Structure effects in $P \to \ell\nu$, $P \to \ell\nu\gamma$, and $\pi^0 \to \gamma\gamma$ decays

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The amplitudes of pseudoscalar meson decays are calculated in a relativistic nonperturbative quark model. One assumes that mesons are made of a quark-antiquark pair and of a scalar neutral component representing the global contribution of the nonelementary fluctuations of the quark gluonic field. The model allows us to treat in a unitary way the weak and weak radiative decays, leading to manifest gauge-invariant amplitudes and to an essential dependence of the form factors on the photon momentum in the radiative decays. The amplitudes and the form factors are expressed in terms of the internal functions of the mesons and of the quark masses. The experimental data can be fitted in a satisfactory manner using current quark masses 0.5 MeV$\leq m_u \leq$ 3 MeV, 3 MeV$\leq m_d \leq$ 15 MeV, 150 MeV$\leq m_s \leq$ 300 MeV.

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I. INTRODUCTION

The existence of some uncertainties both in the experimental and theoretical information concerning the form factors in the amplitudes of various pseudoscalar meson decays [1] stimulated the development of a plethora of models for these processes.

In the last years chiral perturbation theory, appearing in the context of nonperturbative QCD, provided a powerful tool for the investigation of low energy behavior of the form factors [2]. In models of this kind pseudoscalar mesons are treated like Goldstone bosons, and their low energy dynamics is described by effective Lagrangians expressed in terms of physical fields. By adding some meson loop contribution to the standard quark level calculations [3], a good agreement with the experimental data has been obtained.

An alternative point of view is to look at mesons like quark-antiquark bound states. In this case the central problem is how to introduce the internal structure of mesons. This problem is solved either by means of effective Lagrangians coupling the quark and meson fields [1,4], or directly by means of a bound-state wave function such as, for instance, the wave functions given by the Bethe-Salpeter equation [5], or by the bag model [6]. Recently, trying to escape the difficulties related with the relativistic treatment of the binding mechanism, simple models of mesons such as products of free quark and antiquark states have been proposed [7]. Lorentz invariance is ensured by a suitable relativistic coupling of the spins and by the particular form of the internal distribution of momenta. In order to incorporate some effects of the quantum fluctuations inside the meson, Horbatsch and Koniuk added a gluon to the $q\bar{q}$ state [8], and Mishra and Misra [9] introduced the gluon and quark condensates which transform the perturbative vacuum into the physical vacuum having lower energy.

A model similar to the above ones has been proposed by one of us [10] some time ago. It is based on the conjecture that, at low energy, the confinement of the quarks and the hadron interactions can be treated independently, although both of them are effects of the same elementary quark gluon interactions. This allows one to treat the confinement as a mean field effect, independently of the external interaction which is an effect of some quantum fluctuations. Specifically, it has been assumed that the quarks are independent particles whose confinement reflects in the internal wave function of the hadron. As a mean field one introduced an effective, vacuumlike field Φ representing globally the fluctuations of the quark gluonic field which cannot be described in terms of a few elementary excitations. The field Φ has some properties which resemble those of other nonperturbative contributions. For instance, it has a nonvanishing vacuum expectation value like the condensates [11], it contributes with its own energy to the hadron energy, like the bag in the bag models [12], and it can be a source of other particles like the neutral sea in quark-parton models [10,13].

The hadron momentum shares between the valence quarks and the effective field Φ and, according to the fundamental conjecture, in the absence of an external interaction, the internal distribution of momenta does not change. The external interaction as an effect of quantum fluctuations is formally described by a time translation operator $U_e(t,t')$, which produces some modifications in the distribution of flavors and momenta in the interacting hadrons. It is important to emphasize that it is of no use to write the operator $U_e(t,t')$ in terms of some elementary quark gluon Lagrangian, because at low energy it cannot be treated perturbatively. In Ref. [10] its action on the quark operators has been put in the form of a Bogoliubov transformation involving the effective field Φ in order to ensure the energy-momentum conservation.

The model succeeds in giving a unitary view of low

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energy processes involving hadrons, but the numerical results are strongly dependent on the unknown functions introduced by the model, namely the internal wave function and the Bogoliubov transformation. The remarkable feature of the processes studied in this paper is that, in the limit of a given approximation, there is no need to specify the Bogoliubov transformation. Moreover, one can define some ratios of the form factors and decay constants, which are independent of the particular internal function of the meson, making our results more reliable.

In the following section we present the essential features of the model and derive the expressions of the decay amplitudes. The comparison of our results with the experimental data is performed in Sec. III. Our conclusions are presented in Sec. IV. The Appendix contains some useful notation and integrals used throughout the paper, as well as the expression of the partial width of the weak radiative decay and the inadvertencies found by us here.

II. CALCULATION OF THE DECAY AMPLITUDES

The model used for calculating the decay amplitudes is based on the following picture we now have for hadrons. Hadrons look like bags, cavities, or bubbles containing a few bound valence quarks in equilibrium with the excitation of the surrounding quark-gluonic field. The equilibrium is stable and the internal distribution of momenta does not change as long as the hadrons remain free. During the interaction, the equilibrium breaks down, the fluctuations of the quark-gluonic field become very intense, and new states can emerge. A long experience proves that perturbation theory is unable to describe both the binding effects and the hadron interaction at low energy. This fact shows clearly that the fluctuations of the quark-gluonic field generating these effects are far from being elementary. For this reason we avoid considering some elementary excitations such as a small number of quarks or gluons aside from the valence quarks, and we introduce an effective field Φ to describe globally the fluctuations of the quark-gluonic field inside the hadrons. The field Φ has the quantum numbers of the vacuum and carries its own momentum which is not the subject of any mass shell constraint because Φ does not represent an elementary excitation.

According to the ideas outlined above, we propose the following expression for the pseudoscalar meson [10]:

$$|M_a(P)\rangle = \int d^3p\ \frac{m}{e}\ d^3q\ \frac{m'}{\varepsilon}\ d^4Q\ \delta^{(4)}(p+q+Q-P) \times\psi(p,q;Q)\times\bar{u}(\mathbf{p})\gamma_5 v(\mathbf{q})\ \chi^\dagger\lambda_a\varphi \times a^\dagger_\alpha(\mathbf{p})b^\dagger_\beta(\mathbf{q})\ \Phi^\dagger(Q)\,|0\rangle \quad (1)$$

with the notation $p^\mu=(e,\mathbf{p})$ and $q^\mu=(\varepsilon,\mathbf{q})$, $p_\mu p^\mu = m^2$, $q_\mu q^\mu = m'^2$, and where $a^\dagger_\alpha(\mathbf{p})$ and $b^\dagger_\beta(\mathbf{q})$ are the creation operators of the valence quark and antiquark whose color, spin, and flavor are denoted by the collective indices α and β. Here χ and φ denote vectors in flavor space whose coupling is described by a λ matrix. ψ represents the internal wave function of the pion. The quark creation and annihilation operators satisfy the free field canonical anticommutation relations, the only nonvanishing one being

$$\left[a^\dagger_\alpha(\mathbf{p}),a_\beta(\mathbf{q})\right]_+ = \left[b^\dagger_\alpha(\mathbf{p}),b_\beta(\mathbf{q})\right]_+ = (2\pi)^3\frac{e}{m}\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\alpha\beta}. \quad (2)$$

$\Phi^\dagger(Q)$ denotes the creation of a nonelementary excitation having the internal quantum numbers of the vacuum and carrying the momentum Q. It represents the quantum fluctuations of the quark-gluonic field in equilibrium with the valence quarks. It is an independent component of the meson, and hence it is supposed to commute with the quark operators.

The physical meaning of the effective field $\Phi^\dagger(Q)$ allows us to assume that

$$\Phi(Q_1)\Phi(Q_2)\cdots\Phi^\dagger(Q_n) = \Phi^\dagger(-Q_1-Q_2+\cdots+Q_n). \quad (3)$$

At the same time, keeping into account that in any process the conservation of the quark momentum is a consequence of the commutation relations (2) and that $\Phi(Q)$ is an independent component of the hadrons, one must impose from outside the separate conservation of its four-momentum. We assume then

$$\langle 0\,|\Phi(Q)|\,0\rangle = (2\pi)^4\mu^4\delta^{(4)}(Q), \quad (4)$$

where μ is a constant having the dimension of mass introduced for dimensional reasons and can be determined from the normalization of the pion state. Condition (4) is essential for ensuring the overall energy conservation, solving in this way one of the frequent problems of phenomenological quark models [6].

The introduction of the effective field Φ describing all the quantum fluctuations of the quark-gluonic field inside the hadrons other than the valence quarks allows us to assume safely that the valence quarks are bare particles and their masses are current masses. In this respect our model differs from many other models for soft processes where the quarks are supposed to be of constituent type and have their own dressing.

The binding effects in the expression (1) of the pion state are introduced through the internal wave function $\psi(p,q;Q)$ which must cut off the large relative momenta in order to allow the satisfaction of the orthogonality relation

$$\langle M_a(P)|M_b(P')\rangle = \delta_{ab}(2\pi)^3\ 2E\ \delta^{(3)}(\mathbf{P}'-\mathbf{P}) \times\delta(M'^2-M^2). \quad (5)$$

The factor $\delta(M'^2-M^2)$ above is due to Eq. (4) following from the nonelementary character of the phenomenological field Φ. It forces us to modify the phase space density as

$$\frac{1}{(2\pi)^3}d^3P\frac{1}{2E}\to\frac{1}{(2\pi)^3}dM\ d^3\ P\ \frac{1}{2E}\ \rho(M,M_0), \quad (6)$$

where $\rho(M,M_0)$ is the distribution function of the meson mass around a central value M_0. If $\rho(M,M_0)$ has a small width, as in our case, the integral over M in the cross section can be performed from the beginning, and one returns to the old phase space density.

Before passing to the explicit calculation one has to specify the dynamical assumptions of the model.

As stated in the Introduction, the hadron dynamics is introduced formally by means of a time translation operator $U_e(t_1,t_2)$ which describes the modifications in the hadron structure produced by an external interaction.

In the cases studied in this paper, the meson is the single hadron involved, and nothing can change its structure up to the emission of a weak or electromagnetic quanta. Therefore, the time translation operator $U_e(t_0,-\infty)$, where t_0 is the momentum of the first emission, acting on the single meson state leaves it unchanged since there was no external interaction to perturb the internal equilibrium in the quark system. It can then be replaced by unity and the same conjecture can be made on the time translation operator acting on the final state, where there are no more quarks.

In the case of the radiative decays, the emission of the second quanta takes place in a state already perturbed by the recoil of the first emission. Because of the smallness of the involved momenta we assume, however, that the perturbation of the equilibrium state can be neglected. Therefore, the time translation operator $U_e(t_2,t_1)$, describing the effects of the quantum fluctuations in the time interval between the two emissions, will be *approximated* by unity, which prevents us from specifying its action on the quark system.

The calculation is presented in detail for the pion case and is generalized to the kaon case.

(*a*) $P^\pm\to l^\pm\nu_l$. In the lowest order of the weak interaction, the S matrix element of pion decay into leptons is

$$S(\pi^\pm\to l^\pm\nu_l)=i\int d^4x\,\langle l^\pm\nu_l|\,U_e(+\infty,x_0)\times H_w(x)\,U_e(x,-\infty)|\pi^\pm(P)\rangle, \qquad (7)$$

where the weak Hamiltonian

$$\begin{aligned}H_w&=\frac{G_F}{\sqrt{2}}\cos\theta_C[J_\mu^{(q,w)+}(x)J_\nu^{(l,w)-}(x)\;g^{\mu\nu}+\text{H.c.}]\\&=\frac{G_F}{\sqrt{2}}\cos\theta_C[\bar d(x)\gamma_\mu(1-\gamma_5)u(x)\\&\quad\times\bar\nu_l(x)\gamma^\mu(1-\gamma_5)l(x)+\text{H.c.}]\end{aligned} \qquad (8)$$

is written in the limit of local quark-lepton interaction. In Eq. (8) G_F is the Fermi constant, θ_C is the Cabibbo angle, $u(x)$, $d(x)$ are the up and down quark fields, $l(x)$, $\nu_l(x)$ are the lepton fields, and $J_\mu^{(q,w)+}(x)$ and $J_\nu^{(l,w)-}(x)$ represent the weak quark and weak lepton currents, respectively. The time translation operators $U_e(+\infty,x_0)$, $U_e(x_0,-\infty)$ in Eq. (7) represent the global effect of the quantum fluctuations in the quark system due to some external interactions which are absent both in the single pion and in the final state where there are no more quarks. Then, as we have already mentioned above, it follows that the time translation operators $U_e(+\infty,x_0)$ and $U_e(x_0,-\infty)$ in Eq. (7) can be replaced by unity.

After introducing in Eq. (7) the plane wave decomposition of the free fields (A1), (A2), the expression (7) of the S matrix element can be written as

$$\begin{aligned}S(\pi^+\to l^+\nu_l)&=-i\frac{G_F}{\sqrt{2}}\cos\theta_C\;\bar u_{\nu_l}(l')\gamma^\mu\,(1-\gamma_5)\,v_l(l)\\&\quad\times\int d^4x\;e^{i(l+l')x}\\&\quad\times\langle 0\,|\bar d(x)\gamma_\mu\gamma_5u(x)|\,\pi^+(P)\rangle\,,\end{aligned} \qquad (9)$$

where $u_{\nu_l}(l')$ and $v_l(l)$ are the Dirac spinors for the leptons.

The matrix element of the quark current in Eq. (9) is calculated using the expression (1) of the pion state. One gets easily

$$\begin{aligned}\langle 0\,|\bar d(x)\gamma_\mu\gamma_5u(x)|\,\pi^+(P)\rangle&=\langle 0\,|\;A_\mu^-(x)\;|\,\pi^+(P)\rangle\\&=\int d^3p\;\frac{m_u}{e}\;d^3q\;\frac{m_d}{\epsilon}\;d^4Q\;e^{-i(p+q)x}\;\psi(p,q;Q)\delta^{(4)}(p+q+Q-P)\\&\quad\times\langle 0|\Phi^+(Q)|0\rangle\;\mathrm{Tr}\left(\frac{\hat p+m_u}{2m_u}\gamma_5\frac{\hat q-m_d}{2m_d}\gamma_\mu\gamma_5\right)\\&=e^{-iPx}\int d^3p\frac{m_u}{e}d^3q\frac{m_d}{\varepsilon}\;(2\pi)^4\mu^4\psi(p,q;0)\delta^{(4)}(p+q-P)\mathrm{Tr}\left(\frac{\hat p+m_u}{2m_u}\gamma_5\frac{\hat q-m_d}{2m_d}\gamma_\mu\gamma_5\right),\end{aligned} \qquad (10)$$

where $p^\mu=(e,\mathbf{p}),q^\mu=(\epsilon,\mathbf{q})$ are the quark and antiquark momenta, and $\hat p=\gamma_\mu p^\mu$. In Eq. (10), the condition (4) has been used. In this case it means that the annihilation into leptons of the valence quark and antiquark of the pion, proceeds in a "bare" state, i.e., in the absence of any other excitation of the quark-gluonic field. This is a natural result of the present model, which is consistent with our initial assumption that one deals with current quarks. At the same time, it marks the essential difference between the present model and the well-known model of Van Royen and Weisskopf [14], where the decay constant is proportional with the value at the origin of the internal function in configuration space.

Using the definition of the pion decay constant $F_{\pi^\pm}$, (A5), one gets, from Eq. (10),

$$F_{\pi^\pm} = C_{\pi^\pm} \frac{\sqrt{2}\pi p}{M} (m_u + m_d) \left(1 - \frac{(m_d - m_u)^2}{M^2}\right), \tag{11}$$

with

$$p = \frac{1}{2} M \sqrt{\left[1 - \frac{(m_u + m_d)^2}{M^2}\right] \left[1 - \frac{(m_u - m_d)^2}{M^2}\right]},$$

where $C_{\pi^\pm} = -3i(2\pi)^4 \mu^4 \ \psi_{\pi^\pm}(0)$, and where $\psi_{\pi^\pm}(0)$ is the value of internal wave function $\psi(p, q; Q)$ for $Q_\mu = 0$ in the $\pi^\pm$ case, $\psi_{\pi^\pm}(0) = \psi(p, q; 0)$. The factor 3 in $C_{\pi^\pm}$ is due to the quantum number of color and is present whenever one encounters a quark loop.

It is worth noticing that Eq. (10) looks like those appearing in other quark loop models, with the difference that the spin projectors of (10) are replaced there by propagators. In our model the valence quark and antiquark are real, on-mass shell particles and, because of this fact, the integral over their momenta in Eq. (10) leads to the finite result (11). In this respect the present model differs from the usual quark loop models [4], where F_π is infinite and hence cannot be calculated directly in the absence of a consistent renormalization procedure.

Proceeding in the same manner one can calculate any other matrix element of a weak current between the vacuum and a pseudoscalar meson. The neutral pion constant F_{π^0} defined by (A5), which must coincide with that of the charged meson for equal quark masses, is then

$$F_{\pi^0} = C_{\pi^0} \sum_{j=u,d} \frac{\sqrt{2}\pi p_j}{M} m_j, \tag{12}$$

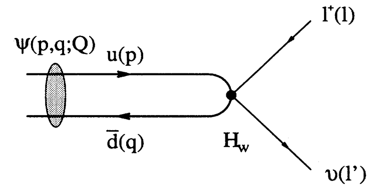

FIG. 1. Quark diagram for the pion decay constant F_π. The pion is represented by the quark and the antiquark lines. The bubble represents the internal wave function of the pion, $\psi(p, q; Q)$.

where $C_{\pi^0} = -3i(2\pi)^4 \ \mu^4 \ \psi_{\pi^0}(0)$ and $p_j = \frac{1}{2} M \sqrt{1 - (2m_j/M)^2}$, $j = u, d$.

Following Eq. (11), the charged kaon decay constant $F_{K^\pm}$ can be written in the same form:

$$F_{K^\pm} = C_{K^\pm} \frac{\sqrt{2}\pi p}{M} (m_u + m_s) \left(1 - \frac{(m_s - m_u)^2}{M^2}\right) \tag{13}$$

with $C_{K^\pm} = -3i(2\pi)^4 \mu^4 \ \psi_{K^\pm}(0)$. As currently done in quark models, one can associate a diagram like that in Fig. 1 to the expression of the matrix element (10). An essential difference is that in our diagram the quark lines correspond to projectors on the states of positive and negative energy, while in most quark-loop models the quark lines correspond to propagators.

(*b*) $P^\pm \to l^\pm \nu_l \gamma$. In the lowest order of perturbation with respect to the weak and electromagnetic interactions, the element of the S matrix for the decay of the positive charged pion can be written

$$\begin{aligned} S(\pi^+ \to l^+ \nu_l \gamma) = (-i)^2 & \int d^4x \ d^4y \\ & \times \langle l^\pm(\mathbf{l}) \nu_l(\mathbf{l}') \gamma(\mathbf{k}) \mid U_e(+\infty, x_0) \ H_w(x) \ U_e(x_0, y_0) \ H_{\rm em}(y) \ U_e(y_0, -\infty) \ \theta(x_0 - y_0) \\ & + U_e(+\infty, y_0) \ H_{\rm em}(y) \ U_e(y_0, x_0) \ H_w(x) \ U_e(x_0, -\infty) \ \theta(y_0 - x_0) \mid \pi^+(P) \rangle , \end{aligned} \tag{14}$$

where the electromagnetic Hamiltonian (written in terms of electromagnetic quark current and electromagnetic lepton current) is

$$\begin{aligned} H_{\rm em}(x) &= e_0 [J_\lambda^{(q,\rm em)}(x) \ + \ J_\lambda^{(l,\rm em)}(x)] A^\lambda(x) \\ &= e_0 \left[\sum_{i=u,d} \sigma_i \ \bar{q}_i(x) \gamma_\lambda q(x) - \ \bar{l}(x) \gamma_\lambda l(x) \right] A^\lambda(x), \end{aligned} \tag{15}$$

and σ_i denote the quark charges in terms of electric charge e_0.

Then, just as in the $P^\pm \to l^\pm \nu_l$ case, the time translation operators $U_e(+\infty, t)$ and $U_e(t, -\infty)$ can be replaced by unity.We also replace by unity the operators $U_e(x_0, y_0)$ and $U_e(y_0, x_0)$ because the perturbation produced in the equilibrium state by the emission of a quanta can be neglected, as outlined in the Introduction, due to the smallness of the involved momenta. We shall comment more about the implications of this approximation in the last section.

The calculation of the S matrix element proceeds then as usual, leading to the expression (with ϵ^λ for the photon polarization vector)

$$\begin{aligned} S(\pi^+ \to l^+ \nu_l \gamma) = -e_0 \frac{G_F}{\sqrt{2}} \cos\theta_C \ \epsilon^\lambda & \int d^4x \ d^4y \ e^{ikx} \\ & \times \left[\left\langle l^+(\mathbf{l}) \nu_l(\mathbf{l}') \left| T \left(J_\lambda^{(l,\rm em)+}(x) J_\mu^{(l,w)}(y) \right) \right| 0 \right\rangle \left\langle 0 \left| J_\nu^{(q,w)}(y) \right| \pi^+(P) \right\rangle \right. \\ & + \left. \left\langle l^+(\mathbf{l}) \nu_l(\mathbf{l}') \left| J_\mu^{(l,w)}(y) \right| 0 \right\rangle \left\langle 0 \left| T \left(J_\lambda^{(q,\rm em)}(x) J_\nu^{(q,w)}(y) \right) \right| \pi^+(P) \right\rangle \right] g^{\mu\nu}, \end{aligned} \tag{16}$$

where the first term is called the bremsstrahlung term and the second one—the structure-dependent term. The first one describes the emission of the photon by the charged lepton, while the second one describes the emission of the photon by the meson itself.

After using Wick's theorem to pass from time ordered to normal ordered products of quark operators, we can write the matrix elements involving two currents in Eq. (16) in the following manner:

$$\left\langle l^+(l)\nu_l(l')\left|T[J_\lambda^{(l,\mathrm{em})}(x)\; J_\mu^{(l,w)}(y)]\right|0\right\rangle = -i\left\langle l^+(l)\nu_l(l')\,\right|\,:\bar\nu_l(y)\;\gamma_\mu\,(1-\gamma_5)\;S_F(y-x)\;\gamma_\lambda\;l(x):\;|\,0\rangle$$
$$= -i\;\bar u(l')\gamma_\mu\,(1-\gamma_5)\;\int\frac{d^4k'}{(2\pi)^4}\frac{e^{ik'(x-y)+ilx+il'y}}{\hat k'-m_l}\;\gamma_\lambda v(l) \tag{17}$$

$$\left\langle 0\left|\;T[J_\lambda^{(q,\mathrm{em})}(x)\;J_\nu^{(q,w)}(y)]\right|\pi^+(P)\right\rangle$$
$$= i\frac{2}{3}\,\langle 0\,|\,:\bar d(y)\gamma_\nu\,(1-\gamma_5)\;\;S_F(y-x)\;\gamma_\lambda u(x)\;:|\,\pi^+(P)\rangle - i\;\frac{1}{3}\,\langle 0\,|:\bar d(x)\gamma_\lambda S_F(x-y)\gamma_\nu\,(1-\gamma_5)\,u(y):|\,\pi^+(P)\;\rangle$$
$$= i\;\int d^3p\;\frac{m_u}{e}d^3q\;\frac{m_d}{\varepsilon}\;d^4Q\;\;\langle 0\;\;|\Phi^+(Q)\;|\,0\rangle\;\;\psi(p,q;Q)\delta^{(4)}(p+q+Q-P)$$
$$\times\left[\;\frac{2}{3}\int\frac{d^4k'}{(2\pi)^4}\;e^{-ik'(y-x)-ipx-iqy}\;\mathrm{Tr}\left(\frac{\hat p+m_u}{2m_u}\gamma_5\frac{\hat q-m_d}{2m'}\gamma_\nu\,(1-\gamma_5)\,\frac{\hat k'+m_u}{k'^2-m_u^2}\gamma_\lambda\right)\right.$$
$$\left.-\frac{1}{3}\int\frac{d^4k'}{(2\pi)^4}e^{-ik'(x-y)-ipy-iqx}\mathrm{Tr}\left(\frac{\hat q-m_d}{2m_d}\gamma_\lambda\frac{\hat k'+m_d}{k'^2-m_d^2}\gamma_\nu(1-\gamma_5)\frac{\hat p+m_u}{2m_u}\gamma_5\right)\right], \tag{18}$$

where iS_F stands for the fermion propagator. The bremsstrahlung and the structure-dependent terms can be represented diagramatically as in Figs. 2(a) and 2(b).

Using (A6), (A7), (A8), and (A9) we perform the integrations over the spatial coordinates and over the internal momenta in Eqs. (16), (17), and (18) and put the amplitude of the weak radiative decay into its standard form [1]:

$$T(\pi^+\to l^+\nu_l\gamma) = e_0\;\frac{G_F}{\sqrt2}\cos\theta_C\;\epsilon^\lambda\left\{i\;m_l\sqrt2F_{\pi^+}\;\bar u_{\nu_l}(l')\left[\frac{P_\lambda}{(P\cdot k)}-\frac{2l_\lambda+\sigma_{\alpha\lambda}k^\alpha}{2(k\cdot l)}\right](1+\gamma_5)v(l)\right.$$
$$\left.+\frac{1}{M}\left[\varepsilon_{\lambda\mu\alpha\beta}k^\alpha P^\beta F_V + i\,(g_{\lambda\mu}\;P\cdot k - P_\lambda k_\mu)\,F_A\right]\;\bar u_{\nu_l}(l')\gamma^\mu\,(1-\gamma_5)\,v(l)\right\}, \tag{19}$$

where F_{π^+} is given in Eq. (11), ϵ denotes the photon polarization vector, and the form factors F_V and F_A are functions of the dynamical variables $s = P\cdot k$ and $t = k^2$. In our case where $t = 0$, they are expressed in terms of the usual adimensional variable $x = 2(P\cdot k)/M^2$ and the quark masses as

$$F_V = C_{\pi^\pm}\;\frac{1}{x}\;f_V, \tag{20a}$$

$$f_V = 2\pi\left[\frac{2}{3}\frac{m_u}{M}\ln\frac{e+p}{e-p}-\frac{1}{3}\frac{m_d}{M}\ln\frac{\varepsilon+p}{\varepsilon-p}+\frac{2p(m_d-m_u)}{M^2}\right], \tag{20b}$$

$$F_A = C_{\pi^\pm}\;\left(\frac{1}{x}f_A^{(1)}+\frac{1}{x^2}f_A^{(2)}\right), \tag{21a}$$

$$f_A^{(1)} = 2\pi\left[\frac{4p(2m_u+m_d)}{3M^2}-\frac{2p(m_u+m_d)(m_d-m_u)^2}{M^4}-\frac{2m_u}{3M}\ln\frac{e+p}{e-p}-\frac{m_d}{3M}\ln\frac{\varepsilon+p}{\varepsilon-p}\right], \tag{21b}$$

$$f_A^{(2)} = 4\pi(m_d-m_u)\left[\frac{p}{3M^2}-\frac{p(m_d^2-m_u^2)}{M^4}-\frac{1}{M^3}\left(\frac{2}{3}m_u^2\ln\frac{e+p}{e-p}-\frac{1}{3}m_d^2\ln\frac{\varepsilon+p}{\varepsilon-p}\right)\right], \tag{21c}$$

where

$$p = \frac{1}{2}M\sqrt{\left[1-\frac{(m_u+m_d)^2}{M^2}\right]\left[1-\frac{(m_u-m_d)^2}{M^2}\right]},\; e = \sqrt{p^2+m_u^2},\; \varepsilon = \sqrt{p^2+m_d^2}.$$

It is important to notice that neither the bremsstrahlung amplitude nor the structure-dependent one in Eq. (16) are separately gauge invariant. The detailed calculation shows that in our model the terms violating gauge invariance in both cases cancel each other. Gauge invariance appears then to be a direct consequence of the unitary treatment of the weak and weak radiative decay, which is a valuable feature of this model.

It must also be mentioned that the form factors (20) and (21) depend essentially on the photon energy, both of them having poles at $x = 0$, i.e., at vanishing photon energy. This leads to an infrared divergence in the axial contribution to the decay width, which requires a careful analysis of the soft photon limit.

Finally, one remarks that, due to the appearance of the same constant in the expressions (11), (20a), and (21a) the ratios F_V/F_{π^+} and F_A/F_{π^+} depend on the quark masses only. This fact is a direct consequence of the neglect of recoil effects which allowed us to replace $U_e(x_0, y_0)$ by unity in Eq. (14). This will enable us to eliminate the unknown value of the constant C_{π^+} by expressing it in terms of F_{π^+} whose value can be taken from the decay rate of the charged pion into leptons [15].

The results obtained up to now can be easily generalized to the charged kaon case, by simply replacing m_d with m_s and the pion mass with the kaon mass.

(*c*) $\pi^0 \to \gamma\gamma$. The calculation of the neutral pion decay amplitude can be treated in the same manner as we did in the case of weak radiative decay of the charged pion. Then, just as in that case, one has

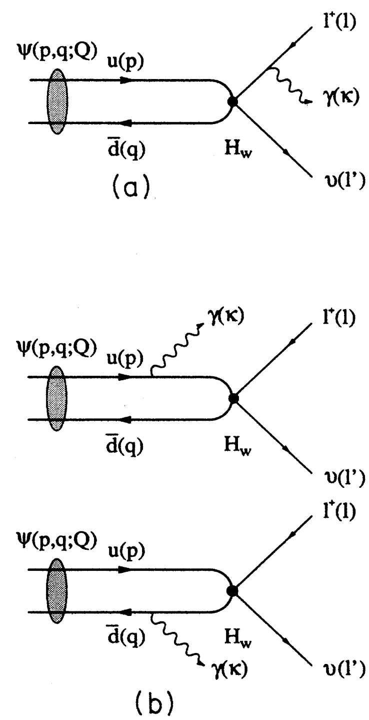

FIG. 2. (a) The diagram of the bremsstrahlung term. (b) Quark diagrams of the structure dependent term.

$$
\begin{aligned}
S(\pi^0 \to \gamma\gamma) &= (-i)^2 \int d^4x d^4y \, \langle \gamma(k_1)\gamma(k_2) \, |[U_e(+\infty, x_0) \; H_{\rm em}(x) U_e(x_0, y_0) H_{\rm em}(y) U_e(y_0, -\infty)\theta(x_0, -y_0) \\
&\quad + U_e(+\infty, y_0) H_{\rm em}(y) U_e(y_0, x_0) H_{\rm em}(x) U_e(x_0, -\infty)\theta(y_0 - x_0)]| \, \pi^0(P)\rangle \\
&= -i \; e_0^2 \int e^{ik_1x + ik_2y} \epsilon^\mu(\mathbf{k}_1)\epsilon^\nu(\mathbf{k}_2) \\
&\quad \times \left\langle 0 \left| \left[\frac{4}{9}\bar{u}(x)\gamma_\mu S_F(x-y)\gamma_\nu u(y) + \frac{1}{9}\bar{d}(x)\gamma_\mu S_F(x-y)\gamma_\nu d(y) \right] + \left[\begin{matrix} x \to y \\ \mu \to \nu \end{matrix} \right] \right| \pi^0(P) \right\rangle \\
&= -(2\pi)^4 \delta^{(4)}(P - k_1 - k_2)\sqrt{2} e_0^2 \sum_{j=u,d} \sigma_j \; i \int d^3p \frac{m_j}{e} d^3p' \frac{m_j}{e'} (2\pi)^4 \mu^4 \; \psi(p,q;0)\epsilon_j^2 \\
&\quad \times \delta^{(4)}(p + q - P) \left[\mathrm{Tr}\left(\gamma_5 \frac{\hat{p} + m_j}{2m_j} \gamma_\mu \frac{\hat{p} - \hat{k}_1 + m_j}{(p-k_1)^2 - m_j^2} \gamma_\nu \frac{\hat{p}' - m_j}{2m_j} \right) + \mathrm{Tr}\left(\begin{matrix} \mu \to \nu \\ k_1 \to k_2 \end{matrix} \right) \right],
\end{aligned}
\tag{22}
$$

where $\sigma_u = 1$ and $\sigma_d = -1$.

Writing the S matrix element of the neutral pion decay into its usual form

$$
S(\pi^0 \to \gamma\gamma) \;=\; i(2\pi)^4 \delta^{(4)}(P - k_1 - k_2) \; e_0^2 \; T(\pi^0 \to \gamma\gamma) \varepsilon_{\mu\nu\alpha\beta} \epsilon^\mu(\mathbf{k}_1)\epsilon^\nu(\mathbf{k}_2) k_1^\alpha k_2^\beta \tag{23}
$$

and performing the trace and the integration over the spatial coordinates and internal momenta, one gets, from Eqs. (22) and (23),

$$
T(\pi^0 \to \gamma\gamma) = C_{\pi^0} \frac{2\sqrt{2}\pi}{M} \left[\frac{4}{9} \frac{m_u}{M} \ln \frac{e_u + p_u}{e_u - p_u} - \frac{1}{9} \frac{m_d}{M} \ln \frac{e_d + p_d}{e_d - p_d} \right], \tag{24}
$$

where $C_{\pi^0} = -3i(2\pi)^4 \mu^4 \psi_{\pi^0}(0)$, $e_j = \frac{1}{2}M$ and $p_j = \sqrt{e_j^2 - m_j^2}$, $j = u, d$. Just as in the case of weak radiative decays, the unknown value of the constant C_{π^0} can be eliminated by using the expression (12) for F_{π^0}. The value of F_{π^0} cannot be taken directly from the experiment as it is the case for $F_{\pi^\pm}$. However, it can be extracted from the experimental value of the neutral pion decay amplitude to which it is related by the Adler-Bell-Jackiw anomaly in the soft pion limit [16].

III. RESULTS

The aim of this section is to test the validity of the model by comparing with the experimental data the values of π^0 decay amplitude, Eq. (24), and of the vector and axial form factors F_V (20), F_A (21) for $\pi^\pm$ and $K^\pm$ case.

As we said before, these quantities depend on the constant C_{π^0} or $C_{\pi^\pm}$,($C_{K^\pm}$), which represents the value of the unknown and model-dependent wave function $\psi(p,q,Q=0)$. This dependence is eliminated by using the decay constants expressions, Eqs. (12), (11), and (13), respectively, for which the numerical values are taken from experimental data. Because in our model the valence quarks are bare particles, we shall compare our theoretical results with the experimental measurements in the following range of masses: 0.5 MeV $\leq m_u \leq 3$ MeV, 3 MeV $\leq m_d \leq$ 15 MeV 150 MeV $\leq m_s \leq 300$ MeV, consistent with the range quoted by the Particle Data Group [15] for current quarks.

In Fig. 3(a) we represented the quantity $\Omega(m_u,m_d) = T(\pi^0\to\gamma\gamma)_{\rm th}/T(\pi^0\to\gamma\gamma)_{\rm expt}$ as a function of the quark mass ratio $\eta = m_d/m_u$ with m_u taken as parameter. $T(\pi^0\to\gamma\gamma)_{\rm expt}$ is about 2.72×10^{-4} MeV^{-1} [15]. The dashed line represents the perfect agreement between theory and experiment, $\Omega=1$. Each parametric curve corresponds to a given value for m_u expressed in MeV with a variation of 0.1 MeV between two successive curves. All parametric curves between the curves $m_u=0.56$ and $m_u=1.2$ can reach the value $\Omega=1$. A small variation of m_u outside this range gives a value for Ω far from experiment, due to logarithms of very small arguments in Eq. (24). (See, for example, the curve $m_u=1.36$ giving theoretical results ≈ 4 times bigger than experiment.) So, the model is very sensitive at small variations of current quark masses, giving perfect agreement with experiment for $m_u=[0.5,1.2]$ and $\eta=[5.8,6.5]$ or $m_d=[2.9,7.8]$ (MeV).

As concerns the form factors occuring in the weak radiative decay, we notice that one cannot perform a direct comparison with the experimental values because, as pointed out above, in our case the form factors F_V and F_A depend significantly, through x, on the photon energy, Eqs. (20), and (21), while the experimental data [15,17] have been fitted with some constant values.

However, in order to test the predictive power of the model, we treat these experimental values for the form factors as mean values of the real form factors over the phase space region where the measurements have been made and compare them with our theoretical mean values over the same phase space region.

Using then $\bar{\frac{1}{x}}$ instead of $\frac{1}{x}$ in Eq. (20a) we define the mean form factor $\langle F_V\rangle$. We have to evaluate the mean value of $\frac{1}{x}$ over the phase space region. Since there are two different kinematical weights SD^+ and SD^- given by (A26) and (A27) which correspond to the left and right photons, one can define two mean values $\left(\bar{\frac{1}{x}}\right)_+$ and $\left(\bar{\frac{1}{x}}\right)_-$:

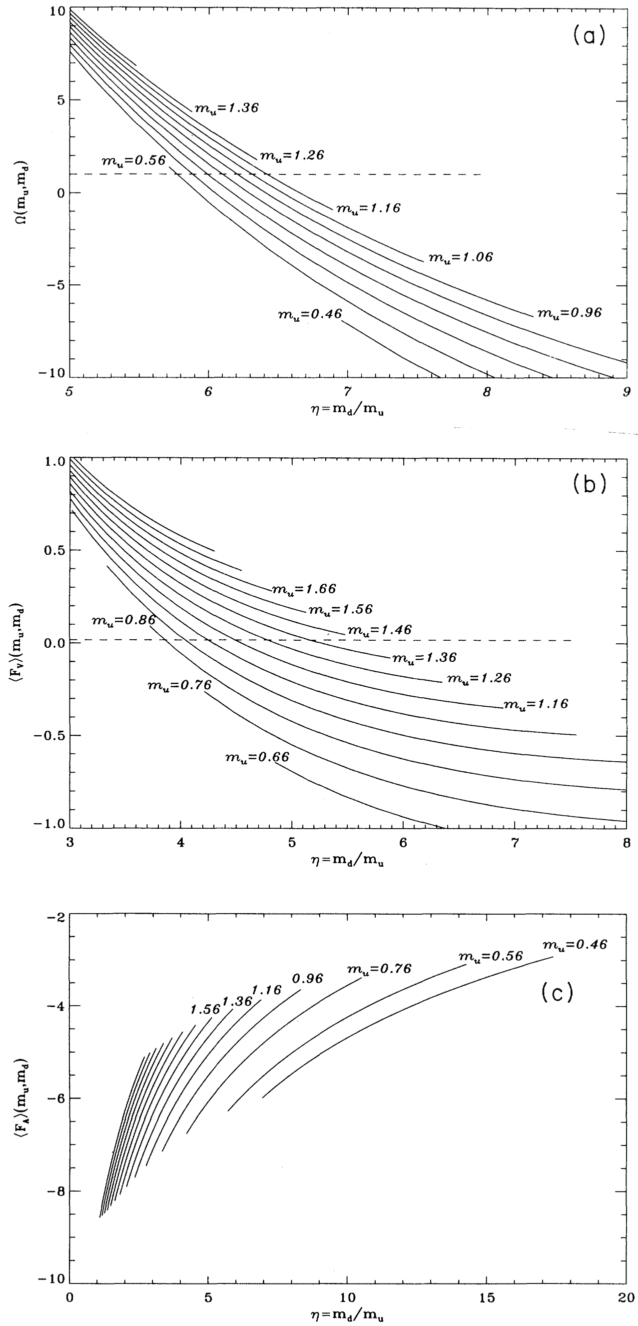

FIG. 3. (a) The function $\Omega(m_u,m_d) = T(\pi^0\to\gamma\gamma)_{th}/T(\pi^0\to\gamma\gamma)_{\rm expt}$ represented as a function of $\eta = m_d/m_u$ with m_u parameter; each curve corresponds to a constant value for m_u (MeV), written on it. The agreement with experiment, $\Omega=1$ (dashed line), exists for a continuous range of masses $m_u=[0.5,1.2]$. (b) $\pi^\pm$ case: $\langle F_V\rangle(m_u,m_d)$ represented as a function of $\eta=m_d/m_u$ with m_u parameter; each curve corresponds to a constant value for m_u (MeV), written on it. The dashed line corresponds to the experimental value 0.017 [15] for the vector form factor. (c) $\pi^\pm$ case: $\langle F_A\rangle(m_u,m_d)$ represented as a function of $\eta=m_d/m_u$ with m_u parameter; each curve corresponds to a constant value for m_u (MeV), written on it; for *unmarked* curves m_u increases by 0.2 MeV from right to left.

$$\left(\frac{1}{x}\right)^2_+ = \frac{\int\int_\Delta \frac{1}{x^2}\, SD^+(x,y)\; dx\; dy}{\int\int_\Delta\, SD^+(x,y)\; dx\; dy},$$

$$\left(\frac{1}{x}\right)^2_- = \frac{\int\int_\Delta \frac{1}{x^2}\, SD^-(x,y)\; dx\; dy}{\int\int_\Delta\, SD^-(x,y)\; dx\; dy}, \qquad (25)$$

where $x = \frac{2(P\cdot k)}{M^2}$, $y = \frac{2(P\cdot l)}{M^2}$ [1], $SD^+(x,y)$ and $SD^-(x,y)$ are given in the Appendix [(A26) and (A27)], and Δ is the volume in phase space. Following Bolotov *et al.* [17], we choose $\Delta = \{x,y|d \leq x \leq e; a+bx \leq y \leq c\}$ with $a = 1.; b = -0.8; c = 1.; d = 0.3; e = 1$, and find that $\left(\frac{1}{x}\right)_+ \approx \left(\frac{1}{x}\right)_- \approx 1.65$.

In Fig. 3 (b) we represented, in the $\pi^\pm$ case, our mean form factor $\langle F_V\rangle$ computed as we mentioned above, as a function of the quark mass ratio m_d/m_u with m_u taken as a parameter. Each curve corresponds to a constant value of m_u written on it. Again, the dashed line shows the experimental "mean" value of the vector form factor which is 0.017 [15]. The agreement with experiment is good for the range $m_u = [0.86, 1.46]$, corresponding to $\eta = [3.85, 5.7]$ or $m_d = [3.3, 8.27]$ (MeV).

We try the same procedure in the axial case where there are two contributions to the form factor: one with a simple pole and the other with a double pole at $x = 0$. Following Eq. (21a) we define the theoretical mean value of the axial form factor by replacing $\left(\frac{1}{x}\right)$ and $\left(\frac{1}{x^2}\right)$ with their mean values over Δ, $\left(\frac{1}{x}\right)$ and $\left(\frac{1}{x^2}\right)$, respectively. Defining as above $\left(\frac{1}{x^2}\right)_+$ and $\left(\frac{1}{x^2}\right)_-$ we find $\left(\frac{1}{x^2}\right)_+ \approx \left(\frac{1}{x^2}\right)_- \approx 3.15$ for the volume Δ. The mean experimental value of the axial form factor must then be compared with the theoretical mean value $\langle F_A\rangle = C_{\pi^\pm}(1.65 f_A^{(1)} + 3.15 f_A^{(2)})$.

With these assumptions, we represent in Fig. 3(c), for the $\pi^\pm$ case, the dependence of $\langle F_A\rangle$ on the ratio $\eta = m_d/m_u$, with m_u taken as a parameter. Again, each curve corresponds to a constant value of m_u, expressed in MeV; we see that it is not possible to put the theoretical prediction in agreement with the experimental data [15], using the values of the quark mass in the range predicted in [15]. The difference is about two orders of magnitude, which is quite bad.

In the kaon case the experimental data are less precise; we compare these data with the theoretical values of the sum and difference of axial and vector form factors with $x = 1$, $(\omega = \frac{1}{2}M_{K^\pm})$, for m_u and m_s in the range quoted above, 0.5 MeV $\leq m_u \leq$ 3 MeV, 150 MeV$\leq m_s \leq$ 300 MeV. In Figs. 4(a) and 4(b), the values of $\langle F_\pm\rangle$ defined as $\langle F_\pm\rangle = \langle F_A \pm F_V\rangle = C_K(f_A^{(1)} + f_A^{(2)} \pm f_V)$ are represented as functions of $\eta = m_s/m_u$ for different values of m_u (MeV). For the above mass range, our values for $\langle F_+\rangle$ [Fig. 4(a)] are in agreement with the experimental value of 0.23 (dashed line) [15]. $\langle F_-\rangle$ [Fig. 4(b)] is situated within the experimental interval which ranges from -2.2 to 0.3 muon case [15], and is below the *upper* limit of 0.49, electron case [15], giving a good agreement with experiment.

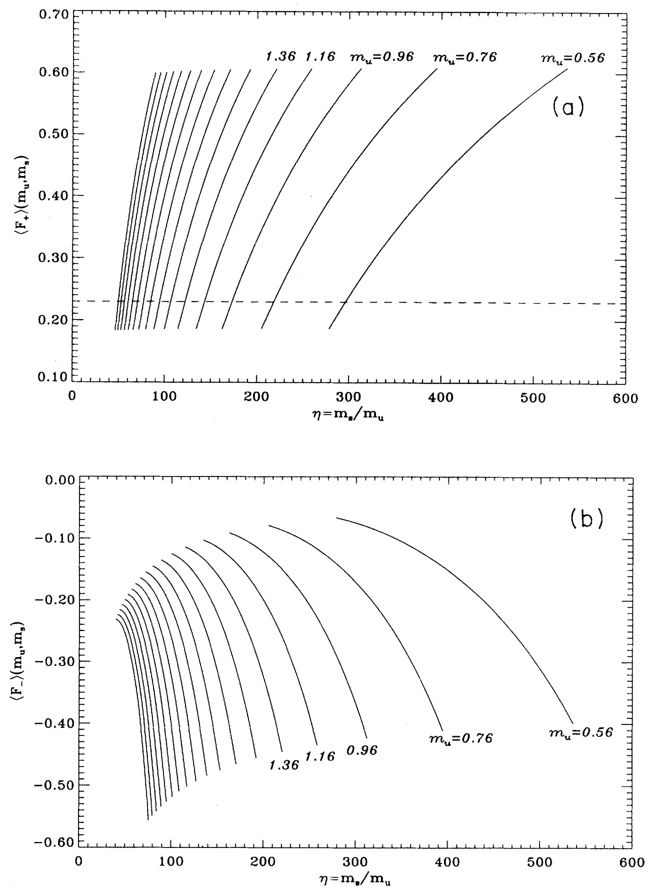

FIG. 4. (a) $K^\pm$ case: $\langle F_+\rangle\,(m_u, m_s)$ represented as a function of $\eta = m_s/m_u$ with m_u (MeV) parameter; the dashed line corresponds to the experimental value $F_A + F_V = 0.23$ [15]; for unmarked curves m_u increases from right to left by 0.2 MeV. (b) $K^\pm$ case: $\langle F_-\rangle\,(m_u, m_s)$ represented as a function of $\eta = m_s/m_u$ with m_u (MeV) parameter; the experimental data for $F_A - F_V$ range between -2.2 and 0.49 [15]; for unmarked curves m_u increases from right to left by 0.2 MeV.

IV. COMMENTS AND CONCLUSIONS

The model presented in this paper has some attractive features when compared with other quark models for meson decays. It is essentially relativistic and allows the consistent treatment of the meson structure and of the electromagnetic interaction, leading to manifest gauge invariant amplitudes for radiative decays. It also ensures the overall energy conservation which is most valuable for a phenomenological quark model [6]. At the same time, we were able to avoid the instabilities related to the choice of a particular internal function of the mesons by expressing its unknown value at $Q_\mu = 0$ entering our results, in terms of decay constants, Eqs. (11), (12), and (13), whose numerical values were taken from experiment.

In the preceding section we pointed out the existence of some narrow regions in the quark mass range where the agreement with the experimental data is satisfactory in the cases analyzed above. However, the values of quark masses found by us from the best fit are not very reliable in the absence of a careful quantitative analysis of the recoil effects on the internal equilibrium in the quark system after a first emission of a quanta. Keeping this in mind, the difference between the best values of the quark mass ratio η in the pion case is not very annoying. In this sense, the major success of the model is that the quark masses giving the best fit are compatible with the quoted values for current quark masses [15].

As concerns the disagreement observed in the pion case for the axial form factor [Fig. 3(c)], in our opinion it is mainly due to the neglect of recoil effects which are more important in the pion case due to the small quark masses. As a proof we mention the qualitative agreement observed in the kaon case, where the large mass of the strange quark diminishes the recoil effects. However, it must be emphasized that in the case of radiative decays a significant comparison with the experiment would require some more information about the real dependence of the form factors on the photon energy. In this sense, a new fit of the measurements with x-dependent form factors, including also the interference term which was neglected until now, would be most interesting and could possibly remove the uncertainty in the quoted values [17].

The last comment concerns the infrared behavior of the form factors. A careful examination of Eqs. (A24)–(A29) in the Appendix shows that only the axial contribution to the structure-dependent part leads to an infrared divergence in the expression of decay width. According to the standard procedure used in quantum electrodynamics, the infinities of this kind in the cross section are eliminated by compensation with those appearing in the interference term between the amplitude with radiative corrections and without them [19,18]. Trying a similar procedure in our case, with quarks instead of electrons, it is easy to see that the compensation mechanism does not work, since the integral over the fermion momenta must be performed in the amplitude, not in the cross section, as in QED [19]. The problem deserves a further study by using, instead of the electric charge, a more appropriate perturbation parameter.

ACKNOWLEDGMENTS

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APPENDIX

(a) Definitions and notation used throughout the paper:

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3}\, \frac{m}{e}[b(\mathbf{k})e^{-ikx}\,\bar{u}(\mathbf{k}) + b^+(\mathbf{k})\, e^{ikx}\, v(\mathbf{k})], \tag{A1}$$

$$A_\mu(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} \times[\, e^{ikx}\epsilon_\mu^{(\lambda)*}\, a_\lambda^+(\mathbf{k}) \,+\, e^{-ikx}\epsilon_\mu^{(\lambda)}a_\lambda(\mathbf{k})], \tag{A2}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=4\,(g^{\mu\nu}\; g^{\rho\sigma} \;-\; g^{\mu\rho}\; g^{\nu\sigma} \;+\; g^{\mu\sigma}\; g^{\nu\rho})\,, \tag{A3}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) \;=\; -4i\;\epsilon^{\;\mu\nu\rho\sigma}, \tag{A4}$$

$$\langle 0\,|\; A_\mu^a(0)\;|\,\pi^b(P)\rangle \;=\; \delta_{ab}\; i\; F_a\; P_\mu\;,\; a=1,...,8, \tag{A5}$$

$$\langle 0\,|A_\mu^\pm(0)|\,\pi^\pm(P)\rangle \;=\; i\,\sqrt{2}\; F_{\pi^\pm}\; P_\mu.$$

(b) Integrals: The evaluation of the structure-dependent terms arising from Eq. (18) implies the calculation of the integrals

$$\int d^3p\; \frac{m}{e}\; d^3q\; \frac{m'}{\varepsilon}\; \delta^{(4)}(p+q-P) \;=\; I_0, \tag{A6}$$

$$\int d^3p\; \frac{m}{e}\; d^3q\; \frac{m'}{\varepsilon}\; \frac{1}{2p\cdot k}\; \delta^{(4)}(p+q-P) \;=\; I, \tag{A7}$$

$$\int d^3p\; \frac{m}{e}\; d^3q\; \frac{m'}{\varepsilon}\; \frac{p_\mu}{2p\cdot k}\; \delta^{(4)}(p+q-P) \;=\; J_\mu, \tag{A8}$$

$$\int d^3p\; \frac{m}{e}\; d^3q\; \frac{m'}{\varepsilon}\; \frac{p_\mu p_\nu}{2p\cdot k}\; \delta^{(4)}(p+q-P) \;=\; K_{\mu\nu}, \tag{A9}$$

with $p^\mu = (e,\mathbf{p})$, $q^\mu = (\varepsilon,\; \mathbf{q})$, $p^\mu p_\mu = m^2$, $q^\mu q_\mu = m'^2$. Elementary calculations give

$$I_0 \;= \frac{4\pi\; p\; m\; m'}{M}, \tag{A10}$$

$$I \;= \frac{\pi\; m\; m'}{M\;\omega}\ln\frac{e+p}{e-p}, \tag{A11}$$

where ω is the energy of the real photon, $e = \frac{1}{2}M(1+\frac{m^2-m'^2}{M^2})$ and $p=\sqrt{e^2-m^2}$.

Using the criterion of covariance we write the integrals J_μ as

$$J_\mu \;=\; A\; P_\mu \;+\; B\; k_\mu. \tag{A12}$$

We observe that

$$P^\mu\; J_\mu \;=\; M\; e\; I \;=\; M^2\; A \;+\; M\;\omega\; B, \tag{A13}$$

$$k^\mu\; J_\mu \;=\; \frac{1}{2}\; I_0 \;=\; M\;\omega\; A \tag{A14}$$

and obtain

$$J_\mu \;=\; \frac{1}{2\omega}\; I_0\; \frac{P_\mu}{M} \;+\; \left(eI - \frac{1}{2\omega}I_0\right)\; \frac{k_\mu}{\omega}. \tag{A15}$$

In a similar way, using covariance, one has

$$K_{\mu\nu} \;=\; aP_\mu P_\nu + b\;(P_\mu k_\nu + k_\mu P_\nu) + ck_\mu k_\nu + d\; g_{\mu\nu} \tag{A16}$$

and, proceeding as above, one gets, for $k^2=0$,

$$P^\mu K_{\mu\nu} = MeI_\nu = (M^2a + M\omega b + d)\;P_\nu + (M^2b + M\omega c)\;k_\nu, \tag{A17}$$

$$k^\mu K_{\mu\nu} = \frac{P_\nu}{2M} I_0 = M\omega\, a\, P_\nu + (M\omega b + d)\, k_\nu, \tag{A18}$$

$$g^{\mu\nu} K_{\mu\nu} = m^2 I = M^2 a + 2M\omega b + 4d. \tag{A19}$$

We solve the above system and write the integrals $K_{\mu\nu}$ as

$$K_{\mu\nu} = \frac{e}{2\omega} I_0 \frac{P_\mu}{M}\frac{P_\nu}{M} + \left(\frac{e}{4\omega} I_0 - \frac{m^2}{2} I\right) \times \left(\frac{P_\mu}{M}\frac{k_\nu}{\omega} + \frac{k_\mu}{\omega}\frac{P_\nu}{M} - g_{\mu\nu}\right) + \left[\left(e^2 + \frac{m^2}{2}\right) I - \frac{3e}{4\omega} I_0\right] \frac{k_\mu}{\omega}\frac{k_\nu}{\omega}. \tag{A20}$$

(*c*) $\pi^+ \to l^+\nu\gamma$ decay width: Following the classic paper of Brown and Bludman [20] and Ref. [1], the amplitude of the $\pi^+ \to l^+\nu\gamma$ decay becomes

$$T = T_{IB} + T_{\mathrm{SDV}} + T_{\mathrm{SDA}},$$

where

$$T_{IB} = ie_0 \frac{G_F \cos\theta_C}{\sqrt{2}} m_l \sqrt{2}\, F_{\pi^+} \bar{u}_{\nu_l} \times \left(\frac{P_\lambda}{P\cdot k} - \frac{2l_\lambda + \sigma_{\mu\lambda}k^\mu}{2k\cdot l}\right) \times \epsilon^\lambda\, (1+\gamma_5)\, v_l, \tag{A21}$$

$$T_{\mathrm{SDV}} = \frac{e_0 G_F \cos\theta_C}{\sqrt{2}M} L^\mu\, \epsilon^\nu\, F_V\, \epsilon_{\mu\nu\rho\sigma} P^\rho k^\sigma \tag{A22}$$

$$T_{\mathrm{SDA}} = \frac{ie_0 G_F \cos\theta_C}{\sqrt{2}M} L^\mu\, \epsilon^\nu\, F_A\, [P\cdot k\, g_{\mu\nu} - k_\mu P_\nu], \tag{A23}$$

with $L^\mu = \bar{u}_{\nu_l}(\mathrm{l}')\gamma^\mu(1-\gamma_5)v_l(\mathrm{l})$ and $\sigma_{\lambda\mu} = \frac{1}{2}[\gamma_\lambda, \gamma_\mu]$.

Taking the square of the modulus of the above amplitude and summing over the polarizations one gets the expression of the partial decay rate [1,19]:

$$\frac{d^2\Gamma_{\pi^+\to l^+\nu\gamma}}{dx\, dy} = \frac{d^2\Gamma_{IB}}{dx\, dy} + \frac{d^2\Gamma_{SD}}{dx\, dy} + \frac{d^2\Gamma_{\mathrm{int}}}{dx\, dy},$$

where

$$\frac{d^2\Gamma_{IB}}{dx\, dy} = \frac{\alpha\Gamma_{\pi\to l\nu}}{2\pi(1-r)} IB,$$

$$\frac{d^2\Gamma_{\mathrm{int}}}{dx\, dy} = \frac{\alpha}{2\pi}\frac{1}{\sqrt{2}F_{\pi^\pm}} \Gamma_{\pi^+\to l^+\nu} [(F_V - F_A)\,\mathcal{F} + (F_V + F_A)\,\mathcal{G}],$$

$$\frac{d^2\Gamma_{SD}}{dx\, dy} = \frac{\alpha}{16\pi} \Gamma_{\pi\to l\nu} \frac{1}{r(1-r)^2}\left(\frac{M}{F_\pi}\right)^2 \times [(F_V - F_A)^2\, SD^+ + (F_V + F_A)^2\, SD^-], \tag{A24}$$

with the notation

$$IB = \frac{1-y+r}{x^2(x+y-1-r)} \times \left[x^2 + 2(1-x)(1-r) - \frac{2xr(1-r)}{x+y-1-r}\right], \tag{A25}$$

$$SD^+ = (x+y-1-r)[(x+y-1)(1-x) - r], \tag{A26}$$

$$SD^- = (1-y+r)[(1-x)(1-y) + r], \tag{A27}$$

$$\mathcal{F} = \frac{1-y+r}{x(x+y-1-r)} [(1-x)(1-x-y) + r], \tag{A28}$$

$$\mathcal{G} = \frac{1-y+r}{x(x+y-1-r)} [x^2 - (1-x)(1-x-y) - r], \tag{A29}$$

where $r = (m_l/M)^2$, $x = 2(P\cdot k)/M^2$, and $y = 2(P\cdot l)/M^2$.

We point out here two inadvertencies appearing in the expression of the partial decay width quoted in Refs. [1,15,20].

The first one concerns the sign of the last term in the square brackets in IB, which is different from that appearing in Ref. [20] and the forthcoming papers. This difference is negligible in the case of pion decay into light leptons, but it can be important in other decays involving heavier leptons. The second one concerns the interchange of SD^+ with SD^- and of $\mathcal{F}$ with $\mathcal{G}$ in (A24) when passing from π^+ decay to π^- decay (Ref. [20]). Reference [15] does not mention this.

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