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Self-consistent approach to neutral-current processes in supernova cores

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The problem of neutral-current processes (neutrino scattering, pair emission, pair absorption, axion emission, etc.) in a nuclear medium can be separated into an expression representing the phase space of the weakly interacting probe, and a set of dynamic structure functions of the medium. For a nonrelativistic medium we reduce the description to two structure functions $S_A(\omega)$ and $S_V(\omega)$ of the energy transfer, representing the axial-vector and vector interactions. S_V is well determined by the single-nucleon approximation while S_A may be dominated by multiply interacting nucleons. Unless the shape of $S_A(\omega)$ changes dramatically at high densities, scattering processes always dominate over pair processes for neutrino transport or the emission of right-handed states. Because the emission of right-handed neutrinos and axions is controlled by the same medium response functions, a consistent constraint on their properties from consideration of supernova cooling should use the same structure functions for both neutrino transport and exotic cooling mechanisms.

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I. INTRODUCTION

The neutrino signal from the supernova (SN) 1987A has confirmed our basic understanding that type-II supernovae result from the core collapse of massive stars [1]. The observed neutrinos were radiated from the "neutrino sphere" with a luminosity and energy spectrum commensurate with expectations based both on broad theoretical grounds and detailed numerical models. Further, the time scale for the neutrino "light curve" was broadly in agreement with the hypothesis that energy transport in the hot, dense core is dominated by neutrino diffusion. Despite these successes there is much that needs to be done to achieve both a quantitative and a qualitative understanding of SN physics.

In this paper, we examine a number of issues concerning neutrino transport in SN cores. We are especially interested in the neutral-current (NC) processes that govern the transport of μ and τ neutrinos. These include scattering $\nu X \to X'\nu$, pair emission $X \to X'\nu\bar\nu$, and pair absorption $\nu\bar\nu X \to X'$ where X and X' are configurations of one or several particles of the medium. These processes are intimately related because the underlying matrix elements are structurally the same. Still, the pair processes have usually been neglected in transport calculations [2], presumably because in a noninteracting gas of nucleons they vanish due to energy-momentum conservation. We would like to know if this naive expectation is born out in a strongly interacting medium.

Although this question has always been present, it has been obscured by the difficulty of calculating any of the relevant processes reliably. Further, the authors that worried about scattering processes and transport in hot SN cores [3–6], could plausibly assume that scattering was the dominant part of neutrino transport; whereas authors worried about pair processes were usually concentrating on the cooling of older, cold, degenerate neutron stars [7]. There is another problem in that the relevant calculations are usually done in the context of perturbation theory—processes with the fewest number of nucleons are assumed to dominate. The lowest-order emission process is usually neutrino pair bremsstrahlung involving two nucleons, $NN \to NN\nu\bar\nu$, whereas the lowest-order scattering process is scattering from a single nucleon $\nu N \to N\nu$. Therefore, if pair processes turned out to be important compared to scattering, the whole perturbative framework of the calculations would seem questionable.

In this light, we find several recent papers on the SN emission of hypothetical right-handed (RH) neutrinos extremely interesting [8–10]. On the basis of perturbative calculations of nucleon-nucleon and pion-nucleon interactions (if there is a pion condensate) it appeared that pair processes of the type $X \to X'\bar\nu_L\nu_R$ would be more important than spin-flip scattering $\nu_L X \to X'\nu_R$ in a nondegenerate nuclear medium. As these processes are structurally very similar to the ones involving only left-handed (LH) neutrinos we are led to wonder if pair processes are also important for the standard neutrino transport in newborn neutron stars, and under what, if any, conditions the perturbative framework can be trusted. Similarly, resolving the question of pair processes in transport calculations will affect the interpretation of the emission rate calculated for several species of hypothetical particles, including RH neutrinos and axions.

General answers to these questions are difficult because of our lack of knowledge about strongly interacting systems. In the end we travel the perturbative road, but before doing so we want to make sure that the relation between scattering and pair processes is kept clear.

To this end, we factorize the problem of neutrino interactions into a "neutrino part" and a "medium part." This allows us to maintain a consistent treatment of the medium while discussing the different neutrino processes. The medium is then described by a small number of response functions, common to all NC processes. In the limit of nonrelativistic medium constituents and ignoring the neutrino momentum transfer to the nucleons, the medium response can be reduced to a single structure function $S(\omega)$ of the energy transfer. S is a linear combination of the density and spin-density dynamic structure functions at vanishing three-momentum transfer. Although we derive S for ordinary LH neutrino interactions it also applies to spin-flip scattering.

In a sufficiently dilute medium the dynamic structure functions can be calculated perfectly well by perturbative methods, so we may borrow readily from other authors' calculations. Our hope is that when extended into the regime of high densities the overall shape of $S(\omega)$ will not abruptly change. In this case we will argue that the *relative* strength of scattering almost certainly exceeds that of pair processes for a medium dominated by nonrelativistic nucleons. Thus, if a pion condensate would substantially add to $X \to X'\nu_R\bar{\nu}_L$ it would add even more to $\nu_L X \to X'\nu_R$. Then, however, it would also strongly affect $\nu_L X \to X'\nu_L$ and thus neutrino transport.

Turn back to the emission of weakly interacting particles. The bremsstrahlung emission of RH neutrinos or axions are sensitive only to the spin-density fluctuations. Even so, the spin-density structure function used for RH neutrino or axion emission is the same as that for ordinary LH neutrino scattering and, therefore, the uncertainties of, for example, the axion emission rate are not unrelated to that of neutrino transport. It follows that a consistent treatment of "exotic" particle emission would have to rely on a common structure function $S(\omega)$ for both neutrino transport and particle emission, whether or not $S(\omega)$ can be reliably calculated. Modifications of the neutrino transport and the emission of novel particles will both affect the observable neutrino signal, and thus they should be implemented on the same footing.

There remains the problem of how to calculate the dynamic structure functions, a problem which is arguably more difficult for the present application then for any other problem concerning neutrino interactions with stellar material. The medium at the core of a photoneutron star is hot, dense, and strongly interacting. There are no parameters that one may use as the basis for a perturbation expansion. Further, the material is neither degenerate nor nondegenerate. In this context, we suggest a phenomenological approach that we believe illustrates some of the features that a full treatment of an interacting medium must possess. This approach uses the interactions of the medium to regulate soft processes, i.e., those where the energy transfer is small, but leaves the hard processes relatively unscathed. Although not totally successful, the technique goes in the direction of regulating the amplitude of the response function calculated in a perturbative series, while modifying its shape in a controlled way.

The rest of this paper is organized as follows. In Sec. II, we develop the main tool for our investigation, the vector and axial structure functions $S_V(\omega)$ and $S_A(\omega)$, respectively, which describe the medium's response to NC's. In Sec. III we present useful properties of these functions and evaluate them perturbatively. In Sec. IV, we examine the relative strength of pair vs scattering processes under a variety of different assumptions about the high-density behavior of S_A. In Sec. V, we consider applications to nonstandard physics, i.e., the emission of novel particles and the impact of a pion condensate. Finally, in Sec. VI, we summarize our discussions and conclusions, with some suggestions as to useful strategies for future research.

II. NEUTRAL-CURRENT NEUTRINO PROCESSES IN A MEDIUM

A. Collision integral

Neutrino transport in a medium is governed by the Boltzmann collision equation, $L[f] = C[f,\bar{f}] \equiv (df/dt)_{\rm coll}$, where f and $\bar{f}$ are the neutrino and antineutrino occupation numbers for LH states. (For now we focus on the standard model without RH neutrinos.) An analogous equation applies to $\bar{f}$. The Liouville operator is given by $L[f] = \partial_t f + \dot{\mathbf{x}}\cdot\nabla_{\mathbf{x}} + \dot{\mathbf{k}}\cdot\nabla_{\mathbf{k}} f$ while the collision integral is

$$\left.\frac{df_{\mathbf{k}_1}}{dt}\right|_{\rm coll} = \int \frac{d^3\mathbf{k}_2}{(2\pi)^3}[W_{k_2,k_1} f_{\mathbf{k}_2}(1-f_{\mathbf{k}_1}) - W_{k_1,k_2} f_{\mathbf{k}_1} \times(1-f_{\mathbf{k}_2}) + W_{-k_2,k_1}(1-f_{\mathbf{k}_1})(1-\bar{f}_{\mathbf{k}_2}) - W_{k_1,-k_2} f_{\mathbf{k}_1}\bar{f}_{\mathbf{k}_2}]\,, \tag{2.1}$$

where we have written the momentum variables as subscripts. The first term corresponds to neutrino scatterings into the mode $\mathbf{k}_1$ from all other modes, the second term is scattering out of mode $\mathbf{k}_1$ into all other modes, the third term is pair production with a final-state neutrino $\mathbf{k}_1$, and the fourth term is pair absorption of a neutrino of momentum $\mathbf{k}_1$ and an antineutrino of any momentum. In this collision integral we only include effective NC processes between neutrinos and the nuclear medium; i.e., we are not considering charged-current processes and NC processes coupling neutrinos to the leptons in the medium.[1]

The expression $W_{k_1 k_2} = W(k_1,k_2)$ is the rate for a neutrino in state k_1 to scatter into k_2 via interaction with the

[1]The temperature T is expected to be of order 50 MeV, whereas the density of baryons is characterized by Fermi momenta in the 400–500 MeV range. Particle species whose density is characterized by T^3 are not nearly as abundant as nucleons. The importance of degenerate e^- and ν_e for neutrino scattering is typically less than a tenth of the nucleons. Even though it is conceivable that there may be situations where they dominate [11], we will always ignore purely leptonic processes.

medium. W is defined in the entire k space for both of its arguments where negative energies indicate the "crossing" of an initial-state neutrino into a final-state antineutrino or vice versa. In thermal equilibrium f and $\bar{f}$ must assume Fermi-Dirac distributions at a given temperature T and chemical potential μ while the collision integral must vanish, leading to the detailed-balance requirement $W(k_1, k_2) = e^{(\omega_1-\omega_2)/T} W(k_2, k_1)$.

It is the pair production and absorption terms under the integral in Eq. (2.1) that are usually neglected in the context of SN neutrino transport. The justification appears to be that the pair terms vanish for a medium of free nucleons. More specifically, $W(k_1, k_2)$ is identically zero if the energy momentum transfer $k = k_1 - k_2$ is timelike ($k_2 > 0$) and if the dispersion relations of the medium excitations are like that of ordinary particles, $E^2 - p^2 = m^2 > 0$.

B. The transition rate $W(k_1, k_2)$

The low-energy NC Hamiltonian for the neutrino field ψ and an effective current operator B^μ for the medium is

$$H_{\text{int}} = \frac{G_F}{2\sqrt{2}} B^\mu \bar{\psi} \gamma_\mu (1 - \gamma_5) \psi \ . \tag{2.2}$$

If the neutrino interactions are dominated by nucleons, the current of the background medium is

$$B^\mu = \sum_{i=n,p} \bar{\psi}_i \gamma^\mu (C_{V,i} - C_{A,i}\gamma_5) \psi_i \ , \tag{2.3}$$

where $\psi_{n,p}$ are the interacting quantum fields for protons and neutrons, and $C_{V,i}$ and $C_{A,i}$ are the relevant vector and axial weak charges.

The current-current structure of H_{int} allows the transition rate to be written in the general form

$$W(k_1, k_2) = \frac{G_F^2 N_B}{8} S_{\mu\nu} N^{\mu\nu} \ , \tag{2.4}$$

where G_F is the Fermi constant and N_B the number density of baryons. $N^{\mu\nu}$ entails the neutrino kinematics, and $S_{\mu\nu}$ describes fluctuations in the medium that produce, scatter, or absorb neutrinos; i.e., it is the dynamic structure function of the medium for NC's.

Up to a normalization factor, $N^{\mu\nu}$ is given by the neutrino part of the squared matrix element for the scattering $\nu_{\mathbf{k}_1} \to \nu_{\mathbf{k}_2}$ [12]:

$$N^{\mu\nu} = \frac{8}{2\omega_1 2\omega_2} [k_1^\mu k_2^\nu + k_2^\mu k_1^\nu - k_1 k_2 g^{\mu\nu} - i\epsilon^{\alpha\beta\mu\nu} k_{1\alpha} k_{2\beta}] \ . \tag{2.5}$$

Then, by the usual crossing relations the expression for the emission of a pair $\bar{\nu}_{\mathbf{k}_1} \nu_{\mathbf{k}_2}$ is given by $k_1 \to -k_1$, while $k_2 \to -k_2$ is for the absorption of $\nu_{\mathbf{k}_1} \bar{\nu}_{\mathbf{k}_2}$. Both operations leave $N^{\mu\nu}$ unchanged because we included a factor $(2\omega_1 2\omega_2)^{-1}$ in its definition.

The dynamic structure factor is then functionally dependent only on the energy momentum transfer so that a single tensor $S_{\mu\nu}(k)$ describes all processes in Eq. (2.1). It can be written directly in terms of the fluctuations in the weak NC's [13],

$$S^{\mu\nu}(k) = \frac{1}{N_B} \int_{-\infty}^{+\infty} dt \ e^{i\omega t} \langle B^\mu(t, \mathbf{k}) B^\nu(0, -\mathbf{k}) \rangle \ , \tag{2.6}$$

where $k = k_1 - k_2 = (\omega, \mathbf{k})$. By the expectation value $\langle \cdots \rangle$ of some operator we mean a trace over a thermal ensemble of the background medium.

Equation (2.4) is a first-order perturbative result in the weak Hamiltonian. The medium, on the other hand, is strongly interacting, so a perturbative calculation of $S_{\mu\nu}$ may or may not be possible. Of course, in principle some insight into its properties can be gained by laboratory experiments, e.g., particle emission from the hot and dense systems produced in heavy-ion collisions.

However, for practical SN calculations one proceeds with a perturbative approach as follows. For a particular term in the expansion one calculates the squared matrix element, which can be written in the form $|\mathcal{M}|^2 = (G_F^2/8) M_{\mu\nu} N^{\mu\nu} 2\omega_1 2\omega_2$, where $M^{\mu\nu}$ is the square of the medium part of the matrix element. Then one performs a phase-space integration over all medium participants and sums over all relevant processes:

$$\begin{aligned} S_{\mu\nu} = \frac{1}{N_B} \sum_{\text{processes}} \prod_{i=1}^{N_i} \int \frac{d^3\mathbf{p}_i}{2E_i(2\pi)^3} f_{\mathbf{p}_i} \\ \times \prod_{f=1}^{N_f} \int \frac{d^3\mathbf{p}_f}{2E_f(2\pi)^3} (1 \pm f_{\mathbf{p}_f}) \\ \times (2\pi)^4 \delta^4 \left(k + \sum_{i=1}^{N_i} p_i - \sum_{f=1}^{N_f} p_f \right) \sum_{\text{spins}} M_{\mu\nu} \ , \end{aligned} \tag{2.7}$$

where N_i and N_f are the number of initial and final medium particles, the $f_{\mathbf{p}}$ are occupation numbers, and the $(1 \pm f_{\mathbf{p}})$ are Pauli-blocking or Bose-stimulation factors.

Unfortunately, for the conditions of a SN core higher-order processes typically yield contributions to $S_{\mu\nu}$ which are of the same order or larger than lower-order ones (see Sec. III below), so one may question the validity of this method. However, because $W(k_1, k_2)$ factorizes, this complication can be separated from a discussion of the relative importance of scattering *vs.* pair processes, which is largely an issue of neutrino phase space.

Therefore, instead of relying too much on perturbative calculations of $S^{\mu\nu}(k)$ we should focus on its general properties. To this end we assume that the medium is homogeneous and isotropic, in which case $S^{\mu\nu}$ can be constructed only from the energy-momentum transfer k and the four-velocity u of the medium [14]:

$$\begin{aligned} S^{\mu\nu} = R_1 u^\mu u^\nu + R_2 (u^\mu u^\nu - g^{\mu\nu}) + R_3 k^\mu k^\nu \\ + R_4 (k^\mu u^\nu + u^\mu k^\nu) + i R_5 \epsilon^{\mu\nu\alpha\beta} u_\alpha k_\beta \ . \end{aligned} \tag{2.8}$$

The structure functions $R_1, \ldots, R_5$ depend on the medium temperature and chemical composition, and on the Lorentz scalars that can be constructed from u and k, namely k^2 and uk. (The third possibility $u^2 = 1$ is a constant.) Instead of k^2 and uk we will use as independent variables the energy and momentum transfer ω and $|\mathbf{k}|$, measured in the rest frame of the medium.

It is convenient to calculate the interaction rate in the rest frame of the medium defined by $u = (1,0,0,0)$. Multiplying Eqs. (2.5) and (2.8), we find

$$W(k_1,k_2) = \frac{G_F^2 N_B}{4}[(1+\cos\theta)R_1 + (3-\cos\theta)R_2 - 2(1-\cos\theta)(\omega_1+\omega_2)R_5] \ , \qquad (2.9)$$

where θ is the neutrino scattering angle. It is specific to the contraction with $N^{\mu\nu}$, relevant for LH neutrinos, that only $R_{1,2,5}$ contribute. For the spin-flip processes discussed in Sec. V we will find that all R's contribute while for axion emission only R_2 survives.

C. Nonrelativistic and long-wavelength limit

Within the context of a nucleonic medium there are two closely related limits that are often taken—the "nonrelativistic" and "the long-wavelength" approximations. The justification for both is that the nucleon mass is larger than any other energy or momentum scale in the problem. (Even though nucleon Fermi momenta are in the 400–500 MeV range and the effective nucleon mass may be as low as 600 MeV, it is still reasonable to treat the nucleons nonrelativistically for the purposes of this paper.) Then, the currents are expanded as a power series in $1/m_N$, after which only the leading term is kept.[2] The long-wavelength approximation assumes that the three-momentum transfer to the nucleons is small compared to typical nucleon momenta and thus can be ignored when calculating the available phase space, thus simplifying those calculations considerably.

As a consequence of the nonrelativistic assumption the structure functions $R_{3,4,5}$ vanish. To see this we first note that the medium current in Eq. (2.2) generally is a sum of vector and an axial-vector piece, $B^\mu = V^\mu + A^\mu$. Then the transformation properties under parity of the five terms in Eq. (2.8) are easily identified to be like $\langle V^\mu V^\nu\rangle$ or $\langle A^\mu A^\nu\rangle$ for terms $1-4$ and $\langle A^\mu V^\nu + V^\mu A^\nu\rangle$ for term 5. In the limit of nonrelativistic nucleons V^μ has only a zero component, $V^0 = C_V\psi^\dagger\psi$ (a sum over nucleon species is implied), while the axial-vector current A^μ has only spatial components, $A^i = C_A\psi^\dagger\sigma^i\psi$, where σ^i are the Pauli matrices and the nonrelativistic ψ are Pauli spinors. If $V^i = 0$ and $A^0 = 0$ all terms involving $\langle V^0 V^i\rangle$ or $\langle A^0 A^i\rangle$ vanish, i.e., terms 3 and 4. Term 5 involves components $\langle A^i V^j\rangle$ which also vanish. Term 1 only has a 00 component and thus corresponds to $\langle V^0 V^0\rangle$, i.e., for only one species of nucleons $R_1 = C_V^2 S_\rho$ with S_ρ, the usual dynamic structure function for nucleon density fluctuations [6]. Term 2 only has spatial components and thus, it corresponds to $\langle A^i A^i\rangle$, i.e., $R_2 = C_A^2 S_\sigma$ with S_σ the dynamic structure function for spin-density fluctuations [6]. For a mixed medium of protons and neutrons, the interpretation of $R_{1,2}$ is more complicated because there are isospin 0 and 1 contributions to both [4].

[2]It may happen that the leading contribution to the current makes no contribution to an interaction, in which case the next term in the nonrelativistic expansion should be kept.

In the nonrelativistic limit we are thus left with only two structure functions $R_{1,2}(k)$ to consider. Turning to the long-wavelength approximation we assume that the three-momentum transferred to the nucleons is negligible. This is justified by considering the perturbative series Eq. (2.7). Each contribution to $S_{\mu\nu}$ "knows" about the energy-momentum transfer only through the overall energy-momentum conserving δ function. If the nucleon mass is very large we may neglect the momenta of the neutrinos (and other relativistic participants) in the overall law of three-momentum conservation so that

$$\delta^4(p) \to \delta^3\left(\sum \mathbf{p}_i - \sum \mathbf{p}_f\right)\delta\left(\omega + \sum E_i - \sum E_f\right) \ . \qquad (2.10)$$

It may seem somewhat strange to drop the momentum transfer and keep the energy transfer. Specifically, when scattering from individual nucleons in the nonrelativistic limit, the energy transfer is smaller than the momentum transfer: $\omega = \mathbf{k}\cdot\mathbf{p}/m_N < |\mathbf{k}|$, where here $\mathbf{p}$ is the nucleon momentum. Although this is true for single-particle scattering, and indeed implies that $\omega = 0$ in the long-wavelength approximation, it is not true for the higher-order terms involving the interaction of a neutrino with a pair (or more) of interacting nucleons. In that case, it is possible to conserve momentum exactly in the nucleon sector while still releasing energy to or absorbing energy from the neutrinos. Thus, in the long-wavelength limit the perturbative response function is a δ function in ω at leading order, but at higher order the medium can transfer any amount of energy subject to the thermal constraints of the medium and the energy available in the incident neutrino; but to all orders it is independent of $\mathbf{k}$.

Thus, in an isotropic medium there remain two structure functions $S_V(\omega) \equiv R_1(\omega,0)$ and $S_A(\omega) \equiv R_2(\omega,0)$ which are given in terms of field correlators as

$$S_V(\omega) = \frac{1}{N_B}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle V^0(t)V^0(0)\rangle \ ,$$
$$S_A(\omega) = \frac{1}{3N_B}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle \mathbf{A}(t)\cdot\mathbf{A}(0)\rangle \ , \qquad (2.11)$$

where

$$V^0(t) = \int d^3\mathbf{x}[C_{V,p}\psi_p^\dagger(t,\mathbf{x})\psi_p(t,\mathbf{x}) + C_{V,n}\psi_n^\dagger(t,\mathbf{x})\psi_n(t,\mathbf{x})] \ ,$$
$$A^i(t) = \int d^3\mathbf{x}[C_{A,p}\psi_p^\dagger(t,\mathbf{x})\sigma^i\psi_p(t,\mathbf{x}) + C_{A,n}\psi_n^\dagger(t,\mathbf{x})\sigma^i\psi_n(t,\mathbf{x})] \ , \qquad (2.12)$$

with the Pauli matrices σ^i.

For an isotropic distribution of neutrinos, after averaging over the scattering or emission angles the $\cos\theta$ terms in Eq. (2.9) average to zero. Therefore, we are left with $W(k_1,k_2) = G_F^2 N_B S_\nu(\omega)$ with a single structure function

$$S_\nu(\omega) \equiv \frac{S_V(\omega)+3S_A(\omega)}{4} , \tag{2.13}$$

which, in the nonrelativistic and long-wavelength limit, is all we need to study neutrino scattering and pair processes.

III. EVALUATION OF $S(\omega)$

A. Simple properties of S_V and S_A

In general, a full evaluation of $S(\omega)$ is not possible. We will present perturbative calculations later in this section, but there are limitations to the range of their reliability. However, even without detailed knowledge of the structure functions we can easily state several useful and simple properties of $S_V(\omega)$ and $S_A(\omega)$.

To begin, the detailed-balance requirement for $W(k_1,k_2)$ translates into

$$S_{V,A}(-\omega) = e^{-\omega/T} S_{V,A}(\omega) \tag{3.1}$$

so that is it enough to specify these functions for positive energy transfers (energy absorbed by the medium).

Next, we consider the overall normalization. By integrating Eq. (2.11) over $d\omega/2\pi$ we extract the total strength. On the RHS $\int e^{i\omega t} d\omega/2\pi = \delta(t)$, allowing us to trivially perform the time integration, and find

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_V(\omega) &= \frac{1}{N_B}\langle V^0(0)V^0(0)\rangle , \\ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_A(\omega) &= \frac{1}{3N_B}\langle \mathbf{A}(0)\cdot\mathbf{A}(0)\rangle . \end{aligned} \tag{3.2}$$

Of course, we could have chosen any time, not just $t=0$, as a reference point.

If there are no spin or isospin correlations between different nucleons of the medium we may interpret $\mathbf{A}$ as the single-nucleon spin operator (apart from overall factors). In this case Eq. (3.2) yields the "sum rule"

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_V(\omega) &= Y_p C_{V,p}^2 + Y_n C_{V,n}^2 \equiv K_V , \\ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_A(\omega) &= Y_p C_{A,p}^2 + Y_n C_{A,n}^2 \equiv K_A , \end{aligned} \tag{3.3}$$

where Y_p and Y_n are the number fractions of protons and neutrons, respectively.[3] With the values of the weak-coupling constants discussed in Appendix A and because $Y_p + Y_n = 1$ one finds $K_A \approx K_V \approx 1$.

For noninteracting nucleons the time evolution of the operators is trivial, e.g., $\mathbf{A}(t) = \mathbf{A}(0)$, and so we find

$$S_{V,A}(\omega) = K_{V,A} 2\pi\delta(\omega) . \tag{3.4}$$

This result yields directly the usual neutrino scattering rate in a medium of nonrelativistic, nondegenerate, quasifree nucleons. $\Gamma = G_F^2 N_B \omega_1^2 (K_V + 3K_A)/4\pi$, where ω_1 is the energy of the incident neutrino.

It is expected that the normalization and shape of the response functions will be altered in an interacting medium, however this primarily affects the axial response S_A. For the vector current Eq. (3.4) should be a good approximation even in the presence of nucleon-nucleon interactions because the quantity V^0 (the zeroth component of the four velocity) does not fluctuate; i.e., it is identically 1 in the nonrelativistic limit where the wavelength of the momentum exchange is large compared to the smearing of the nuclear charge density due to interactions in the medium. The axial charge, on the other hand, is a spatial three-vector quantity. Although a nucleon's location is not smeared out, nucleon interactions in the medium cause the direction of the nuclear spin to fluctuate, altering the effective charge seen by a leptonic probe. We expect that the axial response is weakened for soft energy transfers, $\omega < \Gamma_{\rm int}$, where by $\Gamma_{\rm int}$ we mean some typical rate for the nuclear spin to fluctuate due to collisions.[4]

The counterpoint to these arguments is that there is no vector response for nonzero energy transfers, but in general the axial response gets spread out to higher ω with increasing interactions. Specifically, explicit calculations (see Sec. III B below) show that only the axial-vector current contributes to bremsstrahlung in the nonrelativistic limit.

A related issue concerns the appropriate values to use for the axial and vector charges C_A and C_V. In a dilute medium, one should use the vacuum values, but in a nuclear medium the axial weak-coupling constants may be altered. The vector charges remain unchanged. Specific values are discussed in Appendix A.

Thus, within the nonrelativistic and long-wavelength approximation the vector current is unaffected by multi-

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[3] If the medium is degenerate, but approximated by noninteracting particles, one must account for final-state blocking effects and replace the number fractions by $Y_i \to N_B^{-1} \int d^3\mathbf{p}\, 2f_i(1-f_i)/(2\pi)^3$. This result, however, cannot be extended trivially to interacting nucleons.

[4] Isospin fluctuations do not lead to an equivalent damping of the vector response, due to isospin conservation in the strong interactions. In order to have the equivalent effect a proton must turn into a neutron, but this can only occur if a neutron turns into a proton in the same interaction. In the long-wavelength approximation, the net weak vector charge probed by the neutrinos will remain unchanged.

ple collisions but does not contribute to bremsstrahlung, and the reverse applies to the axial-vector current. It remains to determine the axial-vector structure function $S_A(\omega)$ in a fully interacting medium.

Finally, it should be clear that random spin fluctuations of the sort which happen in a thermal medium may lessen the total scattering interaction rate, but we would be surprised if any significant enhancement could occur through thermal interactions. Thus, the normalization of $S_{V,A}$ in Eq. (3.3) should constitute a rough upper bound. This does not allow, however, for collective effects such as spin waves, or the influence of a pion condensate, which could in principle enhance the interaction rates of neutrinos with the medium.

B. Perturbative result for $S(\omega)$

We proceed with a perturbative calculation of $S_A(\omega)$. After the δ-function contribution from free nucleons, the next lowest-order process that can contribute is nucleon-nucleon scattering. We consider nondegenerate nucleons, model the NN interaction by one-pion exchange, and for simplicity neglect m_π in the pion propagator. A summary of our calculation, which follows the same general lines as Brinkmann and Turner's [15] calculation of axion bremsstrahlung, is given in Appendix B. The result is

$$S_{NN}(\omega) = \frac{\Gamma_A}{\omega^2} s(\omega/T) \ , \tag{3.5}$$

where[5]

$$\Gamma_A \equiv 4\pi^{1/2} \frac{\alpha_\pi^2 N_B T^{1/2}}{m_N^{5/2}} \ . \tag{3.6}$$

and $\alpha_\pi \equiv (2fm_N/m_\pi)^2/4\pi \approx 15$, where $f \approx 1$ is the pion-nucleon coupling. The function $s(\omega/T)$ is given by

$$\begin{aligned} s(x) = {} & Y_n^2 C_{A,n}^2[s_0(x) - s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] + Y_p^2 C_{A,p}^2[s_0(x) - s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] \\ & + \tfrac{4}{3} Y_n Y_p[(7C_{A,+}^2 + 5C_{A,-}^2)s_0(x) \\ & - (6C_{A,+}^2 + 2C_{A,-}^2)s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] \ , \end{aligned} \tag{3.7}$$

where, $x = \omega/T$, and $C_{A,\pm} = \frac{1}{2}(C_{A,n} \pm C_{A,p})$. The functions s_0 and $s_{\mathbf{k}\cdot\boldsymbol{l}}$ are remnants of the nucleon phase-space integration and are given in the nonrelativistic limit by

$$\begin{aligned} s_0 &= \int_0^\infty dv (v^2 + xv)^{1/2} e^{-v} \ , \\ s_{\mathbf{k}\cdot\boldsymbol{l}}(x) &= \int_0^\infty dv \frac{x^2}{2(2v+x)} \ln\left(\frac{\sqrt{v+x}+\sqrt{v}}{\sqrt{v+x}-\sqrt{v}}\right) e^{-v} \ . \end{aligned} \tag{3.8}$$

[5] In a previous version of this paper which we had circulated as a preprint we erroneously gave a coefficient $48\pi^{1/2}$ instead of $4\pi^{1/2}$ in the expression for Γ_A. With the corrected Γ_A a puzzling discrepancy with our estimate of Γ_A in [16] has disappeared.

These formulas are derived in Appendix B, along with more general results for when the pion mass should not be neglected. We also give there analytic approximations that are good to a few percent. In passing, we note that the combination $s_0(x) - s_{\mathbf{k}\cdot\boldsymbol{l}} = 1 \pm 0.2$ for $x < 20$, and that $s(x)$ is typically 2–3 times larger for a mix of protons and neutrons than for pure neutrons.

This result cannot be the complete answer, however, as it diverges at $\omega = 0$. The perturbative calculation is based on the assumption that a given bremsstrahlung event can be viewed as a single collision where *in* states travel unperturbed from the infinite past and are scattered into *out* states which remain unperturbed into the infinite future. In a medium these assumptions are violated as each nucleon participant in a given reaction has emerged from a previous interaction and will interact again in the future. The nucleon field can freely evolve only for an approximate duration $\tau = \Gamma_{\rm int}^{-1}$ where $\Gamma_{\rm int}$ is a typical nucleon-nucleon scattering rate. Hence, we cannot expect such a calculation to yield meaningful results for energy scales $|\omega| \lesssim \tau^{-1} = \Gamma_{\rm int}$. This estimate reveals the approximate range of ω where a naive perturbative approach can be trusted.

In a nondegenerate medium in thermal equilibrium, the scale of a typical energy transfer in any collision or radiation event is given by the temperature T. Hence, as long as $T/\Gamma_{\rm int} \gg 1$ one may expect that a naive application of perturbative methods is justified and we call such a medium "dilute" for the purposes of our discussion. Conversely, the criteria $T/\Gamma_{\rm int} \ll 1$ quantifies our notion of a "dense" medium (the limit of large collision rates) where the naive use of perturbation results is problematic because it is not justified in any obvious way. The hot material of a young SN core appears to be a medium which is approximately nondegenerate and yet "dense" in the spirit of this definition.

Energy conservation in the collisions between neutrinos and nucleons will be satisfied only with a precision $\Delta E \approx \Gamma_{\rm int}$. This observation suggests that the nucleons may be described as resonances with an approximate width $\Gamma_{\rm int}$, rather than as states with a fixed energy [17]. In a previous paper we argued [16] that in this case one should replace the $1/\omega^2$ soft behavior of $S_A(\omega)$ by a Lorentzian $1/(\omega^2 + \Gamma_{\rm int}^2/4)$. It is tempting to identify $\Gamma_{\rm int} = \Gamma_A$, especially since Γ_A as given in Eq. (3.6) has the structure of an interaction rate ($n\sigma v$) for nonrelativistic nucleons of size $1/m_\pi$. We therefore define

$$S'_{NN}(\omega) = \frac{\Gamma_A}{\omega^2 + a^2\Gamma_A^2} s(\omega/T) \ , \tag{3.9}$$

where a is an arbitrary number of order 1 which we will choose for convenience. For example, if we choose $a = s(0)/(2C_{A,n})$ then this form has two attractive features in the limit of a dilute, but interacting, medium. For $\Gamma_A \ll T$, $S_A(\omega) \to C_A^2 2\pi\delta(\omega)$, i.e., it gives the correct single-nucleon scattering rate, and it gives also the correct bremsstrahlung rate which probes $S_A(\omega)$ mostly at $\omega \gg \Gamma_A$.

C. Large collision rates

We have argued that in the limit of small collision rates, defined by the condition $\Gamma_A \lesssim T$, the low-ω behavior can be reasonably approximated by a Lorentzian shape. In a supernova core, however, we are confronted with the opposite case of large collision rates. Numerically, Eq. (3.6) becomes

$$\frac{\Gamma_A}{T} \approx 19\frac{\rho}{\rho_0}\left(\frac{800\ \mathrm{MeV}}{m_N}\right)^{5/2}\left(\frac{50\ \mathrm{MeV}}{T}\right)^{1/2}, \qquad (3.10)$$

where $\rho_0 = 3\times 10^{14}\ \mathrm{g/cm^3}$ is the nuclear density. The numerical value of Γ_A/T is in reasonable agreement with the value we estimated for $\Gamma_{\rm int}/T$ in a previous paper [16] on the basis of low-energy p-p and p-D scattering data. It is the large magnitude of Γ_A/T that makes the interpretation of the perturbative results for S_A very problematic.

If we used the form Eq. (3.9) then both neutrino scattering and pair processes would be substantially suppressed. This will be shown explicitly in Sec. IV. For now, it suffices to note that in a thermal environment, typical values of ω are a few T, and if $\Gamma_A \gg T$ then the denominator in Eq. (3.9) becomes large for reasonable ω. Although there may be scattering strength at large ω that strength will not be sampled by thermal energy transfers.

It is plausible, however, that in a dense medium $S_A(\omega)$ is narrower than indicated by a naive extrapolation of the dilute-medium result. The width of the spin-spin correlation function for a single nucleon is determined essentially by scattering off the spatial spin fluctuations in the medium. But if those spins are all fluctuating rapidly in time, then *their* spatial fluctuations will be damped by the same mechanism. To get a consistent picture, one must self-consistently include the effects of interactions in an evaluation of the interaction rates.

Even though it appears impossible to extract reliable results for S_A from a simple-minded perturbative calculation, we believe that the overall shape of the structure function will be a broad distribution, and that the high-ω wings will be represented by the bremsstrahlung result. However, it is in no way evident what one should use for the soft part of the axial response function. The Lorentzian shape espoused in Eq. (3.9) is possible, but it is also possible that some radically different shape is correct. Indeed, in the next section we will show that extreme changes to the shape are required if pair processes are to dominate over scattering processes.

IV. SCATTERING VS PAIR PROCESSES

There are many ways that one could choose to compare the strength of the scattering and pair processes. For example, we could concentrate on the contribution to various transport coefficients or calculate the Rosseland mean opacity. To illustrate our arguments we will use the rate at which particles are absorbed by either the pair or scattering terms, averaged over the neutrino phase space:

$$\Gamma_{\rm scat} = \frac{1}{N_\nu}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}W(k_1,k_2)f(\omega_1)[1-f(\omega_2)]\,,$$

$$\Gamma_{\rm pair} = \frac{1}{N_\nu}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}W(k_1,-k_2)f(\omega_1)\bar f(\omega_2)\,. \qquad (4.1)$$

These quantities have the advantage of being simple to calculate and interpret, without being too specific to the details in some corner of phase space. We have normalized by the number density of neutrinos $N_\nu = \int f(\omega_1)d^3\mathbf{k}_1/(2\pi)^3$. The occupation numbers will be taken to be Fermi-Dirac distributions without a neutrino chemical potential.

In Sec. II, we argued that for the conditions relevant for a SN core, $W(k_1,k_2)$ depends only on one function S_ν of the energy transfer ω according to Eq. (2.13). Therefore,

$$\Gamma_{\rm scat} = \frac{G_F^2 N_B}{4\pi^4 N_\nu}\int_0^\infty d\omega_1\int_0^\infty d\omega_2\,\omega_1^2\omega_2^2 f(\omega_1)[1-f(\omega_2)]S_\nu(\omega)\,, \qquad (4.2)$$

or transforming to an integral over $\tilde\omega = \omega_2-\omega_1 = -\omega$,

$$\Gamma_{\rm scat} = \frac{G_F^2 N_B}{4\pi^4 N_\nu}\int_{-\omega_1}^\infty d\tilde\omega\int_0^\infty d\omega_1\,\omega_1^2(\tilde\omega+\omega_1)^2 f(\omega_1)[1-f(\tilde\omega+\omega_1)]S_\nu(-\tilde\omega)\,. \qquad (4.3)$$

By means of the detailed-balance requirement $S_\nu(-\omega) = e^{-\omega/T}S_\nu(\omega)$, and relabeling the integration variable $\tilde\omega \to \omega$, we may write this as an integral over positive ω only:

$$\Gamma_{\rm scat} = \int_0^\infty d\omega F_{\rm scat}(\omega)S_\nu(\omega)\,. \qquad (4.4)$$

We have arranged things so that we will need $S_\nu(\omega)$ only for positive-energy transfers (energy absorbed by the medium). The detailed-balance Boltzmann factor $e^{\omega/T}$, relevant for negative-energy transfers (energy given to the neutrinos) has been included in the definition of the phase-space factor $F(\omega)$ which is specific to a given process and chosen type of thermal average, e.g., by number, by energy, etc.

$F_{\rm scat}$ is then expressed as a sum of two terms because we split the ω integration to ensure that ω is always positive:

$$F_{\rm scat}(\omega) = \frac{G_F^2 N_B}{4\pi^4 N_\nu}\left\{\int_\omega^\infty d\omega_1 \omega_1^2(\omega_1-\omega)^2 f(\omega_1)[1-f(\omega_1-\omega)] + \int_0^\infty d\omega_1\omega_1^2(\omega_1+\omega)^2 f(\omega_1)[1-f(\omega_1+\omega)]e^{-\omega/T}\right\} . \tag{4.5}$$

Note that in the first term $\omega = \omega_1 - \omega_2$, while in the second $\omega = \omega_2 - \omega_1$. Also, there is a detailed-balance factor $e^{-\omega/T}$ in the second integral relevant for the case when the medium gives energy to the neutrinos rather than absorbing it. Finally, in the case $S_\nu(\omega) \sim \delta(\omega)$ only one of the terms in Eq. (4.5) should be included, to avoid double counting.

Similarly, $\Gamma_{\rm pair} = \int_0^\infty d\omega\; F_{\rm pair}(\omega)S_\nu(\omega)$ with

$$F_{\rm pair}(\omega) = \frac{G_F^2 N_B}{4\pi^4 N_\nu}\int_0^\omega d\omega_1\omega_1^2(\omega-\omega_1)^2 f(\omega_1)\bar f(\omega-\omega_1) . \tag{4.6}$$

Because we consider thermal neutrino distributions, $F_{\rm pair}$ equally applies to the phase-space averaged pair emittance.

As a further simplification, for nondegenerate neutrinos we may approximate the Fermi-Dirac by Boltzmann distributions, and ignore the Pauli-blocking factor in Eq. (4.1). In $F_{\rm scat}$ and $F_{\rm pair}$ this means that $f(\omega_1) \to e^{-\omega_1/T}$ and $\bar f(\omega_2) \to e^{-\omega_2/T}$ while $1 - f(\omega_2) \to 1$. Then we find analytically

$$F_{\rm scat}(x) \approx \frac{G_F^2 N_B T^2}{2\pi^2}(24 + 12x + 2x^2)e^{-x} ,$$
$$F_{\rm pair}(x) \approx \frac{G_F^2 N_B T^2}{2\pi^2}\frac{x^5}{60}e^{-x} , \tag{4.7}$$

where $x \equiv \omega/T$. These approximations are shown in Fig. 1; they deviate from Eqs. (4.5) and (4.6) by less than 10%, except for $F_{\rm pair}$ at values of x less than 5, where Eq. (4.7) overestimates $F_{\rm pair}$ by up to a factor of 3. Since $F_{\rm pair} \ll F_{\rm scat}$ for these x, we can use Eq. (4.7) without reservation.

From Fig. 1 it is clear that scattering is more sensitive to the structure function near $\omega = 0$ while pair absorption is more sensitive to S_ν at higher ω. Put another way, for pair processes to dominate, $F_\nu(\omega)$ would need to have much more power at large ω than near $\omega = 0$. For example, if $S_\nu(\omega)$ would vary like a power law ω^n, $\Gamma_{\rm pair}$ would exceed $\Gamma_{\rm scat}$ for $n > 3$, but would be subdominant for smaller n.

For a noninteracting medium one can use the δ-function result Eq. (3.4). This limit allows for no pair processes, and scattering dominates. The scattering rates due to this choice of $S_A(\omega)$ and $S_V(\omega)$ are shown as solid horizontal lines in Fig. 2. Including interactions broadens the axial response. For a dilute medium, the Lorentzian shape, Eq. (3.9) seems a reasonable approximation. For large ω, the axial response then behaves as ω^{-2}, a result which is much softer than the $n > 3$ power law required for pair processes to dominate. Again, there is no scope for pair processes to dominate.

In the limit of large collision rates there is more room for speculation. In order to avoid the conclusion that $\Gamma_{\rm scat} > \Gamma_{\rm pair}$ we would have to assume that $S_\nu(\omega)$ is not monotonically decreasing, but rather is "hollowed out" near $\omega = 0$. A radical example would be to take Eq. (3.5) for $S_A(\omega)$ above some cutoff $\omega_{\rm min}$, and to set[6] $S_A(\omega) = 0$ for $\omega < \omega_{\rm min}$. Under this assumption, the rates $\Gamma_{\rm scat}$ and $\Gamma_{\rm pair}$ are shown as the dotted curves in Fig. 2. As can be seen, S_{NN} becomes more important for pair processes than for scattering for $\omega_{\rm min}/T \gtrsim 5$. However, even if such a radical transformation of the axial response occurs, there will still be the vector contri-

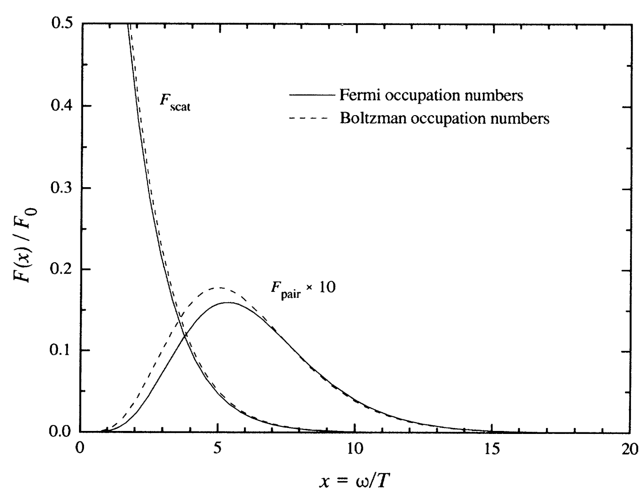

FIG. 1. The functions $F_{\rm scat}$ and $F_{\rm pair}$ defined in Eqs. (4.5) and (4.6), and the analytic approximations defined in Eq. (4.7). The results are normalized to $F_0 \equiv G_F^2 N_B T^2$.

[6]For the sake of this extreme example we ignore the normalization of Eq. (3.3). If that relation is indeed an upper bound then the implied cutoff is $\omega_{\rm min} \gtrsim 130$.

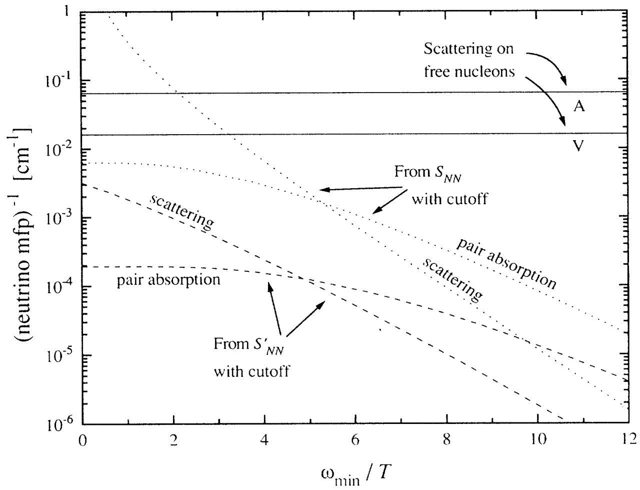

FIG. 2. Comparison of scattering and pair absorption rates, Eq. (4.4), calculated using the F's in Eq. (4.7), for different choices of $S(\omega)$. Horizontal lines are single-nucleon scattering rates. Dotted curves use S_{NN} from Eq. (3.5) with a lower cutoff $\omega_{\rm min}$ while the dashed curves are for S'_{NN} from Eq. (3.9), also cut off below $\omega_{\rm min}$. The parameters used are $\rho = \rho_0 = 3\times10^{14}\,{\rm gm\,cm^{-3}}$, $T = 50$ MeV, $Y_n = 0.7$, $m_N = 800$ MeV, and the vector and axial charges given in Eqs. (A1) and (A2), respectively.

bution, $S_V(\omega) = 2\pi K_V\delta(\omega)$, to scattering. From Fig. 2, for $\rho = \rho_0$ and $\omega_{\rm min}/T \gtrsim 3$, vector current scattering dominates over two nucleon axial current processes. Thus, even with the radical assumption of a total cutoff of the low-ω response there is almost no range where pair processes dominate over scattering.

Not only must the axial response be hollowed out, but in order for the pair processes to dominate, the axial strength at high ω must remain strong. At the same time, we note that for $\omega_{\rm min}/T \lesssim 2$, the scattering response is enhanced over the single-nucleon result, a consequence of the ω^{-2} behavior of S_{NN}. If we use the Lorentzian form, the ω^{-2} behavior is regulated, but the strength of the axial scattering is severely suppressed. For example, using Eq. (3.9), for $\omega > \omega_{\rm min}$ instead of the naive bremsstrahlung result, Eq. (3.5), yields the two dashed curves in Fig. 2. Again, pair processes exceed scattering for a cutoff $\omega_{\rm min}/T \approx 5$; however, now both processes are two orders of magnitude smaller than the vector contribution.

The results shown in Fig. 2 are for $\rho = \rho_0$, and $\Gamma_A \approx 19$. As the density is raised even higher the single-particle contributions to the rates increase linearly with density, while the naive bremsstrahlung rates increase as ρ^2. So it would appear that eventually pair processes could overtake the contribution from S_V. However, if we look at the contribution from S'_{NN} instead of S_{NN} we find that the contribution to Γ is independent of ρ, and the comparison favors scattering even more strongly. Given the wide range of possibilities for S_A, it is not clear whether or not increasing the density can eventually allow pair processes to win out.

We do not think that an extreme behavior such as a sharp cutoff to $S_\nu(\omega)$ is realistic, so we view these exercises as examples of how radically different the high-density behavior of S_ν would have to be in order that pair processes might dominate.

We conclude that, even though we cannot calculate the exact form of the axial response of the medium, a combination of phase-space arguments and a recognition of the stability of the vector response, seems to preclude pair processes dominating over scattering processes in the evaluation of neutrino transport properties. In order to avoid this conclusion the axial response function for a strongly interacting medium would have to be both suppressed at low- and enhanced at high-energy transfers in comparison to the bremsstrahlung result, S_{NN}.

Another effect of increasing density is degeneracy of the nucleons. As the nucleons become degenerate the amount of nucleon phase space available for small energy transfers decreases, effectively creating a harder response function. From Friman and Maxwell's [7] bremsstrahlung rate we can easily extract S_{NN}. The result includes various prefactors, but for our purposes the interesting part is the dependence on the energy transfer,

$$s_{\rm deg}(x) = \frac{x^3 + 4\pi^2 x}{x^2 + \gamma^2/4}\frac{e^x}{e^x - 1}\,, \tag{4.8}$$

where $x = \omega/T$ and $\gamma = \Gamma_{\rm int}/T$ is our phenomenological cutoff, again necessary to regulate an x^{-2} divergence at small ω. Now, however, the large-x behavior is much "stiffer," $s_{\rm deg} \propto x$. This is still not hard enough to provide for the dominance of pair processes, but it is getting closer. In the extreme case where $\Gamma_{\rm int} \gg T$, the effective power law of the response function is $n = 3$. This last value is enough to make pair processes competitive; however, we do not think this extreme case really applies to the SN core since the temperatures are high enough that the nucleons are only mildly degenerate.

In summary, we repeat that it is in no way evident how one ought to extrapolate the dilute-medium approximation for $S_A(\omega)$ into the high-density regime. However, unless the overall shape of S is radically different from its limiting behavior, the pair processes will always be less important than scattering. Further, it appears that both the pair rates and the axial-vector scattering rates will be suppressed relative to their "naive" values.

Finally, one may wonder if these conclusions are not in conflict with the reasonable agreement between the SN 1987A neutrino observations and the expected signal duration. However, as the vector-current appears to remain unsuppressed, the scattering rates would be reduced only to about $\frac{1}{5}$ of their naive values, even if the axial-vector contributions were entirely suppressed. It is an interesting question if, given the freedom to adjust other parameters, such a reduction in neutrino opacities can be excluded on the basis of the detected events. For a first attempt at answering this question see Keil, Janka, and Raffelt [18].

V. NONSTANDARD PHYSICS

We now turn to exploring how our conclusions affect various arguments related to certain nonstandard aspects of SN physics. First, we consider the emission of hypothetical particles weakly coupled to the NC of the medium. This process could provide an anomalous cooling mechanism for the core, which in turn could result in a diminished neutrino signal of SN 1987A, thus allowing one to constrain the properties of the particles in question [19]. In fact, trying to understand the consistency of such arguments was the original motivation for this work. Second, we look at the possible impact of a pion condensate for both neutrino transport and hypothetical particle emission. All these issues are connected through the same medium structure functions that control neutrino transport. It is therefore possible, and necessary, to treat the novel effects and neutrino transport in a consistent fashion.

A. Processes with "flipped" neutrinos

As a first case we consider the possibility that neutrinos have a small Dirac mass m_ν, which allows for the production of "wrong-helicity" states in the deep interior of a SN core [20]. If m_ν is small enough these states can escape freely and thus provide an anomalous sink for the heat in the core. If m_ν is not too small then that heat sink would have had observable effects on the SN 1987A neutrino signature, thus constraining m_ν. Several estimates of these bounds have been made, often with different treatments of the processes which contribute to the emission [8–12,20,21]. Specifically, under certain assumptions, the strength of the pair emission processes $X \to X'\bar{\nu}_L\nu_R$ has been shown to exceed the rate for spin-flip scattering from single nucleons $\nu_L N \to N'\nu_R$ [8–10]. In analogy to our remarks about the importance of pair processes in the transport of ordinary LH neutrinos; we argue that unless the full response is very "hard" spin-flip scattering still dominates over pair emission.

We take m_ν to be very small compared with typical neutrino energies. Therefore, it is not necessary to distinguish carefully between helicity and chirality, and so we shall always refer to wrong-helicity neutrinos or antineutrinos as right-handed and to the correct-helicity states as left-handed. Moreover, if m_ν is sufficiently small, RH states will not be trapped so that their occupation numbers are subthermal. The energy-loss rate per unit volume due to the emission of RH neutrinos is then $Q_{\nu_R} = Q_{\rm scat} + Q_{\rm pair}$ with

$$Q_{\rm scat} = \int \frac{d^3\mathbf{k}_L}{(2\pi)^3}\frac{d^3\mathbf{k}_R}{(2\pi)^3}\widetilde{W}_{k_L,k_R} f_{\mathbf{k}_L}\omega_R \ ,$$
$$Q_{\rm pair} = \int \frac{d^3\mathbf{k}_L}{(2\pi)^3}\frac{d^3\mathbf{k}_R}{(2\pi)^3}\widetilde{W}_{-k_L,k_R}(1-\bar{f}_{\mathbf{k}_L})\omega_R \ , \tag{5.1}$$

where $\widetilde{W}_{k_L,k_R}$ is the transition probability for the scattering process $\nu_L(k_L) + X \to X' + \nu_R(k_R)$ while $\widetilde{W}_{-k_L,k_R}$ refers to $X \to X' + \bar{\nu}_L(k_L) + \nu_R(k_R)$. For the energy loss due to $\bar{\nu}_R$ one obtains an analogous expression; for nondegenerate ν_L we have $Q_{\bar{\nu}_R} = Q_{\nu_R}$.

In order to determine $\widetilde{W}_{k_L,k_R}$ we note that it can be written in the form Eq. (2.4) where the structure function $S_{\mu\nu}$ remains unchanged. For interactions of neutrinos with specified helicities, Gaemers, Gandhi, and Lattimer [12] showed that the expression for $N^{\mu\nu}$ remains of the form Eq. (2.5) if one substitutes $k_i \to \frac{1}{2}(k_i \pm m_\nu s_i)$, $i = 1$ or 2, where the plus sign refers to ν and the minus sign to $\bar{\nu}$, and s is the covariant spin vector. For relativistic neutrinos we may consider a noncovariant lowest-order expansion in terms of m_ν. Then k_i remains unchanged for LH states while

$$k_i = (\omega_i, \mathbf{k}_i) \to \tilde{k}_i = (m_\nu/2\omega_i)^2(\omega_i, -\mathbf{k}_i) \tag{5.2}$$

for RH ones. After this substitution has been performed, all further effects of a nonzero m_ν are of higher order so that one may neglect m_ν everywhere except in the global spin-flip factor.[7] Therefore, the neutrino tensor $\widetilde{N}^{\mu\nu}$ is found from Eq. (2.5) by multiplication with $(m_\nu/2\omega_R)^2$, inserting $k_1 = k_L$ and $k_2 = (\omega_R, -\mathbf{k}_R)$.

Next, we explicitly contract $\widetilde{N}^{\mu\nu}$ with $S_{\mu\nu}$ in the form Eq. (2.8) in order to derive the spin-flip equivalent of Eq. (2.9):

$$\begin{aligned}\widetilde{W}(k_L,k_R) = \frac{G_F^2 N_B}{4}\left(\frac{m_\nu}{2\omega_R}\right)^2 &[(1-\cos\theta)R_1 \\ &+(3+\cos\theta)R_2 \\ &+4\omega_R^2(1-\cos\theta)R_3 - 4\omega_R(1-\cos\theta)R_4 \\ &+2(\omega_R-\omega_L)(1+\cos\theta)R_5] \ .\end{aligned} \tag{5.3}$$

Following Gaemers, Gandhi, and Lattimer [12] we emphasize that this expression differs in more than the factor $(m_\nu/2\omega_R)^2$ from the nonflip rate Eq. (2.9). This difference is related to the changed angular momentum budget of reactions with spin-flipped neutrinos.

In the nonrelativistic limit, however, the contribution of $R_{3,4,5}$ can be neglected as discussed in Sec. II C above. Moreover, in an isotropic medium the cosθ terms average to zero. Then, the spin-flip factor is the only modification necessary to deal with RH neutrinos, and we may write $\widetilde{W}(k_L,k_R) = G_F^2 N_B(m_\nu/2\omega_R)^2 S_\nu(\omega)$, which involves the same structure function as for the nonflip case, i.e., we do not need to define a new function $\widetilde{S}_\nu$. Then, in analogy to Eq. (4.4), the energy-loss rates can be written as

[7]The dispersion relation of neutrinos in a SN differs markedly from the vacuum form; in the core the "effective m_ν" is several 10 keV. However, m_ν in Eq. (5.2) is the vacuum mass which couples LH and RH states and thus leads to spin flip while the medium-induced "mass" only affects the dispersion relation of LH states. This view is supported by a detailed study of Pantaleone [22]. Of course, for nonrelativistic neutrinos the situation is more complicated because an approximate identification of helicity with chirality is not possible.

$$Q_i = \int_0^\infty d\omega \widetilde{F}_i(\omega) S_\nu(\omega) \ , \tag{5.4}$$

where, i stands for "pair" or "scat." If for nondegenerate neutrinos we use a Boltzmann distribution and neglect Pauli blocking in Eq. (5.1), the phase-space functions are

$$\widetilde{F}_{\rm scat}(x) = \frac{\sigma_{\rm flip} N_B T^4}{2\pi^3}(x^2+6x+12)e^{-x} \ ,$$
$$\widetilde{F}_{\rm pair}(x) = \frac{\sigma_{\rm flip} N_B T^4}{2\pi^3}\frac{x^4}{12}e^{-x} \ , \tag{5.5}$$

where $\sigma_{\rm flip} \equiv G_F^2 m_\nu^2/4\pi$ and $x = \omega/T$. In Eqs. (5.5) a factor of 2 was included in both $\widetilde{F}_{\rm scat}$ and $\widetilde{F}_{\rm pair}$ to include the scattering of antineutrinos and the production of flipped antineutrinos, respectively.

A comparison of Eqs. (5.5) and (4.7) shows that the RH neutrino emission bears a resemblance to LH neutrino absorption: in order for pair processes to dominate, the response function must be very hard. In Sec. III C we characterized the "hardness" of S_ν by a power law ω^n, and noted that the "critical exponent" where pair processes were as important as scattering was $n = 3$. Similarly, for RH neutrino emission, the critical exponent is $n \approx 2.57$.

Since the critical spectrum for spin-flip pair emission is slightly softer than for LH pair absorption, it is slightly easier for pair processes to dominate, than in the case of LH neutrino transport. In Fig. 3 we show the LH emission results analogous to the LH scattering results shown in Fig. 2. The emission rates are scaled to $Q_0 = \sigma_{\rm flip} N_B T^4/(2\pi^3)$. The results are similar to those presented for the transport of LH neutrinos. The crossover to pair processes dominating the axial emission occurs at $\omega_{\rm min}/T \approx 4$, but a combination of axial plus vector scattering always dominates even when no attempt (other than the cutoff at $\omega_{\rm min}$) is made to regulate the soft ω response. When the Lorentzian form for S_A is used, vector scattering dominates at nuclear density. Therefore, by any indication that we can obtain from extrapolating perturbative calculations into the high-density regime, the scattering processes are dominant for spin-flip emission of RH states, as well as for the transport of LH neutrinos.

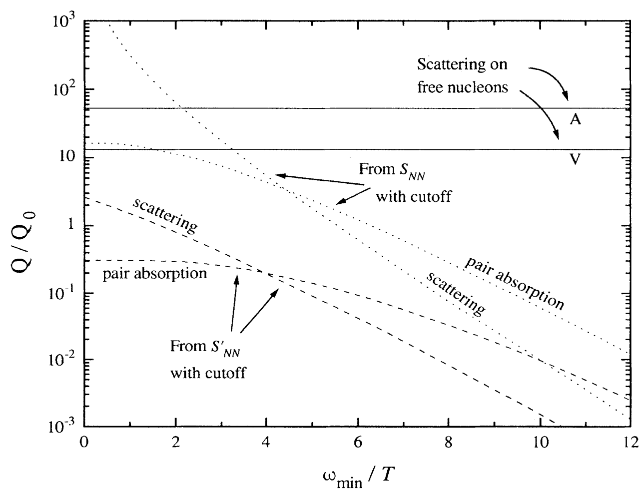

FIG. 3. Comparison of scattering and pair emission contributions to the emission of RH neutrinos from Eqs. (5.4) and (5.5), for the same choices of $S(\omega)$ and parameters as in Fig. 2.

Even though we can be reasonably sure that scattering processes dominate the emission rate of RH neutrinos there is still uncertainty about the total emission rate due to the uncertainty about the axial contribution. If the axial contribution is much suppressed then $Q_{{\rm scat},V}$, should give a good lower bound to the emission rate, whereas if the axial part is enhanced the emission rate may be much larger than $Q_{{\rm scat},V}$. We feel that the latter possibility is remote but cannot be ruled out on the basis of perturbative calculations alone.

Even though perturbative calculations do not seem to allow for a reliable calculation of the absolute magnitude of neutrino scattering rates, the NC transport of ordinary LH neutrinos and the spin-flip emission of RH Dirac-mass neutrinos are linked to each other through a single-response function. In fact, the functions $F_{\rm scat}(x)$ and $\widetilde{F}_{\rm scat}(x)$ in Eqs. (4.7) and (5.5) even have identical shapes.[8] Because modifications of the transport coefficients and of the spinflip rates would both affect the observable neutrino light curve, a consistent attempt to constrain Dirac masses would then depend on letting the NC transport coefficients and the spin-flip emission rate "float" together.

B. Axions

The emission of axions is another interesting possibility for an anomalous energy sink in the inner core of a SN. Again, the expected impact on the SN 1987A neutrino signature was used to constrain the coupling strength and then indirectly the mass of these pseudoscalar bosons [24]. They would couple to the medium according to

$$H_{\rm int} = \frac{1}{f_a} B_a^\mu \partial_\mu \phi \ , \tag{5.6}$$

where ϕ is the axion field and f_a is the axion decay con-

[8]If one of ν_μ or ν_τ had a large enough mass so that spin-flip processes would be important at all, and if it had a not-too-small mixing angle with ν_e, a degenerate sea of this flavor would be populated [8,23]. For such a degenerate flavor the phase-space functions $F_\mu(\omega)$ and $\widetilde{F}_\mu(\omega)$ for ordinary and spin-flip scattering would be different because the former would involve neutrino Pauli-blocking factors for the final-state ν_L. The two processes would then involve different integrals over the common function $S_\nu(\omega)$.

stant which has dimensions of energy. The medium current is presumably dominated by protons and neutrons according to

$$B_a^\mu = \sum_{j=n,p} C_{a,j}\bar{\psi}_j\gamma^\mu\gamma_5\psi_j \ , \tag{5.7}$$

where $C_{a,n}$ and $C_{a,p}$ are model-dependent dimensionless coupling constants of order unity. The structure of this interaction is of the current-current type so that we may apply an analysis similar to that for neutrino processes, except that we replace $N^{\mu\nu}$ by its axionic equivalent $\Phi^{\mu\nu} \equiv k_a^\mu k_a^\nu/2\omega_a$, where k_a is the four-momentum of the axion and ω_a its energy. Note that by our definition of the momentum transfer $k = -k_a$.

If axions are weakly enough coupled they are not trapped in a SN core, so that their occupation numbers are subthermal. In this case the energy-loss rate per unit volume is

$$Q_a = \frac{1}{f_a^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\omega_a S_a^{\mu\nu}\Phi_{\mu\nu} \ . \tag{5.8}$$

The response function $S_a^{\mu\nu}$ is as defined in Eq. (2.6) with the above current B_a^μ. In the nonrelativistic and long-wavelength limit we have

$$Q_a = \frac{N_B}{4\pi^2 f_a^2}\int_0^\infty d\omega\, \omega^4 e^{-\omega/T} S_a(\omega) \ . \tag{5.9}$$

Apart from normalization the phase-space function for axion emission, $\omega^4 e^{-\omega/T}$, is identical to that for spin-flip pair emission. S_a, however, differs from its neutrino counterpart S_ν in a number of ways. (a) $R_{1,a}$ vanishes because the current B_a^μ is purely axial. (b) Since the axion couplings to the nucleons are generally different from the axial vector current couplings of the nucleons in the weak interactions, the isospin structure of $R_{2,a}$ will differ in detail from that of $R_{2,\nu}$ for neutrinos. However, the general considerations behind a calculation of R_2 are entirely analogous in the two cases and have the same soft divergence in the energy transfer ω^{-2}. (c) Because of the contractions with $\Phi_{\mu\nu}$ and $N_{\mu\nu}$, respectively, $S_a = R_{2,a}(\omega,0)$, whereas the axial part of S_ν is equal to $\frac{3}{4}R_{2,\nu}(\omega,0)$.

From our previous discussions several points should be evident. First, naive axion bremsstrahlung calculations rely on an S_a similar in form to the S_{NN} given in Eq. (3.5). Although there is no divergence in the emission rate, that is only because the phase-space factor has sufficient powers of ω to regulate the divergence. However, since that divergence does show up in neutrino transport, it must be regulated in some fashion; e.g., the modification to the Lorentzian form in Eq. (3.9), or the sharp cutoff at $\omega_{\rm min}$ used in Figs. 2 and 3. It seems likely that whatever mechanism is chosen to regulate the divergence at small ω, that regulation will result in a suppression of the axion emission rates compared to the naive bremsstrahlung calculations, perhaps by 1 or 2 orders of magnitude. As an example, consider the effect of the Lorentzian regulation by comparing the curves labeled $Q_{{\rm pair},NN}$ and $Q'_{{\rm pair}NN}$ in Fig. 3. Conversely, if one were to insist that the naive calculation of the axion emission rates were correct, one would have to conclude that the neutrino scattering rates are much enhanced, in violation of our expectation that thermal spin fluctuations can only decrease the net interaction rate. To probe the effects that axion emission would have on the neutrino cooling curves of supernova cores, one must treat axion emission and neutrino transport using the same assumptions about the axial response functions of the medium.

C. Pion-induced processes

It is also interesting to consider nonstandard effects in the nuclear medium that could grossly affect SN cooling. In particular, the conditions are close to forming a pion condensate, and in some equations of state the number of π^- can be much in excess of a thermal distribution [25]. A pion condensate can have an important impact on the late-time cooling of neutron stars [26], and it was recently speculated that under certain assumptions it could have a substantial effect on neutrino processes in a newborn neutron star also [8,10].

We calculate the one-pion contribution to the structure function on the basis of π^-p in the hadronic initial state, and n in the final state, while we neglect π^0 and π^+ as we are mostly interested in the case of a π^- condensate. For the pion-nucleon interaction we use the pseudoscalar form

$$H_{\pi np} = \sqrt{2}\frac{2fm_N}{m_\pi}\bar{\psi}_n\gamma_5\pi^-\psi_p \ . \tag{5.10}$$

We set $m_n = m_p = m_N$, and keep $\alpha_\pi = (2fm_N/m_\pi)^2/4\pi$ constant. Moreover, we calculate in the nonrelativistic and long-wavelength limit, leading to $\delta^3(\mathbf{k}_p - \mathbf{k}_n)$ for momentum conservation. Thus, $E_n = E_p$ and we are left with $\delta(\omega_\pi + \omega)$ for energy conservation. With these simplifications we find, for the one π^- contribution to[9] Eq. (2.7),

$$S_\pi^{\mu\nu} = 16\pi^2\alpha_\pi \int \frac{d^3\mathbf{p}}{(2\pi)^3 N_B} f_p(\mathbf{p})[1-f_n(\mathbf{p})] \int \frac{d^3\mathbf{k}_\pi}{(2\pi)^3}\frac{f_\pi(\mathbf{k}_\pi)}{2\omega_\pi}\frac{\delta(\omega_\pi+\omega)}{[(p+k_\pi/2)\cdot k_\pi]^2}$$
$$\times\{C_{A,-}^2 k_\pi^\mu k_\pi^\nu - C_{A,+}^2[P^{\mu\nu}|\mathbf{k}_\pi|^2 + P^{\mu\alpha}P^{\nu\beta}k_\pi^\alpha k_\pi^\beta] - C_{V,-}^2[P^{\mu\nu}\omega_\pi^2 - u^\mu u^\nu|\mathbf{k}_\pi|^2 - (u^\mu P^{\nu\alpha} + u^\nu P^{\mu\alpha})k_\pi^\alpha\omega_\pi]\} \ , \tag{5.11}$$

[9]In deriving Eq. (5.11) we neglected a diagram where the Z boson interacts with the π^- before it is absorbed by the nucleon. We also neglected Δ degrees of freedom and made no attempt to modify the π properties for medium effects. These issues are discussed in Migdal *et al.* [26] in the context of a degenerate nuclear medium. Although there is no real justification for neglecting the same effects here, it does reduce the complexity of the result and allows a simple comparison to other work [8].

where $P^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$. The isospin couplings are defined by $C_{A,\pm} = \frac{1}{2}(C_{A,n} \pm C_{A,p})$, and similarly for the vector couplings. The $C_{V,+}$ coupling does not contribute in the nonrelativistic approximation. Also, we have dropped a term proportional to $C_{A,+}C_{V,-}$ which is formally of the same order of magnitude but vanishes when averaged over neutrino directions since it behaves as $(\mathbf{k}_1 \times \mathbf{k}_2) \cdot \mathbf{k}_\pi$.

The term $[(p + k_\pi/2) \cdot k_\pi]^2$ in the denominator of Eq. (5.11) comes from the propagator of the intermediate nucleon. It differs slightly between the two amplitudes which contribute to Eq. (5.11), but we dropped the neutrino four-momentum from the nucleon propagator so that the two diagrams would be more similar. In the nonrelativistic approximation and for thermal pions, it is equal to $(m_N\omega_\pi)^2$; then the integral over the nucleon phase space gives 1. However, for a pion condensate we will have to be more careful as this denominator can diverge.

Note that the tensor structure of $S_\pi^{\mu\nu}$ reduces to the general form of Eq. (2.8) if the pions are distributed isotropically. However, if the pion condensate has a nonzero momentum, the covariant Lorentz structure will involve the condensate momentum and so, the number of distinct response functions increases.

As a first specific case we discuss thermal pions. In order to determine the contribution $S_\pi(\omega)$ to our general response function we contract $S_\pi^{\mu\nu}$ with $N^{\mu\nu}$, and integrate over neutrino angles. After this, any angular dependence on $\mathbf{k}_\pi$ has disappeared so that the pion phase-space integration can now be done just over ω_π. For nondegenerate nucleons the result is

$$S_\pi(\omega) = \frac{4\alpha_\pi}{m_N^2}(\omega^2 - m_\pi^2)^{1/2}\left[C_{A,-}^2\left(1 - \frac{m_\pi^2}{2\omega^2}\right) + C_{A,+}^2\left(1 - \frac{m_\pi^2}{\omega^2}\right) + C_{V,-}^2\left(2 - \frac{m_\pi^2}{2\omega^2}\right)\right] \times \begin{cases} Y_p f_{\pi^-}(|\omega|) & \text{for } \omega < -m_\pi\,, \\ 0 & \text{for } -m_\pi < \omega < m_\pi, \\ Y_n[1 + f_{\pi^-}(|\omega|)] & \text{for } \omega > m_\pi\,. \end{cases} \tag{5.12}$$

If one takes $m_\pi = 0$ and $C_{A,+} = 0$, this agrees with Turner's result [8]. Note that the term with $\omega > m_\pi$ corresponds to n in the initial and $p\pi^-$ in the final state, i.e., to the creation of a pion. In thermal and chemical equilibrium one must satisfy $\mu_\pi = \mu_n - \mu_p$ so that $Y_p f_{\pi^-}(|\omega|) = e^{-|\omega|/T} Y_n[1 + f_{\pi^-}(|\omega|)]$, and detailed balance is satisfied.

Since $S_\pi \propto \Theta(|\omega| - m_\pi)$, it is an explicit example of a "hard" S with all power above some threshold, so it is interesting to compare the scattering and pair process strength as a function of m_π. Evaluating $\Gamma_i = \int_0^\infty F_i(\omega)S_\pi(\omega)d\omega$, with F_i given in Eq. (4.7) for i =scat or pair, and using Eq. (5.12) we find that $\Gamma_{\text{pair}}(x_\pi) = \Gamma_{\text{scat}}(x_\pi)$ for $x_\pi \equiv m_\pi/T = 3.9$. At this point the scattering processes are suppressed by about a factor of 6 from their maximum if we had ignored the pion mass, i.e., $\Gamma_{\text{scat}}(3.9) = 1/6\Gamma_{\text{scat}}(0)$; but the pair processes which are sensitive to higher values of ω are essentially unchanged from their $x_\pi = 0$ value.

Next, we compare the thermally average scattering rate to the single-nucleon scattering rate Γ_{scat}^N, i.e., the scattering rate expected if there were no multiple-scattering suppression. Using $m_\pi = 0$, which maximizes S_π, and approximating $1 + f_{\pi^-} \to 1$ for thermal pions we have $S_\pi(\omega) = (4\alpha_\pi/m_N^2)Y_n C_\pi^2 \omega$ where $C_\pi^2 = C_{A,-}^2 C_{A,+}^2 + 2C_{V,-}^2$. Then,

$$\frac{\Gamma_{\text{scat}}^\pi}{\Gamma_{\text{scat}}^N} = \frac{10\alpha_\pi T^2}{\pi m_N^2} \approx 0.17\left(\frac{T}{50\text{ MeV}}\right)^2\left(\frac{800\text{ MeV}}{m_N}\right)^2, \tag{5.13}$$

where we have set $Y_n C_\pi^2/[\frac{1}{4}(K_V + 3K_A)] = 1$. By using the single-nucleon result, we have probably overestimated the contribution to Γ_{scat} from nucleons; however, any multiple scattering suppression of S_{NN} should also affect S_π. Further, we have overestimated the contribution from pions by taking $m_\pi = 0$. Therefore, the interaction of thermal pions with nucleons yields only a modest correction to the scattering rate for the conditions pertaining to a SN core.

Now turn to the possibility of a pion condensate which implies that $\mu_\pi = \mu_n - \mu_p = \omega_{\pi 0}$ where $\omega_{\pi 0}$ is the lowest-energy value for the π^-. The interaction with nucleons typically causes a dispersion relation for pions where this minimum occurs for a nonzero momentum $\mathbf{k}_{\pi 0}$ so that the four-momentum describing the condensate is $k_{\pi 0} = (\omega_{\pi 0}, \mathbf{k}_{\pi 0})$. The occupation numbers for the condensate are given by $f_\pi(\mathbf{k}_\pi) = (2\pi)^3 N_\pi \delta^3(\mathbf{k}_\pi - \mathbf{k}_{\pi 0})$, reducing the π^- phase-space integration in $S_\pi^{\mu\nu}$ to $N_\pi/(2\omega_{\pi 0})$ and leaving a factor $\delta(\omega_{\pi 0} - \omega)$.

Most authors use a condensate with $\mathbf{k}_{\pi 0} \neq 0$, based upon consideration of cold, degenerate nuclear matter characteristic of an older neutron star. It is conceivable, however, that thermal effects may modify the pion dispersion relation so that $\omega_\pi(\mathbf{k}_\pi)$ has its minimum at $\mathbf{k}_{\pi 0} = 0$. We will first consider such a zero-momentum condensate. The contraction $S_\pi^{\mu\nu}N_{\mu\nu}$ is identical to that for thermal pions except that we have to set both $\omega_\pi = \omega_{\pi 0}$ and $m_\pi = \omega_{\pi 0}$. Then we find

$$S_\pi(\omega) = 16\pi^2\alpha_\pi(C_{A,-}^2 + 3C_{V,-}^2) \times \frac{N_\pi}{\omega_{\pi 0} m_N^2}[Y_p\delta(\omega_{\pi 0} + \omega) + Y_n\delta(\omega_{\pi 0} - \omega)]\,. \tag{5.14}$$

We have used the fact that $1+f_\pi \simeq f_\pi$ for the condensate. Then $Y_p/Y_n = e^{-\omega_{\pi 0}/T}$, and detailed balance is satisfied. Curiously, the axial and vector charges have exchanged roles from the single-nucleon case, a fact which is related to the γ_5 in the pion coupling to nucleons.

It is easy to compare scattering and pair absorption for LH neutrinos by comparing $F_{\rm scat}$ with $F_{\rm pair}$ in Eq. (4.7) at $x = \omega_{\pi 0}/T$. The pion contribution to the pair process exceeds that to scattering for $\omega_{\pi 0} > 6.4T$. Equally, we may compare the energy-loss rate in RH neutrinos from $\pi^- p \to n\bar{\nu}_L\nu_R$ with that from $\nu_L\pi^- p \to n\nu_R$ or $\nu_L n \to \pi^- p\nu_R$. Here, the comparison is between $\tilde{F}_{\rm scat}$ with $\tilde{F}_{\rm pair}$ of Eq. (5.5). Pair emission exceeds spin-flip scattering for $\omega_{\pi 0} \gtrsim 5.5T$. Therefore, in both cases scattering is more important than the pair processes because for $T \approx 50$ MeV a condensate with $\omega_{\pi 0} \gtrsim 5T$ is highly unlikely.

Next, we compare the pion-induced scattering rate with the single-nucleon one. In analogy with Eq. (5.13) we now find

$$\frac{\Gamma^\pi_{\rm scat}}{\Gamma^N_{\rm scat}} = \frac{8\alpha_\pi N_\pi}{m_N^2 T}\frac{x^2+6x+12}{6x}e^{-x} \qquad (5.15)$$

with $x \equiv \omega_{\pi 0}/T$. For example, taking the number density of pions to be $N_\pi = 0.1N_B$, $m_N = 800$ MeV, $T = 50$ MeV, and $\omega_{\pi 0} = 100$ MeV we get $\Gamma^\pi_{\rm scat}/\Gamma^N_{\rm scat} \approx 0.2$. Even for small values of $\omega_{\pi 0}$ it is difficult to get pion-induced scattering to dominate over that from single nucleons. Further, as $\omega_{\pi 0}$ is decreased the result will become more sensitive to the scheme used to include multiple-scattering effects for soft processes. To conclude, a zero-momentum condensate is unlikely to dominate over single-nucleon scattering for neutrino transport or the emission of RH neutrinos.

Finally, we consider finite momentum condensates. As long as $(\omega_{\pi 0}, \mathbf{k}_{\pi 0})$ is timelike, we may always go to a Lorentz frame with $\mathbf{k}_{\pi 0} = 0$ at the expense of exact isotropy of the nucleon and neutrino distributions. If $|\mathbf{k}_{\pi 0}|/\omega_{\pi 0}$ is not large, then in the new frame the fluid momentum will be small, number densities will be of order their values in the fluid rest frame, and we may still use the long-wavelength approximation. Apart from an overall factor of order unity involving $\omega_{\pi 0}$ and $\mathbf{k}_{\pi 0}$, there will be no dramatic changes from the case of a zero-momentum condensate.

However, when the condensate dips below the light-cone so that $(\omega_{\pi 0}, \mathbf{k}_{\pi 0})$ is spacelike, $\omega_{\pi 0}^2 < |\mathbf{k}_{\pi 0}|^2$, a different approach is required. In this case the denominator $[(p + k_\pi/2)\cdot k_\pi]^2$ in Eq. (5.11) can become zero, corresponding to on-shell intermediate nucleon states. Put differently, the process $p\pi^- \leftrightarrow n$ is now possible without any other particles involved. However, because the pions have one fixed momentum, we may consider a Lorentz frame where $\omega'_{\pi 0} = 0$. There, the pion condensate looks like a static pion field with a fixed wave number $\mathbf{k}'_{\pi 0}$. The nucleons, then, should be described as Bloch waves in this periodic potential, i.e., the nucleon quasiparticles will be certain superpositions of n and p, involving spatial Fourier components of typical nucleon momenta and of $\mathbf{k}'_{\pi 0}$.

The NC scattering of neutrinos from these quasiparticles in the nonrelativistic and long-wavelength limits should be very similar to the scattering off quasifree single nucleons. Therefore, in the nondegenerate limit it should be given essentially by the number density of baryons times a typical weak cross section on a nucleon. We do not expect an anomalously enhanced scattering rate. (We note that in old neutron stars a pion condensate leads to a strong enhancement because the pion momentum is available to conserve momentum and thus, processes can go which otherwise are suppressed by degeneracy effects. In our case, there are no barriers from momentum conservation to overcome, and scattering can occur with full strength anyway.)

To summarize, we arrive at two conclusions. First, pions would always enhance the scattering rates more than they would enhance pair processes, whether or not they are in a condensate. For RH neutrino emission that was considered in [8] this means that its effect would be most important for the spin-flip scattering channel $\nu_L p\pi^- \to n\nu_R$. Second, however, we find that typically the effect of pions will not yield an anomalous enhancement of the scattering rate relative to the single-nucleon case. The overall uncertainty of the neutrino interaction rates at high densities appears to be much larger than the uncertainty introduced by the question of whether or not there is a pion condensate.

We must comment that our conclusions differ somewhat from Mayle *et al.* [10], who find that a pion condensate can have a dramatic effect, especially for the emission of RH neutrinos. There are two essential differences. First, they estimate the emission of *RH* neutrinos by multiplying the emissivity per thermal pion times the number of pions in the condensate, which we believe significantly overestimates the emissivity. For example, we can compare the emissivity, Q, per pion by using Eqs. (5.12) or (5.14) for S_π, for thermal pions or for a $\mathbf{k}_{\pi 0} = 0$ condensate, respectively. Plugging into Eq. (5.4) and using $\tilde{F}_{\rm pair}$, the ratio of the emissivity per thermal π divided by the emissivity per condensate π is

$$\frac{(Q/N_\pi)_{\rm th}}{(Q/N_\pi)_{\rm cond}} = \frac{30C_\pi^2}{(C_{A,-}^2 + 3C_{V,-}^2)x_0^3(1+e^{-x_0})}\ . \qquad (5.16)$$

Unless the condensate if "hard," use of the thermal pion formula overestimates the emissivity by an order of magnitude; but, if $\omega_{\pi 0} > T$ then Eq. (5.15) shows that the rates will be small compared to those from single nucleons. Further, as argued above, for a $\mathbf{k}_{\pi 0} \neq 0$ condensate we expect little enhancement over the single-nucleon emissivity, if one uses an appropriate set of Bloch states to describe the nucleons.

We have conducted our entire discussion of the pion-induced effects as if we were in a dilute medium where different contributions to the emissivities add linearly. The pion interactions, however, are just another contribution to the nucleon spin fluctuations and so, they essentially add to the width of $S_A(\omega)$ without adding to the overall strength, at least as long as we may treat the normalization in Eq. (3.3) as an upper bound. If we take our conclusions concerning S_{NN} and S'_{NN} as a guide, it is quite possible that a large number of pions could actually have the effect of *decreasing* the effective emissivities of axions or RH neutrinos.

The second distinction is that Mayle *et al.* [10] stress that the presence of charge in a pion condensate will reduce the electron degeneracy, which in turn results in the release of entropy and heating of the core material. The resulting increase in temperature increases all emission rates. In this picture, the presence of pions affects the emission rates indirectly through their impact on the equation of state. This aspect of the work of Mayle *et al.*'s is precisely in the spirit of what we advocate: When adding novel physics to the model one must be careful to include all ramifications, not just focusing on a single aspect of the new phenomena. In the present context, if one adds a pion condensate to the model to increase the emission of RH neutrinos then one must also allow for a change in the transport properties of the ordinary LH neutrinos.

VI. DISCUSSION AND SUMMARY

We have studied weak NC processes in the environment of the newly born neutron star in a SN core. They are crucial for understanding the cooling of the core for the first tens of seconds after collapse, the time during which the neutrino flux from a galactic SN would be observable in a large underground detector. The most important application is the role NC's play in ordinary neutrino transport of lepton number and heat. In addition, they also couple weakly to a variety of hypothetical particles, which would provide for an anomalous loss of heat from the core by particle emission. Constraints on exotic particle properties can be derived from a comparison between their calculated effect on the neutrino flux from the "surface" of the star with the SN 1987A observations.

Understanding these transport and emission processes is made difficult because of the breakdown of perturbation theory, the usual tool for studying particle interactions. Nonetheless, it is possible to relate different aspects of a particular problem, e.g., the transport of lepton number or energy, to a common medium structure function, $S_\nu(\omega)$, where ω is the energy transfer to the medium. The rate for any weak process can then be put into the form $\int d\omega F(\omega) S_\nu(\omega)$, where F is specific to that process. The difficulty with perturbation theory is confined to the function $S_\nu(\omega)$, so that once a particular $S_\nu(\omega)$ has been determined, the rates for all related processes can be calculated in a consistent fashion. Of course, different processes have F's which weigh the frequency dependence of S_ν in different ways, so there remains some sensitivity to the shape of $S_\nu(\omega)$. Within this framework we have compared neutrino scattering and pair absorption and emission as they apply to the transport of ordinary neutrinos. We have also studied how the same function $S_\nu(\omega)$ relates to the emission of RH neutrinos and how a very similar function S_a applies to axions.

In the limit that the medium is dominated by nonrelativistic nucleons, the medium response consists of the sum of vector and axial-vector response functions, $S_V(\omega)$ and $S_A(\omega)$, respectively. In the context of perturbative calculations of S, we have discussed two contributions: one from quasifree nucleons $S_0\delta(\omega)$, and one from interactions in the presence of two nucleons $S_{NN}(\omega)$. In the nonrelativistic long-wavelength limit the former is a δ function, whereas the latter naively has a soft divergence $S_{NN}(\omega) \sim \omega^{-2}$. In Sec. III, we argued that, based on physical arguments and explicit calculation, $S_{NN}(\omega)$ contributes only to S_A, i.e., the vector response is well approximated by quasifree nucleons.

The axial response, on the other hand, requires a full calculation of the spin fluctuations in the medium, and is not well determined. Nucleon-nucleon interactions randomize the nuclear spin at a rate comparable to the collision frequency, which has a twofold effect. They soften $S_A\delta(\omega)$ to something like a Lorentzian with a width of about the nucleon-nucleon interaction rate Γ_{int} and, by the same token, they regulate the divergence of $S_{NN}(\omega)$ at $\omega = 0$. It is possible to choose the width of the Lorentzian so that it reproduces the quasifree nucleon δ-function result in the limit of a very dilute medium, while at the same time for large values of ω the bremsstrahlung result for $S_{NN}(\omega)$ is obtained. Although this formulation is appealing (it replaces the one and two nucleon results by a single function), if extended into the regime of large collision rates it implies a very strong suppression to the axial response function and, as discussed in Sec. III, the internal consistency of the approach may be questioned, as well as its justification based on a comparison with laboratory data.

We conclude that there is no unique way to extrapolate the S_{NN} to high densities, and that different prescriptions lead to significantly different results for quantities such as neutrino opacities or axion emissivities. We stress that this inadequacy of perturbation theory cannot be ignored. We are interested in the evolution of a neutron star for, say, the first 10 sec after the collapse of the progenitor star. During this phase, all numerical models show maximum temperatures around $T = 50$ MeV. Even though large parts of the core may be cooler, the neutrino opacity increases with temperature, and so the hot regions should act as a bottleneck to neutrino transport. There, the nuclear medium is on the verge of degeneracy, but still essentially nondegenerate. If the medium were sufficiently dilute, then even for strong interactions one could proceed perturbatively with some confidence, while if the degeneracy were much higher only nucleons near their Fermi surfaces participate, in effect rendering the medium more dilute. In the middle, perturbation theory fares worst. Still, because one can calculate reasonable answers both for very high and for very low degeneracies, we find it difficult to believe that the neutrino scattering rate should be *very* different from its naive lowest-order result in the intermediate range. In particular, an anomalously *enhanced* scattering rate seems rather unlikely, especially as it would require a violation of the upper limit in Eq. (3.3), which is based on the presumption that the different momentum, spin, and isospin parts of the nucleonic wave function are uncorrelated.

We speculate that the overall shape of $S(\omega)$ given by the low-density approximation will not be radically modified at high densities. In this case scattering rates definitely exceed the rates for pair processes. This conclusion applies to the transport of LH neutrinos as well as to the emission of RH ones. Moreover, extreme modifications to the shape of S_A seem likely to suppress the total interaction rate for both scattering and pair processes. Since the vector response is essentially unmodified and contributes only to scattering, it seems unlikely that any reasonable form for S_A will make pair processes dominate over scattering. This conclusion holds equally for the scattering of LH neutrinos and the emission of RH neutrinos.

We caution, however, that there seems to be a trend that as the medium becomes less dilute the response function gets suppressed for low ω while it increases on the wings. We have assumed that for ω above a typical nucleon-nucleon interaction rate, S_A is still reasonably represented by a perturbative result. If this were not the case, the true shape of $S(\omega)$ could be so "hard" that pair processes could be important for neutrino transport after all. Equally, if the medium had strongly excited collective modes, their decay into neutrino pairs could be stronger than their contribution to scattering, depending on the nature of their dispersion relation. In this light we note that Iwamoto and Pethick [6] have discussed the importance of sound waves with a strong NC coupling, while Haensel and Jerzak [27] have shown that in degenerate matter quasifree nucleons still seem to make the dominant contribution. Further, in the explicit case of a timelike π^- condensate we find that pair emission does not dominate.

Given the uncertainty in the basic transport rates, how is one to understand the good agreement between the SN 1987A neutrino signature with theoretical expectations? One answer is that other parameters may have been adjusted to achieve good agreement with the SN 1987A neutrino signal, hiding the sensitivity to the neutrino transport. Another possibility is that somehow the cooling curve is rather insensitive to the details of neutrino transport. Or, perhaps the transport in the numerical models, just be chance, is close to the real thing. For example, if the axial response were eliminated entirely, the vector response would still contribute $\approx \frac{1}{5}$ of the naive single-nucleon scattering rate. Perhaps even a fifth of the standard scattering rates is sufficient to reproduce the observed cooling time scale of SN 1987A. In this light, we feel it is time to undertake a systematic survey of how variations in the neutrino diffusion affect the cooling of the core on time scales from a second to a minute.[10] This certainly seems important in light of the efforts to prepare experiments in advance of the next galactic supernova neutrino burst.

We have also commented on the derivation of bounds on exotic particle interactions. In such calculations, the nonstandard processes should be treated on the same footing with the ordinary neutrino transport, i.e., all NC-type processes should be based on the same function S. Any regulation applied to keep the ordinary scattering rates bounded must be applied consistently to exotic emission rates also, which essentially has the effect of decreasing them. Further, new processes proposed to increase emission rates must be studied for their effects on neutrino transport as well. In the end, only calculations where the transport and emission are performed with the same approximations for $S(\omega)$ may be used for quantitatively precise constraints, and even then only after allowing for different choices of $S(\omega)$ and variation of all other unknown parameters.

Finally, we have evaluated the contribution to the response function from the interaction of single pions with nucleons, $S_\pi(\omega)$. Contrary to previous speculations in the literature, we find no indication for an anomalously large contribution by this latter process for the conditions relevant for the cooling of a SN core.

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APPENDIX A: EFFECTIVE NUCLEON WEAK CHARGES

If the nucleon weak current is written in the form Eq. (2.3) the vector charge is given by $C_V = \tau_3 - 4\sin^2\theta_W Q$, where $\tau_3 = +1$ for protons and -1 for neutrons and Q is the nucleon electric charge in units of e. Therefore,

$$C_V^n = -1 \text{ and } C_V^p = 0.07\,, \tag{A1}$$

where we have used $\sin^2\theta_W = 0.2325$ for the weak mixing angle.

There remain problems, however, with the value of the axial charge. If axial vector currents were conserved we would have $C_A = \tau_3$, but axial charge conservation is known to be violated by the strong interactions, leading to a charged-current axial coupling of bare nucleons of $C_{A,\mathrm{CC},0} = 1.26$. In large nuclei this value is suppressed somewhat, and the commonly used value [29] for nuclear matter calculations is $C_{A,\mathrm{CC}} = 1.0$. Based on the naive quark model it was common practice until recently to take the NC axial charges to be isovector in character and so, in a nuclear medium $C_{A,\mathrm{NC}} = 1.0\tau_3$ was a reasonable choice. It is now realized, however, that the neutral axial current has an isoscalar piece as well, which is associated with the polarization of the strange quark sea of nucleons.

[10] By the time we had prepared a revised version of the present paper, a first attempt at such a numerical study had been completed [18]. The modification of the neutrino opacities implemented in that study were relatively schematic. Still, it appears that a complete suppression of the axial-vector opacities is not compatible with the SN 1987A signal if the late-time events in the Kamiokande and IMB detectors are taken to represent the tail end of the Kelvin-Helmholtz cooling of the neutron star. Some implications of [18] for the issues raised in our present paper will be discussed in a forthcoming joint article [28].

The axial charge of nucleon i can be written as a sum over the contributions from different quarks, $C_A^i = \Delta u^i - \Delta d^i - \Delta s^i$. The contributions from u and d quarks include both valence and sea contributions, whereas the s quark only has a sea contribution. It is expected that the heavy quarks do not contribute significantly. Isospin symmetry dictates that $\Delta u^n = \Delta d^p$ and $\Delta d^n = \Delta u^p$. Charged-current processes determine $\Delta u^p - \Delta d^p = 1.26$ for bare nucleons. Scattering experiments of polarized muons on polarized hadronic targets by the European-Muon Collaboration (EMC) [30] and Spin Muon Collaboration (SMC) [31] groups at CERN, and by polarized electrons at SLAC (E142 [32]) give a range of values for Δs^p between 0 and -0.2. In a review of the phenomenology and theory, Ellis and Karliner [33] suggest $\Delta s = -0.11 \pm 0.04$, leading to $C_{A,0}^p = 1.37$ and $C_{A,0}^n = -1.15$ in vacuum. For nuclear matter we find

$$C_A^n = -0.91 \text{ and } C_A^p = 1.09 \tag{A2}$$

if Δu, Δd, and Δs are suppressed by an equal factor $\frac{1}{1.26}$. We do not give uncertainties because of the many possible sources of systematic errors, both in the interpretation of the laboratory results and in the extrapolation to nuclear matter.

For the scattering of neutrinos on quasifree nucleons the relevant combination of coupling constants is $\overline{C^2} = \frac{1}{4}(C_V^2 + 3C_A^2)$ for which we find

$$\overline{C^2} = \begin{cases} 0.89 & \text{for protons (1.41 in vacuum)}, \\ 0.87 & \text{for neutrons (1.24 in vacuum)}. \end{cases} \tag{A3}$$

Therefore, the scattering rate is suppressed by about a factor of 0.63 for protons and 0.70 for neutrons relative to their vacuum values.

As a last point, we note that the reductions in C_A are a property of the medium, not the weak interactions. It seems reasonable, then, that if axions exist, their couplings would also be reduced at nuclear density compared to their vacuum values. This would introduce an additional uncertainty in relating the constraints on axion models from supernovae to axion properties as measured in other experiments.

APPENDIX B: PERTURBATIVE ESTIMATES OF $S_A(\omega)$

In order to determine the perturbative $S_A(\omega)$ from NN interactions we find it convenient to study the axion emission rate $NN \to NNa$. To apply these results to neutrinos one must take care to use the appropriate couplings, cf. Eq. (B15) below and Eq. (3.7).

The axionic energy loss rate from a medium is given by the usual phase-space integral

$$Q_a = \int d\Pi f_1 f_2 S \sum_{\text{spins}} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - k_a)\omega , \tag{B1}$$

where $p_{1,2}$ are the four-momenta of the initial-state nucleons, $p_{3,4}$ are for the final states, and k_a is for the axion. The symmetry factor S is equal to $\frac{1}{4}$ for nn or pp interactions, and equal to 1 for np interactions. The $d\Pi$ is a product of phase-space factors for all particles involved: $d^3\mathbf{k}_a/[2\omega(2\pi)^3]$ for the axion and $d^3\mathbf{p}_j/[2E_j(2\pi)^3]$ for each nucleon. The initial-state occupation numbers $f_{1,2}$ are given by the Maxwell-Boltzmann distribution for nonrelativistic particles

$$f(\mathbf{p}) = (N_B/2)(2\pi/m_N T)^{3/2} e^{-\mathbf{p}^2/2m_N T} \tag{B2}$$

so that $\int 2f(\mathbf{p}) d^3\mathbf{p}/(2\pi)^3 = N_B$ gives the nucleon (baryon) density where the factor 2 accounts for the two spin states. Final-state Pauli-blocking factors may be omitted for nondegenerate conditions.

To estimate $|\mathcal{M}|^2$ we assume nonrelativistic nucleons and use the one-pion-exchange approximation to model the NN interaction. We follow Brinkman and Turner [15] and agree with their result for $|\mathcal{M}|^2$. For nn interactions,

$$\sum_{\text{spins}} |\mathcal{M}|_n^2 = \frac{16(4\pi)^3 \alpha_\pi^2 \alpha_a}{3m_N^2} C_{a,n}^2 \left(\frac{\mathbf{k}^4}{(\mathbf{k}^2 + m_\pi^2)^2} + \frac{\boldsymbol{l}^4}{(\boldsymbol{l}^2 + m_\pi^2)^2} + \frac{\mathbf{k}^2\boldsymbol{l}^2 - 3|\mathbf{k}\cdot\boldsymbol{l}|^2}{(\mathbf{k}^2 + m_\pi^2)(\boldsymbol{l}^2 + m_\pi^2)} \right) , \tag{B3}$$

where $\alpha_a = (2m_N/f_a)^2/4\pi$ is the axion-nucleon "fine-structure constant," $C_{a,n}$ is defined in Eq. (5.7), $\alpha_\pi = (2fm_N/m_\pi)^2/4\pi \approx 15$ for $f \approx 1$, and here $\mathbf{k} = \mathbf{p}_1 - \mathbf{p}_3$ and $\boldsymbol{l} = \mathbf{p}_1 - \mathbf{p}_4$ are the three-momentum transfers in the direct and exchange diagrams. The first term constitutes the result from the direct diagrams, the second from the exchange diagrams, and the third term is the interference between them. For pp interactions one just replaces n by p in Eq. (B3), but for np interactions the result is more cumbersome:

$$\sum_{\text{spins}} |\mathcal{M}|_{np}^2 = \frac{16(4\pi)^3 \alpha_\pi^2 \alpha_a}{3m_N^2} \left[(C_{a,+}^2 + C_{a,-}^2) \frac{\mathbf{k}^4}{(\mathbf{k}^2 + m_\pi^2)^2} + (4C_{a,+}^2 + 2C_{a,-}^2) \frac{\boldsymbol{l}^4}{(\boldsymbol{l}^2 + m_\pi^2)^2} + 2\frac{(C_{a,+}^2 + C_{a,-}^2)\mathbf{k}^2\boldsymbol{l}^2 - (3C_{a,+}^2 + C_{a,-}^2)|\mathbf{k}\cdot\boldsymbol{l}|^2}{(\mathbf{k}^2 + m_\pi^2)(\boldsymbol{l}^2 + m_\pi^2)} \right] , \tag{B4}$$

where $C_{a,\pm} = \frac{1}{2}(C_{a,n} \pm C_{a,p})$. The larger coefficients in the np result are due to the stronger coupling (by a factor $\sqrt{2}$) of charged rather than neutral pions to nucleons. We note that Eq. (B4) is at odds with the corresponding result for bremsstrahlung of $\nu\bar{\nu}$ in np interactions derived by Friman and Maxwell [7] who give in their Eq. (70) the coefficient of the $C_{A,-}^2\mathbf{k}^2\boldsymbol{l}^2$ term as -1 instead of $+2$. There are other differences, but they may all be attributed to the fact that Friman and Maxwell work in the degenerate limit where $\mathbf{k}\cdot\boldsymbol{l} = 0$ or that they use $C_{A,n} = -C_{A,p}$ so that $C_{A,+} = 0$.

It remains to perform the phase-space integrations. In the nonrelativistic limit one may use $d^3\mathbf{p}_j/[2E_j(2\pi)^3] = d^3\mathbf{p}_j/[2m_N(2\pi)^3]$ and $E_j = \mathbf{p}_j^2/2m_N$ in the energy δ function. Moreover, the axion momentum can be ignored in the momentum δ function. The integration is simplified by introducing center-of-mass momenta such that $\mathbf{p}_{1,2} = \mathbf{P} \pm \mathbf{p}_i$ and $\mathbf{p}_{3,4} = \mathbf{P} \pm \mathbf{p}_f$, where $\mathbf{P} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_4)$ and the last equality is ensured by using the momentum δ function. The integral over $d^3\mathbf{P}$ may be done separately since neither the energy δ function or $|\mathcal{M}|^2$ depend on $\mathbf{P}$. Since we have already averaged $|\mathcal{M}|^2$ over nucleon polarizations and axion emission angles, all that remains is a four dimensional integral over $|\mathbf{p}_i|, |\mathbf{p}_f|, \omega_a$, and z, where $z \equiv (\mathbf{p}_i\cdot\mathbf{p}_f)/(|\mathbf{p}_i|\,|\mathbf{p}_f|)$ is the cosine of the nucleon scattering angle. Finally, one introduces a dimensionless axion energy $x \equiv \omega/T$ and initial- and final-state nucleon energies $u \equiv \mathbf{p}_i^2/m_NT$ and $v \equiv \mathbf{p}_f^2/m_NT$. In terms of these variables the energy δ function becomes $\delta(u-v-x)/T$. The various terms in $|\mathcal{M}|^2$ then take on the form

$$\begin{aligned}\mathbf{k}^2/m_NT &= u+v-2z\sqrt{uv}\ ,\\ \boldsymbol{l}^2/m_NT &= u+v+2z\sqrt{uv}\ ,\\ \mathbf{k}\cdot\boldsymbol{l}/m_NT &= u-v\ .\end{aligned} \tag{B5}$$

Defining $y \equiv m_\pi^2/m_NT$, we rewrite the energy loss in terms of reduced phase-space integrals, $I(y)$. For the case of nn bremsstrahlung

$$Q_{a,nn} = \frac{\alpha_a\alpha_\pi^2}{\pi^{1/2}}\frac{N_B^2T^{7/2}}{m_N^{9/2}}\frac{128}{105}C_{a,n}^2[I_\mathbf{k}(y) + I_\boldsymbol{l}(y) + I_{\mathbf{k}\boldsymbol{l}}(y) - 3I_{\mathbf{k}\cdot\boldsymbol{l}}(y)]\ , \tag{B6}$$

where

$$I(y) = \frac{35}{128}\int_0^\infty du\int_0^\infty dv\int_0^\infty dx\,x^2\sqrt{uv}e^{-u}\delta(u-v-x)\frac{1}{2}\int_{-1}^1 dz\left\{\begin{array}{ll}\left(\frac{u+v-2z\sqrt{uv}}{u+v-2z\sqrt{uv}+y}\right)^2 & \text{for } \mathbf{k}\ ,\\ \left(\frac{u+v+2z\sqrt{uv}}{u+v+2z\sqrt{uv}+y}\right)^2 & \text{for } \boldsymbol{l}\ ,\\ \frac{(u+v)^2-4uvz^2}{(u+v+y)^2-4uvz^2} & \text{for } \mathbf{k}\boldsymbol{l}\ ,\\ \frac{(u-v)^2}{(u+v+y)^2-4uvz^2} & \text{for } \mathbf{k}\cdot\boldsymbol{l}\ .\end{array}\right. \tag{B7}$$

Clearly, $I_\mathbf{k}(y) = I_\boldsymbol{l}(y)$ by symmetry under $z \to -z$. The normalization arises from

$$\int du dv dx x^2\sqrt{uv}e^{-u}\delta(u-v-x) = \tfrac{128}{35}$$

and ensures that $I_\mathbf{k}(0) = I_\boldsymbol{l}(0) = I_{\mathbf{k}\boldsymbol{l}}(0) = 1$.

For the high temperatures of a SN core it is reasonable to neglect the pion mass and set $y = 0$. Even in this limit the $\mathbf{k}\cdot\boldsymbol{l}$ term must still be averaged over z. To this end we use the δ function to eliminate the integration over the axion energy and perform the z integration analytically. After defining $t \equiv (v/u)^{1/2}$, the u integration can be separated out leaving only the t integral so that

$$\begin{aligned}\beta \equiv 3I_{\mathbf{k}\cdot\boldsymbol{l}}(0) &= 3\frac{105}{16}\int_0^1\frac{t(1-t^2)^4}{1+t^2}\ln\left(\frac{1+t}{1-t}\right)dt\\ &= 1.3078\ .\end{aligned} \tag{B8}$$

In this notation $I_\mathbf{k} + I_\boldsymbol{l} + I_{\mathbf{k}\boldsymbol{l}} - 3I_{\mathbf{k}\cdot\boldsymbol{l}} = 3 - \beta$ for the present case of a vanishing m_π. Equation (B8) differs from Brinkmann and Turner who found $\beta = 1.0845$ because they gave $(1+t^2)^2$ instead of $1+t^2$ in the denominator of the integrand (see the unnumbered formula at the top of p. 2347 of [15]).

If one wants to extract a perturbative expression for the function $S_A(\omega)$ the x integration must be left undone. Using the energy δ function to eliminate the u integration we may write

$$\begin{aligned}I_\mathbf{k}(0) = I_\boldsymbol{l}(0) &= I_{\mathbf{k}\boldsymbol{l}}(0)\\ &= \frac{35}{128}\int_0^\infty dx\,x^2e^{-x}s_0(x)\end{aligned} \tag{B9}$$

with

$$s_0(x) = \int_0^\infty dv\ e^{-v}(xv+v^2)^{1/2}\ . \tag{B10}$$

Similarly, for the $\mathbf{k}\cdot\boldsymbol{l}$ part of the interference term we define

$$s_{\mathbf{k}\cdot\boldsymbol{l}}(x) = \int_0^\infty dv\frac{x^2}{2(2v+x)}\ln\left(\frac{\sqrt{v+x}+\sqrt{v}}{\sqrt{v+x}-\sqrt{v}}\right)e^{-v}\ . \tag{B11}$$

Combining these results with Eq. (B6) we may write

$$Q_{a,nn} = \frac{\alpha_a \alpha_\pi^2}{\pi^{1/2}} \frac{N_B^2 T^{7/2}}{m_N^{9/2}} \int_0^\infty dx\, x^2 e^{-x} s_{nn}(x) \qquad \text{(B12)}$$

with $s_{nn} = C_{a,n}^2 (s_0 - s_{\mathbf{k}\cdot\boldsymbol{l}})$.

We may now compare with Eq. (5.9), the representation of Q_a derived in the main text from more general considerations:

$$Q_a = \frac{\alpha_a}{4\pi} \frac{N_B}{m_N^2} \int_0^\infty d\omega\, \omega^4 e^{-\omega/T} S_a(\omega) \ . \qquad \text{(B13)}$$

We conclude that $S_a(\omega)$ may be written as

$$S_a(\omega) = \frac{\Gamma_A}{\omega^2} s(\omega/T) \qquad \text{(B14)}$$

with $\Gamma_A = 4\pi^{1/2}\alpha_\pi^2 N_B T^{1/2} m_N^{-5/2}$ as stated in Eq. (3.6) of the main text.

For a mixed medium of protons and neutrons (both nondegenerate) one must include pp and np interactions as well. Generalizing the above analysis for massless pions, one need only modify s:

$$\begin{aligned} s(x) = {} & Y_n^2 C_{a,n}^2 [s_0(x) - s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] + Y_p^2 C_{a,p}^2 [s_0(x) - s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] \\ & + \tfrac{4}{3} Y_n Y_p [(7C_{a,+}^2 + 5C_{a,-}^2) s_0(x) \\ & - (6C_{a,+}^2 + 2C_{a,-}^2) s_{\mathbf{k}\cdot\boldsymbol{l}}(x)] \ , \end{aligned} \qquad \text{(B15)}$$

where Y_n and Y_p are the numbers of neutrons and protons per baryon; we assume $Y_p + Y_n = 1$. The total emission rate depends on both the mix of baryons and the details of the couplings. If we ignore the low ω cutoff considerations of the main text, the naive axion emission rate scales as

$$\begin{aligned} Q_a \propto {} & (Y_n^2 C_{a,n}^2 + Y_p^2 C_{a,p}^2)(1 - \tfrac{1}{3}\beta) \\ & + \tfrac{4}{3} Y_n Y_p [7C_{a,+}^2 + 5C_{a,-}^2 - \tfrac{1}{3}\beta(6C_{a,+}^2 + 2C_{a,-}^2)] \ . \end{aligned}$$

Typically the axion emission rate is some 2.5 times higher for a baryon mix of $Y_n = 0.7$ than for pure neutrons.

We give analytic approximations for s_0 and $s_{\mathbf{k}\cdot\boldsymbol{l}}$ that may be useful when one must perform integrations involving $s(x)$ in the integrand:

$$s_0(x) \approx \left(1 + \frac{\pi}{4}x\right)^{1/2} \ ,$$

$$s_{\mathbf{k}\cdot\boldsymbol{l}}(x) \approx \left[1 + \left(\frac{\pi}{4}x\right)^{5/4}\right]^{2/5} - \left[1 + \left(\frac{64}{169\pi}x\right)^{5/4}\right]^{-2/5} \ . \qquad \text{(B16)}$$

These approximations were chosen to give the leading behavior at large and small x, and for $s_{\mathbf{k}\cdot\boldsymbol{l}}$ the next to leading order at large x and a pleasing fit at small x as well. For all, x, $s_0 - s_{\mathbf{k}\cdot\boldsymbol{l}}$ differs from the integral representation by less than 3%.

As a last item, we discuss the situation where m_π is not small. Now, the I's are functions of $y = m_\pi^2/m_N T$. For the nn case the "reduced structure function" is

$$s_{nn}(x,y) = 2s_{\mathbf{k}}(x,y) + s_{\mathbf{k}\boldsymbol{l}}(x,y) - 3s_{\mathbf{k}\cdot\boldsymbol{l}}(x,y) \ , \qquad \text{(B17)}$$

where it was used that $s_{\boldsymbol{l}} = s_{\mathbf{k}}$. Explicitly we find

$$s(x,y) = \int_0^\infty dv \sqrt{uv}\, e^{-v} \int_{-1}^{1} dz \begin{cases} \left(\frac{u+v-2z\sqrt{uv}}{u+v-2z\sqrt{uv}+y}\right)^2 & \text{for } \mathbf{k} \ , \\ \frac{(u+v)^2 - 4uvz^2}{(u+v+y)^2 - 4uvz^2} & \text{for } \mathbf{k}\boldsymbol{l} \ , \\ \frac{(u-v)^2}{(u+v+y)^2 - 4uvz^2} & \text{for } \mathbf{k}\cdot\boldsymbol{l} \ , \end{cases} \qquad \text{(B18)}$$

where is it understood that $u = v + x$ as determined by energy conservation.

We were able to perform analytically the z integrals, but not all of the resulting v intervals. To qualitatively show the effects of the pion mass, we plot the functions $I_{\mathbf{k}}(y)/I_{\mathbf{k}}(0)$, $I_{\mathbf{k}\boldsymbol{l}}(y)/I_{\mathbf{k}\boldsymbol{l}}(0)$, and $I_{\mathbf{k}\cdot\boldsymbol{l}}(y)/I_{\mathbf{k}\cdot\boldsymbol{l}}(0)$ in Fig. 4. Because $I_{\mathbf{k}}(0) = I_{\mathbf{k}\boldsymbol{l}}(0) = 1$ the former two expressions are simply $I_{\mathbf{k}}(y)$ and $I_{\mathbf{k}\boldsymbol{l}}(y)$, respectively. Even for temperatures as low as 10 MeV the $I(y)$ are reduced by only about 50% relative to the $I(0)$ values. We conclude that even for nondegenerate nucleons, pion mass effects do not seriously suppress Q_a (or $\Gamma_{\rm scat}$, etc., for neutrino processes) until $T \lesssim 10$ MeV. For degenerate nucleons the typical momentum transfer will be higher and pion mass effects will be even less important. See Burrows, Ressell, and Turner [34] for a similar discussion of the suppression due to a finite pion mass.

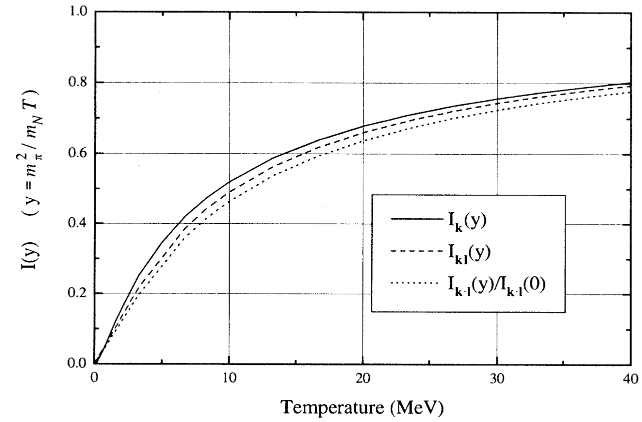

FIG. 4. Functions $I(y)$ as defined in Eq. (B7), plotted as a function of T according to $y = m_\pi^2/m_N T$. Recall that $I_{\boldsymbol{l}}(y) = I_{\mathbf{k}}(y)$ and that $I_{\boldsymbol{l}}(0) = I_{\mathbf{k}}(0) = I_{\mathbf{k}\boldsymbol{l}}(0) = 1$.

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