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Scattering off an SO(10) cosmic string

Anne-Christine Davis* and Rachel Jeannerot
Department of Applied Mathematics and Theoretical Physics, Cambridge University, Silver Street, Cambridge, CB3 9EW, United Kingdom
and Isaac Newton Institute for Mathematical Sciences, Cambridge University, 20 Clarkson Road, Cambridge, CB3 0EH, United Kingdom
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The scattering of fermions from the Abelian string arising during the phase transition $SO(10) \to SU(5) \times Z_2$ induced by the Higgs field in the **126** representation is studied. Elastic cross sections and baryon-number-violating cross sections due to the coupling to gauge fields in the core of the string are computed by both a first-quantized method and a perturbative second-quantized method. The elastic cross sections are found to be Aharonov-Bohm-type. However, there is a marked asymmetry between the scattering cross sections for left- and right-handed fields. The catalysis cross sections are small, depending on the grand unified scale. If cosmic strings were observed our results could help tie down the underlying gauge group.

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I. INTRODUCTION

Modern particle physics and the hot big-bang model suggest that the Universe underwent a series of phase transitions at early times at which the underlying symmetry changed. At such phase transitions topological defects [1] could be formed. Such topological defects, in particular cosmic strings, would still be around today and provide a window into the physics of the early Universe. In particular, cosmic strings arising from a grand unified phase transition are good candidates for the generation of density perturbations in the early Universe, which lead to the formation of large scale structure [2]. They could also give rise to the observed anisotropy in the microwave background radiation [3].

Cosmic strings also have interesting microphysical properties. Like monopoles [4], they can catalyze baryon-violating processes [5,6]. This is because the full grand unified symmetry is restored in the core of the string, and hence grand unified, baryon-violating processes are unsuppressed. In [6] it was shown that the cosmic string catalysis cross section could be a strong interaction cross section, independent of the grand unified scale, depending on the flux on the string. Unlike the case of monopoles, where there is a Dirac quantization condition, the string cross section is highly sensitive to the flux, and is a purely quantum phenomena. Defect catalysis is potentially important. It has already been used to bound the monopole flux [9], and could erase a primordial baryon asymmetry [10]. It is, thus, important to calculate the string catalysis cross section in a realistic grand unified theory. In [6] a toy model based on a U(1) theory was used. In a grand unified theory the string flux is given by the gauge group, and cannot be tuned.

A cosmic string is essentially a flux tube. Hence the elastic cross section [11] is just an Aharonov-Bohm cross section [12], depending on the string flux. This gives the dominant energy loss in a friction-dominated universe [13]. Since the string flux is fixed for any given particle species it is important to check that the Aharonov-Bohm cross section persists in a realistic grand unified theory.

In this paper we calculate the elastic and inelastic cross sections for cosmic strings arising from an SO(10) grand unified theory [14]. Cosmic strings arise in the breaking scheme [15] $SO(10) \to SU(5) \times Z_2$ where the breaking is due to the **126** representation of the Higgs field, the self-dual antisymmetric 5-index tensor of SO(10). These stable strings survive the subsequent transitions to $SU(3) \times SU(2) \times U(1) \times Z_2$ [15]. They have been studied elsewhere [17].

Now the SO(10) symmetry is restored inside the string core, and therefore there are baryon-number-violation processes mediated by the gauge fields X, Y, X', Y', and X_s of SO(10). We therefore expect a nonzero inelastic cross section which we will determine. This cross section should be running from a small cross section $O(\eta^{-1})$, where η is the grand unified scale $\sim 10^{15}$ GeV, to a much larger cross section of the order of the strong interaction.

The plan of this paper is as follows: In Sec. II we define an SO(10) string model. We give "top-hat" forms for the Higgs and gauge fields forming the string, since the "top-hat" core model does not affect the cross sections of interest [6]. Looking at the microscopic structure of the string core, we introduce the baryon-number-violating gauge fields of SO(10) present in the core of the string.

In Sec. III A we review the method used to calculate the scattering cross sections. There are two different approaches. A fundamental quantum-mechanical one and a perturbative second-quantized method [5,6]. The latter consists in calculating the geometrical cross section, i.e.,

*Also at King's College, Cambridge, England.

the scattering cross section for free fermionic fields. The catalysis cross section is then enhanced by an amplification factor to the power of four.

In Sec. III B we derive the equations of motion. In order to simplify the calculations and to get a fuller result, we also consider a "top-hat" core model for the gauge fields mediating quark to lepton transitions.

In Secs. III C and III D we calculate the solutions to the equations of motion outside and inside the string core, respectively, and in Sec. III E we match our solutions at the string radius. In Sec. III F we calculate the scattering amplitude for incoming plane waves of linear combinations of the quark and electron fields.

We use these results in Secs. IV and V in order to calculate the scattering cross sections of incoming beams of pure single fermion fields. In Sec. IV we calculate the elastic cross sections. And in Sec. VII we calculate the baryon-number-violation cross sections.

In Sec. VI we derive the catalysis cross section using the second-quantized method of Refs. [5,6]. The second-quantized cross sections are found to agree with the first-quantized cross section of Sec. V.

There are four Appendixes. Appendix A gives a brief review on SO(10) theory, and gives an explicit notation used everywhere in this paper. Appendixes B and C contain the technical details of the external and internal solutions calculations. Finally, Appendix D is a discussion of the matching conditions at the core radius.

II. AN SO(10) STRING

In Appendix A we give a brief review of SO(10) theory. With that notation, the Lagrangian is

$$L = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi_{126})^\dagger(D^\mu\Phi_{126}) - V(\Phi) + L_F \ , \tag{1}$$

where $F_{\mu\nu} = -i\,F^a_{\mu\nu}\tau_a$, τ_a $a = 1,\ldots,45$ are the **45** generators of SO(10). Φ_{126} is the Higgs **126**, the self-dual antisymmetric 5-index tensor of SO(10). L_F is the fermionic part of the Lagrangian. In the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, $A_\mu = A^a_\mu\tau_a$, where A^a_μ $a = 1,\ldots,45$ are **45** gauge fields of SO(10).

We assume that the universe undergoes the breaking scheme

$$\mathrm{SO}(10) \overset{\langle\phi_{126}\rangle}{\to} \mathrm{SU}(5)\times Z_2 \overset{\langle\phi_{45}\rangle}{\to} \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times Z_2 \overset{\langle\phi_{10}\rangle}{\to} \mathrm{SU}(3)\times\mathrm{U}(1)_Q\times Z_2 \ ,$$

giving vacuum expectation values to the components of the **10** which correspond to the usual Higgs doublet. The decomposition of the **126** representation under SU(5) × U(1) is given by

$$\mathbf{126} = \mathbf{1}_{10} + \cdots \ . \tag{2}$$

The first transition is achieved by giving vacuum expectation value to the component of the **126** in the $\mathbf{1}_{10}$ direction. The first homotopy group $\pi_1[\mathrm{SO}(10)/\mathrm{SU}(5)\times Z_2]$ is Z_2, and therefore Z_2 strings are formed. In terms of SU(5), the **45** generators of SO(10) can be decomposed as

$$\mathbf{45} = \mathbf{24} + \mathbf{1} + \mathbf{10} + \overline{\mathbf{10}} \ . \tag{3}$$

From the **45** generators of SO(10), 24 belong to SU(5), 1 generator corresponds to the U(1)′ symmetry in SO(10) not embedded in SU(5), and there are 20 remaining ones. Therefore, the breaking of SO(10) to SU(5) × Z_2 induces the creation of two types of strings. An Abelian one, corresponding to the U(1)′ symmetry, and a non-Abelian one made with linear combinations of the 20 remaining generators. In this paper we are interested in the Abelian strings since the non-Abelian version are Alice strings, and would result in global quantum number being ill defined, and hence unobservable [7]. We note that there is a wide range of parameters where the non-Abelian strings have lower energy [17]. However, since the Abelian string is topologically stable, there is a final probability that it could be formed by the Kibble mechanism [8].

If we call $\tau_{\rm str}$ the generator of the Abelian string, $\tau_{\rm str}$ will be given by the diagonal generators of SO(10) not lying in SU(5), that is,

$$\tau_{\rm str} = \frac{1}{5}\left(M_{12} + M_{34} + M_{56} + M_{78} + M_{9\,10}\right) \ , \tag{4}$$

where M_{ij} : $i,j = 1,\ldots,10$ are the 45 SO(10) generators defined in Appendix A in terms of the generalized γ matrices. Numerically, this gives

$$\tau_{\rm str} = \mathrm{diag}\left(\frac{1}{2},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{-3}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{-3}{10},\frac{1}{10},\frac{-3}{10},\frac{-3}{10},\frac{-3}{10}\right) . \tag{5}$$

The results of Perkins *et al.* [6] find that the greatest enhancement of the cross section is for fermionic charges close to integer values. Thus, from Eq. (5), we expect no great enhancement; the most being due to the right-handed neutrino.

We are going to model our string as is usually done for an Abelian U(1) string. That is, we take the string along the z axis, resulting in the Higgs field Φ_{126} and the gauge field A_μ of the string to be independent of the z coordinate, depending only on the polar coordinates (r,θ). Here A_μ is the gauge field of the string, obtained from the product $A_\mu = A_{\mu,\rm str}\tau_{\rm str}$. The solution for the Abelian string can be written as

$$\Phi_{126} = f(r)\,e^{i\tau_{\rm str}\theta}\,\Phi_0 = f(r)\,e^{i\theta}\,\Phi_0 \ , \tag{6}$$

$$A_\theta = -\frac{g(r)}{er}\tau_{\rm str}, \quad A_r = A_z = 0 \ , \tag{7}$$

where Φ_0 is the vacuum expectation value of the Higgs field **126** in the $\mathbf{1}_{10}$ direction. The functions $f(r)$ and $g(r)$ describing the behavior the Higgs and gauge fields forming the string are given by

$$f(r) = \begin{cases} \eta, & r \geq R, \\ \eta(\frac{r}{R}), & r < R, \end{cases} \quad g(r) = \begin{cases} 1, & r \geq R, \\ (\frac{r}{R})^2, & r < R, \end{cases} \tag{8}$$

where R is the radius of the string. $R \sim \eta^{-1}$, where η is the grand unified scale, assumed to be $\eta \sim 10^{15}$ GeV. In order to simplify the calculations and to get a fuller result we use the top-hat core model, since it has been shown not to affect the cross sections of interest. The top-hat core model assumes that the Higgs and gauge fields forming the string are zero inside the string core. Hence, $f(r)$ and $g(r)$ are now given by

$$f(r) = \begin{cases} \eta, & r \geq R, \\ 0, & r < R, \end{cases} \quad g(r) = \begin{cases} 1, & r \geq R, \\ 0, & r < R. \end{cases} \tag{9}$$

The full SO(10) symmetry is restored in the core of the string. SO(10) contains five gauge bosons leading to baryon decay. These are the bosons X and Y of SU(5) plus three other gauge bosons usually called X', Y', and X_s. Therefore inside the string core, there are quark-to-lepton transitions mediated by the gauge bosons X, X', Y, Y', and X_s, and we expect the string to catalyze baryon-number-violating processes in the early Universe.

The X, X', Y, Y', and X_s gauge bosons are associated with nondiagonal generators of SO(10). For the electron family, the relevant part of the Lagrangian is given by

$$L_x = \bar{\Psi}_{16} \left[ie\gamma^\mu (X_\mu \tau^X + X'_\mu \tau^{X'} + Y_\mu \tau^Y + Y'_\mu \tau^{Y'} + X_{s\mu} \tau^{X_s}) \right] \Psi_{16} \ , \tag{10}$$

where τ^X, $\tau^{X'}$, τ^Y, and $\tau^{Y'}$ and τ^{X_s} are the nondiagonal generators of SO(10) associated with the X, X', Y, Y', and X_s gauge bosons, respectively.

Expanding Eq. (10) gives [19]

$$\begin{aligned} L_x = {} & \frac{g}{\sqrt{2}} X^\alpha_\mu [-\epsilon_{\alpha\beta\gamma} \bar{u}^{c\gamma}_L \gamma^\mu u^\beta_L + \bar{d}_{L\alpha} \gamma^\mu e^+_L + \bar{d}_{R\alpha} \gamma^\mu e^+_R] + \frac{g}{\sqrt{2}} Y^\alpha_\mu [-\epsilon_{\alpha\beta\gamma} \bar{u}^{c\gamma}_L \gamma^\mu d^\beta_L - \bar{d}_{R\alpha} \gamma^\mu \nu^c_e - \bar{u}_{L\alpha} \gamma^\mu e^+_L] \\ & + \frac{g}{\sqrt{2}} X^{\alpha'}_\mu [-\epsilon_{\alpha\beta\gamma} \bar{d}^{c\gamma}_L \gamma^\mu d^\beta_L - \bar{u}_{R\alpha} \gamma^\mu \nu^c_R - \bar{u}_{L\alpha} \gamma^\mu \nu^c_L] + \frac{g}{\sqrt{2}} Y^{\alpha'}_\mu [\epsilon_{\alpha\beta\gamma} \bar{d}^{c\gamma}_L \gamma^\mu u^\beta_L - \bar{u}_{R\alpha} \gamma^\mu e^+_R - \bar{d}_{L\alpha} \gamma^\mu \nu^c_L] \\ & + \frac{g}{\sqrt{2}} X^\alpha_{s\mu} [\bar{d}_{L\alpha} \gamma^\mu e^-_L + \bar{d}_{R\alpha} \gamma^\mu e^-_R + \bar{u}_{L\alpha} \gamma^\mu \nu_L + \bar{u}_{R\alpha} \gamma^\mu \nu_R] \ , \end{aligned} \tag{11}$$

where α, β, and γ are color indices. The X_s does not contribute to nucleon decay except by mixing with the X' because there is no vertex qqX_s. We consider baryon-violating processes mediated by the gauge fields X, X', Y, and Y' of SO(10). In previous papers [6,11], baryon-number-violating processes resulting from the coupling to scalar condensates in the string core have been considered. In our SO(10) model we do not have such a coupling.

III. SCATTERING OF FERMIONS FROM THE ABELIAN STRING

A. The scattering cross section

Here, we will briefly review the two methods used to calculate the scattering cross section. The first is a quantum-mechanical treatment. From the fermionic Lagrangian L_F, we derive the equations of motion inside and outside the string core. We then find solutions to the equations of motion inside and outside the string core and we match our solutions at the string core. Considering incoming plane waves of pure quarks, we then calculate the scattering amplitude. The matching conditions together with the scattering amplitude enable us to calculate the elastic and inelastic scattering cross sections. The second method is a quantized one, where one calculates the geometrical cross section $(\frac{d\sigma}{d\Omega})_{\text{geom}}$, i.e., using free fermions spinors ψ_{free}. The catalysis cross section is enhanced by a factor $\mathcal{A}^4$ over the geometrical cross section,

$$\sigma_{\text{inel}} = \mathcal{A}^4 \left(\frac{d\sigma}{d\Omega} \right)_{\text{geom}} \ , \tag{12}$$

where the amplification factor $\mathcal{A}$ is defined by

$$\mathcal{A} = \frac{\psi(R)}{\psi_{\text{free}}(R)} \ , \tag{13}$$

where R is the radius of the string, $R \sim \eta^{-1}$. This method has been applied in Refs. [6,18].

B. The equations of motion

The fermionic part of the Lagrangian L_F is given in terms of 16-dimensional spinors as defined in Appendix A. We shall consider only one family in this work, and in particular the electron family. The fermionic Lagrangian for only one family,

$$L_F = L_F^{(e)} = \bar{\Psi}_{16} \gamma^\mu D_\mu \Psi_{16} + L_M + L_x \ , \tag{14}$$

where L_M is the mass term and L_x is the Lagrangian describing quark-to-lepton transitions through the X, X', Y, Y', and X_s gauge bosons in SO(10) and given by Eq. (11). The covariant derivative is given by $D_\mu = \partial_\mu - ieA_{\mu,\text{str}}\tau_{\text{str}}$, where $A_{\mu,\text{str}}$ is the gauge field forming the string and τ_{str} is the string generator given by Eq. (5). Therefore, since τ_{str} is diagonal, there will be no mixing of fermions around the string. The Lagrangian L_F will split in a sum of eight terms, one for each fermion of the family. In terms of four-spinors, this is

$$L_F = \sum_{i=1}^{8} L^i_f + L_x \tag{15}$$

where $L^i_f = i\bar{\psi}^i_L \gamma^\mu D^L_\mu \psi^i_L + i\bar{\psi}^{c,i}_L \gamma^\mu D^{Lc}_\mu \psi^{c,i}_L + L^i_m$, and i

runs over all fermions of the given family. One can show that $i\bar{\psi}_L^{c,i}\gamma^\mu D_\mu^L \psi_L^{c,i} = i\bar{\psi}_R^i \gamma^\mu D_\mu^R \psi_R^i$ and $\tau_{\rm str}^{Lc\,i} = \tau_{\rm str}^{R\,i}$. Finally, L_x is given by Eq. (11). It is easy to generalize to more families.

From Eqs. (15) and (11) we derive the equations of motions for the fermionic fields. We take the fermions to be massless inside and outside the string core. This is a relevant assumption since our methods apply for energies above the confinement scale. We consider the case of free quarks scattering from the string and coupling with electrons inside the string core. Outside the string core, the fermions feel the presence of the string only by the presence of the gauge field. We are interested in the elastic cross sections for all fermions and in the cross section for these quark decaying into electron. The fermionic Lagrangian given by Eqs. (15) and (11) becomes

$$\begin{aligned} L_F(e,q) &= i\bar{e}_L\gamma^\mu D_\mu^{e,L} e_L + i\bar{e}_R\gamma^\mu D_\mu^{e,R} e_R + i\bar{q}_L\gamma^\mu D_\mu^{q,L} q_L \\ &\quad + i\bar{q}_R\gamma^\mu D_\mu^{q,R} q_R - \frac{gG^\mu}{2\sqrt{2}}\bar{q}_L\gamma_\mu e_L^+ - \frac{g{G'}^\mu}{2\sqrt{2}}\bar{q}_R\gamma_\mu e_R^+ \\ &\quad + \text{H.c.}\ , \end{aligned} \tag{16}$$

giving the equations of motion

$$\begin{aligned} &i\gamma^\mu D_\mu^{e,L} e_L + \frac{gG'_\mu}{2\sqrt{2}}\gamma^\mu q_L^c = 0\ , \\ &i\gamma^\mu D_\mu^{e,R} e_R + \frac{gG_\mu}{2\sqrt{2}}\gamma^\mu q_R^c = 0\ , \\ &i\gamma^\mu D_\mu^{q^c,L} q_L^c + \frac{gG'_\mu}{2\sqrt{2}}\gamma^\mu e_L^- = 0\ , \\ &i\gamma^\mu D_\mu^{q^c,R} q_R^c + \frac{gG_\mu}{2\sqrt{2}}\gamma^\mu e_R^- = 0\ , \end{aligned} \tag{17}$$

which are valid everywhere. The covariant derivatives $D_\mu^{e,(L,R)} = \partial_\mu + ieA_{\mu,{\rm str}}\tau_{\rm str}^{e,(L,R)}$ and $D_\mu^{q^c,(L,R)} = \partial_\mu + ieA_{\mu,{\rm str}}\tau_{\rm str}^{q,(L,R)}$. We have $\tau_{\rm str}^{R,u} = \tau_{\rm str}^{L,u} = \tau_{\rm str}^{L,e} = \tau_{\rm str}^{L,d} = \frac{1}{10}$ and $\tau_{\rm str}^{R,e} = \tau_{\rm str}^{R,d} = \frac{-3}{10}$, together with $\tau_{\rm str}^{Lc,\,i} = \tau_{\rm str}^{R,\,i}$ and $\tau_{\rm str}^{L,\,i} = \tau_{\rm str}^{Rc,\,i}$. G_μ and G'_μ stand for X_μ, X'_μ, Y'_μ, or Y'_μ, depending on the chosen quark.

Since these equations involve quarks and lepton mixing, we do not find an independent solution for the quark and lepton fields. However, we can solve these equations taking linear combinations of the quark and lepton fields, $q_L^c \pm e_L$ and $q_R^c \pm e_R$. In this case, the effective gauge fields are

$$e\left(A_{\mu,{\rm str}}\tau_{\rm str}^{f_L} \pm G_\mu\right) \tag{18}$$

and

$$e\left(A_{\mu,{\rm str}}\tau_{\rm str}^{f_R} \pm G'_\mu\right)\ , \tag{19}$$

respectively.

In order to make the calculations easier, we use a top-hat θ component for G and G' within the string core, since Perkins *et al.* [6] have shown that the physical results are insensitive to the core model used for the gauge fields mediating baryon-violating processes.

C. The external solution

Outside the string core, the gauge field of the string $A_{\mu,{\rm str}}$ has only, from Eqs. (7) and (9), a nonvanishing component $A_\theta = \frac{1}{er}\tau_{\rm str}$, and the effective gauge fields G and G' are set to zero. Therefore the equations of motion (17) for $r > R$ become

$$\begin{aligned} &i\gamma^\mu D_\mu^{e,L} e_L = 0,\quad i\gamma^\mu D_\mu^{e,R} e_R = 0\ , \\ &i\gamma^\mu D_\mu^{u,L} q_L^c = 0,\quad i\gamma^\mu D_\mu^{u,R} q_R^c = 0\ , \end{aligned} \tag{20}$$

where the covariant derivatives $D_\mu^{e,(L,R)} = \partial_\mu + ieA_{\mu,{\rm str}}\tau_{\rm str}^{e,(L,R)}$ and $D_\mu^{q^c,(L,R)} = \partial_\mu + ieA_{\mu,{\rm str}}\tau_{\rm str}^{q,(L,R)}$.

We take the usual Dirac representation $e_L = (0,\xi_e)$, $e_R = (\chi_e, 0)$, $q_L^c = (0,\xi_{q^c})$, and $q_R^c = (\chi_{q^c},0)$ and the mode decomposition for the spinors ξ_{q^c}, ξ_e, χ_{q^c}, and χ_e,

$$\begin{aligned} \chi_{(e,q^c)}(r,\theta) &= \sum_{n=-\infty}^{n=+\infty}\begin{pmatrix} \chi^n_{1\,(e,q^c)}(r) \\ i\ \chi^n_{2\,(e,q^c)}(r)\,e^{i\theta}\end{pmatrix} e^{in\theta}\ , \\ \xi_{(e,q^c)}(r,\theta) &= \sum_{n=-\infty}^{n=+\infty}\begin{pmatrix} \xi^n_{1\,(e,q^c)}(r) \\ i\ \xi^n_{2\,(e,q^c)}(r)\,e^{i\theta}\end{pmatrix} e^{in\theta}\ . \end{aligned} \tag{21}$$

From Appendix B we see that the fields $\xi^n_{1,(e,q^c)}$, $\xi^n_{2,(e,q^c)}$, $\chi^n_{1,(e,q^c)}$, and $\chi^n_{2,(e,q^c)}$ satisfy Bessel equations of order $n-\tau_{\rm str}^{R\,(e,q^c)}$, $n+1-\tau_{\rm str}^{R\,(e,q^c)}$, $n-\tau_{\rm str}^{L\,(e,q^c)}$, and $n-\tau_{\rm str}^{R\,(e,q^c)}$, respectively. The external solution becomes

$$\begin{pmatrix} \xi_{(e,q^c)}(r,\theta) \\ \chi_{(e,q^c)}(r,\theta)\end{pmatrix} = \sum_{n=-\infty}^{n=+\infty}\begin{pmatrix} [v_n^{(e,q^c)} J_{n-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r) & + & v_n^{(e,q^c)'} J_{-(n-\tau_{\rm str}^{R\,(e,q^c)})}(\omega r)] & e^{in\theta} \\ i\ [v_n^{(e,q^c)} J_{n+1-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r) & - & v_n^{(e,q^c)'} J_{-(n+1-\tau_{\rm str}^{R\,(e,q^c)})}(\omega r)] & e^{i(n+1)\theta} \\ [w_n^{(e,q^c)} J_{n-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r) & + & w_n^{(e,q^c)'} J_{-(n-\tau_{\rm str}^{L\,(e,q^c)})}(\omega r)] & e^{in\theta} \\ i\ [w_n^{(e,q^c)} J_{n+1-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r) & - & w_n^{(e,q^c)'} J_{-(n+1-\tau_{\rm str}^{L\,(e,q^c)})}(\omega r)] & e^{i(n+1)\theta}\end{pmatrix}. \tag{22}$$

Therefore, outside the string core, we have independent solutions for the quark and electron fields.

D. The internal solution

Inside the string core, the gauge field of the string, A_μ, is set to zero whereas G_θ and G'_θ take the values $2\sqrt{2}A$ and $2\sqrt{2}A'$, respectively. Therefore, the equations of motion (17) become

$$\begin{aligned} &i\gamma^\mu\partial_\mu e_L + \frac{gG'_\mu}{2\sqrt{2}}\gamma^\mu q_L^c = 0\ , \\ &i\gamma^\mu\partial_\mu e_R + \frac{gG_\mu}{2\sqrt{2}}\gamma^\mu q_R^c = 0\ , \\ &i\gamma^\mu\partial_\mu q_L^c + \frac{gG'_\mu}{2\sqrt{2}}\gamma^\mu e_L^- = 0\ , \\ &i\gamma^\mu\partial_\mu q_R^c + \frac{gG_\mu}{2\sqrt{2}}\gamma^\mu e_R^- = 0\ . \end{aligned} \tag{23}$$

Since these equations of motions involve quark-lepton mixings, there are no independent solutions for the quarks and electron fields. However, we get solutions for the fields $\rho^{\pm}$ and $\sigma^{\pm}$ which are linear combinations of the quarks and electron fields:

$$\rho^{\pm} = \chi_{q^c} \pm \chi_e \tag{24}$$

and

$$\sigma^{\pm} = \xi_{q^c} \pm \xi_e \,. \tag{25}$$

Using the mode decomposition (21) for the fields $\rho^{\pm}$ and $\sigma^{\pm}$, the internal solution becomes

$$\begin{pmatrix} \rho^{\pm}_{n1}\, e^{in\theta} \\ i\;\rho^{\pm}_{n2}\, e^{i(n+1)\theta} \\ \sigma^{\pm}_{n1}\, e^{in\theta} \\ i\;\sigma^{\pm}_{n2}\, e^{i(n+1)\theta} \end{pmatrix} , \tag{26}$$

where $\rho^{\pm}_{n1}$ ($\rho^{\pm}_{n2}$) and $\sigma^{\pm}_{n1}$ ($\sigma^{\pm}_{n2}$) are the upper (lower) components of the fields $\rho^{\pm}$ and $\sigma^{\pm}$, respectively. They are given in terms of hypergeometric functions. From Appendix C we get

$$\rho^{\pm}_{n1} = (kr)^{|n|} e^{-ikr} \sum_{j=0}^{n=+\infty} \alpha^{\pm}_j \frac{(2ikr)^j}{j!} , \tag{27}$$

where $k^2 = w^2 - (eA)^2$, $e = \frac{g}{2\sqrt{2}}$. $\alpha^{\pm}_{j+1} = \frac{(a^{\pm}+j)}{(b+p)} \alpha^{\pm}_j$ with $a^{\pm} = \frac{1}{2} + |n| \pm \frac{eA(2n+1)}{2ik}$ and $b = 1 + 2|n|$. $\rho^{\pm}_{n2}$ can be obtained using the coupled equation (C1b) of Appendix C. We find

$$\rho^{\pm}_{n2} = -\frac{1}{w} (kr)^{|n|} e^{-ikr} \times \sum_{j=0}^{n=+\infty} \alpha^{\pm}_j \frac{(2ikr)^j}{j!} \left(\frac{|n|-n}{r} - ik + \frac{j}{r} \pm eA \right) . \tag{28}$$

We get similar hypergeometric functions for the fields $\sigma^{\pm}_{n1}$ and $\sigma^{\pm}_{n2}$.

E. Matching at the string core

From now on we will do calculations for the right-handed fields, the calculations for the left-handed ones being straightforward. Once we have our internal and external solutions, we match them at the string core. We must take the same linear combinations of the quark and lepton fields outside and inside the core, and must have continuity of the solutions at $r = R$. The continuity of the solutions at $r = R$ implies

$$(\chi^n_{1,q} \pm \chi^n_{1,e})^{\text{out}} = \rho^{\pm\,\text{in}}_{n1} , \tag{29}$$

$$(\chi^n_{2,q} \pm \chi^n_{2,e})^{\text{out}} = \rho^{\pm\,\text{in}}_{n2} . \tag{30}$$

Nevertheless, we will have discontinuity of the first derivatives:

$$\left(\frac{d}{dr} \mp eA\right) \rho^{\pm\,\text{in}}_{n2} = \left(\frac{d}{dr} - \frac{\tau^{R\,(e,q^c)}_{\text{str}}}{R}\right) (\chi^n_{2,q} \pm \chi^n_{2,e})^{\text{out}} \tag{31}$$

$$\left(\frac{d}{dr} \pm eA\right) \rho^{\pm\,\text{in}}_{n1} = \left(\frac{d}{dr} + \frac{\tau^{R\,(e,q^c)}_{\text{str}}}{R}\right) (\chi^n_{1,q} \pm \chi^n_{1,e})^{\text{out}} . \tag{32}$$

These equations lead to a relation between the coefficients of the Bessel functions for the external solution, as derived in Appendix D:

$$\frac{v^{q'}_n \pm v^{e'}_n}{v^q_n \pm v^e_n} = \frac{w\, l^{\pm}_n J_{n+1-\tau_R}(wR) + J_{n-\tau_R}(wR)}{w\, l^{\pm}_n J_{-(n+1-\tau_R)}(wR) + J_{-(n-\tau_R)}(wR)} , \tag{33}$$

where

$$l^{\pm}_n = \frac{\sum_{j=0}^{n=+\infty} \alpha^{\pm}_j \frac{(2ikr)^j}{j!}}{\sum_{j=0}^{n=+\infty} \alpha^{\pm}_j \frac{(2ikr)^j}{j!} \left(\frac{|n|-n}{r} - ik + \frac{j}{r} \pm eA\right)} . \tag{34}$$

The relations (33) and (34) are the matching conditions at $r = R$.

F. The scattering amplitude

In order to calculate the scattering amplitude, we match our solutions to an incoming plane wave plus an outgoing scattered wave at infinity. However, since the internal solution, and therefore the matching conditions at $r = R$, are given in terms of linear combinations of quarks and leptons, we consider incoming waves of such linear combinations. Let $f^{\pm}_n$ denote the scattering amplitude for the mode n, f^{+}_n if we consider the scattering of (quarks + electrons) and f^{-}_n if we consider the scattering of (quarks − electrons). Then the matching conditions at infinity are

$$(-i)^n \begin{pmatrix} J_n \\ iJ_{n+1}\, e^{i\theta} \end{pmatrix} + \frac{f^{\pm}_n e^{ikr}}{\sqrt{r}} \begin{pmatrix} 1 \\ i\;e^{i\theta} \end{pmatrix} = \begin{pmatrix} (v^q_n \pm v^e_n) J_{n-\tau_R} + (v^{q'}_n \pm v^{e'}_n) J_{-(n-\tau_R)} \\ i\;[(v^q_n \pm v^e_n) J_{n+1-\tau_R} + (v^{q'}_n \pm v^{e'}_n) J_{-(n+1-\tau_R)}]\; e^{i\theta} \end{pmatrix} . \tag{35}$$

Using then the large r forms for the Bessel functions,

$$J_\mu(\omega r) = \sqrt{\frac{2}{\pi\omega r}} \cos\left(\omega r - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) , \tag{36}$$

and matching the coefficients of $e^{i\omega r}$, we find

$$\sqrt{2\pi\omega} f_n^{\pm} e^{i\frac{\pi}{4}} = \begin{cases} e^{-in\pi}(e^{i\tau_R\pi} - 1) + (v_n^{q'} \pm v_n^{e'})e^{i(n-\tau_R)\frac{\pi}{2}}(1 - e^{-2i(n-\tau_R)\pi}), \\ e^{in\pi}(e^{i(n-\tau_R)\pi} - e^{-in\pi}) + (v_n^{q} \pm v_n^{e\pm})e^{-i(n-\tau_R)\frac{\pi}{2}}(1 - e^{2i(n-\tau_R)\pi}) . \end{cases} \tag{37}$$

Matching the coefficients $e^{-i\omega r}$, we get relations between the Bessel functions coefficients:

$$(v_n^q \pm v_n^e) = [1 - (v_n^{q'} \pm v_n^{e'})e^{-i(n-\tau_R)\frac{\pi}{2}}]\, e^{-i(n-\tau_R)\frac{\pi}{2}} . \tag{38}$$

The relations (37), (38), (33), and (34) determine the scattered wave.

IV. THE ELASTIC CROSS SECTION

When there are no baryon-number-violating processes inside the string core, when the gauge fields mediating quark-to-lepton transitions are set to zero, we have elastic scattering. In this case, the scattering amplitude reduces to

$$f_n^{\text{elast}} = \frac{1}{\sqrt{2\pi\omega}} e^{-i\frac{\pi}{4}} \begin{cases} e^{-in\pi}\,(e^{i\tau_R\pi} - 1), & n \geq 0, \\ e^{in\pi}\,(e^{-i\tau_R\pi} - 1), & n \leq -1. \end{cases} \tag{39}$$

The elastic cross section per unit length is given by

$$\sigma_{\text{elast}} = \left| \sum_{n=-\infty}^{+\infty} f_n^{\text{elast}} e^{in\theta} \right|^2 . \tag{40}$$

Using the relations $\sum_{n=a}^{+\infty} e^{inx} = \frac{e^{iax}}{1-e^{ix}}$ and $\sum_{n=-\infty}^{b} e^{inx} = \frac{e^{ibx}}{1-e^{-ix}}$, we find the elastic cross section to be

$$\sigma_{\text{elast}} = \frac{1}{2\pi\omega} \frac{\sin^2 \tau_R\pi}{\cos^2 \frac{\theta}{2}} . \tag{41}$$

This is an Aharonov-Bohm cross section, and τ_R is the flux in the core of the string.

Now, remember that $\tau_{\text{str}}^{Lc,u} = \tau_{\text{str}}^{L,u} = \tau_{\text{str}}^{Lc,e} = \tau_{\text{str}}^{L,d} = \frac{1}{10}$, $\tau_{\text{str}}^{L,e} = \tau_{\text{str}}^{Lc,d} = \frac{-3}{10}$, and $\tau_{\text{str}}^{Lc,\,i} = \tau_{\text{str}}^{R,\,i}$ and $\tau_{\text{str}}^{L,\,i} = \tau_{\text{str}}^{Rc,\,i}$. Hence,

$$\sigma_{\text{elast}}^{e_L} = \sigma_{\text{elast}}^{d_R} > \sigma_{\text{elast}}^{e_R} = \sigma_{\text{elast}}^{u_R} = \sigma_{\text{elast}}^{d_L} = \sigma_{\text{elast}}^{u_L} . \tag{42}$$

We therefore have a marked asymmetry between fermions. We have a marked asymmetry between left- and right-handed electrons, left- and right-handed down quarks or, since $\sigma_{\text{elast}}^{i_{Lc}} = \sigma_{\text{elast}}^{i_R}$ and $\sigma_{\text{elast}}^{i_{Rc}} = \sigma_{\text{elast}}^{i_L}$, between left-handed particles and antiparticles, respectively, right-handed, for the electron and the down quark. But we have equal cross sections for right-handed particles and left-handed antiparticles for the electrons and the down quark, and equal cross sections for both left-handed and right-handed up quarks and antiquarks. This is a marked feature of grand unified theories. If cosmic strings are found, it may be possible to use this asymmetry to identify the underlying gauge symmetry.

V. THE INELASTIC CROSS SECTION

The gauge fields X, X', Y, and Y' are now "switched on." In this case we are calculating the baryon-number-violating cross section. If we consider identical beams of incoming pure ρ^+ and ρ^-, recalling that $\rho^{\pm} = \chi_{q^c} \pm \chi_e$, this will ensure that we will have an incoming beam of pure quark. Therefore, the scattering amplitude for the quark field is given by half the difference of f_n^+ and f_n^-, and the scattering amplitude for the electron field is given by half the sum of f_n^+ and f_n^-. From Eq. (37) we get

$$\frac{1}{2}\sqrt{2\pi\omega}\,(f_n^+ - f_n^-)\, e^{i\frac{\pi}{4}} = v_n^e\, e^{-i(n-\tau_R)\frac{\pi}{2}}\,(1 - e^{-\tau_R 2\pi}) . \tag{43}$$

The inelastic cross section for the quark field is given by

$$\sigma_{\text{inel}} = \left| \sum_{n=-\infty}^{+\infty} (f_n^+ - f_n^-)\, e^{in\theta} \right|^2 . \tag{44}$$

Hence, from Eq. (43),

$$\sigma_{\text{inel}} \sim \frac{1}{\omega} \left| \sum_{n=-\infty}^{+\infty} v_n^e\, e^{-in(\frac{\pi}{2}-\theta)} \right|^2 . \tag{45}$$

Using Eqs. (33), (34), and (38), we find

$$v_n^e = \frac{e^{i(n-\tau_R)\frac{\pi}{2}}}{2} \left(\frac{1}{\delta_n^+ + e^i(n - \tau_R\pi)} - \frac{1}{\delta_n^- + e^i(n - \tau_R\pi)} \right) , \tag{46}$$

where

$$\delta_n^{\pm} = \frac{w\, l_n^{\pm} J_{n+1-\tau_R}(wR) + J_{n-\tau_R}(wR)}{w\, l_n^{\pm} J_{-(n+1-\tau_R)}(wR) + J_{-(n-\tau_R)}(wR)} \tag{47}$$

and $\lambda^{\pm}$ are given by Eqs. (34). Equations (45), (46), and (47) determine the inelastic cross section. This is given in terms of a power series. However, using small argument expansions for Bessel functions, we conclude that this power series involves always one dominant term, the other terms being suppressed by a factor $(\omega R)^n$, where n is an integer such that $n \geq 1$. Therefore the inelastic cross section involves one dominant mode, the other modes being exponentially suppressed. If d denotes the dominant mode we get $\sigma_{\text{inel}} \sim \frac{1}{\omega}|v_d^e|^2$. The value of the dominant mode depends on the sign of the fractional flux τ_{str}. Our results can be summarized as follows.

For $0 < \tau_R < 1$, the mode $n = 0$ is enhanced, and the other modes are exponentially suppressed. Hence,

$$\sigma_{\text{inel}} \sim \frac{1}{\omega}\,|v_0^e|^2 . \tag{48}$$

Using small argument expansions for Bessel functions, we find this yields

$$\sigma_{\text{inel}} \sim \frac{1}{\omega}\,(eAR)^2\,(\omega R)^{4(1-\tau_R)} , \tag{49}$$

where A is the value of the gauge field inside the string core, e is the gauge coupling constant, and $R \sim \eta$, η being the the grand unified scale $\sim 10^{15}$ GeV. The greater amplification occurs for $eAR \sim 1$, giving $\sigma_{\rm inel} \sim \frac{1}{\omega}(\omega R)^{4(1-\tau_R)}$.

For $-1 < \tau_R < 0$, the mode $n = -1$ is enhanced, and the other modes are exponentially suppressed. Hence,

$$\sigma_{\rm inel} \sim \frac{1}{\omega}\left|v^e_{-1}\right|^2 . \tag{50}$$

Using small argument expansions for Bessel functions, this yields

$$\sigma_{\rm inel} \sim \frac{1}{\omega}(eAR)^2(\omega R)^{4(1+\tau_R)} . \tag{51}$$

The greater amplification occurs for $eAR \sim 1$, giving $\sigma_{\rm inel} \sim \frac{1}{\omega}(\omega R)^{4(1+\tau_R)}$. Thus, the baryon-number-violating cross section is not a strong interaction cross section, but is suppressed by a factor depending on the grand unified scale $\eta \sim R^{-1} \sim 10^{15}$ GeV. The baryon-number-violation cross sections are very small. For u_L and d_L we obtain

$$\sigma_{\rm inel} \sim \frac{1}{\omega}(\omega R)^{3.6} , \tag{52}$$

whereas for d_R we get

$$\sigma_{\rm inel} \sim \frac{1}{\omega}(\omega R)^{2.8} . \tag{53}$$

Here again we have a marked asymmetry between left- and right-handed fields. We find an indeterminate solution for the left-conjugate up quark because its phase around the string $(\frac{1}{10})$ differs from the phase of the left-handed electron $(\frac{-3}{10})$ by a fractional value different from a half.

VI. THE SECOND QUANTIZED CROSS SECTION

We now derive the baryon-number-violating cross sections using the perturbative method introduced in Sec. III A.

Firstly, we calculate the geometrical cross section. This is the cross section for free fields $\psi_{\rm free}$, where $\psi_{\rm free}$ is a two-spinor. In the case of gauge fields mediating catalysis it is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm geom} = \frac{1}{\omega}(\omega R)^4(eAR)^2 , \tag{54}$$

where ω is the energy of the massless field $\psi_{\rm free}$, A is the value of the gauge field mediating quark-to-lepton transitions, e is the gauge coupling constant, and R is the radius of the string with $R \sim \eta^{-1}$ with $\eta \sim 10^{15}$ GeV.

The second step is to calculate the amplification factor $\mathcal{A} = \frac{\psi}{\psi_{\rm free}}$, ψ and $\psi_{\rm free}$ being two two-spinors. The catalysis cross section is enhanced by a factor $\mathcal{A}^4$ over the geometrical cross section:

$$\sigma_{\rm inel} \sim \mathcal{A}^4\left(\frac{d\sigma}{d\Omega}\right)_{\rm geom} . \tag{55}$$

We now use the results of Secs. III C, III D, and III E, where we have solved the equations of motion for the fields ψ and calculated the matching conditions. Using Eq. (22), we get the wave function ψ at the string core, and, for the mode n,

$$\psi^n = \begin{pmatrix} [(v^q_n \pm v^e_n)\,J_{n-\tau_{\rm str}}(\omega R) & + & (v^{q'}_n \pm v^{e'}_n)\,J_{-(n-\tau_{\rm str})}(\omega R)] & e^{in\theta} \\ i\;[(v^q_n \pm v^e_n)\,J_{n+1-\tau_{\rm str}}(\omega R) & + & (v^{q'}_n \pm v^{e'}_n)\,J_{-(n+1-\tau_{\rm str})}(\omega R)] & e^{i(n+1)\theta} \end{pmatrix} . \tag{56}$$

Using Eqs. (33) and (34) and using small argument expansions for Bessel functions, we conclude that for $n \geq 0$, $(v^q_n \pm v^e_n) \gg (v^{q'}_n \pm v^{e'}_n)$, and for $n < 0$, $(v^q_n \pm v^e_n) \ll (v^{q'}_n \pm v^{e'}_n)$. Now, from Eq. (38), we see there is one coefficient dominates that will be of order 1. Hence, for $n \geq 0$, $(v^q_n \pm v^e_n) \sim 1$, and for $n < 0$, $(v^{q'}_n \pm v^{e'}_n) \sim 1$. Therefore, using small argument expansions for Bessel functions we get, for $n \geq 0$,

$$\psi^n \sim \begin{pmatrix} (\omega R)^{n-\tau_{\rm str}} \\ (\omega R)^{n+1-\tau_{\rm str}} \end{pmatrix} , \tag{57}$$

which is to be compared with $\psi_2^{\rm free} \sim 1$ for free spinors. The upper component of the spinor is amplified while the other one is suppressed by a factor $\sim (\omega R)$. For $n < 0$ we have

$$\psi^n \sim \begin{pmatrix} (\omega R)^{-(n-\tau_{{\rm str},R})} \\ (\omega R)^{-(n+1-\tau_{{\rm str},R})} \end{pmatrix} . \tag{58}$$

Hence we conclude that for $n < 0$ the lower component is amplified while the upper one is suppressed by a factor $\sim \omega R$.

Therefore, for $\tau_{\rm str} = \frac{-3}{10}$, the amplification occurs for the lower component and for the mode $n = -1$. The amplification factor is

$$\mathcal{A} \sim (\omega R)^{\tau_{\rm str}} , \tag{59}$$

leading to the baryon-number-violating cross section

$$\sigma_{\rm inel} \sim \frac{1}{\omega}(eAR)^2(\omega R)^{4(1+\tau_{\rm str})} . \tag{60}$$

In the case $\tau_{\rm str} = \frac{1}{10}$, the amplification occurs for the upper component and for the mode $n = 0$. The amplification factor is

$$\mathcal{A} \sim (\omega R)^{-\tau_{\rm str}} , \tag{61}$$

leading to the baryon-number-violating cross section

$$\sigma_{\rm inel} \sim \frac{1}{\omega}(eAR)^2(\omega R)^{4(1-\tau_{\rm str})} . \tag{62}$$

This method shows explicitly which component of the spinor and which mode are enhanced. The results agree with scattering cross sections derived using the first-quantized method.

VII. CONCLUSION

We have investigated elastic and inelastic scattering off Abelian cosmic strings arising during the phase transition $SO(10) \stackrel{\langle\phi_{126}\rangle}{\rightarrow} SU(5) \times Z_2$ induced by the Higgs field in the **126** representation in the early Universe. The cross sections were calculated using both first-quantized and second-quantized methods. The results of the two methods are in good agreement.

During the phase transition $SO(10) \rightarrow SU(5) \times Z_2$, only the right-handed neutrino gets a mass. This together with the fact that we are interested in energies above the confinement scales allows us to consider massless particles.

The elastic cross sections are found to be Aharonov-Bohm-type cross sections. This is as expected, since we are dealing with fractional fluxes. We found a marked asymmetry between left-handed and right-handed fields for the electron and the down quark fields. But there is no asymmetry for the up quark field. This is a general feature of grand unified theories. If cosmic strings were observed, it might be possible to use Aharonov-Bohm scattering to determine the underlying gauge group.

The inelastic cross sections result from quark-to-lepton transitions via gauge interactions in the core of the string. The catalysis cross sections are found to be quite small, and here again we have a marked asymmetry between left- and right-handed fields. They are suppressed from a factor $\sim \eta^{-3.6}$ for the left-handed up and down quark fields to a factor $\sim \eta^{-2.8}$ for the right-handed down quark field.

Previous calculations have used a toy model to calculate the catalysis cross section. Here the string flux could be "tuned" to give a strong interaction cross section. In our case the flux is given by the gauge group, and is fixed for each particle species. Hence, we find a strong sensitivity to the grand unified scale. Our small cross sections make it less likely that grand unified cosmic strings could erase a primordial baryon asymmetry, though they could help generate it [16]. If cosmic strings are observed our scattering results, with the distinctive features for the different particle species, could help tie down the underlying gauge group.

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APPENDIX A: BRIEF REVIEW OF SO(10)

The fundamental representation of SO(10) consists of 10 generalized γ matrices. They can be written in an explicit notation, in terms of cross products;

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_2 &= \sigma_2 \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_3 &= I \times \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_4 &= I \times \sigma_2 \times \sigma_3 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_5 &= I \times I \times \sigma_1 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_6 &= I \times I \times \sigma_2 \times \sigma_3 \times \sigma_3 \ , \\
\Gamma_7 &= I \times I \times I \times \sigma_1 \times \sigma_3 \ , \\
\Gamma_8 &= I \times I \times I \times \sigma_2 \times \sigma_3 \ , \\
\Gamma_9 &= I \times I \times I \times I \times \sigma_1 \ , \\
\Gamma_{10} &= I \times I \times I \times I \times \sigma_2 \ , \qquad \text{(A1)}
\end{aligned}$$

where the σ_i are the Pauli matrices and I denotes the two-dimensional identity matrix. They generate a Clifford algebra defined by the anticommutation rules

$$\{\Gamma_i, \Gamma_j\} = 2\,\delta_{ij} \quad i = 1, ..., 10 \ . \qquad \text{(A2)}$$

One can define the chirality operator χ, which is the generalized γ_5 of the standard model by

$$\chi = (-i)^5 \prod_{i=1}^{10} \Gamma_i \ . \qquad \text{(A3)}$$

In terms of the cross product notation, χ has the form

$$\chi = \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \ . \qquad \text{(A4)}$$

The 45 generators of SO(10) are also given in terms of the generalized γ matrices:

$$M_{ab} = \frac{1}{2i}\,[\Gamma_i, \Gamma_j] \;\; i, j = 1, ..., 10 \ . \qquad \text{(A5)}$$

They are antisymmetric, purely imaginary 32×32 matrices. One can write the diagonal M:

$$\begin{aligned}
M_{12} &= \frac{1}{2}\,\sigma_3 \times I \times I \times I \times I \ , \\
M_{34} &= \frac{1}{2}\,I \times \sigma_3 \times I \times I \times I \ , \\
M_{56} &= \frac{1}{2}\,I \times I \times \sigma_3 \times I \times I \ , \\
M_{78} &= \frac{1}{2}\,I \times I \times I \times \sigma_3 \times I \ , \\
M_{910} &= \frac{1}{2}\,I \times I \times I \times I \times \sigma_3 \ . \qquad \text{(A6)}
\end{aligned}$$

In SO(N) gauge theories fermions are conventionally assigned to the spinor representation. For N even, the spinor representation is $2^{\frac{N}{2}}$ dimensional and decomposes into two equivalent spinors of dimension $2^{\frac{N}{2}-1}$ by means of the projection operator $P = \frac{1}{2}\,(1 \pm \chi)$, where 1 is the $2^{\frac{N}{2}} \times 2^{\frac{N}{2}}$ identity matrix. Thus SO(10) has two irreducible representations:

$$\sigma^{\pm} = \frac{1 \pm \chi}{2} \qquad \text{(A7)}$$

of dimension 16. Therefore SO(10) enables us to put all the fermions of a given family in the same spinor. Indeed, since each family contains eight fermions, we can put all left- and right-handed particles of a given family in the same 16-dimensional spinor. This is the smallest grand unified group which can do so. However, gauge interactions conserve chirality. Indeed,

$$\bar{\psi}\gamma_\mu A^\mu \psi = \bar{\psi}_L \gamma_\mu A^\mu \psi_L + \bar{\psi}_R \gamma_\mu A^\mu \psi_R \, . \tag{A8}$$

Therefore ψ_L and ψ_R cannot be put in the same irreducible representation. Hence, instead of choosing ψ_L and ψ_R, we chose ψ_L and ψ_L^c. The fields ψ_L and ψ_L^c annihilate left-handed particles and antiparticles, respectively, or create right-handed antiparticles and particles. The fields ψ_L and ψ_L^c are related to the fields ψ_R and $\bar{\psi}_R$ by the relations

$$\psi_L^c \equiv P_L \psi^c = P_L C \bar{\psi}^T = C(\bar{\psi} P_L)^T = C \bar{\psi}_R{}^T = C \gamma_0^T \psi_R^* \, , \tag{A9}$$

$$\bar{\psi}_L^c \equiv \psi_L^{c\dagger} \gamma_0 = \psi_R^{*\dagger} \gamma_0^{T\dagger} C^\dagger \gamma_0 = -\psi_R^T C^{-1} = \psi_R^T C \, , \tag{A10}$$

where the projection operators $P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)$ and C is the usual charge conjugation matrix. For the electron family we get

$$\Psi_L^{(e)} = (\nu_{(e)}^c \, , u_r^c \, , u_y^c \, , u_b^c \, , d_b \, , d_y \, , d_r \, , e^- \, , u_b \, , u_y \, , u_r \, , \nu_{(e)} \, , e^+ \, , d_r^c \, , d_y^c \, , d_b^c)_L \, , \tag{A11}$$

where the upper index c means conjugate, and the subindices refer to quark color. We find similar spinor $\Psi^{(\mu)}$ and $\Psi^{(\tau)}$ associated with the μ and the τ family, respectively:

$$\Psi^{(\mu)} = (\nu_{(\mu)}^c \, , c_r^c \, , c_y^c \, , c_b^c \, , s_b \, , s_y \, , s_r \, , \mu^- \, , c_b \, , c_y \, , c_r \, , \nu_{(\mu)} \, , \mu^+ \, , s_r^c \, , s_y^c \, , s_b^c)_L \tag{A12}$$

$$\Psi^{(\tau)} = (\nu_{(\tau)}^c \, , t_r^c \, , t_y^c \, , t_b^c \, , b_b \, , b_y \, , b_r \, , \tau^- \, , t_b \, , t_y \, , t_r \, , \nu_{(\tau)} \, , \tau^+ \, , b_r^c \, , b_y^c \, , b_b^c)_L \, . \tag{A13}$$

APPENDIX B: THE EXTERNAL SOLUTION

We want to solve Eqs. (20). We set $\partial_t = -i\omega$, where ω is the energy of the electron and take the usual Dirac representation $e_L = (0, \xi_e)$, $e_R = (\chi_e, 0)$, $q_L^c = (0, \xi_q)$, and $q_R^c = (\chi_q, 0)$. We use the usual mode decomposition for the spinors ξ_q, ξ_e, χ_q, and χ_e :

$$\chi_{(e,q^c)}(r,\theta) = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix} \chi_{1\,(e,q^c)}^n(r) \\ i \;\; \chi_{2\,(e,q^c)}^n(r)\, e^{i\theta} \end{pmatrix} e^{in\theta} \, ,$$

$$\xi_{(e,q^c)}(r,\theta) = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix} \xi_{1\,(e,q^c)}^n(r) \\ i \;\; \xi_{2\,(e,q^c)}^n(r)\, e^{i\theta} \end{pmatrix} e^{in\theta} \, . \tag{B1}$$

Then, using the basis

$$\gamma^j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \, , \tag{B2}$$

the equations of motion (20) become

$$\begin{aligned} \omega\chi_{1,(e,q^c)}^n - \left(\frac{d}{dr} + \frac{n+1}{r} - \frac{\tau_{\rm str}^{R\,(e,q^c)}}{r}\right) \chi_{2,(e,q^c)}^n &= 0 \, , \\ \omega\chi_{2,(e,q^c)}^n + \left(\frac{d}{dr} - \frac{n}{r} + \frac{\tau_{\rm str}^{R\,(e,q^c)}}{r}\right) \chi_{1,(e,q^c)}^n &= 0 \, , \\ \omega\xi_{1,(e,q^c)}^n + \left(\frac{d}{dr} + \frac{n+1}{r} - \frac{\tau_{\rm str}^{L\,(e,q^c)}}{r}\right) \xi_{2,(e,q^c)}^n &= 0 \, , \\ \omega\xi_{2,(e,q^c)}^n - \left(\frac{d}{dr} - \frac{n}{r} + \frac{\tau_{\rm str}^{L\,(e,q^c)}}{r}\right) \xi_{1,(e,q^c)}^n &= 0 \, . \end{aligned} \tag{B3}$$

It is easy to show that the fields $\xi_{1,(e,q^c)}^n$, $\xi_{1,(e,q^c)}^n$, $\chi_{1,(e,q^c)}^n$, and $\chi_{2,(e,q^c)}^n$ satisfy Bessel equations of order $n-\tau_{\rm str}^{R\,(e,q^c)}$, $n+1-\tau_{\rm str}^{R\,(e,q^c)}$, $n-\tau_{\rm str}^{L\,(e,q^c)}$, and $n-\tau_{\rm str}^{R\,(e,q^c)}$, respectively. Hence the external solution is

$$\begin{pmatrix} \xi_{(e,q^c)}(r,\theta) \\ \chi_{(e,q^c)}(r,\theta) \end{pmatrix} = \sum_{n=-\infty}^{n=+\infty} \begin{pmatrix} [v_n^{(e,q^c)} Z^1_{n-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r) + v_n^{(e,q^c)\prime} Z^2_{n-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r)] \; e^{in\theta} \\ i\;[v_n^{(e,q^c)} Z^1_{n+1-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r) + v_n^{(e,q^c)\prime} Z^2_{n+1-\tau_{\rm str}^{R\,(e,q^c)}}(\omega r)] \; e^{i(n+1)\theta} \\ [w_n^{(e,q^c)} Z^1_{n-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r) + w_n^{(e,q^c)\prime} Z^2_{n-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r)] \; e^{in\theta} \\ i\;[w_n^{(e,q^c)} Z^1_{n+1-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r) + w_n^{(e,q^c)\prime} Z^2_{n+1-\tau_{\rm str}^{L\,(e,q^c)}}(\omega r)] \; e^{i(n+1)\theta} \end{pmatrix} . \tag{B4}$$

The order of the Bessel functions will always be fractional. We therefore take $Z_\nu^1 = J_\nu$ and $Z_\nu^2 = J_{-\nu}$.

APPENDIX C: THE INTERNAL SOLUTION

We get solutions for fields which are linear combinations of the quark and electron fields. Indeed, we get solutions for the fields $\sigma^\pm = \xi_q \pm \xi_e$ and $\rho^\pm = \chi_q \pm \chi_e$. Using the mode decomposition (21), the upper components of the fields $\rho^\pm$ and $\sigma^\pm$ are, respectively, $\rho_{n1}^\pm = \chi_{1\,q^c}^n \pm \chi_{1\,e}^n$ and $\rho_{n2}^\pm = \chi_{2\,q^c}^n \pm \chi_{2\,e}^n$ while the lower components are $\sigma_{n1}^\pm = \xi_{1\,q^c}^n \pm \xi_{1\,e}^n$ and $\sigma_{n2}^\pm = \xi_{2\,q^c}^n \pm \xi_{2\,e}^n$, respectively. The equations of motions (23) become

$$\omega\rho_{n1}^\pm - \left(\frac{d}{dr} + \frac{n+1}{r} \mp eA'\right) \rho_{n2}^\pm = 0 \, , \tag{C1a}$$

$$\omega\rho_{n2}^\pm + \left(\frac{d}{dr} - \frac{n}{r} \pm eA'\right) \rho_{n1}^\pm = 0 \, , \tag{C1b}$$

$$\omega\sigma_{n1}^\pm + \left(\frac{d}{dr} + \frac{n+1}{r} \mp eA\right) \sigma_{n2}^\pm = 0 \, , \tag{C1c}$$

$$\omega\sigma_{n2}^{\pm} - \left(\frac{d}{dr} - \frac{n}{r} \pm eA\right)\sigma_{n1}^{\pm} = 0 \; . \tag{C1d}$$

Combining (C1a) and (C1b), one can see that $\rho_{n1}^{\pm}$ satisfy a hypergeometric equation giving

$$\rho_{n1}^{\pm} = (kr)^{|n|}e^{-ikr}\sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikr)^j}{j!} \; , \tag{C2}$$

where $k^2 = w^2 - (eA)^2$, $e = \frac{g}{2\sqrt{2}}$. $\alpha_{j+1}^{\pm} = \frac{(a^{\pm}+j)}{(b+p)}\alpha_j^{\pm}$ with $a^{\pm} = \frac{1}{2} + |n| \pm \frac{eA(2n+1)}{2ik}$ and $b = 1 + 2|n|$. $\rho_{n2}^{\pm}$ can be obtained using the coupled equation (C1b). We find

$$\rho_{n2}^{\pm} = -\frac{1}{w}(kr)^{|n|}e^{-ikr} \times \sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikr)^j}{j!}\left(\frac{|n|-n}{r} - ik + \frac{j}{r} \pm eA\right) . \tag{C3}$$

$\sigma_{n2}^{\pm}$ are also solutions of hypergeometric equations, and using the coupled equation (C1d) we get

$$\sigma_{n1}^{\pm} = (kr)^{|n|}e^{-ikr}\sum_{j=0}^{n=+\infty}\beta_j^{\pm}\frac{(2ikr)^j}{j!} \; , \tag{C4}$$

$$\sigma_{n2}^{\pm} = -\frac{1}{w}(kr)^{|n|}e^{-ikr} \times \sum_{j=0}^{n=+\infty}\beta_j^{\pm}\frac{(2ikr)^j}{j!}\left(\frac{|n|-n}{r} - ik + \frac{j}{r} \pm eA'\right) , \tag{C5}$$

where $k^2 = w^2 - (eA')^2$, $\beta_{j+1}^{\pm} = \frac{(c^{\pm}+j)}{(b+p)}\beta_j^{\pm}$ with $c^{\pm} = \frac{1}{2} + |n| \pm \frac{eA'(2n+1)}{2ik}$. And the internal solution is

$$\begin{pmatrix} \rho_{n1}^{\pm}\, e^{in\theta} \\ i\ \rho_{n2}^{\pm}\, e^{i(n+1)\theta} \\ \sigma_{n1}^{\pm}\, e^{in\theta} \\ i\ \sigma_{n2}^{\pm}\, e^{i(n+1)\theta} \end{pmatrix} . \tag{C6}$$

Therefore the internal solution is giving by a linear combination of the quark and electron fields.

APPENDIX D: THE MATCHING CONDITIONS

The continuity of the solutions at $r = R$ leads to

$$(kR)^{|n|}\,e^{-ikR}\sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikR)^j}{j!} = (v_n^q \pm v_n^e)J_{n-\tau_R}(\omega R) + (v_n^{q'} \pm v_n^{e'})J_{-(n-\tau_R)}(\omega R) \; , \tag{D1}$$

$$-\frac{1}{w}(kR)^{|n|}\,e^{-ikR}\sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikr)^j}{j!}\left(\frac{|n|-n}{R} - ik + \frac{j}{R} \pm eA\right) = (v_n^q \pm v_n^e)J_{n+1-\tau_R}(\omega R) + (v_n^{q'} \pm v_n^{e'})J_{-(n+1-\tau_R)}(\omega R) \; . \tag{D2}$$

Nevertheless, we will have discontinuity of the first derivatives. Indeed, inside we have

$$\begin{aligned} \omega\rho_{n1}^{\pm} &- \left(\tfrac{d}{dr} + \tfrac{n+1}{r} \mp eA'\right)\rho_{n2}^{\pm} = 0 \; , \\ \omega\rho_{n2}^{\pm} &+ \left(\tfrac{d}{dr} - \tfrac{n}{r} \pm eA'\right)\rho_{n1}^{\pm} = 0 \; , \end{aligned} \tag{D3}$$

whereas outside we have

$$\omega(\chi_{1,q^c}^n \pm \chi_{1,e}^n) - \left(\frac{d}{dr} + \frac{n+1}{r} - \frac{\tau_{\rm str}^{R\,(e,q^c)}}{r}\right) \times(\chi_{2,q^c}^n \pm \chi_{2,e}^n) = 0 \; ,$$

$$\omega(\chi_{2,q}^n \pm \chi_{2,e}^n) + \left(\frac{d}{dr} - \frac{n}{r} + \frac{\tau_{\rm str}^{R\,(e,q^c)}}{r}\right) \times(\chi_{1,q^c}^n \pm \chi_{1,e}^n) = 0 \; . \tag{D4}$$

Now,

$$(\chi_{1,q^c}^n \pm \chi_{1,e}^n)^{\rm out} = \rho_{n1}^{\pm\,\rm in}, \tag{D5}$$

$$(\chi_{2,q^c}^n \pm \chi_{2,e}^n)^{\rm out} = \rho_{n2}^{\pm\,\rm in}, \tag{D6}$$

giving us the relations for the first derivatives:

$$\left(\frac{d}{dr} \mp eA\right)\rho_{n2}^{\pm\,\rm in} = \left(\frac{d}{dr} - \frac{\tau_{\rm str}^{R\,(e,q^c)}}{R}\right)(\chi_{2,q^c}^n \pm \chi_{2,e}^n)^{\rm out}, \tag{D7}$$

$$\left(\frac{d}{dr} \pm eA\right)\rho_{n1}^{\pm\,\rm in} = \left(\frac{d}{dr} + \frac{\tau_{\rm str}^{R\,(e,q^c)}}{R}\right)(\chi_{1,q^c}^n \pm \chi_{1,e}^n)^{\rm out} . \tag{D8}$$

Dividing Eq. (D1) by Eq. (D2) or either replacing Eq. (D1) in Eq. (D7), we get the relations

$$\frac{v_n^{q'} \pm v_n^{e'}}{v_n^q \pm v_n^e} = \frac{w\,l_n^{\pm}J_{n+1-\tau_R}(wR) + J_{n-\tau_R}(wR)}{w\,l_n^{\pm}J_{-(n+1-\tau_R)}(wR) + J_{-(n-\tau_R)}(wR)} \; , \tag{D9}$$

where

$$l_n^{\pm} = \frac{\sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikr)^j}{j!}}{\sum_{j=0}^{n=+\infty}\alpha_j^{\pm}\frac{(2ikr)^j}{j!}\left(\frac{|n|-n}{r} - ik + \frac{j}{r} \pm eA\right)} \; . \tag{D10}$$

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