Scattering of massless scalar waves by a Schwarzschild black hole: A phase-integral study

Nils Andersson
Department of Physics and Astronomy, University of Wales College of Cardiff, P.O. Box 913, Cardiff CF2 3YB, United Kingdom
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Plane scalar waves scattered off a Schwarzschild black hole are studied in the partial-wave picture. A new approximate formula for the relevant phase shifts is derived using the phase-integral method. This formula is, in principle, uniformly valid for all frequencies and agrees with well-known approximations for frequencies well above and below the top of the curvature potential barrier. The reliability of the phase-integral formula is assessed in two different ways. First we use the fact that higher orders of approximation are easily implemented in the phase-integral method. The accuracy of each phase shift can be estimated by the contribution to it by the following order of approximation. Second, we use the approximate phase shifts to construct physically meaningful quantities, such as the deflection function and cross section, for several scattering frequencies. The features of these quantities, especially those associated with the prominent black-hole glory in the backward direction, are in excellent agreement with results of previous studies of the problem. The new phase-shift formula is thus shown to be reliable and provides a useful and efficient tool, especially for intermediate and high frequencies.

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I. INTRODUCTION

In standard quantum scattering plane waves are used to probe the details of a physical system such as, for example, an atom or a molecule. Such investigations have proved extremely fruitful, experimentally as well as theoretically, and have led to a better understanding of some of the most beautiful phenomena in nature. In fact, the theoretical framework that is used to explain the features of a rainbow can be brought to bear also on the description of the interatomic forces. One may wonder whether the scattering approach could not prove illuminating also for larger bodies such as black holes (or neutron stars). Indeed it can, basically because these systems are very clean and "easily" modeled, and important work in this respect has been carried out since the late 1960s (see [1] for an exhaustive description).

The present investigation addresses the simplest scattering problem that involves black holes, i.e., the case when massless scalar waves are scattered off a Schwarzschild hole. We reexamine this "old" problem (for previous work see [2–6]) in order to find out whether various descriptions of scattering developed for quantum problems can prove useful in general relativity. In this paper we discuss the scattering of a monochromatic scalar wave in terms of phase shifts and derive a new approximate phase-integral formula for those that are valid for a wide range of frequencies. We assess the reliability and usefulness of the new formula in two different ways. The first of these is based on the fact that higher orders of approximation are easily implemented in a phase-integral study. The arbitrary-order phase-integral scheme [7,8] is inherently asymptotic and it makes sense to consider each phase shift as an asymptotic expansion when higher orders are included. Then one would expect the accuracy of the approximate phase shifts to first improve and eventually diverge as the order of approximation is increased. The best possible result would be achieved by truncating the expansion at the order of approximation that contributes the least, i.e., immediately before divergence. However, it seems unlikely that phase shifts of extreme accuracy are actually needed to explore the underlying physics of the black-hole scattering problem. We will therefore perform calculations only for the lowest two orders of approximation here. The higher order will be used to provide an estimate of the accuracy of the lowest order of approximation. However, it should be mentioned that our final formula for the phase shifts is valid in *any* order of approximation. The reliability of the new approximate formula can also be inferred in a more indirect way. We can use the resultant phase shifts to construct physically relevant scattering quantities (such as the deflection function and the differential cross section) for a range of frequencies. The detailed features of these quantities should be compared to the results of previous, analytical as well as numerical, work in this field.

As already stated, we want to see whether various descriptions of scattering can be used in the context of black holes. That the standard partial-wave paradigm is useful is already well established [1], but some alternatives to this picture do not seem to have been discussed previously. One such alternative is based on the use of complex angular momenta. We have recently approached the black-hole problem within that framework [9,10]. The present study is complementary to that one and the results discussed here were, in fact, used to establish the reliability of the complex angular momentum approach.

It may be argued that black-hole scattering problems

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have been exhaustively studied before, and that there is no need for them to be considered further. This may, indeed, be true for the relatively simple problem studied here: Our understanding of scattering of scalar, electromagnetic, and gravitational waves from Schwarzschild black holes is satisfactory [1]. However, the situation is different for Kerr black holes. Although numerical calculations have been performed also for such problems, the physical "observables," e.g., the cross sections, are not as transparent as for nonrotating holes. In fact, the features that occur for rapidly rotating black holes are not at all well understood. In order to improve on the situation alternative descriptions and new approaches are needed. Although the problem can immediately be approached numerically [1], it seems likely that an approximate study will enable a better understanding of the actual physics involved. It seems natural to approach the Kerr black-hole problem via the phase-integral method, which has proved useful and reliable in atomic and molecular scattering. However, before any "new" tool, such as the approximate phase-shift formula of this paper or the complex angular momentum description [9,10], is used in a more challenging situation it must be tested under somewhat controlled circumstances. This is the main reason why the present study is restricted to massless scalar waves scattered by a nonrotating black hole. Most of the formulas discussed below should generalize to Kerr black holes, and they are certainly applicable to scattering of electromagnetic and gravitational waves from Schwarzschild black holes.

II. SCATTERING FROM BLACK HOLES

A. Equations describing a massless scalar field

It is well known that the Klein-Gordon equation in the Schwarzschild geometry can be written

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V(r)\right]\phi_\ell = 0 \ . \tag{1}$$

Here it is assumed that the contribution of the scalar field to the curvature of spacetime can be neglected and we use geometrized units (in which $c = G = 1$). The effective potential $V(r)$ is partly due to the centrifugal barrier that is familiar from other problems with spherical symmetry, but a part of it arises because of the curvature of spacetime in the vicinity of the hole:

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right] . \tag{2}$$

M is the black-hole mass and ℓ is the integer index of the standard spherical harmonics ($\ell \geq 0$).

The tortoise coordinate r_* in (1) is related to the Schwarzschild radius r by

$$\frac{d}{dr_*} = \left(1 - \frac{2M}{r}\right)\frac{d}{dr} \tag{3}$$

or

$$r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right) + \text{const.} \tag{4}$$

The integration constant will be left unspecified for the moment, but it should be mentioned that we will later choose it to be nonzero (see Sec. IV C).

Once a solution ϕ_ℓ to (1) is known for all non-negative integers ℓ, the scalar field Φ can be calculated through the partial-wave sum

$$\Phi = \sum_{\ell=0}^{\infty}(2\ell+1)\frac{\phi_\ell(r,t)}{r}P_\ell(\cos\theta) \ . \tag{5}$$

B. Scattering amplitude and phase shifts

Let us now assume that we study monochromatic waves with a certain frequency ω. Then ϕ_ℓ will have a time dependence $\exp(-i\omega t)$ and (for each value of ℓ) a general solution to (1) that satisfies the causal boundary condition of purely ingoing waves crossing the horizon is

$$\phi_\ell \sim \begin{cases} e^{-i\omega r_*} \ , & r_* \to -\infty \ , \\ A_{\text{in}}e^{-i\omega r_*} + A_{\text{out}}e^{i\omega r_*} \ , & r_* \to \infty \ . \end{cases} \tag{6}$$

(In the following the appropriate dependence on time will not be written out explicitly.) The scattering problem involves finding such a solution to (1), i.e., identifying A_{in} and A_{out}. Then we can extract the scattered wave by discarding the part of the solution that corresponds to the original plane wave. The physical information we are interested in is contained within the scattering amplitude $f(\theta)$:

$$\Phi \sim \Phi_{\text{plane}} + \frac{1}{r}f(\theta)e^{i\omega r_*} \quad \text{as } r_* \to +\infty \ . \tag{7}$$

This program seems straightforward, but it involves at least one far from trivial question: What exactly do we mean by a *plane wave* in a curved spacetime?

This problem has been discussed in several papers by other authors (see Matzner [2] and Chrzanowski *et al.* [11]). They conclude that, in the case of a scalar field, the desired expression for a plane wave at infinity is

$$\Phi_{\text{plane}} \sim \frac{1}{\omega r}\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)P_\ell(\cos\theta)\sin\left[\omega r_* - \frac{\ell\pi}{2}\right] \quad \text{as } r_* \to +\infty \ . \tag{8}$$

This is, at first sight, a remarkable result. We retain the standard flat spacetime description by replacing r_* in (8) by r. In retrospect it becomes more obvious. When described in terms of the Schwarzschild coordinate r the wave equation (1) is a Schrödinger-like equation with a long-range effective potential (that falls off as $1/r$ for large r). The problem is therefore analogous to that of Coulomb scattering. In such problems the effect of the long-range potential is accounted for by a modification of the scattering amplitude similar to that above [12].

The remaining road is clear and we can extract the scattered wave by subtracting the plane-wave contribution (8) from our total solution (6). In doing so we need only make sure that no incoming wave remains at infinity. Letting

$$\Phi - \Phi_{\text{plane}} \sim \frac{1}{2i\omega r} e^{i\omega r_*} \sum_{\ell=0}^{\infty} (2\ell+1) \left[e^{2i\delta_\ell} - 1\right] P_\ell(\cos\theta) \quad \text{as } r_* \to +\infty \tag{9}$$

define the (complex-valued) phase shifts δ_ℓ it is straightforward to show that

$$e^{2i\delta_\ell} = (-1)^{\ell+1} \frac{A_{\text{out}}}{A_{\text{in}}} \ . \tag{10}$$

From the above it follows that the scattering amplitude, which contains all the physical information, is given by

$$f(\theta) = \frac{1}{2i\omega} \sum_{\ell=0}^{\infty} (2\ell+1) \left[e^{2i\delta_\ell} - 1\right] P_\ell(\cos\theta) \ . \tag{11}$$

The next step in our investigation is to develop a technique for calculating the phase shifts. That is, we want to find a solution to the ordinary differential equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] \phi_\ell = 0 \tag{12}$$

that satisfies the physical condition that no waves come out of the horizon (at $r_* = -\infty$). Then we can extract A_{in} and A_{out} from the asymptotic behavior of this solution as $r_* \to +\infty$ and the phase shifts follow from (10). Since (12) is similar to the one-dimensional Schrödinger equation with an effective potential corresponding to a single barrier, we can use any standard method of quantum mechanics to approach it. Approximate formulas valid for low and high frequencies (well below and above the top of the potential barrier) can be found in the literature [13–15]. The purpose of the following section is to derive an approximate phase-shift formula that is formally valid for all frequencies ω (to a certain extent including complex ones).

In the analysis that follows a global solution to (12) is obtained by means of the phase-integral method devised by Fröman and Fröman (see [7] or [8] for an introduction and many of the original references). This method has already, with some success, been applied to the calculation of the complex quasinormal-mode frequencies of black holes [16–18]. An introductory review of the method and its use in the context of complex frequencies with a relatively large imaginary part (rapidly damped modes) has recently been written [17].

III. PHASE-INTEGRAL METHOD

A. Local approximations in any order

In order to approach the black-hole problem we define a new dependent variable (as in, for example, [17]):

$$\phi_\ell = \left(1 - \frac{2M}{r}\right)^{-1/2} \psi_\ell \ . \tag{13}$$

The differential equation (12) then becomes

$$\left[\frac{d^2}{dr^2} + R(r)\right] \psi_\ell = 0 \ , \tag{14}$$

where

$$R = \left(1 - \frac{2M}{r}\right)^{-2} \left[\omega^2 - \left(1 - \frac{2M}{r}\right) \frac{\ell(\ell+1)}{r^2} + \frac{M^2}{r^4}\right] \ . \tag{15}$$

In the phase-integral method (see [7,8,17] for alternative introductions) two linearly independent solutions to (14) are given by

$$f_{1,2}(r, t_j) = q^{-1/2}(r) \exp\left[\pm i \int_{t_j}^{r} q(\xi) d\xi\right] \ . \tag{16}$$

Inserting these functions in (14) it is easy to show that the function $q(r)$ should be a solution to the nonlinear equation

$$\frac{R - q^2}{q^2} - \frac{1}{2q^3}\frac{d^2 q}{dr^2} + \frac{3}{4q^4}\left(\frac{dq}{dr}\right)^2 = 0 \ . \tag{17}$$

Although it is hardly ever possible to find an exact solution to this equation, one can usually find an approximate one; let us call it Q. For example, if R is "slowly varying" in the sense that the derivatives in (17) can be neglected, the standard WKB approximation $Q^2 = R$ follows (see, for example, [19]). Although this approximation may be useful, there are situations (such as in regions of the r plane where R has a second order pole) where a more flexible approximation is preferable. The phase-integral method, developed by Fröman and Fröman [7,8], provides such a scheme. The fundamental idea is that the desired function Q may be equal to $R^{1/2}$ but that is not at all necessary. By choosing a slightly different functional form for Q the approximation can often be improved, especially in cases where it would otherwise break down [20], as will be shown below. The arbitrary-order phase-integral approximation can be viewed as an expansion in terms of the nonzero, but small quantity (ϵ) that occurs on the right-hand side of (17) for a certain choice of the first approximation Q. In the $(2N+1)$th order of approximation we get the local (asymptotic) expansion

$$q(r) = Q(r) \sum_{n=0}^{N} Y_{2n} \ , \tag{18}$$

where $Y_0 = 1$ and the second term is

$$Y_2 = \frac{1}{2}\epsilon \ , \tag{19}$$

with

$$\epsilon = \frac{R - Q^2}{Q^2} - \frac{1}{2Q^3}\frac{d^2 Q}{dr^2} + \frac{3}{4Q^4}\left(\frac{dQ}{dr}\right)^2 \ . \tag{20}$$

Although the phase-integral scheme can formally be extended to any order, we will only use the lowest two orders of approximation in this paper. Should a higher numerical accuracy than that obtained here be desired, our final formulas can be directly extended to any order since the fundamental analysis below is independent of the order of approximation used (cf. [7,8,17]). One will generally find that the use of higher orders leads to very accurate numerical results as long as the lowest order of approximation is reasonably good.

In the lowest order of approximation the lower limit t_j of integration for the phase in (16) is usually taken as one of the (possibly complex) zeros of Q^2. These are generalizations of the classical turning points of a standard WKB analysis and will be refered to as "transition points" below. These points play an important role in a global analysis of (14). It is easy to see from (16) that

$$f_{1,2}(r,t_j) = e^{\pm i\gamma_{jk}} f_{1,2}(r,t_k) \ , \tag{21}$$

where γ_{jk} is a line integral between t_j and t_k. A cut must be introduced from each transition point in order to keep Q single valued, but in most cases these cuts can be placed in such a way that they do not obscure the actual analysis. In higher orders of approximation the function q becomes singular at the transition points; cf. (20). Consequently, the integral in (16) must in higher orders be replaced by a contour integral encircling the transition point t_j.

The freedom to choose the function Q, which generates the approximation, freely is one of the great advantages of the phase-integral technique (see Sec. IV in [8] and [20]). This freedom is, in fact, essential for a study of the black-hole problem. Hence, it is worthwhile to discuss it in some detail here. It is important to clarify why the standard WKB approximation $Q^2 = R$ is unsuitable.

It is easy to see that a choice $Q \neq R$ will certainly be preferred for $\ell = 0$ in the black-hole problem. When $\ell = 0$ and $Q^2 = R$, Eq. (14) corresponds to superbarrier scattering for all frequencies ω. There are no transition points on the real r axis in the interval $[2M, +\infty]$. This is, however, not the case for the original equation (12): It corresponds to sub-barrier scattering for low frequencies. These two cases could clearly lead to quite different physical results. Thus it would seem that a different choice of Q^2, one that gives rise to two transition points on the real r axis for low frequencies, should be used.

The above argument is, however, not the main reason why one should choose Q^2 different from R in this problem. The usual motivation is the qualitative behavior of the approximate solutions f_1 and f_2. As is clear from (19) the two functions f_1 and f_2 (with $q = Q$) will be exact solutions to (14) if one can find a function Q such that $\epsilon = 0$. Although it is impossible to find a function that makes ϵ vanish globally, it is always possible to choose Q such that ϵ vanishes at a certain point. That is, the two phase-integral solutions can be used to construct the exact solution to (14) at any specific point, at least as long as that point is not a zero of Q^2. For example, we see from (15) that $R \to \omega^2$ as $r \to +\infty$. It immediately follows from (20) that, if we choose Q^2 in such a way that it approaches ω^2 for large r, the phase-integral solutions f_1 and f_2 tend to exactness at infinity (ϵ vanishes). This is clearly advantageous since the scattering problem consists of finding the asymptotic amplitudes of the outgoing and ingoing waves. The situation is slightly different close to the horizon of the black hole (at $r = 2M$). Using $Q^2 = R$ one can show that

$$\epsilon \to -\frac{1}{16\omega^2 M^2 + 1} \quad \text{as } r \to 2M \ . \tag{22}$$

For high frequencies ϵ will be small and the fact that it does not really vanish may not be a problem. But $|\epsilon| \to 1$ as $\omega M \to 0$ and it is clear that our approximation will not be reliable at all in the vicinity of the horizon for low frequencies. This is potentially disastrous since we need to impose a condition of purely ingoing waves at $r = 2M$. It would clearly be preferable to generate the approximation from a function that makes sure that ϵ vanishes as one approaches the horizon. Such a choice is

$$Q^2 = R - \frac{1}{4(r-2M)^2} \ , \tag{23}$$

or any other function that has the same behavior close to $r = 2M$. This choice was used in, for example, [17], where it was shown that it leads to functions f_1 and f_2 that behave as the power-series solutions to (14) close to the second order pole at $2M$. In Fig. 1 we compare $|\epsilon|$ for the choice $Q^2 = R$ to that obtained using (23). It is clear that the choice (23) is advantageous close to the horizon. This kind of graph indicates for which values of r the phase-integral solution will be useful. One would generally expect the expansion (18) to provide a high accuracy if the lowest order is reasonably good, i.e., if $|\epsilon|$ is considerably smaller than 1. It is evident from the graph that the phase-integral solutions cannot be used to construct a local solution to (14) close to the transition points.

B. Global solution for a general (complex) potential barrier

As already mentioned, the exact solution to (14) can always be represented by a linear combination of the two functions (16) at a point r (away from all transition points). That is, in matrix notation it can be written

$$\Psi = \alpha_n f_n \ , \tag{24}$$

where summation over repeated indices is assumed. It is quite obvious that the coefficients α_n cannot be global constants. Different linear combinations of the approximate solutions (16) must be used in different parts of the r plane. This is a manifestation of the so-called Stokes phenomenon. In a sense it is the price we are paying for expressing the exact solution to (14), which should be single valued, in terms of the multivalued functions f_1 and f_2 [21]. On the other hand, the exponential form is convenient because we want to draw conclusions about the asymptotic behavior of the approximate solutions, and we are discussing wave phenomena.

Without yet specifying in detail how the approximate solution to (14) changes as it is extended from r to another point r_0 we note the formal relation (see [13,7])

$$\alpha_m(r_0) = F_{mn}(r_0, r)\, \alpha_n(r) \ . \tag{25}$$

The diagonal elements of the 2×2 matrix F are often equal to unity. Because of their origin in the Stokes phenomenon, the elements of the F matrix are often referred to as Stokes constants. It is worth noticing that the determinant of F is 1, which means that it is a trivial task to invert (25).

The scattering situation we are interested in here is that of a single potential barrier. This problem involves only two transition points t_1 and t_2. Below we describe this situation in general terms. The given formulas can be used for any situation that is qualitatively similar. What we need to do is clear: We are looking for a detailed description of the Stokes phenomenon. A knowledge of the Stokes constants would enable us to continue a given solution from one part of the complex r plane to any other by means of (25), and thus handle more or less any problem that involves a global solution to (14). Specifically, we will be able to extend a solution that satisfies the boundary condition of purely ingoing waves falling across the horizon to spatial infinity and thus infer the asymptotic behavior. Finally, the corresponding phase shift follows from (10).

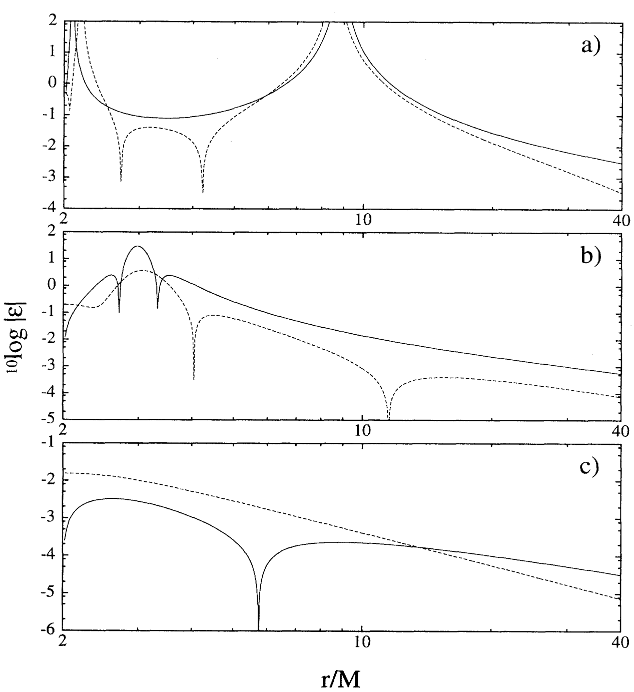

FIG. 1. The local "accuracy" of the lowest-order phase-integral approximation (as indicated by $|\epsilon|$) is compared for two choices of generating function: $Q^2 = R$ is indicated by dashed lines and $Q^2 = R - 1/4(r-2M)^2$ corresponds to the solid lines. The three cases shown correspond to $\ell = 2$ and (a) $\omega M = 0.25$, i.e., sub-barrier scattering, (b) $\omega M = 0.5$, i.e., close to the barrier top, (c) $\omega M = 2.0$, i.e., superbarrier scattering. It is clear [from cases (b) and (c)] that it is advantageous to choose Q^2 different from R close to the horizon (at $r = 2M$), and also [from case (a)] that the phase-integral solutions are not accurate at all close to the transition points. In fact, ϵ should diverge as one approaches a zero of Q^2.

In order to extract the desired information we study what happens when the approximate solution (24) is continued around the region of the r plane that contains the two transition points. In doing this we consider the region close to t_1 and t_2 as a "black box," the details of which we need not know. The analysis consists basically of constructing the pattern of Stokes and anti-Stokes lines that is associated with the generating function Q. From this pattern and general principles we can infer much of the information that we need.

The anti-Stokes lines are contours along which the quantity Qdr is purely real (on the real r axis they correspond to the classically allowed regions of a standard WKB analysis). It is straightforward to show that three such lines emerge from each zero of Q^2. The approximate solutions f_1 and f_2 are of the same order of magnitude on these lines, and both functions are oscillatory without exponential growth or decay. At a general point of the complex r plane, on the other hand, one of the functions f_1 and f_2 is dominant (exponentially large) while the other is subdominant. This property changes as an anti-Stokes line is crossed: If f_1 is dominant in one region of the coordinate plane, it becomes subdominant after crossing an anti-Stokes line. The dominance is maximal on curves where Qdr is imaginary. These are the so-called Stokes lines. It is when a solution to (14) is continued across one of these curves that the Stokes phenomenon occurs. That is, the three Stokes lines that emanate from each transition point divide the complex r plane into sectors where different linear combinations of f_1 and f_2 should be used to represent the exact solution to (14). As Berry has shown [22,23], the Stokes phenomenon corresponds to a smooth change in the coefficient α_n of the solution f_n that is subdominant on the Stokes line. At the same time the coefficient of the dominant solution remains unchanged. Far away from all transition points the change is confined to an infinitesimal interval including the Stokes line.

In the case of two transition points anti-Stokes lines emerge in four asymptotic directions. Similarly, there are four asymptotic Stokes lines. This situation is shown in Fig. 2 (which contains the same information as Fig. 5 in [7]). Assuming that there is a cut between t_1 and t_2, Q is single valued far away from the transition points. Then we need not cross any cuts when continuing a solution to (14) one full turn around t_1 and t_2. Let us denote the four unknown Stokes constants by a_{-1}, b_0, a_1, and b_2 as in Fig. 2(a). It turns out that these are related, and Fröman, Fröman, and Lundborg [7] have shown that

$$b_2 = e^{2i\gamma_{21}} b_0 \ , \tag{26}$$

$$a_1 = e^{-2i\gamma_{21}} a_{-1} \ , \tag{27}$$

$$a_{-1} b_0 = -\left[1 + e^{2i\gamma_{21}}\right] \ . \tag{28}$$

These relations are obtained by comparing the product of all four F matrices in Fig. 2(a) with the proper matrix that takes the solution one turn around the region containing the transition points (this should lead to a single-valued solution in the initial point) [13]. The quantity γ_{21} is

$$\gamma_{21} = \int_{t_2}^{t_1} Q(\xi) d\xi \ , \tag{29}$$

in the lowest order of approximation. In higher orders this must be replaced by

$$\gamma_{21} = \frac{1}{2} \oint_\Gamma q(\xi) d\xi \ , \tag{30}$$

where Γ is a contour encircling the two transition points. Anyway, it is clear that only *one* of the four Stokes constants remains unspecified.

An expression for the remaining Stokes constant can be obtained by the comparison equation technique [24]. Equation (14) can be uniformly mapped onto an equation that has a similar local structure and for which exact solutions are known. In the present case a suitable comparison equation is that for a parabolic barrier (since it involves two transition points), the solutions of which can be expressed in terms of Weber functions. By expressing the asymptotic behavior of these functions in phase-integral quantities one can infer the form of the desired Stokes constant. This is a fairly technical procedure, but the resulting formulas are uniformly valid (in this case they can be used not only for frequencies above and below the barrier top, but also for frequencies close to the top of the barrier when the two transition points coalesce) which makes the labor worthwhile. We get [25,26]

$$a_{-1} = i \left[1 + e^{-2i\gamma_{21}}\right]^{1/2} e^{-2i\sigma} \ , \tag{31}$$

where, in the $(2N+1)$th order of phase-integral approximation,

$$\sigma = -\frac{1}{4i} \ln \frac{\Gamma(1/2 - \gamma_{21}/\pi)}{\Gamma(1/2 + \gamma_{21}/\pi)} + \frac{i\gamma_{21}}{2\pi} \ln \frac{i\gamma_{21}^{(0)}}{\pi} + \sum_{n=0}^{N} \sigma^{(2n)} \ . \tag{32}$$

Here $\gamma_{21}^{(0)}$ represents the lowest-order contribution to γ_{21} and

$$\sigma^{(0)} = -\frac{i\gamma_{21}^{(0)}}{2\pi} \ , \tag{33}$$

$$\sigma^{(2)} = -\frac{i\pi}{48\gamma_{21}^{(0)}} \ . \tag{34}$$

Higher-order terms are listed in, for example, [26]. Given (31) and (32) it can be verified that a_{-1} approaches the expected value ($= i$) as the two transition points move away from each other. Specifically, we will have

$$\gamma_{21} \to -i\infty \ , \tag{35}$$

$$\sigma \to 0 \ , \tag{36}$$

far below the barrier top, and

$$\gamma_{21} \to +i\infty \ , \tag{37}$$

$$\sigma \to -\frac{1}{2}\gamma_{21} \ , \tag{38}$$

high above the barrier.

It should be mentioned that formulas similar to those discussed in this section have been used by several other authors. The studies of Fröman and Fröman [27] and of Amaha and Thylwe [28] are closely related to the present one. For other examples of applications involving the uniform barrier approximation see [15,29,30]. A study of special interest is that by Lundborg [31] where the above approximation is tested for an exactly solvable model.

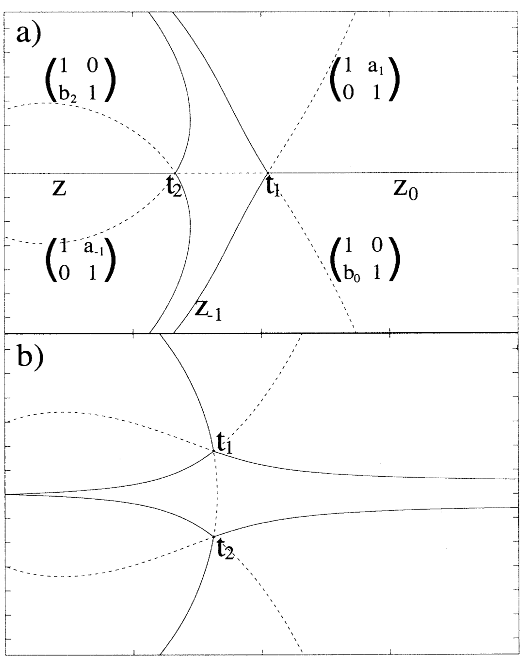

FIG. 2. The pattern of Stokes (dashed) and anti-Stokes lines (solid) for a typical barrier problem. The two transition points are denoted by t_1 and t_2. (a) adheres to frequencies below the barrier top whereas (b) corresponds to frequencies above the barrier. The relevant F matrices that are used to continue an approximate phase-integral solution around the transition points are indicated in (a). In the analysis of the black-hole problem, the point z represents the event horizon and z_0 is spatial infinity. It is assumed that the phase of Q is such that f_1 represents an outgoing wave at z_0.

IV. BLACK-HOLE PHASE SHIFTS

A. Approximate formula

After the general considerations of the previous section we are equipped to construct a solution to (14). Choosing the phase of Q such that f_1 describes an outgoing wave at $+\infty$, we have

$$\int q dr \sim \omega r + 2M\omega \ln\left(\frac{r}{2M} - 1\right) \sim \omega r_* \quad \text{as } r \to +\infty \ . \tag{39}$$

Moreover, it turns out that this is the behavior also as $r \to 2M$ (as long as the two relevant transition points lie on opposite sides of the real r axis or on the axis itself). Comparing this situation with that in Fig. 2(a), we can identify $r = 2M$ with the point z and $r = +\infty$ with z_0 in that figure. It follows immediately that a solution given at the horizon of the black hole can be extended to spatial infinity by means of the matrix

$$F(+\infty, 2M) = \begin{pmatrix} 1 & a_{-1} \\ b_0 & 1 + a_{-1}b_0 \end{pmatrix} \ . \tag{40}$$

Since the solution that corresponds to purely ingoing waves crossing the event horizon is proportional to f_1 [according to (39)], we can infer [by means of (24) and (25)] that an approximate solution is

$$\psi_\ell \approx f_1 + b_0 f_2 \ , \tag{41}$$

when r is considerably larger than t_1 and t_2.

In order to get an expression for the phase shifts we need to study the behavior of this solution as $r \to +\infty$ and extract $A_{\rm in}$ and $A_{\rm out}$ by comparing to (6). Let, in the lowest order of approximation,

$$\eta = \int_{t_2}^{+\infty} \left[q - \left(1 - \frac{2M}{r}\right)^{-1} \omega \right] dr \\ -\omega \left[t_2 + 2M \ln\left(\frac{t_2}{2M} - 1\right)\right] \tag{42}$$

(in higher orders this must be replaced by a contour integral encircling t_2 in the appropriate way) represent the constant "asymptotic phase." Then we get, from (16) and (41),

$$f_{1,2} \to \frac{1}{\sqrt{\omega}} e^{\pm i(\omega r_* + \eta)} \quad \text{as } r \to +\infty \ . \tag{43}$$

Now it is easy to identify

$$\frac{A_{\rm out}}{A_{\rm in}} = \frac{e^{2i\eta}}{b_0} \ . \tag{44}$$

From this and (10) follows the final formula for the phase shift:

$$\delta_\ell = \eta - \frac{1}{2i} \ln b_0 + (\ell + 1)\frac{\pi}{2} \ . \tag{45}$$

This formula can be written in a more explicit way using (28) and (31):

$$\delta_\ell = \eta - \frac{(3 \mp 1)}{4}\gamma_{21} - \frac{1}{4i} \ln\left[1 + e^{\pm 2i\gamma_{21}}\right] \\ -\sigma + (2\ell + 1)\frac{\pi}{4} \ . \tag{46}$$

It is natural to use the upper sign for frequencies above the barrier and the lower sign below the barrier [because of (35) and (37)]. As already mentioned, this formula provides a uniform approximation in the sense that it is equally valid above and below the top of the potential barrier [see the discussion preceding (31)].

B. High- and low-frequency limits

In order to check whether our phase-shift formula (45) makes sense we can compare its predictions for the reflection coefficient

$$\mathcal{R} = \left| \frac{A_{\rm out}}{A_{\rm in}} \right|^2 \tag{47}$$

to standard WKB formulas. For frequencies far below the top of the barrier we can use (35) and (36). That is, we need only figure out what the imaginary part of η is in order to get an expression for $\mathcal{R}$. This is, however, trivial since we are dealing with a real potential barrier. The imaginary part of η must be equal to γ_{21} [cf. (42) and Fig. 2(a)]. Then it follows immediately from (44) that

$$\mathcal{R} = \frac{e^{2|\gamma_{21}|}}{1 + e^{2|\gamma_{21}|}} \quad \text{when } (\omega M)^2 \ll V_{\rm max} \ . \tag{48}$$

This is exactly Eq. (9.12a) in [13].

For high frequencies the appropriate limits are (37) and (38), but the imaginary part of η is now equal to $\gamma_{21}/2$ [due to symmetry that is apparent in Fig. 2(b)]. Then we have

$$\mathcal{R} = \frac{1}{1 + e^{2|\gamma_{21}|}} \quad \text{when } (\omega M)^2 \gg V_{\rm max} \ , \tag{49}$$

which is Eq. (9.27a) in [13].

Finally, if formula (45) is to be uniformly valid, it should also agree with the well-known one-turning-point WKB formula for phase shifts (used by Sanchez in the context of black-hole scattering [3]) in the low-frequency limit. In that limit one can safely neglect transmission, since the barrier becomes very thick, and consider a wave as totally reflected at the outer transition point t_1. From (35), (36), and (45) it then follows that

$$\delta_\ell = \int_{t_1}^{+\infty} \left[q - \left(1 - \frac{2M}{r}\right)^{-1} \omega \right] dr \\ -\omega \left[t_1 + 2M \ln\left(\frac{t_1}{2M} - 1\right)\right] + (2\ell + 1)\frac{\pi}{4} \ , \tag{50}$$

when ω is small. It is easy to verify that this is indeed the formula used by Sanchez [Eq. (9) in [3]]. This is of some computational importance. It means that we need not evaluate the integral γ_{21} in the low-frequency limit (i.e., when $\ell >> \omega M$) and can thus save a considerable amount of computing time.

C. Digression: Quasinormal modes

A problem that is closely related to the scattering one concerns the so-called quasinormal modes of black holes. These modes have attracted a lot of interest in recent years (see [17] for a discussion of the literature). They are expected to be excited in all dynamical processes that involve a black hole and will dominate the emerging radiation at relatively late times. Each black hole has a spectrum of complex-frequency modes that depends only on the black-hole parameters: mass, angular momentum, and electric charge. In view of the ongoing attempts to detect gravitational radiation [32] the quasinormal modes are of special importance. They may offer a more or less direct way of identifying black holes. The quasinormal modes correspond to solutions to (12) that satisfy a condition of no waves coming in from infinity, i.e., $A_{\rm in} = 0$. They can be viewed as waves that are temporarily trapped in the region of the peak of the curvature potential ($r \approx 3M$), i.e., as scattering resonances. As already indicated, our analysis of the scattering problem should remain valid also for complex frequencies, at least as long as only two transition points need be considered. Hence, we can infer a phase-integral condition that determines quasinormal modes from (41) and (43). We get

$$A_{\rm in} = \frac{b_0}{\sqrt{\omega}} e^{-i\eta} \ . \tag{51}$$

That is, the only way that we can achieve $A_{\rm in} = 0$ is by b_0 vanishing. According to (28) this means that we must have

$$b_0 = i e^{2i\sigma} e^{i\gamma_{21}} \left[1 + e^{2i\gamma_{21}}\right]^{1/2} = 0 \ , \tag{52}$$

i.e.,

$$\gamma_{21} = \left(n + \frac{1}{2}\right)\pi \ , \tag{53}$$

where n is an integer. This is the so-called Bohr-Sommerfeld formula that was discussed in detail in [17]. It has been shown to generate the slowest damped quasinormal-mode frequencies with high accuracy. It eventually fails when Im $\omega \approx$ Re ω [26]. One should not be very surprised that we can derive it in this way. It was first derived (within the phase-integral method) by Fröman *et al.* [16], and they used an approach formally similar to the present one. However, this clearly demonstrates the extent to which the present analysis of the scattering problem remains valid also for complex frequencies.

D. Numerical results

Typical results of numerical calculations for $\ell = 0 - 3$ using our approximate phase-shift formula are given in Fig. 3. Figure 3(a) shows that the real part of the phase shift [as computed directly from (45) with Q^2 according to (23)] has two zeros and approaches a constant as $\omega M \to \infty$. In order to get agreement with the "Newtonian analogue," which is discussed in detail in Sec. 6.1.1 of [1], we should add an "integration constant" $-2\omega M \ln 4\omega M + \omega M$ to our phase shifts. That we are allowed to do so is clear from (4) which defines the tortoise coordinate r_*. The function of ω that is to be added is also indicated in Fig. 3(a). When this "constant" is added to our results they are in good qualitative agreement with those of Sanchez (see Fig. 1 of [6]). It is important to remember that the introduction of this "constant" is purely conventional. As long as an addition to the phase shifts is independent of ℓ it will not affect the physical results at all [33].

The imaginary part of δ_ℓ vanishes as $\omega M \to 0$; see Fig. 3(b). This is in accordance with the results of Starobinskii [34] (see also [35]) in the low-frequency limit. Moreover, from Fig. 3(b) we can see that the imaginary part of δ_ℓ increases monotonically with ωM. This means that

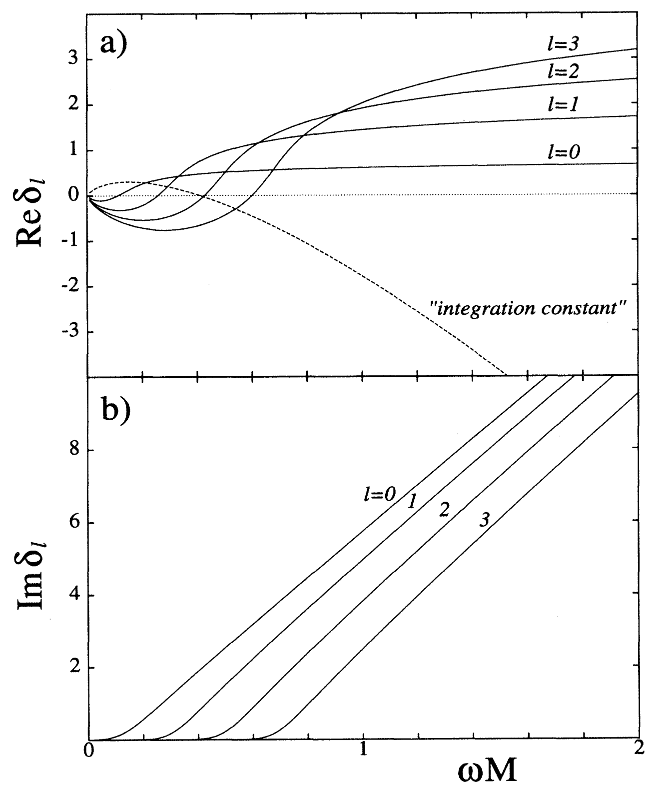

FIG. 3. Phase shifts for $\ell = 0 - 3$, as computed from the approximate phase-integral formula, are drawn as functions of ωM. (a) It is clear that the real parts approach zero as the frequency vanishes and that they pass through another zero when the frequency increases. For high frequencies the real part approaches a constant value. This value increases with ℓ. In order to compare the approximate results with the exactly solvable Newtonian analogue we must add $-2\omega M \ln 4\omega M + \omega M$ to our results. This function is also shown (as a dashed curve). (b) The imaginary parts of δ_ℓ also vanish when the frequency becomes small, but as the frequency increases they increase linearly with ωM. The onset of increase occurs for higher frequencies for larger ℓ.

the present results agree well with those shown in Fig. 1 of [5].

We have tested the two choices for Q discussed in Sec. III A numerically. They both lead to phase shifts that depend on the frequency in a qualitatively similar way (cf. Fig. 3). The agreement between the two improves for higher values of ℓ and as the frequency increases. But we need a more quantitative assessment of how accurate these phase shifts are. Interestingly such an estimate is readily achievable within the phase-integral method. The higher-order phase integral approximation is based on the local asymptotic expansion (18). Hence, it makes some sense to assume that any formula obtained within this method, such as the present one for phase shifts, consists of an asymptotic expansion (in some sense) as well. Then one would expect the accuracy of the approximate phase shifts, for a certain value of ℓ and a given frequency, to first improve and eventually diverge as the order of approximation is increased. The best possible result would be obtained if the expansion were truncated at the order of approximation that gives the smallest contribution, i.e., the one that precedes the divergence of the series. Up to that point the contribution by a certain order of approximation provides a useful estimate of the error in the preceding order. A similar idea to this has previously been used by Andersson *et al.* [18]. Hence, it makes sense to extend the present study to at least the first two orders of approximation. It seems likely that all relevant physical effects in the scattering problem can be recovered using the lowest order, but the following one provides an estimate of the actual error of each phase shift. The result of such an error analysis is shown in Fig. 4. Generally, one finds that the phase shifts are

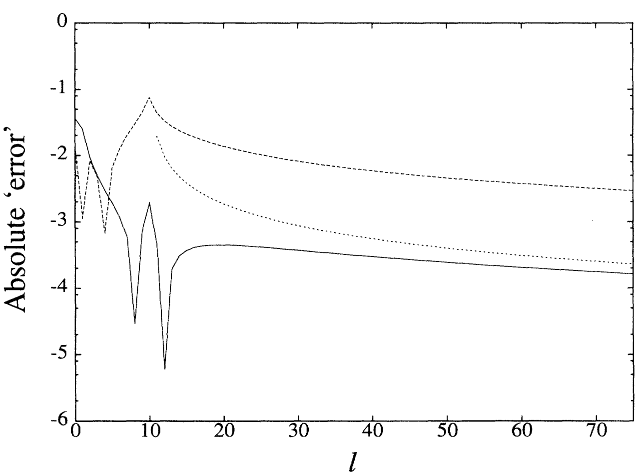

FIG. 4. Estimating the accuracy of the approximate phase shifts. The error in the lowest order is assumed to be similar in magnitude to the contribution from the next order of approximation. We plot the absolute difference between the first and the third order phase shifts here. $Q^2 = R$ is indicated by a dashed line and $Q^2 = R - 1/4(r-2M)^2$ corresponds to the solid line. We also show results obtained from the one-transition point formula that can be used for large ℓ. These are only calculated for the second choice of Q^2 and are indicated by a short-dashed line.

less accurate for low frequencies, but that the accuracy improves with increasing frequency for a fixed ℓ (or for a fixed frequency and increasing ℓ). The choice (23) is also typically at least one order of magnitude more accurate than that prescribed by the standard WKB approximation ($Q^2 = R$). Given this information we will only discuss the choice (23) in the following.

It should be pointed out that we cannot expect our formulas to be reliable when $\omega M \to 0$. The reason for this is the following: When $\omega M \to 0$ the innermost of the two transition points (t_2) that we are considering will move closer to $r = 2M$. Then the assumption that t_1 and t_2 are far away from all other transition points (zeros and poles of Q^2) is clearly no longer valid. This situation can only be studied by means of uniform phase-integral approximations including also the pole at $r = 2M$ in the analysis; see [26]. This deficiency may not be crucial, however. We have already seen that we can replace (45) with (50) when $\ell \gg \omega M$. In effect, we need only worry about our analysis failing for low frequencies when $\ell = 0$. This is, in fact, a typical deficiency of WKB-type formulas in problems with long-range potentials. One would generally find that the approximation improves as ℓ increases (for a fixed frequency) in such situations. That this is true also in the black-hole case is clear from the present results (see also [36]).

V. PHYSICAL IMPLICATIONS

From the previous section it would seem as if our approximate phase-shift formula is reliable in most situations. Hence, it is reasonable to assess whether it can be of any practical use in black-hole physics. For that to be the case we must establish two things: that it can reproduce accepted physical results and that it can help us proceed beyond what has been achieved previously. The first of these tasks is, in fact, the reason why we chose the scalar-wave problem in the first place. Being the simplest scattering problem that involves black holes it is relatively well understood and therefore an ideal testing ground for a study of this kind.

It is worth pointing out that in the context of scattering the possible breakdown of our approximate analysis as $\omega M \to 0$ for the lowest value(s) of ℓ is of hardly any consequence. One would be mainly interested in diffraction effects and for low frequencies the wavelength is such that the infalling plane wave will hardly be affected at all by the presence of a black hole. Furthermore, the low ℓ partial waves are essentially absorbed, and so any deficiency in the corresponding phase shifts would not be detectable in the differential cross section. Rather, one would expect such effects to show up in the absorption cross section.

A. Deflection function

When discussing the physical quantities that follow from a set of phase shifts it is natural to use Ford and Wheeler's excellent description of semiclassical scattering from the late 1950s [37,38]. It can be brought to bear also on the black-hole problem (see the discussion by Handler and Matzner [39]). In the semiclassical paradigm one introduces an impact parameter b that is related to ℓ (the angular momentum) via

$$b = \left(\ell + \frac{1}{2}\right)\frac{1}{\omega} \ . \tag{54}$$

That is, each partial wave is considered as impinging on the black hole from an initial distance b away from the axis.

In the semiclassical picture, much physical information can be extracted from the so-called deflection function. It corresponds to the angle by which a certain partial wave is scattered by the black hole, and is related to the real part of our approximate phase shifts:

$$\Theta(\ell) = 2\frac{d}{d\ell}\mathrm{Re}\ \delta_\ell \ . \tag{55}$$

Here ℓ is allowed to assume continuous real values. It is certainly possible to derive a phase-integral formula to determine Θ for any real value of ℓ, especially since our phase-shift formula remains valid for noninteger ℓ, but it is probably not a worthwhile exercise. As it turns out, the simple difference formula $\Theta(\ell) \approx \mathrm{Re}\,\delta_{\ell+1} - \mathrm{Re}\,\delta_{\ell-1}$ enables us to understand the features of the differential cross sections.

First of all, we can use the deflection function as yet another check that our phase-shift formula gives reasonable results. For large values of the impact parameter b, one would expect the value of the deflection function to agree with Einstein's classic result $\Theta_E = -4M/b$. As can be seen in Fig. 5 this is, indeed, the case. The phase-integral phase shifts lead to a deflection function that rapidly approaches Θ_E as ℓ increases.

Whenever the classical cross section diverges in either the forward or the backward direction a diffraction phenomenon called a "glory" arises. This phenomenon is well known in optics, but it also arises in quantum scattering [37,38]. Moreover, glories arise in the black-hole case [1,40]. In general, backward glories can occur if $\Theta < -\pi$ for some values of b. Whenever the deflection function passes through zero or a multiple of π we have a glory. In the black-hole case one would expect glory scattering to be associated with the unstable photon orbit at $r = 3M$ [41]. This essentially means that we would expect a logarithmic singularity (in other words, a resonance) in the deflection function to be associated with the critical impact parameter $b_c = 3\sqrt{3}M$. This feature is obvious in Fig. 5. Darwin [42] has deduced an approximate relation between the impact parameter and the deflection function close to this singularity:

$$b(\Theta) \approx 3\sqrt{3}M + 3.48Me^{-\Theta} \ . \tag{56}$$

If we invert this formula and use (54) we get Θ as a function of ℓ. As can be seen in Fig. 5 this approximation is in excellent agreement with the deflection function obtained from the approximate phase shifts.

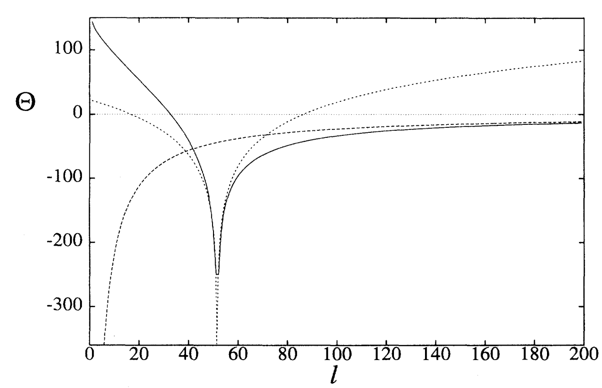

FIG. 5. The deflection function Θ (solid line) is shown as a function of ℓ for $\omega M = 10$. For large impact parameters (large ℓ) the approximate results approach the Einstein deflection angle $-4M/b$ (dashed line). A logarithmic singularity in Θ is apparent at the critical impact parameter ($\ell_c \approx 51.5$ here). This feature is associated with the unstable photon orbit at $r = 3M$. Also shown (as a dashed curve) is an approximation obtained by inverting Darwin's formula (56).

B. Elastic scattering and differential cross sections

Although (11) suggests that we can readily calculate the scattering amplitude as a partial-wave sum once we know the phase shifts δ_ℓ, this is not the case. Because we are dealing with a long-range potential, the sum will diverge. This problem is well known since it also arises in Coulomb scattering [43]. (An interesting historical account of misunderstandings in the Coulomb case can be found in [44].) Some ways of avoiding this difficulty, basically by introducing a cutoff where the remainder of the true partial-wave sum is replaced by analytic results for a limiting case (in the black-hole case the Newtonian analogue discussed in Sec. 6.1.1 of [1]), were discussed by Connor and Thylwe [45]. We approach this difficulty by expressing the scattering amplitude as a sum of two parts. In essence, we extract the contribution from large impact parameters from (11), i.e., replace it by

$$f(\theta) = f_N(\theta) + f_D(\theta) \ . \tag{57}$$

The long-range (Newtonian) contribution is [1,12]

$$f_N(\theta) = M\frac{\Gamma(1-2iM\omega)}{\Gamma(1+2iM\omega)}\left[\sin\frac{\theta}{2}\right]^{-2+4iM\omega} \ , \tag{58}$$

and

$$f_D(\theta) = \frac{1}{2i\omega}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{2i\delta_\ell} - e^{2i\delta_\ell^N}\right]P_\ell(\cos\theta) \tag{59}$$

is the part of the scattering amplitude that is expected to give rise to diffraction effects. The Newtonian phase shifts δ_ℓ^N follow from

$$e^{2i\delta_\ell^N} = \frac{\Gamma(\ell+1-2iM\omega)}{\Gamma(\ell+1+2iM\omega)} \ . \tag{60}$$

Now one would certainly expect the sum in (59) to converge. However, in reality we still have to truncate the sum at a (be it very large) value $\ell_{\max}$. Consequently, one would expect some oscillations due to interference (roughly with a wavelength $2\pi/\ell_{\max}$ [37]) to remain in $f(\theta)$. Although this is a minor effect that does not degrade the actual results, it is not aesthetically pleasing. Hence, we follow Handler and Matzner [39] and, for a given $\ell_{\max}$, add a constant γ to all the approximate phase shifts in (59). The constant should be such that the last term in the sum does not give a contribution, i.e., $\delta_{\ell_{\max}} + \gamma = \delta^N_{\ell_{\max}}$. It is important to note that this means that the partial waves $\ell > \ell_{\max}$ have no effect whatsoever on the cross section.

It should be stressed that, although convenient from the computational point of view, the split of the scattering amplitude is unphysical [45]. Hence, one should avoid drawing conclusions from the two terms in (57) separately. The quantity of physical importance in elastic scattering is the differential cross section, the "intensity" that is scattered into a certain solid angle. It follows from the well-known relation

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \ . \tag{61}$$

We have constructed cross sections for several frequencies. A representative case is shown in Fig. 6. This figure can be compared to Sanchez's results (Fig. 4 in [6] or Fig. 8.12b in [1]).

In general, one would expect the long-range attraction of the gravitational interaction to give rise to a divergent focusing ($\sim \theta^{-4}$) in the forward direction. That is, the first term in (57) should dominate the second for small deflection angles. Similarly, large deflection should only arise for waves that probe the region close to the black hole, and so the second term in (57) should be dominant for large angles. That this is, indeed, the case can be seen in Fig. 6(b) where we show the relation between the two terms in (57).

Our cross sections are generally somewhat different from those of Sanchez [6]. This is perhaps surprising since our approximate phase shifts depend on the frequency in a way similar to that of the phase shifts that Sanchez generated by repeated analytic continuation. However, the diffraction oscillations that are apparent in our figures agree, as we will see, perfectly with what one would expect. The scattering of scalar waves is the simplest possible problem involving black holes and one would not be too surprised to find that the results are perfectly regular (as in Fig. 6). Furthermore, Anninos *et al.* have recently studied the scalar-wave problem in the context of orbiting [46]. We have used phase shifts generated from (45) to reproduce their Fig. 9 (for $\omega M = 2.5$). The result is a perfect match (and therefore not shown here). Hence, it would certainly seem as if our phase-shift formula can be trusted.

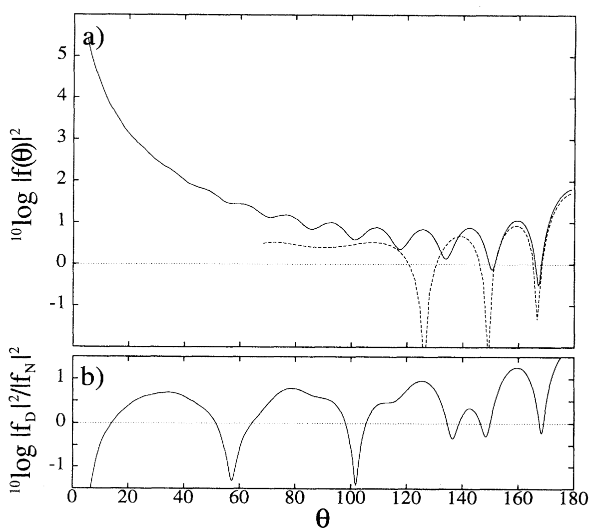

FIG. 6. (a) The differential cross section for $\omega M = 2.0$ (solid line). Shown is $\log_{10}|f(\theta)|^2$. For comparison we also show the glory approximation (62) (dashed line). It should be noted that this approximation agrees remarkably well with our results. (b) The relation between the two terms $f_D(\theta)$ and $f_N(\theta)$ in (57). One hundred approximate phase shifts were used to generate the figure.

C. Black-hole glory

As already mentioned above, we expect to find that interference between partial waves associated with the unstable photon orbit, i.e., with impact parameters $b \approx b_c = 3\sqrt{3}M$, gives rise to a glory effect in the backward direction [41]. One can argue that this effect should only be seen for frequencies considerably larger than $1/6\sqrt{3}M$ since no partial waves get close to the critical impact parameter for lower frequencies (the wavelength is so large that the wave cannot "see" the black hole). We have verified that this is, indeed, the case. For example, for $\omega M = 0.25$ no diffraction oscillations could be seen in the backward direction. As we increase the frequency the cross section changes and Fig. 6 provides a beautiful example of a backward glory.

How can we be sure that we actually do see a glory? First of all, it has been known for a long time that the effect of glory scattering on the cross section can be described in terms of Bessel functions [37] (for a discussion of uniform glory approximations see [47]). It is relatively easy to convince oneself that the oscillations in Fig. 6 agree well with the expected position of the zeros of $J_0(x)$. Naturally, this is only an *ad hoc* argument and it would be nice to compare our results to an approximation of the glory expected from interference of partial waves close to b_c. Such an approximation has been derived (for scattering of waves with arbitrary helicity) by means of functional integration [48,49]. In the case of scalar-wave scattering the resultant formula, which can also be derived by partial-wave decomposition [37], is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{glory}} = 2\pi\omega b^2 \left|\frac{db}{d\theta}\right| J_0^2(\omega b \sin\theta) \ . \tag{62}$$

When combined with the Darwin formula (56) this should provide a good approximation whenever $\omega M \gg 1$

and $|\theta - \pi| \ll 1$ (see [40]). As can be seen from Fig. 6 this approximation agrees nicely with our results. We have constructed cross sections for a variety of frequencies $\leq 10M^{-1}$. We find that the agreement between the resultant cross sections and the glory approximation gets better as the frequency increases. That is, the glory approximation becomes accurate for a wider range of angles in the backward direction. There is thus no doubt that our cross sections are dominated by a focusing divergence in the forward direction and glory diffraction in the backward direction.

According to the predictions of geometrical optics [41] one might expect to find glory oscillations not only in the backward direction, but also in the forward direction. Partial waves associated with the critical impact parameter may be deflected by any multiple of π and so give rise to diffraction close to both $\theta = 0$ and π. Moreover, in Fig. 5 we see that the deflection function generally passes through $\Theta = 0$ for a value of ℓ lower than that associated with the unstable photon orbit. This means that there does exist a forward glory. However, this feature drowns in the divergence of the cross section that is due to the large ℓ partial waves. Moreover, partial waves corresponding to $\ell < \ell_c$ are to a large extent absorbed. The forward glory is therefore exceptionally faint in the black-hole case.

D. Absorption cross section

An obvious and important feature of black-hole scattering is absorption. The contribution that each partial wave makes to absorption is related to the amplitude C_ℓ of the waves that cross the event horizon:

$$\sigma_\ell^{\text{abs}} = 4\pi(2\ell+1)\,|C_\ell|^2 \ . \tag{63}$$

An expression for C_ℓ in terms of the phase shifts can be obtained by employing the solution (6), from which (8) was subtracted, to get an expression for the scattering amplitude:

$$\phi_\ell \to \frac{(-1)^{\ell+1}}{2i\omega}\left[e^{-i\omega r_*} + \frac{A_{\text{out}}}{A_{\text{in}}}e^{i\omega r_*}\right] \quad \text{as } r_* \to +\infty \ . \tag{64}$$

Noting that a linearly independent solution to (1) is the complex conjugate of ϕ_ℓ, and using the fact that the Wronskian of these two solutions must be constant we get

$$|C_\ell|^2 = \frac{1}{4\omega^2}\left[1 - \left|\frac{A_{\text{out}}}{A_{\text{in}}}\right|^2\right] \ . \tag{65}$$

Hence, the contribution of each partial wave to absorption is

$$\sigma_\ell^{\text{abs}} = \frac{\pi}{\omega^2}(2\ell+1)\left[1 - e^{-4\text{Im}\,\delta_\ell}\right] \ ; \tag{66}$$

see also [5]. In Fig. 7 we show σ_ℓ^{abs} as a function of ωM for the first few values of ℓ. We have already mentioned above that the phase-integral analysis will break down for $\ell = 0$ and low frequencies. That this is the case is clear when Fig. 7 is compared to Fig. 2 in [5]. Instead of approaching the expected value $16\pi M^2$ [50] as $\omega M \to 0$ the approximate σ_0^{abs} goes to infinity in that limit. According to Fig. 7 our phase-shift formula is not reliable when $\omega M < 0.1$ or so.

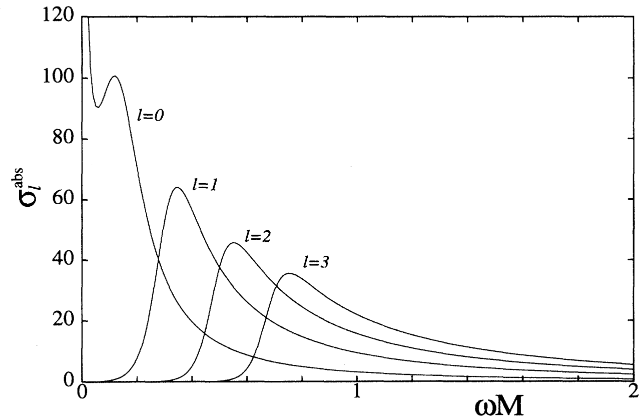

FIG. 7. The contribution σ_ℓ^{abs} for $\ell = 0$–3 to the absorption cross section as a function of ωM. Each contribution attains a maximum value for a frequency slightly higher than that associated with the critical impact parameter.

VI. CONCLUDING DISCUSSION

We have seen that physical results obtained from the approximate phase-shift formula (45) agree very well with the established predictions of geometrical optics [41] and previous studies of the black-hole scattering problem [1,5,6,40]. The only drawback is that our formula cannot be trusted when $\ell = 0$ *and* the frequency is lower than $\omega M = 0.1$ or so. But, as already mentioned above, this is a typical deficiency of WKB-type formulas in Coulomb-like problems.

It seems safe to conclude that our phase-shift formula is reliable, but is it a useful tool for black-hole physics? One would be inclined to give that question an affirmative answer. The reason for this becomes clear when one considers alternative approaches to the problem. In their study of electromagnetic and gravitational waves, Matzner and Ryan [51] integrated the Teukolsky equation [the analogue of (1)] numerically. Since the desired solution is an oscillating function this calculation becomes increasingly difficult (and time consuming) as the frequency increases. Consequently, Matzner and Ryan restricted their study to $\omega M \leq 0.75$ and $\ell \leq 10$. In order to avoid difficulties, Handler and Matzner [39] combined a numerical solution in the region where the potential varies rapidly with approximate WKB solutions for relatively large values of r. With this technique they performed calculations for $\ell \leq 20$ and ωM up to 2.5. While it is not likely that our approximate phase shifts are more accurate than ones generated by numerical integration, the present approach is computationally much more effective than any purely numerical approach. This is basically because (45) does not directly involve the (rapidly oscillating) solution to (1). We can easily compute the first 100 phase shifts for a given frequency in a few minutes on a standard workstation. For the same reason, the present approach should remain reliable also for much higher frequencies than those studied before (we have done calculations for frequencies up to and including $\omega M = 10$). Hence, we conclude that the phase-shift formula discussed in the present paper provides a valuable complement to the previous techniques used to study black-hole scattering. It could certainly prove useful also in the more challenging case of Kerr black holes.

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