Perturbative Ward identity in finite-temperature quantum electrodynamics

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The Ward identity (zero momentum transfer) is examined up to two loop order in finite-temperature quantum electrodynamics. It is shown that, to one loop order, zero momentum transfer limits of the vertex part have different limit values depending on the approach and the Ward identity does not hold in the spacelike limit; to two loop order, the vertex part diverges (simple pole) in the limit and the Ward identity does not hold.

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I. INTRODUCTION

In the previous work [1] we calculated the two loop order self-energies of finite temperature ϕ^4 and ϕ^3 coupling theory using the imaginary time formalism, and have shown that there is a singularity (simple pole) at the zero momentum limit of the ϕ^3 coupling case. Such a divergence does not exist in one loop order, although there appear discrepancies between the spacelike limit and timelike limit [2].

In this work we study this problem on the vertex part of finite temperature quantum electrodynamics, especially in the case that the photon momentum goes to zero. Before proceeding to our calculations, we review the mathematical structure of the appearance of a pole in two loop order while not in one loop. In our imaginary time calculations we decompose each propagators into partial fractions:

$$\frac{1}{q^2-m^2}=\frac{1}{2E_q}\left(\frac{1}{q_0-E_q}-\frac{1}{q_0+E_q}\right),$$

$$E_q=\sqrt{\mathbf{q}^2+m^2},\quad q_0=2ni\pi T\ (n=0,\pm1,\pm2,\ldots),$$

where T is temperature. Thus, in one loop calculations of Fig. 1(a) we encounter the summation

$$\sum_{n=-\infty}^{\infty}\frac{1}{n+x_1}\frac{1}{n+x_2}=\pi\frac{\cot(\pi x_1)-\cot(\pi x_2)}{x_2-x_1},\tag{1}$$

where $1/(n+x_1)$ and $1/(n+x_2)$ emerge from propagators $1/(q^2-m^2)$ and $1/[(p-q)^2-m^2]$, respectively. After analytical continuations, if external momentum p goes to zero, x_2 goes to x_1 and Eq. (1) gives the finite limit. In the two loop calculation of Fig. 1(b) we encounter the summation

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left(\frac{1}{n+x_1}\right)^2\frac{1}{n+x_2}\frac{1}{n+x_3}&=\frac{1}{x_2-x_1}\sum_n\left(\frac{1}{n+x_1}\right)^2\frac{1}{n+x_3}-\frac{1}{x_2-x_1}\sum_n\frac{1}{n+x_1}\frac{1}{n+x_2}\frac{1}{n+x_3}\\&=\frac{1}{x_2-x_1}\left\{\frac{1}{x_3-x_1}\frac{\pi^2}{\sin^2(\pi x_1)}+\pi\frac{\cot(\pi x_3)-\cot(\pi x_1)}{(x_3-x_1)^2}\right.\\&\left.\quad+\pi\frac{\cot(\pi x_1)-\cot(\pi x_2)}{(x_3-x_2)(x_1-x_2)}+\pi\frac{\cot(\pi x_1)-\cot(\pi x_3)}{(x_2-x_3)(x_1-x_3)}\right\},\end{aligned}\tag{2}$$

where $1/(n+x_3)$ comes from the inner loop. When $p\to0$, $x_2\to x_1$ but x_3 does not vary, and Eq. (2) leads to the simple pole.

In contrast, in calculations of Fig. 1(c) we encounter the summation

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}\frac{1}{n+x_1}\frac{1}{n+x_2}\frac{1}{n+x_3}\frac{1}{n+x_4}&=\frac{\pi}{(x_2-x_1)(x_4-x_3)}\left\{\frac{\cot(\pi x_1)-\cot(\pi x_3)}{x_3-x_1}-\frac{\cot(\pi x_1)-\cot(\pi x_4)}{x_4-x_1}\right.\\&\left.\quad-\frac{\cot(\pi x_2)-\cot(\pi x_3)}{x_3-x_2}+\frac{\cot(\pi x_2)-\cot(\pi x_4)}{x_4-x_2}\right\}.\end{aligned}\tag{3}$$

In this case, when $p\to0$, x_2 goes to x_1 and x_4 goes to x_3 and the limit is finite.

Thus, we may infer that, when the external momentum goes to zero, there appears a pole if three propagators coincide, but if only two coincide, no pole appears.

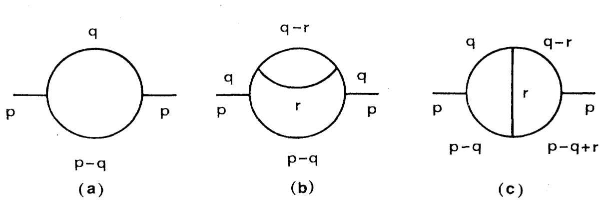

FIG. 1. One loop and two loop self-energies of ϕ^3 coupling scalar theory.

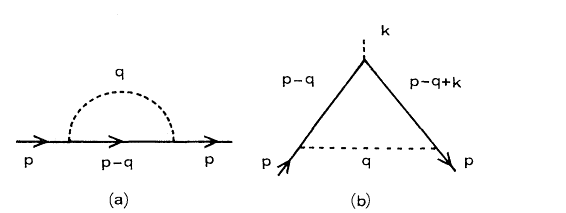

FIG. 2. One loop electron self-energy and one loop photon-electron vertex part.

II. WARD IDENTITY IN ONE LOOP ORDER

At first we calculate one loop contributions to the electron self-energy and the vertex part in finite temperature quantum electrodynamics, as shown in Fig. 2. As usual we decompose each propagator into partial fractions:

$$S_F(p) = \frac{1}{2E_p}\left(\frac{\gamma_0 E_p - \gamma_i p_i + m}{p_0 - E_p} + \frac{\gamma_0 E_p + \gamma_i p_i - m}{p_0 + E_p}\right),$$

$$D_F(k) = \frac{1}{2E_k}\left(\frac{1}{k_0 - E_k} - \frac{1}{k_0 + E_k}\right),$$

$$E_p = \sqrt{\mathbf{p}^2 + m^2},\ \ E_k = \sqrt{\mathbf{k}^2}$$

for the electron and photon, respectively. We take Feynman gauge and zero chemical potential. After rather lengthy calculations we obtain the following results.

For the electron self-energy of Fig. 2(a),

$$\Sigma(p) = \int d^3\mathbf{q}\frac{1}{4E_qE_{p-q}}\left\{\gamma_\alpha X(p-q)\gamma_\alpha\left[\frac{\tanh(E_{p-q}/2T) - \coth(E_q/2T)}{p_0 + E_q - E_{p-q}} - \frac{\tanh(E_{p-q}/2T) + \coth(E_q/2T)}{p_0 - E_q - E_{p-q}}\right]\right.$$
$$\left. + \gamma_\alpha \bar{X}(p-q)\gamma_\alpha\left[\frac{\tanh(E_{p-q}/2T) - \coth(E_q/2T)}{p_0 - E_q + E_{p-q}} - \frac{\tanh(E_{p-q}/2T) + \coth(E_q/2T)}{p_0 + E_q + E_{p-q}}\right]\right\} \tag{4}$$

where $X(p) = \gamma_0 E_p - \gamma_i p_i + m$, and $\bar{X}(p) = \gamma_0 E_p + \gamma_i p_i - m$.

For the vertex part of Fig. 2(b),

$$\Lambda_\mu(p,k) = e^2\int d^3\mathbf{q}\frac{1}{8E_qE_{p-q}E_{p-q+k}}$$
$$\times\left\{\frac{Y_1^\mu}{E_{p-q+k} - E_{p-q} - k_0}\left[\frac{\coth(E_q/2T) + \tanh(E_{p-q}/2T)}{p_0 - E_q - E_{p-q}} - \frac{\coth(E_q/2T) + \tanh(E_{p-q+k}/2T)}{p_0 + k_0 - E_q - E_{p-q+k}}\right.\right.$$
$$\left. + \frac{\coth(E_q/2T) - \tanh(E_{p-q}/2T)}{p_0 + E_q - E_{p-q}} - \frac{\coth(E_q/2T) - \tanh(E_{p-q+k}/2T)}{p_0 + k_0 + E_q - E_{p-q+k}}\right]$$
$$+\frac{Y_2^\mu}{E_{p-q+k} + E_{p-q} + k_0}\left[-\frac{\coth(E_q/2T) + \tanh(E_{p-q}/2T)}{p_0 - E_q - E_{p-q}} + \frac{\coth(E_q/2T) - \tanh(E_{p-q+k}/2T)}{p_0 + k_0 + E_{p-q+k}}\right.$$
$$\left. - \frac{\coth(E_q/2T) - \tanh(E_{p-q}/2T)}{p_0 + E_q - E_{p-q}} + \frac{\coth(E_q/2T) + \tanh(E_{p-q+k}/2T)}{p_0 + k_0 + E_q + E_{p-q+k}}\right]$$
$$+\frac{Y_3^\mu}{E_{p-q+k} + E_{p-q} - k_0}\left[\frac{\coth(E_q/2T) - \tanh(E_{p-q}/2T)}{p_0 - E_q + E_{p-q}} - \frac{\coth(E_q/2T) + \tanh(E_{p-q+k}/2T)}{p_0 + k_0 - E_q - E_{p-q+k}}\right.$$
$$\left. + \frac{\coth(E_q/2T) + \tanh(E_{p-q}/2T)}{p_0 + E_q + E_{p-q}} - \frac{\coth(E_q/2T) - \tanh(E_{p-q+k}/2T)}{p_0 + k_0 + E_q - E_{p-q+k}}\right]$$
$$+\frac{Y_4^\mu}{E_{p-q+k} - E_{p-q} + k_0}\left[-\frac{\coth(E_q/2T) - \tanh(E_{p-q}/2T)}{p_0 - E_q + E_{p-q}} + \frac{\coth(E_q/2T) - \tanh(E_{p-q+k}/2T)}{p_0 + k_0 - E_q + E_{p-q+k}}\right.$$
$$\left.\left. - \frac{\coth(E_q/2T) + \tanh(E_{p-q}/2T)}{p_0 + E_q + E_{p-q}} + \frac{\coth(E_q/2T) + \tanh(E_{p-q+k}/2T)}{p_0 + k_0 + E_q + E_{p-q+k}}\right]\right\}, \tag{5}$$

where

$$Y_1^\mu = \gamma_\alpha X(p-q)\gamma_\mu X(p-q+k)\gamma_\alpha,\ \ Y_2^\mu = \gamma_\alpha X(p-q)\gamma_\mu \bar{X}(p-q+k)\gamma_\alpha,$$
$$Y_3^\mu = \gamma_\alpha \bar{X}(p-q)\gamma_\mu X(p-q+k)\gamma_\alpha,\ \ Y_4^\mu = \gamma_\alpha \bar{X}(p-q)\gamma_\mu \bar{X}(p-q+k)\gamma_\alpha$$

Using these expressions we can examine the Ward identity in the limit $k \to 0$. Equations (4) and (5) can be easily separated to temperature-dependent and -independent parts:

$$\Sigma(p) = \Sigma^0(p) + \Sigma^T(p),\ \ \Lambda_\mu(p,k) = \Lambda_\mu^0(p,k) + \Lambda^T(p,k).$$

In the limit $k \to 0$, the temperature-dependent part of the vertex function approaches a different value in the spacelike limit (first $k_0 \to 0$ and then $\mathbf{k} \to 0$) and timelike limit ($\mathbf{k} \to 0$ then $k_0 \to 0$).

In the timelike limit we obtain the relation

$$\Lambda_0^T(p,0) = -\frac{\partial}{\partial p_0}\Sigma^T(p); \tag{6}$$

however, in the spacelike limit we get

$$\Lambda_0^T(p,0) = -\frac{\partial}{\partial p_0}\Sigma^T(p) + e^2\int d^3\mathbf{q}\frac{n_F'(E_{p-q})}{E_{p-q}}\left\{\frac{\gamma_0 E_{p-q} + 3\gamma_i(p-q)_i + 2m}{(p_0 + E_{p-q})^2 - E_q^2} + \frac{\gamma_0 E_{p-q} - 3\gamma_i(p-q)_i - 2m}{(p_0 - E_{p-q})^2 - E_q^2}\right\}, \tag{7}$$

where n_F' is the differentiation of Fermi-Dirac distribution function with respect to its argument. As for the temperature-independent part we can confirm the usual zero temperature Ward identity.

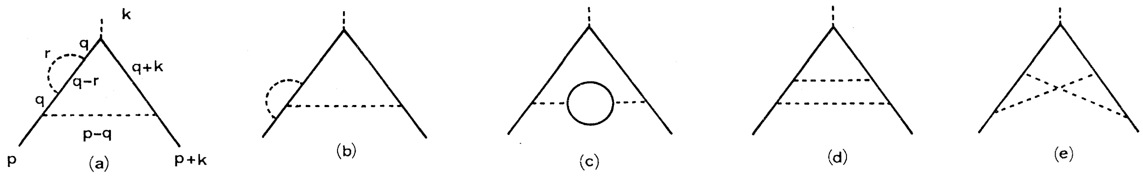

FIG. 3. Two loop photon-electron vertex part.

III. TWO LOOP CALCULATIONS

In two loop order, there exist five diagrams for the vertex part, which are shown in Fig. 3. There may arise divergence in the zero limit of external photon momentum. According to the inference stated in the Introduction, only Fig. 3(a) would have a singularity in the $k \to 0$ limit. In other graphs only two propagators coincide in the $k \to 0$ limit. In fact, we calculated all graph contributions in Fig. 3 and, after rather lengthy calculations, confirmed that contributions from Figs. 3(b)–3(e) do not lead to a divergence. In the intermediate stage of calculations, there appear several singular terms; however, in the final results they all cancel out. We show the explicit expression of the contribution from graph Fig. 3(a) in the Appendix. The singular part is given as follows.

In the spacelike limit (see Appendix),

$$\Lambda_\mu(p,0)_{\text{singular}} \sim \frac{e^4}{2^8}\int d^3\mathbf{q}\int d^3\mathbf{r}\frac{1}{E_q^3 E_{q-r}E_r}\left\{\left(\lim_{k\to 0}\frac{1}{E_{q+k}-E_q}\right)\right\}\frac{\operatorname{sech}^2(E_q/2T)}{2T} \times\sum_{i=1}^{4}\left\{\frac{Z_{1i}^\mu(k=0)}{(E_q+E_i)[(p_0-E_q)^2-E_{p-q}^2]}+\frac{Z_{8i}^\mu(k=0)}{(E_q-E_i)[(p_0+E_q)^2-E_{p-q}^2]}\right\} \tag{8}$$

and, in the timelike limit,

$$\Lambda_\mu(p,k=0)_{\text{singular}} \sim \frac{-e^4}{2^8}\int d^3\mathbf{q}\int d^3\mathbf{r}\frac{1}{E_q^3 E_{q-r}E_r}\left\{\lim_{k_0\to 0}\frac{1}{k_0}\right\} \times\left\{\frac{\operatorname{sech}^2(E_q/2T)}{T}\sum_{i=1}^{4}\left[\frac{Z_{1i}^\mu(k=0)}{(E_q+E_i)[(p_0-E_q)^2-E_{p-q}^2]}-\frac{Z_{8i}^\mu(k=0)}{(E_q+E_i)[(p_0-E_q)^2-E_{p-q}^2]}\right]\right. \left.+\frac{\tanh(E_q/2T)}{E_q}\sum_i\left[\frac{Z_{4i}^\mu(k=0)}{(E_i-E_q)[(p_0+E_q)^2-E_{p-q}^2]}+\frac{Z_{5i}^\mu(k=0)}{(E_i+E_q)[(p_0-E_q)^2-E_{p-q}^2]}\right]\right\}. \tag{9}$$

Thus, in this order the Ward identity cannot hold because the electron self-energy $\Sigma(p)$ does not show any singular behavior for finite p.

APPENDIX

Here we show the explicit expression of the contribution from Fig. 3(a):

$$\Lambda_\mu^{(a)}(p,k) = \frac{e^4}{2^8}\int d^3\mathbf{q}\int d^3\mathbf{r}\frac{1}{E_q^2 E_{q-r}E_r E_{q+k}E_{p-q}} \times\sum_{i=1}^{4}\left\{Z_{1i}^\mu\left[\frac{1}{2T\cosh^2(E_q/2T)(p_0+E_{p-q}-E_q)(k_0-E_{q+k}+E_q)(E_i+E_q)}\right.\right. -\frac{\tanh(E_q/2T)+\tanh(E_i/2T)}{(p_0+E_{p-q}+E_i)(k_0-E_{q+k}-E_i)(E_i+E_q)^2} +\frac{\tanh(E_q/2T)-\tanh(E_{q+k}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(k_0-E_i-E_{q+k})(k_0+E_q-E_{q+k})^2} \left.+\frac{\tanh(E_q/2T)-\coth(E_{p-q}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(p_0+E_i+E_{p-q})(p_0-E_q+E_{p-q})^2}\right] +Z_{2i}^\mu\{\text{all terms for } Z_{1i}^\mu \text{with } E_{q+k}\to -E_{q+k}\} +Z_{3i}^\mu\left[\frac{\tanh(E_q/2T)+\tanh(E_i/2T)}{(p_0+E_{p-q}+E_i)(k_0-E_{q+k}-E_i)(E_i+E_q)(E_q-E_i)}\right. +\frac{\tanh(E_q/2T)}{E_q(p_0+E_{p-q}+E_q)(k_0-E_{q+k}-E_q)(E_i-E_q)} +\frac{\tanh(E_q/2T)-\tanh(E_{q+k}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(k_0-E_i-E_{q+k})(k_0-E_q-E_{q+k})(k_0+E_q-E_{q+k})}$$

$$
+\frac{\tanh(E_q/2T)-\coth(E_{p-q}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(p_0+E_i+E_{p-q})(p_0+E_q+E_{p-q})(p_0-E_q+E_{p-q})}\Bigg]
$$

$$
+Z^\mu_{4i}\{(\text{1st term for } Z_{3i})+(2,3,4\text{th terms with } E_{q+k}\to -E_{q+k}\}
$$

$$
+Z^\mu_{5i}\Bigg[\frac{\tanh(E_q/2T)+\tanh(E_i/2T)}{(p_0+E_{p-q}+E_i)(k_0-E_{q+k}-E_i)(E_i+E_q)(E_q-E_i)}
+\frac{\tanh(E_q/2T)}{E_q(p_0+E_{p-q}-E_q)(k_0-E_{q+k}+E_q)(E_i+E_q)}
-\frac{\tanh(E_q/2T)+\tanh(E_{q+k}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(k_0-E_i-E_{q+k})(k_0+E_q-E_{q+k})(k_0-E_q-E_{q+k})}
-\frac{\tanh(E_q/2T)+\coth(E_{p-q}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(p_0+E_i+E_{p-q})(p_0-E_q+E_{p-q})(p_0+E_q+E_{p-q})}\Bigg]
$$

$$
+Z^\mu_{6i}\{-(\text{1st term for } Z^\mu_{5i})+(2,3,4\text{th terms with } E_{q+k}\to -E_{q+k}\}
$$

$$
+Z^\mu_{7i}\Bigg[\frac{1}{2T\cosh^2(E_q/2T)(p_0+E_{p-q}+E_q)(k_0-E_{q+k}-E_q)(E_i-E_q)}
+\frac{\tanh(E_q/2T)-\tanh(E_i/2T)}{(p_0+E_{p-q}+E_i)(k_0-E_{q+k}-E_i)(E_q-E_i)^2}
-\frac{\tanh(E_q/2T)+\tanh(E_{q+k}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(k_0-E_i-E_{q+k})(k_0-E_q-E_{q+k})^2}
-\frac{\tanh(E_q/2T)+\coth(E_{p-q}/2T)}{(p_0+k_0+E_{p-q}-E_{q+k})(p_0+E_i+E_{p-q})(p_0+E_q+E_{p-q})^2}\Bigg]
$$

$$
+Z^\mu_{8i}\{\text{all terms for } Z^\mu_{7i} \text{ with } E_{q+k}\to -E_{q+k}\}-\text{terms with } E_{p-q}\to -E_{p-q}\Bigg\}
$$

where

$$
Z^\mu_{1i}=\gamma_\alpha X(q)\gamma_\beta W_i\gamma_\beta X(q)\gamma_\mu X(q+k)\gamma_\alpha,\quad Z^\mu_{2i}=\gamma_\alpha X(q)\gamma_\beta W_i\gamma_\beta X(q)\gamma_\mu \bar{X}(q+k)\gamma_\alpha,
$$
$$
Z^\mu_{3i}=\gamma_\alpha X(q)\gamma_\beta W_i\gamma_\beta \bar{X}(q)\gamma_\mu X(q+k)\gamma_\alpha,\quad Z^\mu_{4i}=\gamma_\alpha X(q)\gamma_\beta W_i\gamma_\beta \bar{X}(q)\gamma_\mu \bar{X}(q+k)\gamma_\alpha,
$$
$$
Z^\mu_{5i}=\gamma_\alpha \bar{X}(q)\gamma_\beta W_i\gamma_\beta X(q)\gamma_\mu X(q+k)\gamma_\alpha,\quad Z^\mu_{6i}=\gamma_\alpha \bar{X}(q)\gamma_\beta W_i\gamma_\beta X(q)\gamma_\mu \bar{X}(q+k)\gamma_\alpha,
$$
$$
Z^\mu_{7i}=\gamma_\alpha \bar{X}(q)\gamma_\beta W_i\gamma_\beta \bar{X}(q)\gamma_\mu X(q+k)\gamma_\alpha,\quad Z^\mu_{8i}=\gamma_\alpha \bar{X}(q)\gamma_\beta W_i\gamma_\beta \bar{X}(q)\gamma_\mu \bar{X}(q+k)\gamma_\alpha,
$$
$$
W_1=-X(q-r)\{\tanh(E_{q-r}/2T)+\coth(E_r/2T)\},\quad W_2=X(q-r)\{\tanh(E_{q-r}/2T)-\coth(E_r/2T)\},
$$
$$
W_3=\bar{X}(q-r)\{\tanh(E_{q-r}/2T)-\coth(E_r/2T)\},\quad W_4=-\bar{X}(q-r)\{\tanh(E_{q-r}/2T)+\coth(E_r/2T)\},
$$
$$
E_1=-E_r-E_{q-r},\quad E_2=E_r-E_{q-r}, E_3=-E_r+E_{q-r},\quad E_4=E_r+E_{q-r}.
$$

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[2] H. A. Welden, Phys. Rev. D **47**, 544 (1993).