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Fermions on the lattice by means of Mandelstam-Wilson phase factors

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We propose a Mandelstam-Wilson phase factor approach to solve the problem of handling correctly fermion fields on a lattice. We apply this approach to fermionize exactly QCD [SU(∞)] at the leading limit of the strong coupling limit

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One of the long-standing unsolved problems in the lattice approach to QCD is how to handle discretized massless fermionic fields [1]. In this Brief Report we propose a solution for the above-mentioned problem by considering as the QCD natural field variable to be discretized on the lattice the Mandelstam-Wilson phase factor defined by the color-singlet quark currents, instead of the fermion field as proposed by previous studies. Additionally, we show the usefulness of this propose by obtaining, in an unambiguous way, the associated QCD Nambu–Jona-Lasinio fermionic model, which, upon being bosonized, leads to a low-energy theory of the meson and baryon of QCD.

Let us start our study by considering the Euclidean QCD [SU(N_c)] generating functional for the color-singlet scalar and vectorial quark currents:

$$Z[\sigma+\gamma_5\beta, J_\mu+\gamma_5\tilde{A}_\mu]=\int D^F[A_\mu(x)]\exp\left(-\frac{1}{4}\int d^4x\mathrm{Tr}[F^2_{\mu\nu}(A)](x)\right)\times\left\{\int D^F[\bar{\psi}(x)]D^F[\psi(x)]\exp\left(-\int d^4x(\bar{\psi}[i\gamma^\mu\partial_\mu+ig\gamma^\mu A_\mu+\sigma+\gamma_5\beta+\gamma^\mu j_\mu+\gamma^\mu\gamma^5\tilde{A}_\mu]\psi)(x)\right)\right\}, \tag{1}$$

where $\psi(x), \bar{\psi}(x)$ are the independent Euclidean quark fields, $\sigma(x)+\gamma_5\beta(x)$ and $J_\mu(x)+\gamma^5\tilde{A}_\mu(x)$ are the external sources for the scalar, scalar-axial, and axial-vectorial QCD quark currents. $A_\mu(x)$ denotes the SU(N_c) gluon field.

In order to obtain effective quark field theories from Eq. (1) we propose to integrate out their gluon degrees of freedom in the lattice; i.e., let us first consider the pure gluonic functional integral

$$I[\psi,\bar{\psi}]=\int D^F[A_\mu(x)]\exp\left(-\frac{1}{4}\int d^4x\mathrm{Tr}[F^2_{\mu\nu}(A)](x)\right)\times\exp\left(ig\int d^4x(\bar{\psi}\gamma^\mu\psi)(x)A_\mu(x)\right). \tag{2}$$

Our procedure to evaluate Eq. (2) is, first, to introduce a lattice space-time. At this point we put forward our idea to handle fermonic fields on the lattice. As was shown in [1], it is impossible to have a well-defined procedure to define massless fermion fields on the usual lattice $\{x_\mu=[n_\mu], n_\mu\in\mathbb{Z}\}$ (with spacing a) [1]. We propose, thus, to consider directly the bosonic quark fermion current on the lattice by means of its associated Mandelstam-Wilson phase factor defined on each lattice link $([n_\mu],[n_\mu]+\alpha)$

$$\Phi_\alpha([n_\mu])=\exp(ia(\bar{\psi}\gamma^\alpha\psi)([n_\mu]))\ . \tag{3}$$

Note that the above-written phase factor has indices (i,j) on the group SU(N_c) and an index α related to the Lorentz group as it should be.

The associated gluon U(N) group-valued Mandelstam-Wilson phase factor is still given by the link lattice gluon variable

$$U_\mu([n_\alpha])=\exp(iaA_\mu([n_\alpha]))\ . \tag{4}$$

At this point of our study, we point out that the quark-gluon coupling on the lattice may be written as a product of the Mandelstam-Wilson phase factor given by Eqs. (3) and (4) since we have the formal continuum limit at the lattice space going to zero as one can see by expanding the exponentials

$$\lim_{a\to 0}\left[\sum_{\{[n_\alpha]\}}a^2\mathrm{Tr}(\{U_\mu([n_\alpha])-\mathbb{1}\}\{\phi_\mu([n_\alpha])-\mathbb{1}\})\right]=ig\int d^4xA_\mu(x)(\bar{\psi}\gamma^\mu\psi)(x)\ . \tag{5}$$

Our proposed gauge-invariant lattice version of the gluon functional integral, Eq. (2), is, thus, given by

$$I[\psi,\bar{\psi}] = \int D^H[U_\mu([n_\alpha])]\exp\left(-\frac{1}{4g^2}\sum_{\{[n_\alpha]\}}\mathrm{Tr}\{U_\mu([n_\alpha])U_\nu([n_\alpha+\mu])U_\mu^\dagger([n_\alpha+\nu])U_\nu^\dagger([n_\alpha])\}\right)$$
$$\times\exp\left(-\sum_{\{[n_\alpha]\}}a^2\mathrm{Tr}\{[U_\mu([n_\alpha])-\mathbb{1}][\Phi_\mu([n_\alpha])-\mathbb{1}]\}\right) . \quad (6)$$

The advantage of this lattice phase factor approach to analyze the gluonic path integral, Eq. (2), is its allowance for an exact integration of the lattice gluon phase factors in both the perturbative and the nonperturbative regimes. Let us show its usefulness by evaluating in closed form Eq. (6) in the leading limit of the number of colors and in the leading limit of strong coupling as in [2] [Eq. (3.17)]:

$$I[\psi,\bar{\psi}]_{\substack{g^2\to\infty \\ N_c\to\infty}} = \lim_{N_c\to\infty}\int D^H[U_\mu([n_\alpha])]\exp\left(-a^2\sum_{\{[n_\alpha]\}}\mathrm{Tr}(U_\mu([n_\alpha])\{\Phi^\mu([n_\alpha])-\mathbb{1}\})\right)$$
$$= \exp\left(\frac{\Lambda^{(a)}_{\mathrm{QCD}}a^4}{N_c}\sum_{\{[n_\alpha]\}}\mathrm{Tr}(\{\phi^\mu([n_\alpha])-\mathbb{1}\}\{\phi_\mu([n_\alpha])-\mathbb{1}\}^\dagger)\right) , \quad (7)$$

where $\Lambda_{\mathrm{QCD}}(a)$ is the QCD strong-coupling phenomenological scale with dimension of inverse area (the gluon nonperturbative condensate) which by its turn is lattice spacing dependent.

It is very important to remark that the Jacobian J of the variable change $U_\mu([n_\alpha]) \to U_\mu([n_\alpha]) + \mathbb{1}$ on the lattice functional integrals, Eqs. (6) and (7), is unity only at the continuum limit $a \to 0$ (or at large N_c) since it is explicitly given by the ratio

$$J_{(a)} = \prod_{([n_\mu])}\left\{\frac{\det^{1/2}(M_{ij}\{U_\mu([n_\alpha])+\mathbb{1}\})(a)}{\det^{1/2}(M_{ij}\{U_\mu([n_\alpha])\})(a)}\right\} \quad (8)$$

and for $a \to 0$ we have that $\mathbb{1} + U_\mu^\dagger([n_\alpha]) \to \mathbb{1}$. Here the Haar measure $\prod_{[n_\alpha]} D^H\{U_\mu([n_\alpha])\}$ on the group $\prod_{[n_\alpha]} U(N_c)$ follows from the metric tensor group on each factor $U(N_c)$ [3]:

$$M_{ij} = \mathrm{Tr}\left(U^{-1}([n_\alpha])\frac{\partial}{\partial t_i}U([n_\alpha])\right.$$
$$\left.\times U^{-1}([n_\alpha])\frac{\partial}{\partial t_i}U([n_\alpha])\right) , \quad (9a)$$

$$D^H[U_\mu([n_\alpha])] = \prod_{(i)}(dt_i \det^{1/2}[M_{ij}(t)]) , \quad (9b)$$

where the derivatives are with respect to the group parameters $\{t_i\}$; i.e.,

$$U_\mu([n_\alpha]) = \exp(it^l([n_\mu])\lambda_l) . \quad (10)$$

The formal continuum limit $a \to 0$ of the result, Eq. (7), after a Fierz transformation, leads to the following quartic fermionic action in the continuum:

$$I_{\mathrm{continuum}}[\psi,\bar{\psi}]_{g^2\to\infty,N_c\to\infty} = \exp\left\{\frac{g_F^2}{N_c}\int d^4x[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2 + \tfrac{1}{2}(\bar{\psi}\gamma^\mu\psi)^2 - \tfrac{1}{2}(\bar{\psi}\gamma^\mu\gamma^5\psi)^2](x)\right\} . \quad (11)$$

Here the fermionic effective coupling constant g_F^2 is defined in the continuum by the formal limit $g_F^2 = \lim_{a\to 0}\Lambda^{(a)}_{\mathrm{QCD}}a^2$ and signaling the usual QCD dimensional transmutation phenomenon.

After substituting Eq. (11) into Eq. (1) we get our proposed fermionization for quantum chromodynamics in the very low-energy region with the gluon field $U(N_c)$ integrated out for large N_c in the sense of [2]. We remark that by introducing the Hubbard-Stratonovich ansatz to linearize the quartic fermion interactions, we obtain the U(1) chiral scalar and vectorial bosonized QCD $[U(\infty)]$ meson theory which improves that considered in [4] which was deduced by using phenomenological guessing arguments:

$$Z[\sigma+\gamma_5\beta, J_\mu+\gamma_5\tilde{A}_\mu] = \int D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{A}_\mu]\exp\left(-\frac{N_c}{g_F^2}\int d^4x[(\tfrac{1}{2}\hat{\sigma}^2+\tfrac{1}{2}\hat{\beta}^2+\tfrac{1}{2}\hat{J}_\mu^2+\tfrac{1}{2}\hat{A}_\mu^2)](x)\right)$$
$$\times\{\det{}^{N_c}[i\gamma\partial+(\sigma+i\hat{\sigma})+\gamma_5(\beta+i\hat{\beta})+\gamma_\mu(J_\mu+i\hat{J}_\mu)+\gamma^5\gamma^\mu(i\hat{A}_\mu+\tilde{A}_\mu)]\} . \quad (12)$$

Note that in Eq. (12), $(\hat{\sigma} + i\gamma_5\hat{\beta})$ and $(\hat{J}_\mu + i\gamma_5\hat{A}_\mu)$ should be identified with the U(1) chiral scalar and vectorial low-energy physical meson fields. Let us comment that the dynamics for the meson fields above comes from the evaluation of the quark functional determinant [4]. In the limit of the heavy scalar meson mass $\langle\hat{\sigma}\rangle \to \infty$, one can easily implement the technique of [5] to get the full effective hadronic action in terms of $1/\langle\hat{\sigma}\rangle$ power series.

In the case of baryonlike field excitations of the form $\Omega(x) = \epsilon_{ijk}\psi_i(x)\psi_j(x)\psi_k(x)$ it is still possible to analyze them in our proposed framework. For this task we consider a Hubbard-Stratonovich ansatz to write the generating functional for the baryonlike excitation $B(x)$: namely,

$$Z[B(x)] = \int D^F[\Delta]D^f[\lambda]D^F[A_\mu]D^F[\psi]D^F[\bar{\psi}]\exp\left(-\int d^4x\{\bar{\psi}_p[i\gamma^\mu\partial_\mu + i\lambda_{pq} + \gamma^\mu(A_\mu)_{qp}]\psi_q\}(x)\right) \times\exp\left(-\int d^4x[B(x)\epsilon_{ijk}\psi_i(x)\Delta_{jk}(x)]\right)\exp\left(-i\int d^4x[\lambda_{pq}(x)\Delta_{qp}(x)]\right), \tag{13}$$

where (p,q) are $U(N_c)$ indices and the auxiliary fields (Δ,λ) belong to the adjoint $U(N_c)$ representation.

After integrating out the gluon field $A_\mu(x)$ following the steps leading to Eq. (7) and the quark field as in Eq. (9), we get our proposed effective QCD-baryon field theory:

$$Z[B(x)] = \int D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{A}_\mu]D^F[\Delta]D^F[\lambda]\exp\left(-\frac{N_c}{g_F^2}\int d^4x[\tfrac{1}{2}\hat{\sigma}^2 + \tfrac{1}{2}\hat{\beta}^2](x) + [\tfrac{1}{2}\hat{J}_\mu^2 + \tfrac{1}{2}\hat{A}_\mu^2](x)\right) \times\exp\left(-i\int d^4x\mathrm{Tr}(\lambda\Delta)(x)\right)\det\{[i\gamma\partial + (\hat{\sigma} + i\gamma_5\hat{\beta}) + \gamma^\mu(\hat{J}_\mu + i\gamma_5\hat{A}_\mu)]_{pq} - i\lambda_{pq}\} \times\exp\left\{-\int d^4x d^4y B(x)\epsilon_{ijk}\Delta_{jk}(x)[(i\gamma\partial + (\hat{\sigma} + i\gamma_5\hat{\beta}) + \gamma^\mu\hat{J}_\mu + i\gamma^\mu\gamma_5\hat{A}_\mu - \lambda]^{-1}_{ii'}(x,y)\epsilon_{i'j'k'}\Delta_{j'k'}(y)B(y)\right\}. \tag{14}$$

It is instructive to remark that Eq. (14) indicates the impossibility to consider baryon excitations in isolation from the meson excitations in our proposed bosonized effective QCD field theory.

It is worth pointing out that strong-coupling corrections from the neglected gluon field kinetic action in Eq. (7) are straightforwardly implemented on the lattice by using the usual quantum field theory perturbation theory with the external lattice gluon source coupling [2]: $\sum_{[n_\mu]} J_\mu([n_\alpha])U_\mu([n_\alpha])$;

$$I[\psi,\bar{\psi}]_{N_c\to\infty} = \lim_{J_\nu([n_\alpha])\to 0}\left\{\exp\left[-\frac{1}{4g^2}\sum_{([n_\alpha])}\left(\frac{\delta}{\delta J_\mu([n_\alpha])} + \mathbb{1}\right)\left(\frac{\delta}{\delta J_\nu([n_\alpha+\mu])} + \mathbb{1}\right)\left(\frac{\delta}{\delta J_\mu^\dagger([n_\alpha+\nu])} + \mathbb{1}\right) \times\left(\frac{\delta}{\delta J_\nu^\dagger([n_\alpha])} + \mathbb{1}\right)\right]\tilde{I}[\psi,\bar{\psi}]_{\substack{g^2\to\infty \\ N_c\to\infty}}\right\}, \tag{15}$$

where [see Eq. (7)]

$$\tilde{I}[\psi,\bar{\psi}]_{\substack{g^2\to\infty \\ N_c\to\infty}} = \exp\left(\frac{\Lambda^{(a)}_{\mathrm{QCD}}a^4}{N_c}\sum_{\{[n_\alpha]\}}\mathrm{Tr}\{(\Phi_\mu + J_\mu - \mathbb{1})([n_\alpha])(\Phi_\mu + J_\mu - \mathbb{1})^\dagger([n_\alpha])\}\right). \tag{16}$$

The associated $1/4g^2$ corrected fermionized QCD $[U(\infty)]$ effective theory will, thus, be given by nonlocal current-current quark correlation functions averaged with the leading Nambu–Jona-Lasinio quark field theory, Eq. (11). Unfortunately, only at the limit of large mass of [5] it is possible to implement reliable approximate calculations useful for nuclear physics at low energy. Work on these applications for very low-energy nuclear hadron dynamics will be reported elsewhere.

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