\frac{3\pi}{4G}\right)^{1/2}\tilde{N},$$
$$\Lambda = \left(\frac{3\pi}{4G}\right)^{-1}\frac{\tilde{\Lambda}}{3}. \qquad (4.2)$$

So

$$g(\Omega) = -\tfrac{1}{4}e^{-3\Omega}, \quad v(\Omega) = -Ke^{\Omega} + \Lambda e^{3\Omega}. \qquad (4.3)$$

We are going to look for a global phase time in the minisuperspaces resulting from the combinations of Λ and K, which make sense in the constraint equation. Thus, the cases $(\Lambda < 0; K = 0, 1)$ and $(\Lambda = 0; K = 1)$ are excluded.

In the Minkowski case $(K = 0 = \Lambda)$, the constraint (4.1) is satisfied only if $\pi_\Omega = 0$; the Hamiltonian vector is null in $\pi_\Omega = 0$: the system does not evolve. So this case is not an ideal clock.

Almost all of the remaining cases will prove to be ideal clocks. According to the method described in the former section, the potential $V(\Omega)$ is

$$V(\Omega) = \mathrm{sgn}(v)\left[3\int_{\Omega_0}^{\Omega} e^{2\Omega}\sqrt{|-K + \Lambda e^{2\Omega}|}d\Omega\right]^{2/3}. \qquad (4.4)$$

For the cases $(\Lambda \geq 0;\ K = -1)$ and $(\Lambda > 0;\ K = 0)$ it is $v(\Omega) > 0$, $\forall\Omega$; so the lower boundary in (4.4) should be $-\infty$ [thus $V \to 0$ when $\Omega \to -\infty$ as $-v/g$, in accordance with Eq. (3.5)]. When sgn $(\Lambda) = \mathrm{sgn}(K)$, it is

$$\Omega_0 = \tfrac{1}{2}\ln\left(\frac{K}{\Lambda}\right).$$

What follows is a summary of the results for each minisuperspace.

(1) $\Lambda = 0;\ K = -1$:

$$Q = V(\Omega) = \left(\tfrac{3}{2}\right)^{2/3}e^{4\Omega/3}, \quad Q \in (0, \infty),$$
$$F = \left(\tfrac{3}{2}\right)^{-2/3}e^{-\Omega/3} = \sqrt{2/3}Q^{-1/4}.$$

The global phase time is

$$t(\Omega, \pi_\Omega) = -P = -\tfrac{1}{2}\left(\tfrac{3}{2}\right)^{-1/3}e^{-(4/3)\Omega}\,\pi_\Omega.$$

t_0 should be taken to be positive $(P < 0)$ because F is not well behaved at $Q = 0$.

(2) $\Lambda > 0;\ K = 0, -1$:

$$Q = V(\Omega) = \Lambda^{-2/3}\left[K + (\Lambda e^{2\Omega} - K)^{3/2}\right]^{2/3}, \quad Q \in (0, \infty),$$

$$F = \Lambda^{2/3}e^{\Omega}[1 + K(\Lambda e^{2\Omega} - K)^{-3/2}]^{-2/3}$$
$$= \Lambda^{-1/2}Q^{-1}(\Lambda Q^{2/3} - K)^{2/3} \times\left[K + (\Lambda Q^{2/3} - K)^{2/3}\right]^{1/2},$$

and

$$t(\Omega, \pi_\Omega) = -P = -\tfrac{1}{2}\Lambda^{-1/3}[1 + K(\Lambda e^{2\Omega} - K)^{-3/2}]^{1/3} \times e^{-2\Omega}\,\pi_\Omega.$$

t_0 should be positive because F is not well behaved at $Q = 0$.

(3) $(\Lambda > 0;\ K = 1)$, $(\Lambda < 0;\ K = -1)$:

$$Q = V(\Omega) = |\Lambda|^{-2/3}(\Lambda e^{2\Omega} - K),$$
$$F = |\Lambda|^{2/3}e^{\Omega} = \left(\Lambda Q + |\Lambda|^{1/3}\right)^{1/2}.$$

In the case ($\Lambda < 0$; $K = -1$) the potential $V(\Omega)$ is bounded from above; therefore this case is not an ideal clock. In the case ($\Lambda > 0$; $K = 1$), $Q \in (-\Lambda^{-2/3}, \infty)$, and the global phase time is

$$t(\Omega, \pi_\Omega) = -P = -\tfrac{1}{2}\Lambda^{-1/3}e^{-2\Omega}\, \pi_\Omega, \quad -\infty < t_0 < \infty.$$

As we can see, it is possible to match several cosmological models with the ideal clock by means of an extrinsic time. We remark that an extrinsic time was already introduced by York [13] in the context of superspace for closed three-manifolds without cosmological constant. The York time is $t_{\rm York} = 2/3\gamma^{-1/2}\gamma^{ab}\pi_{ab}$; it is canonically conjugated to minus the volume scale: $p_{t_{\rm York}} = -\gamma^{1/2}$. On a hypersurface $t_{\rm York} = $ const, the Hamiltonian constraint determines the scale factor γ, once the traceless part p_{ab} of momenta π_{ab} satisfying the supermomentum constraints, and the unimodular metric $\sigma_{ab} = \gamma^{-1/3}\gamma_{ab}$, conjugated to p_{ab}, are given on the hypersurface. So $t_{\rm York}$ and σ_{ab} are good dynamic variables, and the Hamiltonian of the system is $h = -p_{t_{\rm York}} = \gamma^{1/2}(\sigma_{ab}, p_{ab}, t_{\rm York})$ [13,1].

The homogeneous and isotropic minisuperspaces studied in this section have the scale factor as their only dynamic variable; σ_{ab} and p_{ab} are frozen, and no such geometry exists in the superspace considered by York [$\Lambda = 0$, $K = 1$ does not make sense in the constraint coming from (4.1)]. So no comparison is possible with York's time.

V. GENUINE DEGREES OF FREEDOM

So far we have studied models that are too simple: they have no degrees of freedom; they are nothing but time. The following step should be the inclusion of genuine degrees of freedom, such as a matter field. In order to glance at how an extrinsic time should be managed in such a case, let us consider a homogeneous scalar field minimally coupled to a Robertson-Walker geometry with $K = 0$, $\Lambda > 0$. The action for a scalar field $\tilde{\phi}$ is

$$S_{\rm matter} = \tfrac{1}{2}\int dt\, d^3x\sqrt{-g}(g^{\mu\nu}\tilde{\phi}_{,\mu}\tilde{\phi}_{,\nu} - \tilde{m}^2\tilde{\phi}^2),$$

which turns out to be

$$S_{\rm matter} = \int \tilde{N}\, dt\, \pi^2 R^3\left(\frac{\dot{\tilde{\phi}}^2}{\tilde{N}^2} - \tilde{m}^2\tilde{\phi}^2\right),$$

in the minisuperspace under consideration.

The Hamiltonian form of this action is

$$S_{\rm matter} = \int d\tau[\dot{\phi}\,\pi_\phi + \dot{\Omega}\,\pi_\Omega - N(\tfrac{1}{4}e^{-3\Omega}\pi_\phi^2 + m^2e^{3\Omega}\phi^2)],$$

where

$$\phi = \pi\left(\frac{3\pi}{4G}\right)^{-1/2}\tilde{\phi}, \quad m = \left(\frac{3\pi}{4G}\right)^{-1/2}\tilde{m}.$$

The constraint for the entire system $S_{\rm grav} + S_{\rm matter}$ is

$$\bar{\mathcal{H}} - \tfrac{1}{4}e^{-3\Omega}(\pi_\phi^2 - \pi_\Omega^2) + m^2e^{3\Omega}\phi^2 + \Lambda e^{3\Omega}. \tag{5.1}$$

For simplicity, let us choose $m = 0$. Using the variables

$$t = -P = -\tfrac{1}{2}\Lambda^{-1/3}e^{-2\Omega}\pi_\Omega, \quad p_t = Q = \Lambda^{1/3}e^{2\Omega} > 0, \tag{5.2}$$

the constraint becomes

$$\bar{\mathcal{H}} = \Lambda^{1/2}p_t^{-3/2}\left(p_t^3 - t^2p_t^2 + \tfrac{1}{4}\pi_\phi^2\right). \tag{5.3}$$

We want to check that t is still time:

$$\{t, \bar{\mathcal{H}}\}|_{\bar{\mathcal{H}}=0} = \Lambda^{1/2}p_t^{-1/2}(3p_t - 2t^2).$$

So, in order that t remains time, it should be granted that $3p_t > 2t^2$ on all the dynamic trajectories. By introducing the variables

$$y \equiv \frac{3p_t}{2t^2} > 0, \quad b \equiv \frac{27\pi_\phi^2}{16t^2} \geq 0,$$

the constraint is written as

$$\bar{\mathcal{H}} = \frac{4}{27}\Lambda^{1/2}t^6p_t^{-3/2}(2y^3 - 3y^2 + b).$$

Thus the dynamics lies on the positive roots of a cubic polynomial parametrized by b. This polynomial has a maximum at $y_{\rm max} = 0$, and a minimum at $y_{\rm min} = 1$. One of its roots is $y_1(b) \leq 0$. If $b \leq 1$, there are two positive roots: $y_2(b) < 1 < y_3(b)$ ($b < 1$), and $y_2(b) = 1 = y_3(b)$ ($b = 1$). When $\pi_\phi = 0$ (i.e., $b = 0$) it is $y_1 = 0 = y_2$, $y_3 = \frac{3}{2}$. Therefore the dynamics lies in the root y_3, because $y_3 = \frac{3}{2}$ means $p_t = t^2$, which is the constraint equation of the former sections. So let us concentrate on the trajectories satisfying $y = y_3(b)$. As was said, t is time only if $y > 1$; since $y = y_3 \geq 1$, we shall only prove that y never reaches 1 along a dynamic trajectory by showing that y increases when b is near to 1:

$$\{y, \bar{\mathcal{H}}\}|_{\bar{\mathcal{H}}=0} = \tfrac{4}{27}\Lambda^{1/2}t^6p_t^{-3/2}\{y, b\}|_{\bar{\mathcal{H}}=0} = \tfrac{4}{3}\Lambda^{1/2}t^3b\; p_t^{-3/2}.$$

As in the case without matter, t should be intended to be positive. In fact, when t goes to zero, then p_t also goes to zero (because $1 \leq y = y_3 \leq \frac{3}{2}$); so we would be faced with a singular point of the Hamiltonian vector, where the dynamics is ill defined. Then y increases when $b \neq 0$, and stops when $b = 0$ at the value $y = y_3(b = 0) = 3/2$ [16]. Thus y remains bigger than 1, so proving that t is time if the constraint is satisfied through the root $y_3(b)$. The constraint $y = y_3(b)$ means

$$\mathcal{H} \equiv p_t - \tfrac{2}{3}t^2y_3(b) = 0. \tag{5.4}$$

The function $h(t, \pi_\phi) \equiv -\frac{2}{3}t^2y_3(b)$ is the Hamiltonian for the reduced system (associated with the chosen time t), so one could quantize the matter field by means of a Schrödinger equation,

$$\hat{\mathcal{H}}\psi(\phi, t) = 0,$$

achieving a unitary theory with the usual inner product:

$$(\psi_1, \psi_2) = \int d\phi\, \psi_1^*(\phi, t_0)\psi_2(\phi, t_0)$$
$$= \int dt\, d\phi\, \psi_1^* \delta(t - t_0)\psi_2, \quad t_0 > 0.$$

Unfortunately the root y_3 is such a complicated function of b as to desist from solving the Schrödinger equation.

As was shown, the internal Schrödinger interpretation can be connected with the solutions of the Wheeler-DeWitt equation

$$\hat{\bar{\mathcal{H}}}\left(\Omega, -i\frac{\partial}{\partial\Omega}, \phi, -i\frac{\partial}{\partial\phi}\right)\bar{\varphi} = 0,$$

where $\hat{\bar{\mathcal{H}}}$ is an operator associated with the constraint (5.1). This equation is much more tractable than the Schrödinger equation for the reduced system. The problem here is how the space of solutions obeying suitable boundary conditions is turned into a Hilbert space. In Secs. II and III we solved this issue in light of the trivial ideal clock: the physical inner product requires the insertion of a singular operator $\hat{\bar{\mu}}_{t_0}$:

$$(\bar{\varphi}_1, \bar{\varphi}_2) = \langle \bar{\varphi}_1, \hat{\bar{\mu}}_{t_0}\bar{\varphi}_2 \rangle.$$

In order to know the singular operator, we should factorize out the physical root in the constraint (5.3), in such a way that

$$\bar{\mathcal{H}} = F(t, p_t, \phi, \pi_\phi)\mathcal{H},$$

where $\mathcal{H}$ is the one of Eq. (5.4). Then

$$\hat{\bar{\mu}}_{t_0} = \hat{F}^{1/2}\hat{\mu}_{t_0}\hat{F}^{1/2},$$

where $\hat{\mu}_{t_0}$ is the one of Eq. (2.9), which results from applying the canonical transformation $(t, p_t) \to (Q(\Omega), P(\Omega, \pi_\Omega))$ to the insertion $\delta(t - t_0)$.

This completes the scheme of quantization based on the solutions of the Wheeler-DeWitt equation and an extrinsic time. In spite of appearances, difficulties are still present. In fact, the operator $\hat{F}$ is complicated enough to prevent one from computing the probabilities.

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[14] Q should be always intended to be unbounded from above, since this requirement warrants that $t = -P$ reaches infinity on the classical trajectory $P^2 = Q$, but there is no obstacle to regarding Q as belonging to an interval (A, ∞), rather than $(-\infty, \infty)$. In such a case one should select the wave function fulfilling $\varphi(A) = 0$, in order that $\hat{P} = -i\partial/\partial Q$ be Hermitian.

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[16] In the case $m \neq 0$, one can still prove that y never reaches 1 if $\Lambda^{-1}m^2 < 12$.