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One-loop QED vertex in any covariant gauge: Its complete analytic form

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The one-loop vertex in QED is calculated in arbitrary covariant gauges as an analytic function of its momenta. The vertex is decomposed into a longitudinal part, which is fully responsible for ensuring that the Ward and Ward-Takahashi identities are satisfied, and a transverse part. The transverse part is decomposed into eight independent components each being separately free of kinematic singularities in *any* covariant gauge in a basis that modifies that proposed by Ball and Chiu. Analytic expressions for all 11 components of the $O(\alpha)$ vertex are given explicitly in terms of elementary functions and one Spence function. These results greatly simplify in particular kinematic regimes.

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I. INTRODUCTION

This paper presents the calculation of the one-loop vertex in QED in an arbitrary covariant gauge. Why should we want to compute this? Needless to say, it is the interactions that determine the structure and properties of any theory. In QED the fermion-boson vertex is this basic interaction. Many, if not most, physical phenomena are controlled by kinematic regimes in which the interactions become strong. This determines, for instance, the spectrum of hadrons and the nature of confinement in QCD or the existence of e^+e^- bound states in the strong electromagnetic fields of heavy nuclei. Such phenomena can only be studied by nonperturbative techniques using (as appropriate) the Schwinger-Dyson or Bethe-Salpeter equations in the continuum or on the lattice. In undertaking studies of the nonperturbative nature of gauge theories [1], we immediately have to confront the issue of what is the nonperturbative form of the fundamental fermion-boson interaction. *Ansätze* for this are needed to accomplish a truncation of the hierarchy of the field equations that are the Schwinger-Dyson equations. It is known that the much used *rainbow* approximation with its bare vertex, γ^μ, while seductively simple, fails to respect the gauge invariance and multiplicative renormalizability so crucial in determining the structure of the theory and the characteristics of observables. Thus one must seek more sophisticated *Ansätze* that do respect these key properties [2–5].

The only truncation of the complete set of Schwinger-Dyson equations that we know of that maintains the gauge invariance and multiplicative renormalizability of a gauge theory at every level of approximation is perturbation theory. Physically meaningful solutions of the Schwinger-Dyson equations must agree with perturbative results in the weak-coupling regime. Perturbation theory can thus serve as a guide to allowed nonperturbative forms. To be concrete, we know that the complete fermion propagator, S_F, of momentum p involves two functions of p^2. This follows from the spin structure of the fermion propagator. These two can be chosen to be $F(p^2)$, the wave function renormalization, and $M(p^2)$, the mass function, so that

$$iS_F(p) = i\frac{F(p^2)}{\not p - M(p^2)} \ . \tag{1}$$

[This can be (and is often) written in a variety of other ways, e.g., $S_F(p)^{-1} = \alpha(p^2)\not p + \beta(p^2)$, etc., always involving two independent scalar functions.] Since $S_F(p)$ is a gauge-variant quantity, these functions $F(p^2)$, $M(p^2)$ will in general depend on the gauge. They can be calculated, in principle, at each order in perturbation theory. At lowest order $F(p^2) = 1$, $M(p^2) = m$, the bare mass. Now these same functions must also occur in the fermion-boson vertex, since the Ward-Takahashi identity relates the three-point Green's function to the fermion propagator in a well-known way. This is satisfied at every order of perturbation theory. Indeed, such identities are true nonperturbatively. Thanks to the work of Ball and Chiu [4] we know how to express the nonperturbative structure of the part of the vertex (a part conventionally called the longitudinal component) that satisfies the Ward-Takahashi identity in terms of the two nonperturbative functions describing the fermion propagator. We have also learned that multiplicative renormalizability of the fermion propagator imposes further constraints on the vertex but these have yet to be fully exploited, though a start has been made [2,3,6]. While the bare fermion-boson vertex in a minimal coupling gauge theory is simply γ^μ, in general the vertex involves 12 spin amplitudes that can be constructed from γ^μ and the 2 independent four-momenta at the vertex as elucidated by Bernstein [7]. This would suggest that the complete fermion-boson vertex involved a large number of independent functions. However, some of these at least must be related to the fermion functions $F(p^2)$, $M(p^2)$, not to mention the analogous boson renormalization function $G(p^2)$. It is to the nature of these forms that perturbation theory can be a guide, but only if we calculate in an arbitrary gauge. For instance, if we calculated the vertex in massless QED merely in the Landau gauge we would find the γ^μ component was like its bare form just γ^μ. This would serve little as a pointer to the form $\frac{1}{2}[F^{-1}(k^2)+F^{-1}(p^2)]\gamma^\mu$ as its nonperturbative structure. Only by calculating the vertex in an arbitrary gauge does this result become clearer. Ball and Chiu have performed this $O(\alpha)$ calculation of the vertex in the Feynman gauge and we will be able to check their result and correct a couple of minor misprints in their published work.

Thus our aim is to compute the fermion-boson vertex to one loop in perturbation theory in any covariant gauge and to decompose it into its 12 spin components, of these all but 11 are nonzero. This full vertex is by its very nature free of kinematic singularities. We then divide the vertex into two parts: the longitudinal and transverse pieces. The longitudinal component alone satisfies the Ward-Takahashi and Ward identities. The way to ensure this without introducing kinematic singularities was fully described by Ball and Chiu. We then investigate the transverse part and decompose it into the basis of eight vectors proposed by Ball and Chiu [4]. We examine each coefficient of these and find that two have singularities in arbitrary gauges. These are not present in the Feynman gauge in which Ball and Chiu work. We propose a straightforward modification of their basis that ensures each transverse component is separately free of kinematic singularities in any covariant gauge. This makes this basis a natural one for future nonperturbative studies. We divide the discussion into five parts.

The one-loop calculation of the vertex, its decomposition into spin amplitudes and the expression of these in terms of known functions, including one Spence function with ten different arguments are all presented in Sec. II.

The one-loop calculation of the fermion propagator to determine the functions $F(p^2)$, $M(p^2)$, which fix the $O(\alpha)$ longitudinal part of the vertex is in Sec. III A.

The extraction of the transverse part of the one-loop vertex and its decomposition into 8 independent components in the Ball-Chiu basis are described in the rest of Sec. III.

Checking the singularity structure of each of the components of the vertex is given in Sec. IV. This leads to the proposal of a new basis for the transverse vertex, which has coefficients that have only the singularities of the full vertex. We deduce the form of the vertex in specific kinematic regimes in Sec. IV C. In Sec. V we give our brief conclusions.

II. PERTURBATIVE CALCULATION

A. Definitions: Feynman rules and basis vectors

For the most part the definitions given here are standard, but they are stated here to make this paper self-contained. The perturbative calculation involves the use of bare quantities defined as follows in Minkowski space:

$$\text{bare vertex}: \quad -ie\Gamma^0_\mu = ie\gamma^0_\mu\ , \tag{2}$$

$$\text{fermion propagator}: \quad iS^0_F(p) = i(\not p + m)/(p^2 - m^2)\ , \tag{3}$$

$$\text{photon propagator}: \quad i\Delta^0_{\mu\nu}(p) = -i[p^2 g_{\mu\nu} + (\xi - 1)p_\mu p_\nu]/p^4\ , \tag{4}$$

where e is the usual QED coupling and the parameter ξ specifies the covariant gauge.

The vertex, Fig. 1, $\Gamma^\mu(k,p)$ can be expressed in terms of 12 spin amplitudes formed from the vectors γ^μ, k^μ, p^μ and the spin scalars $1, \not k, \not p$, and $\not k \not p$. Thus we can write

$$\Gamma^\mu = \sum_{i=1}^{12} P^i V_i^\mu\ , \tag{5}$$

where we number the V_i^μ as

$$\begin{aligned} &V_1^\mu = k^\mu \not k,\quad V_2^\mu = p^\mu \not p,\quad V_3^\mu = k^\mu \not p,\quad V_4^\mu = p^\mu \not k\ ,\\ &V_5^\mu = \gamma^\mu \not k \not p,\quad V_6^\mu = \gamma^\mu,\quad V_7^\mu = k^\mu,\quad V_8^\mu = p^\mu\ ,\\ &V_9^\mu = p^\mu \not k \not p,\quad V_{10}^\mu = k^\mu \not k \not p,\quad V_{11}^\mu = \gamma^\mu \not k,\quad V_{12}^\mu = \gamma^\mu \not p\ . \end{aligned} \tag{6}$$

The vertex satisfies the Ward-Takahashi identity

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p)\ , \tag{7}$$

where $q = k - p$, and the Ward identity

$$\Gamma^\mu(p,p) = \frac{\partial}{\partial p^\mu} S_F^{-1}(p) \tag{8}$$

as the nonsingular $k \to p$ limit of Eq. (7). With the fermion propagator given to any order by Eq. (1), we follow Ball and Chiu and define the longitudinal component of the vertex by

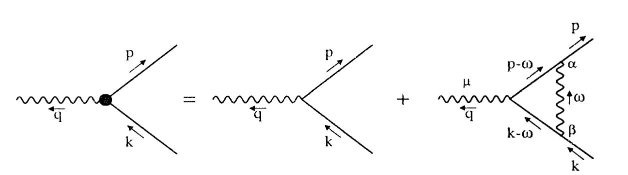

FIG. 1. The fermion-boson vertex to one-loop order showing the definition of momenta and Lorentz indices.

$$\Gamma_L^\mu = \frac{\gamma^\mu}{2}\left(\frac{1}{F(k^2)}+\frac{1}{F(p^2)}\right) + \frac{1}{2}\frac{(\not p+\not k)(k+p)^\mu}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right) - \frac{(p+k)^\mu}{k^2-p^2}\left(\frac{M(k^2)}{F(k^2)}-\frac{M(p^2)}{F(p^2)}\right) . \tag{9}$$

Γ_L^μ alone then satisfies the Ward-Takahashi identity, Eq. (7), and being free of kinematic singularities the Ward identity, Eq. (8), too.

The full vertex can then be written as

$$\Gamma^\mu(k,p) = \Gamma_T^\mu(k,p) + \Gamma_L^\mu(k,p) , \tag{10}$$

where the transverse part satisfies

$$q_\mu \Gamma_T^\mu(k,p) = 0 \text{ and } \Gamma_T^\mu(p,p) = 0 . \tag{11}$$

The Ward-Takahashi identity fixes four coefficients of the 12 spin amplitudes in terms of the fermion functions—the three combinations explicitly given in Eq. (9), while the coefficient of $\sigma_{\mu\nu}k^\mu p^\nu$ must be zero [4]. The transverse component $\Gamma_T^\mu(k,p)$ thus involves eight vectors, which can be expressed in Ball-Chiu form:[1]

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau^i(k^2,p^2,q^2) T_i^\mu(k,p) , \tag{12}$$

where

$$\begin{aligned} T_1^\mu &= p^\mu(k\cdot q) - k^\mu(p\cdot q) , \\ T_2^\mu &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)](\not k+\not p) , \\ T_3^\mu &= q^2\gamma^\mu - q^\mu \not q , \\ T_4^\mu &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)]k^\lambda p^\nu \sigma_{\lambda\nu} , \\ T_5^\mu &= q_\nu \sigma^{\nu\mu} , \\ T_6^\mu &= \gamma^\mu(p^2-k^2) + (p+k)^\mu \not q , \\ T_7^\mu &= \tfrac{1}{2}(p^2-k^2)[\gamma^\mu(\not p+\not k) - p^\mu - k^\mu] \\ &\quad + (k+p)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} , \\ T_8^\mu &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not p - p^\mu \not k , \end{aligned} \tag{13}$$

with

$$\sigma_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu] . \tag{14}$$

The coefficients τ_i are Lorentz scalar functions of k and p, i.e., functions of k^2, p^2, q^2.

A general constraint on the eight τ_i's comes from C-parity transformations. The full vertex must transform under charge conjugation C in the same way as the bare vertex [7,8], so that

$$C\Gamma_\mu(k,p)C^{-1} = -\Gamma_\mu^T(-p,-k) . \tag{15}$$

From the Ward-Takahashi identity, Eq. (7), it is clear that $\Gamma_L^\mu(k,p)$ must be symmetric under $k \leftrightarrow p$ interchange. The symmetry of the transverse part depends on its γ-matrix structure. Thus Eq. (15) together with

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \tag{16}$$

we have, from Eqs. (12) and (13),

$$\tau_i(k^2,p^2,q^2) = \tau_i(p^2,k^2,q^2) \text{ for } i=1,2,3,4,5,7,8 ,$$

$$\tau_6(k^2,p^2,q^2) = -\tau_6(p^2,k^2,q^2) . \tag{17}$$

B. The one-loop calculation

The vertex of Fig. 1 is naturally expressed as

$$\Gamma^\mu(k,p) = \gamma^\mu + \Lambda^\mu(k,p) . \tag{18}$$

Analogous to Eq. (5) we will express Λ^μ as

$$\Lambda^\mu(k,p) = \sum_{i=1}^{12} P_1^i V_i^\mu , \tag{19}$$

where the subscript on the P^i indicates this calculation is only to first order in α.

From the Feynman rules specified in Sec. II A, Λ^μ to $O(\alpha)$ is simply given by

$$-ie\Lambda^\mu = \int_M \frac{d^4w}{(2\pi)^4}(-ie\gamma^\alpha)iS_F^0(p-w)(-ie\gamma^\mu)iS_F^0(k-w)(-ie\gamma^\beta)i\Delta^0_{\alpha\beta}(w) , \tag{20}$$

where M denotes the loop integration is to be performed in Minkowski space. Substituting Eqs. (3) and (4) for $S_F^0(p)$, we have, with $\alpha \equiv e^2/4\pi$,

$$\Lambda^\mu = -\frac{ie^2}{16\pi^4}\int_M d^4w\,\gamma^\alpha \frac{\not p - \not w + m}{(p-w)^2-m^2}\gamma^\mu \frac{\not k - \not w + m}{(k-w)^2-m^2}\gamma^\beta \left[\frac{g_{\alpha\beta}}{w^2} + (\xi-1)\frac{w_\alpha w_\beta}{w^4}\right] , \tag{21}$$

[1]Ball and Chiu [4] use a different notation for the momenta in Fig. 1. They have p, p' as the incoming and outgoing fermion momenta and q as the incoming boson momentum. They define their inverse fermion propagator $S_F^{-1}(p) = \not p F(p^2) + G(p^2)$ and what we here call the τ_i they denote by A_i.

$$\Lambda^\mu = -\frac{i\alpha}{4\pi^3}\int_M d^4w \frac{\gamma^\alpha(\not p - \not w + m)\gamma^\mu(\not k - \not w + m)\gamma_\alpha}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}$$
$$-\frac{i\alpha}{4\pi^3}(\xi-1)\int_M d^4w \frac{\not w(\not p - \not w + m)\gamma^\mu(\not k - \not w + m)\not w}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{22}$$

on separating the $g_{\alpha\beta}$ and $w_\alpha w_\beta$ parts of the photon propagator.

What makes the present calculation in an arbitrary covariant gauge significantly longer and more complicated than that of Ball and Chiu in the Feynman gauge ($\xi = 1$) is the form of the photon propagator Eq. (4). The decomposition of the loop integrals of Eqs. (20)–(22) into scalar forms in the general case brings greater complexity because of the potential appearance of infrared divergences in Eq. (22). Nevertheless, having the Feynman gauge calculation is a most helpful check on our results.

Our first step is to perform a little γ matrix algebra and rewrite Eq. (22) as

$$\Lambda^\mu = -\frac{i\alpha}{4\pi^3}\int_M d^4w\left\{\frac{A^\mu}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]} + (\xi-1)\frac{B^\mu}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}\right\}, \tag{23}$$

where

$$A^\mu = \gamma^\alpha(\not p - \not w)\gamma^\mu(\not k - \not w)\gamma_\alpha + m\gamma^\alpha[(\not p - \not w)\gamma^\mu + \gamma^\mu(\not k - \not w)]\gamma_\alpha + m^2\gamma^\alpha\gamma^\mu\gamma_\alpha, \tag{24}$$

$$B^\mu = \not w(\not p - \not w)\gamma^\mu(\not k - \not w)\not w + m\not w[(\not p - \not w)\gamma^\mu + \gamma^\mu(\not k - \not w)]\not w + m^2\not w\gamma^\mu\not w. \tag{25}$$

To proceed we introduce the following seven basic integrals over the loop momentum d^4w: $J^{(0)}$, $J^{(1)}_\mu$, $J^{(2)}_{\mu\nu}$, I_0, $I^{(1)}_\mu$, $I^{(2)}_{\mu\nu}$, and $K^{(0)}$;

$$J^{(0)} = \int_M d^4w \frac{1}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{26}$$
$$J^{(1)}_\mu = \int_M d^4w \frac{w_\mu}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{27}$$
$$J^{(2)}_{\mu\nu} = \int_M d^4w \frac{w_\mu w_\nu}{w^2[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{28}$$
$$I^{(0)} = \int_M d^4w \frac{1}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{29}$$
$$I^{(1)}_\mu = \int_M d^4w \frac{w_\mu}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{30}$$
$$I^{(2)}_{\mu\nu} = \int_M d^4w \frac{w_\mu w_\nu}{w^4[(p-w)^2-m^2][(k-w)^2-m^2]}, \tag{31}$$
$$K^{(0)} = \int_M d^4w \frac{1}{[(p-w)^2-m^2][(k-w)^2-m^2]}. \tag{32}$$

Λ^μ of Eq. (22) can then be reexpressed in terms of five of these as

$$\Lambda^\mu = -\frac{i\alpha}{4\pi^3}\{\left[\gamma^\alpha\left(\not p\gamma^\mu\not k + m\not p\gamma^\mu + m\gamma^\mu\not k + m^2\gamma^\mu\right)\gamma_\alpha\right]J^{(0)}$$
$$-\left[\gamma^\alpha\left(\not p\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\not k + m\gamma^\nu\gamma^\mu + m\gamma^\mu\gamma^\nu\right)\gamma_\alpha\right]J^{(1)}_\nu + \gamma^\alpha\gamma^\nu\gamma^\mu\gamma^\lambda\gamma_\alpha J^{(2)}_{\nu\lambda}$$
$$+(\xi-1)[(-\gamma^\nu\not p\gamma^\mu - \gamma^\mu\not k\gamma^\nu - m\gamma^\mu\gamma^\nu - m\gamma^\nu\gamma^\mu)J^{(1)}_\nu + \gamma^\mu K^{(0)}$$
$$+\left(\gamma^\nu\not p\gamma^\mu\not k\gamma^\lambda + m\gamma^\nu\not p\gamma^\mu\gamma^\lambda + m\gamma^\nu\gamma^\mu\not k\gamma^\lambda + m^2\gamma^\nu\gamma^\mu\gamma^\lambda\right)I^{(2)}_{\nu\lambda}]\}. \tag{33}$$

Our next step is to compute the basic integrals of Eqs. (26)–(32) [9–11], each of which is a function of k and p. We relegate to Appendix A the tabulation of each of the intermediate integrals.

1. $J^{(1)}_\mu$ and $J^{(2)}_{\mu\nu}$ calculated

The method of relating Lorentz vector and tensor integrals to scalar integrals is by now standard [4], so we will not dwell on this but merely give one example to serve as a reminder to the reader. $J^{(1)}_\mu$ of Eq. (27) can as a Lorentz vector only have components in the directions of the four-moments k_μ and p_μ. Thus we can write

$$J_\mu^{(1)} = \frac{i\pi^2}{2}[k_\mu J_A(k,p) + p_\mu J_B(k,p)] \ , \tag{34}$$

where J_A, J_B must be scalar functions of k and p. The factor of $i\pi^2/2$ is taken out purely for later convenience. It is then easy to see that

$$J_A(k,p) = \frac{1}{i\pi^2\Delta^2}[2k\cdot pp^\mu J_\mu^{(1)} - 2p^2k^\mu J_\mu^{(1)}] \ , \tag{35}$$

with a similar expression for J_B, where

$$\Delta^2 = (k\cdot p)^2 - k^2p^2 \tag{36}$$

is the ubiquitous triangle function of k, p, and q. One then rewrites the integrand numerators using, for instance,

$$2p\cdot w = p^2 + w^2 - m^2 - [(p-w)^2 - m^2] \ , \tag{37}$$

so that, in d dimensions,

$$\begin{aligned} J_A(k,p) = \frac{1}{i\pi^2\Delta^2}\Bigg\{ & [k\cdot p(p^2-m^2) - p^2(k^2-m^2)]\int \frac{d^dw}{w^2[(k-w)^2-m^2][(p-w)^2-m^2]} \\ & +(k\cdot p - p^2)\int \frac{d^dw}{[(k-w)^2-m^2][(p-w)^2-m^2]} \\ & -k\cdot p\int \frac{d^dw}{w^2[(k-w)^2-m^2]} + p^2\int \frac{d^dw}{w^2[(p-w)^2-m^2]}\Bigg\} \ . \end{aligned} \tag{38}$$

The basic 16 scalar integrals, of which 4 appear in this equation, $Q_7(k,p)$, $Q_8(k,p)$, $Q_{14}(k,p)$, and $Q_1(p)$, are given in Appendix A. We thus deduce

$$J_A(k,p) = \frac{1}{\Delta^2}\left\{\frac{J_0}{2}(-m^2p\cdot q - p^2k\cdot q) + k\cdot pL' - p^2L - 2p\cdot qS\right\} \ , \tag{39}$$

$$J_B(k,p) = J_A(p,k) \ , \tag{40}$$

where

$$J^{(0)} = \frac{i\pi^2}{2}J_0 \ , \tag{41}$$

$$L = \left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right) \ , \tag{42}$$

$$L' = L(p\leftrightarrow k) \ , \tag{43}$$

$$S = \frac{1}{2}\left(1 - 4\frac{m^2}{q^2}\right)^{1/2}\ln\frac{[(1-4m^2/q^2)^{1/2}+1]}{[(1-4m^2/q^2)^{1/2}-1]} \ , \tag{44}$$

with J_0 being expressed in terms of Spence functions [9–12]—see Appendix A, Eqs. (A15)–(A18). In an analogous fashion, the tensor integral $J_{\mu\nu}^{(2)}$ of Eq. (28) can be expressed in terms of scalar integrals K, J_C, J_D, and J_E by

$$\begin{aligned} J_{\mu\nu}^{(2)} = \frac{i\pi^2}{2}\Bigg\{ & \frac{g_{\mu\nu}}{d}K_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{4}\right)J_C \\ & + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}\frac{(k\cdot p)}{2}\right)J_D \\ & + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{4}\right)J_E\Bigg\} \ . \end{aligned} \tag{45}$$

All but $K(k,p)$ are ultraviolet finite and so the number of dimensions d has been set equal to 4. In $d \equiv 4+\epsilon$ dimensions, with μ the usual scale parameter introduced to ensure the coupling α remains dimensionless for any d, we have

$$K_0(k,p) = \frac{2}{i\pi^2}K^{(0)} \ , \tag{46}$$

$$K_0(k,p) = 2\mu^\epsilon[C - 2S + 2] \ , \tag{47}$$

where

$$C = -\frac{2}{\epsilon} - \gamma - \ln(\pi) - \ln(m^2/\mu^2) \ . \tag{48}$$

Then

$$J_C(k,p) = \frac{1}{4\Delta^2}\left\{\left(2p^2 + 2k\cdot p\frac{m^2}{k^2}\right) - 4k\cdot pS + 2k\cdot p\left(1-\frac{m^2}{k^2}\right)L' + [2k\cdot p(p^2-m^2) + 3p^2(m^2-k^2)]J_A + p^2(m^2-p^2)J_B\right\}, \quad (49)$$

$$J_D(k,p) = \frac{1}{4\Delta^2}\left\{2k\cdot p[(k^2-m^2)J_A + (p^2-m^2)J_B - 1] - k^2\left[2\frac{m^2}{k^2} - 2S + \left(1-\frac{m^2}{k^2}\right)L' + (p^2-m^2)J_A\right] - p^2\left[-2S + \left(1-\frac{m^2}{p^2}\right)L + (k^2-m^2)J_B\right]\right\}, \quad (50)$$

$$J_E(k,p) = J_C(p,k)\ , \quad (51)$$

all of which involve the previously defined J_A, J_B, L, L', and S of Eqs. (42)–(44).

2. $I_\mu^{(1)}$ and $I_{\mu\nu}^{(2)}$ calculated

In a way analogous to the computation of $J_\mu^{(1)}$ and $J_{\mu\nu}^{(2)}$ the ultraviolet finite integrals $I_\mu^{(1)}$ and $I_{\mu\nu}^{(2)}$ [11] of Eqs. (30), (31) can be reexpressed in terms of scalar integrals, I_A, I_B, I_C, I_D, I_E, that in turn involve the same functions we have already computed. Thus

$$I_\mu^{(1)} = \frac{i\pi^2}{2}[k_\mu I_A(k,p) + p_\mu I_B(k,p)]\ , \quad (52)$$

where

$$I_A(k,p) = \frac{1}{\Delta^2}\left\{-\frac{k\cdot q}{2}J_0 - \frac{2q^2}{\chi}\{(m^2-p^2)k^2 - (m^2-k^2)k\cdot p\}S + \frac{1}{(m^2-p^2)}\left[p^2 - k\cdot p + \frac{p^2q^2}{\chi}(k^2-m^2)(m^2+k\cdot p)\right]L + \frac{k^2q^2}{\chi}(m^2+k\cdot p)L'\right\} \quad (53)$$

and

$$I_B(k,p) = I_A(p,k)\ , \quad (54)$$

with the denominator

$$\begin{aligned}\chi &= (q^2-2m^2)(p^2-m^2)(k^2-m^2) + m^2(p^2-m^2)^2 + m^2(k^2-m^2)^2\\ &= p^2k^2q^2 + 2[(p^2+k^2)k\cdot p - 2p^2k^2]m^2 + m^4q^2\ ,\end{aligned} \quad (55)$$

$$I_{\mu\nu}^{(2)} = \frac{i\pi^2}{2}\left\{\frac{g_{\mu\nu}}{4}J_0 + \left(k_\mu k_\nu - g_{\mu\nu}\frac{k^2}{4}\right)I_C + \left(p_\mu k_\nu + k_\mu p_\nu - g_{\mu\nu}\frac{(k\cdot p)}{2}\right)I_D + \left(p_\mu p_\nu - g_{\mu\nu}\frac{p^2}{4}\right)I_E\right\}\ , \quad (56)$$

$$I_C(k,p) = \frac{1}{4\Delta^2}\left\{2p^2J_0 - 4\frac{k\cdot p}{k^2}\left(1+\frac{m^2}{(k^2-m^2)}L'\right) + \{2k\cdot p - 3p^2\}J_A - p^2J_B + \{-2k\cdot p(m^2-p^2) + 3p^2(m^2-k^2)\}I_A + p^2(m^2-p^2)I_B\right\}, \quad (57)$$

$$I_D(k,p) = \frac{1}{4\Delta^2}\left\{-2(k\cdot p)J_0 + 2\left(1+\frac{m^2}{(k^2-m^2)}L'\right) + 2\left(1+\frac{m^2}{(p^2-m^2)}L\right) + (2k\cdot p - k^2)J_A + (2k\cdot p - p^2)J_B + [k^2(m^2-p^2) - 2k\cdot p(m^2-k^2)]I_A + [p^2(m^2-k^2) - 2k\cdot p(m^2-p^2)]I_B\right\}, \quad (58)$$

$$I_E(k,p) = I_C(p,k) \ . \tag{59}$$

The $1/\chi$ terms in I_A, I_B, I_C, and I_D arise from the extra $1/w^2$ factor that occurs in the second integral of Eq. (23). Notice that the $1/\chi$ term arises in all but the Feynman gauge. The possibility of singularities at $\chi = 0$ has consequences as we shall see later.

3. Λ^μ collected

In terms of the basic functions J_0, J_A, J_B, J_C, J_D, J_E, I_A, I_B, I_C, I_D, I_E, and the ultraviolet divergent K_0, all of which depend on the momenta k and p, i.e., are functions of the Lorentz scalars k^2, p^2, and q^2, Λ^μ can be written completely with its γ matrix and Lorentz index structure displayed explicitly:

$$\Lambda^\mu(k,p) = \sum_{i=1}^{12} \overline{P}_1^i V_i^\mu \ , \tag{19'}$$

where

$$\begin{aligned}
\overline{P}_1^i &= \frac{\alpha}{4\pi} P_1^i \ ,\\
P_1^1 &= 2J_A - 2J_C + (\xi-1)(m^2 I_C + p^2 I_D) \ ,\\
P_1^2 &= 2J_B - 2J_E + (\xi-1)(k^2 I_D + m^2 I_E) \ ,\\
P_1^3 &= -2J_0 + 2J_A + 2J_B - 2J_D + (\xi-1)\left(-\frac{J_0}{2} - \frac{k^2}{2} I_C - k\cdot p I_D + m^2 I_D + \frac{p^2}{2} I_E + J_A\right) \ ,\\
P_1^4 &= -2J_D + (\xi-1)(k^2 I_C + m^2 I_D - J_A) \ ,\\
P_1^5 &= J_0 - J_A - J_B + (\xi-1)\left(\frac{J_0}{4} + \frac{k^2}{4} I_C + \frac{k\cdot p}{2} I_D + \frac{p^2}{4} I_E - \frac{1}{2} J_A - \frac{1}{2} J_B\right) \ ,\\
P_1^6 &= \left[-m^2 J_0 - k^2 J_A - p^2 J_B + \frac{k^2}{2} J_C + k\cdot p J_D + \frac{p^2}{2} J_E + \frac{1}{2}\left(1 + \frac{3\epsilon}{4}\mu^\epsilon\right) K_0\right]\\
&\quad +(\xi-1)\left(-m^2\frac{J_0}{4} - \frac{m^2}{4} k^2 I_C - \frac{m^2}{2} k\cdot p I_D - \frac{m^2}{4} p^2 I_E - \frac{k^2}{2} J_A - \frac{p^2}{2} J_B + \mu^\epsilon[C + 2 - 2S]\right) \ ,\\
P_1^7 &= 2mJ_0 - 4mJ_A + (\xi-1)m\left(\frac{J_0}{2} - 2k\cdot p I_C + \frac{k^2}{2} I_C - p^2 I_D - k\cdot p I_D - \frac{p^2}{2} I_E - J_A\right) \ ,\\
P_1^8 &= 2mJ_0 - 4mJ_B + (\xi-1)m\left(\frac{J_0}{2} + \frac{k^2}{2} I_C - k\cdot p I_D + k^2 I_D - \frac{p^2}{2} I_E - J_B\right) \ ,\\
P_1^9 &= (\xi-1)m(I_D + I_E) \ ,\\
P_1^{10} &= (\xi-1)m(I_D + I_C) \ ,\\
P_1^{11} &= (\xi-1)m\left(p^2 I_D + k\cdot p I_C + \frac{p^2}{2} I_E - \frac{k^2}{2} I_C\right) \ ,\\
P_1^{12} &= (\xi-1)m\left(-k^2 I_D - k\cdot p I_E + \frac{p^2}{2} I_E - \frac{k^2}{2} I_C\right) \ .
\end{aligned} \tag{60}$$

Notice that both the integrals I_A, I_B cancel out in this result. Though this expression appears to involve all 12 spin vectors, one of their coefficients is not independent. The Ward-Takahashi identity, Eq. (7), only involves $\not k, \not p, 1$ as spin structure on the right-hand side. This means that $\not k\not p$ and $\not p\not k$ terms that occur in $q_\mu\Gamma^\mu$ from Eq. (7) must occur in the form of the anticommutator $\{\not k, \not p\} = 2k\cdot p$. Consequently, the coefficients P_i of Eq. (19) are related by

$$P_1^{12} = P_1^9(p^2 - k\cdot p) + P_1^{10}(k\cdot p - k^2) - P_1^{11} \ . \tag{61}$$

Formally, this completes our calculation of the one-loop corrections to the QED vertex in any covariant gauge.

III. ANALYTIC STRUCTURE OF THE VERTEX

A. Longitudinal vertex

As explained in Sec. II A, the longitudinal component of the vertex is determined by the fermion functions, $F(p^2)$, $M(p^2)$, thanks to the Ward-Takahashi identity. In this section we compute these functions to $O(\alpha)$ by calculating the one-loop corrections to the fermion propagator, Fig. 2. Straightforward calculation yields

$$F^{-1}(p^2) = 1 + \frac{\alpha\xi}{4\pi}\left[C\mu^\epsilon + \left(1+\frac{m^2}{p^2}\right)(1-L)\right] , \tag{62}$$

$$M(p^2) = m + \frac{\alpha m}{\pi}\left[\left(1+\frac{\xi}{4}\right) + \frac{3}{4}(C\mu^\epsilon - L) - \frac{\xi}{4}\frac{m^2}{p^2}(1-L)\right] , \tag{63}$$

with the same factors C and L of Eqs. (48) and (42). Simple substitution into Eq. (9) gives the longitudinal vertex, which we write out as

$$\begin{aligned}\Gamma^\mu_L = {}& \frac{\alpha\xi}{8\pi}\gamma^\mu\left[2C\mu^\epsilon + \left(1+\frac{m^2}{k^2}\right)(1-L') + \left(1+\frac{m^2}{p^2}\right)(1-L)\right]\\ &+\frac{\alpha\xi}{4\pi}(k^\mu \not p + k^\mu \not k + p^\mu \not p + p^\mu \not k)\frac{1}{2(k^2-p^2)}\left[m^2\left(\frac{1}{k^2}-\frac{1}{p^2}\right) - \left(1+\frac{m^2}{k^2}\right)L' + \left(1+\frac{m^2}{p^2}\right)L\right]\\ &-\frac{\alpha m}{4\pi}(3+\xi)\frac{(p+k)^\mu}{(k^2-p^2)}[L-L'] .\end{aligned} \tag{64}$$

B. Transverse vertex

Having calculated the vertex $O(\alpha)$, Eq. (60), we can subtract from it the longitudinal vertex of Sec. III A, Eq. (64), and obtain [Eq. (12)] the transverse vertex to $O(\alpha)$. This is given by a rather lengthy expression,

$$\begin{aligned}\Gamma^\mu_T(k,p) = \frac{\alpha}{4\pi}\Bigg\{\sum_{i=1}^{12} V_i^\mu\Bigg(&\frac{1}{2\Delta^2}[a_1^{(i)} + (\xi-1)a_2^{(i)}]J_A + \frac{1}{2\Delta^2}[b_1^{(i)} + (\xi-1)b_2^{(i)}]J_B\\ &+\frac{1}{2\Delta^2}[c_1^{(i)} + (\xi-1)c_2^{(i)}]I_A + \frac{1}{2\Delta^2}[d_1^{(i)} + (\xi-1)d_2^{(i)}]I_B\\ &+\frac{1}{2p^2(k^2-p^2)(p^2-m^2)\Delta^2}[e_1^{(i)} + (\xi-1)e_2^{(i)}]L\\ &+\frac{1}{2k^2(k^2-p^2)(k^2-m^2)\Delta^2}[f_1^{(i)} + (\xi-1)f_2^{(i)}]L' + \frac{1}{\Delta^2}[g_1^{(i)} + (\xi-1)g_2^{(i)}]S\\ &+\frac{1}{2\Delta^2}[h_1^{(i)} + (\xi-1)h_2^{(i)}]J_0 + \frac{1}{\Delta^2}[l_1^{(i)} + (\xi-1)l_2^{(i)}]\Bigg)\Bigg\} ,\end{aligned} \tag{65}$$

in terms of the 12 vectors V_i^μ of Eq. (5) with the coefficients which are listed in Appendix B. Our task is then to express this result in terms of the eight basis vectors defining $\Gamma^\mu_T(k,p)$, Eq. (12). Thus from Eq. (13) we can alternatively write out

$$\begin{aligned}\Gamma^\mu_T = {}& k^\mu \not k[\tau_2(p^2 - k\cdot p) - \tau_3 + \tau_6] + p^\mu \not p[\tau_2(k^2 - k\cdot p) - \tau_3 - \tau_6] + k^\mu \not p[\tau_2(p^2 - k\cdot p) + \tau_3 - \tau_6 + \tau_8]\\ &+p^\mu \not k[\tau_2(k^2 - k\cdot p) + \tau_3 + \tau_6 - \tau_8] + \gamma^\mu[\tau_3 q^2 + \tau_6(p^2-k^2) + \tau_8(k\cdot p)] + \gamma^\mu \not k \not p[-\tau_8]\\ &+p^\mu\left[\tau_1(k^2 - k\cdot p) - \tau_4(k^2 - k\cdot p)(k\cdot p) - \tau_5 + \frac{\tau_7}{2}(k^2 - p^2 - 2k\cdot p)\right]\\ &+k^\mu\left[\tau_1(p^2 - k\cdot p) - \tau_4(p^2 - k\cdot p)(k\cdot p) + \tau_5 + \frac{\tau_7}{2}(k^2 - p^2 - 2k\cdot p)\right] + p^\mu \not k \not p[\tau_4(k^2 - k\cdot p) + \tau_7]\\ &+k^\mu \not k \not p[\tau_4(p^2 - k\cdot p) + \tau_7] + \gamma^\mu \not k\left[-\tau_5 + \frac{\tau_7}{2}(p^2 - k^2)\right] + \gamma^\mu \not p\left[\tau_5 + \frac{\tau_7}{2}(p^2 - k^2)\right] .\end{aligned} \tag{66}$$

Comparing Eqs. (5) and (65), we have 12 equations for the 8 unknown τ_i. Since Γ^μ_T is transverse to the vector q_μ, Eq. (11), only eight of these equations are independent. A laborious solution yields expressions for the eight transverse coefficients τ_i. Each is a function of k^2, p^2, q^2, and ξ. The results are

$$\tau_1 = \frac{\alpha}{4\pi}\frac{(3+\xi)}{\Delta^2} m \left\{ -\frac{1}{2}[m^2 + k\cdot p]J_0 - 2S + \frac{[p^2 + k\cdot p]L}{p^2 - k^2} - \frac{[k^2 + k\cdot p]L'}{p^2 - k^2} \right\} , \tag{67}$$

$$\begin{aligned}
\tau_2 = &-\frac{3}{4\Delta^2}[m^2 + k\cdot p]\tau_8 + \frac{\alpha}{4\pi\Delta^2}\Bigg\{ \frac{1}{8}(q^2 - 4m^2)J_0 + \frac{L+L'}{4} - \frac{1}{2} - \frac{m^2}{2p^2k^2}k\cdot p \\
&+\frac{1}{2(p^2-k^2)}\left[[k^2 + k\cdot p]\left(1 + \frac{m^2}{k^2}\right)L' - [p^2 + k\cdot p]\left(1 + \frac{m^2}{p^2}\right)L\right]\Bigg\} \\
&+\frac{\alpha}{8\pi\Delta^2}(\xi - 1)\Bigg\{ \left[\frac{1}{2}(p^2 + k^2) + \frac{3q^2}{4\Delta^2}(p^2k^2 - m^4)\right] J_0 + 1 - \frac{m^2}{p^2k^2}k\cdot p \\
&+\frac{1}{\chi}\Bigg[\frac{-2m^2}{p^2 - k^2}\left\{\left(p^2 + m^2\frac{k^2}{p^2}\right)L - \left(k^2 + m^2\frac{p^2}{k^2}\right)L'\right\}\Delta^2 \\
&-q^2\frac{[p^2 + 2k\cdot p + k^2]}{2(p^2-k^2)}\left\{m^6\left(\frac{L}{p^2} - \frac{L'}{k^2}\right) + p^2k^2(L - L')\right\} \\
&+\frac{q^2}{2}m^4\left\{\left(1 - \frac{m^2}{p^2}\right)L + \left(1 - \frac{m^2}{k^2}\right)L'\right\} - \frac{m^2}{2}(p^4 - k^4)(L - L') \\
&+\frac{q^2}{2}[k^2p^2(L + L') - m^2(k^2L + p^2L')] + \frac{3m^2}{2}(p^2 - k^2)[(m^2 - p^2)L - (m^2 - k^2)L'] \\
&+\frac{3q^4}{4\Delta^2}(m^4 - p^2k^2)[(m^2 - p^2)L + (m^2 - k^2)L'] + \frac{3q^2}{4\Delta^2}(p^2 - k^2)(m^4 - p^2k^2)[(m^2 + p^2)L - (m^2 + k^2)L'] \\
&+\Bigg[-8q^2(m^4 + k^2p^2) + 12p^2k^2(q^2 - m^2) + 2q^2(k^2 + p^2)m^2 \\
&-\frac{3}{\Delta^2}m^4q^4(m^2 + k\cdot p) + \frac{3}{\Delta^2}k^2p^2q^2k\cdot p(q^2 - m^2) + 2(p^2 - k^2)^2m^2 \\
&-\frac{9}{\Delta^2}m^2p^2k^2q^2k\cdot p + \frac{3}{\Delta^2}m^2p^2k^2(p^2 - k^2)^2\Bigg]S\Bigg]\Bigg\} ,
\end{aligned} \tag{68}$$

$$\begin{aligned}
\tau_3 = &-\frac{3}{32m\Delta^2}[m^2 + k\cdot p](p^2 - k^2)^2\tau_1(\xi = 1) + \frac{\alpha}{8\pi\Delta^2}\Bigg\{\left[-\Delta^2 - \frac{(p^2 + k^2 - 2m^2)^2}{8}\right]J_0 - 2[m^2 + k\cdot p]S \\
&+\frac{k\cdot p}{2}\left[\left(1 - \frac{m^2}{p^2}\right)L + \left(1 - \frac{m^2}{k^2}\right)L'\right] + \frac{1}{4}(p^2 - k^2)(L - L') \\
&+[m^2 + k\cdot p] + \frac{1}{2p^2k^2}(p^2 + k^2)[p^2k^2 + m^2k\cdot p]\Bigg\} \\
&+\frac{\alpha(\xi - 1)}{8\pi\Delta^2}\Bigg\{\left[\frac{p^2k^2 - m^4}{2} - \frac{(p^2 - k^2)^2}{4} - \frac{3}{8\Delta^2}(p^2 - k^2)^2(p^2k^2 - m^4)\right]J_0 \\
&+\frac{m^2}{2p^2k^2}k\cdot p(k^2 + p^2) - \frac{p^2 + k^2}{2} - (k\cdot p - m^2) + \frac{1}{\chi}\Bigg[-m^2\left\{\left(p^2 + m^2\frac{k^2}{p^2}\right)L + \left(k^2 + m^2\frac{p^2}{k^2}\right)L'\right\}\Delta^2 \\
&-\frac{m^6}{2}q^2k\cdot p\left(\frac{L}{p^2} + \frac{L'}{k^2}\right) - \frac{1}{2}p^2k^2q^2k\cdot p(L + L') + \frac{1}{4}q^2m^2[p^2(p^2 - m^2)L + k^2(k^2 - m^2)L'] \\
&-\frac{1}{4}q^2m^2[p^2(k^2 + m^2)L + k^2(p^2 + m^2)L'] + \frac{1}{2}(p^2 - k^2)q^2(m^4 + p^2k^2)(L - L') \\
&+\frac{m^2}{4}(p^4 - k^4)[(p^2 - m^2)L - (k^2 - m^2)L'] + \frac{m^2}{2}(p^4 - k^4)[k\cdot p - m^2](L - L') + m^2(p^2 - k^2)(p^4L - k^4L') \\
&-\frac{m^4}{4}(p^2 - k^2)[p^2 + 2k\cdot p + k^2](L - L') \\
&+\frac{3}{8\Delta^2}q^2(p^2 - k^2)[p^2 + 2k\cdot p + k^2][(p^4k^2 - m^6)L - (k^4p^2 - m^6)L']
\end{aligned}$$

$$-\frac{3m^2}{8\Delta^2}(p^2-k^2)^3[p^2(m^2-k^2)L-k^2(m^2-p^2)L'] + \frac{3m^4}{8\Delta^2}q^2(p^2-k^2)^2[(p^2-m^2)L+(k^2-m^2)L']$$
$$-\frac{3}{8\Delta^2}p^2k^2q^2(p^2-k^2)^2[(p^2-m^2)L+(k^2-m^2)L']$$
$$+\Bigg[-8m^4\Delta^2+4m^2(p^2+k^2)\Delta^2+2m^2(p^2-k^2)^2[m^2-(p^2+k^2)]$$
$$-2m^4q^2[m^2+k\cdot p]+2p^2k^2q^2[m^2+k\cdot p]+2m^4(p^2-k^2)^2$$
$$+\frac{3}{2\Delta^2}q^2(p^2-k^2)^2(m^4-p^2k^2)[m^2+k\cdot p]\Bigg]S\Bigg]\Bigg\}\,, \qquad (69)$$

$$\tau_4 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\Bigg\{-\left[1+\frac{3q^2}{2\Delta^2}[m^2+k\cdot p]\right]J_0 - \frac{2}{p^2k^2}k\cdot p$$
$$+\frac{1}{\chi}\Bigg[-\frac{4m^2}{p^2-k^2}\left(\frac{k^2}{p^2}L-\frac{p^2}{k^2}L'\right)\Delta^2+3m^2(p^2-k^2)(L-L')$$
$$-\frac{m^4q^2}{p^2-k^2}[k^2+2k\cdot p+p^2]\left(\frac{L}{p^2}-\frac{L'}{k^2}\right)-q^2\left\{p^2\left(\frac{m^4}{p^4}-1\right)L+k^2\left(\frac{m^4}{k^4}-1\right)L'\right\}$$
$$+\frac{3m^2q^2}{\Delta^2}(p^2-k^2)\{(m^2+p^2)L-(m^2+k^2)L'\}$$
$$-\frac{3q^2}{2\Delta^2}(p^2-k^2)(m^4-p^2k^2)(L-L')-\frac{3q^4}{2\Delta^2}(m^4-k^2p^2)(L+L')$$
$$-\frac{3m^2q^4}{\Delta^2}\{(p^2-m^2)L+(k^2-m^2)L'\}$$
$$+\left\{-20m^2q^2-2q^2(p^2+k^2)-\frac{12m^2q^4}{\Delta^2}[m^2+k\cdot p]+\frac{6q^4}{\Delta^2}(m^4-p^2k^2)\right\}S\Bigg]\Bigg\}\,, \qquad (70)$$

$$\tau_5 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\Bigg\{-\left[\Delta^2-\frac{1}{4}(p^2-k^2)^2+\frac{q^2}{2}[m^2+k\cdot p]\right]J_0+\frac{p^2+k^2}{p^2k^2}\Delta^2$$
$$+\frac{1}{\chi}\Bigg[-m^2(p^2-k^2)\left(\frac{p^2}{k^2}L'-\frac{k^2}{p^2}L\right)\Delta^2+2m^2(p^2-k^2)(L-L')\Delta^2$$
$$+2m^2q^2\left\{\left(1-\frac{m^2}{p^2}\right)L+\left(1-\frac{m^2}{k^2}\right)L'\right\}\Delta^2+m^2q^2\left\{(m^2+k^2)\frac{L}{p^2}+(m^2+p^2)\frac{L'}{k^2}\right\}\Delta^2$$
$$+\frac{m^2q^2}{2}\left\{q^2[m^2+k\cdot p]+q^2k\cdot p-\frac{1}{2}(p^2-k^2)^2\right\}(L+L')$$
$$+\frac{m^2}{2}(p^2-k^2)\left\{q^2m^2+2q^2k\cdot p-\frac{1}{2}(p^2-k^2)^2\right\}(L-L')$$
$$+\frac{1}{4}q^4(p^2+k^2)(p^2L+k^2L')-\frac{1}{4}q^2(p^4-k^4)(p^2L-k^2L')$$
$$+q^2(p^2+k^2-2m^2)(q^2m^2+q^2k\cdot p+2\Delta^2)S\Bigg]\Bigg\}\,, \qquad (71)$$

$$\tau_6 = \frac{p^2-k^2}{2}\tau_2(\xi=1)+\frac{\alpha}{8\pi\Delta^2}(\xi-1)\Bigg\{(p^2-k^2)\left[\frac{q^2}{4}-\frac{3q^2}{8\Delta^2}(m^4-p^2k^2)\right]J_0-\frac{p^2-k^2}{2p^2k^2}[m^2k\cdot p-p^2k^2]$$
$$+\frac{1}{\chi}\Bigg[m^2\Delta^2\left\{\left(p^2-m^2\frac{k^2}{p^2}\right)L-\left(k^2-m^2\frac{p^2}{k^2}\right)L'\right\}$$
$$+\frac{1}{2}m^2(k^4-p^4)[m^2-k\cdot p](L+L')-2m^2(k\cdot p)(p^4L-k^4L')$$
$$-\frac{m^6q^2}{2}\left\{\left(1+\frac{k\cdot p}{p^2}\right)L-\left(1+\frac{k\cdot p}{k^2}\right)L'\right\}$$

$$
\begin{aligned}
&-\frac{m^2}{2}k\cdot p(p^2-k^2)[(m^2-p^2)L+(m^2-k^2)L'] \\
&-\frac{1}{2}p^2k^2q^2\{[k^2-k\cdot p]L-[p^2-k\cdot p]L'\} \\
&-q^2[p^4(m^2-k^2)L-k^4(m^2-p^2)L']+2m^2k^2p^2(k^2L-p^2L') \\
&+\frac{3m^2}{8\Delta^2}(p^4-k^4)(p^2-k^2)[p^2(m^2-k^2)L-k^2(m^2-p^2)L'] \\
&-\frac{3m^2}{4\Delta^2}p^2k^2q^2(p^2-k^2)[(m^2+p^2)L+(m^2+k^2)L'] \\
&-\frac{3m^2}{4\Delta^2}p^2k^2(p^2-k^2)^2[(m^2-p^2)L-(m^2-k^2)L'] \\
&-\frac{3m^2}{8\Delta^2}p^2q^2(p^4-k^4)[(m^2+k^2)L-(m^2+p^2)L'] \\
&+\frac{3}{8\Delta^2}q^4(p^2-k^2)[(m^6+p^4k^2)L+(m^6+p^2k^4)L'] \\
&+\frac{3q^2}{8\Delta^2}(p^2-k^2)^2[(m^6-p^4k^2)L-(m^6-p^2k^4)L'] \\
&+(p^2-k^2)\Bigg[-\frac{3m^4}{2\Delta^2}q^4[m^2+k\cdot p]+\frac{3}{2\Delta^2}k^2p^2q^4[m^2+k\cdot p] \\
&-4m^4q^2+2m^2q^2(p^2+k^2)\Bigg]S\Bigg]\Bigg\}\,, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
\tau_7 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\Bigg\{&-\frac{q^2}{2}J_0-\frac{2}{p^2k^2}\Delta^2+\frac{1}{\chi}\Bigg[\frac{-2m^2q^2}{p^2-k^2}\left\{\frac{m^2-k^2}{p^2}L-\frac{m^2-p^2}{k^2}L'\right\}\Delta^2 \\
&-2m^2\left\{\left(1-\frac{k^2}{p^2}\right)L+\left(1-\frac{p^2}{k^2}\right)L'\right\}\Delta^2-q^2(k\cdot p)[(m^2-p^2)L+(m^2-k^2)L'] \\
&+q^2[p^2(m^2-k^2)L+k^2(m^2-p^2)L']-2q^2\{q^2[m^2+k\cdot p]+2\Delta^2\}S\Bigg]\Bigg\}\,, \qquad (73)
\end{aligned}
$$

and

$$
\tau_8 = \frac{\alpha}{4\pi\Delta^2}\left\{\frac{q^2}{2}[k\cdot p+m^2]J_0+2q^2S+[-p^2+k\cdot p]L+[-k^2+k\cdot p]L'\right\}\,. \qquad (74)
$$

These τ_i's are given in an arbitrary covariant gauge specified by ξ. The Ball-Chiu calculation was performed in the Feynman gauge ($\xi = 1$), where these results greatly simplify, see Eq. (68), for example. If we compare our τ's with the result of Ball-Chiu, then their result needs a few corrections: in their equation (3.8e) there should be a $(+)$ sign instead of a $(-)$ sign in front of the second term; in (3.8e) $p{\cdot}p'/2[(1-m^2)/p^2]L$ should be $p{\cdot}p'/2(1-m^2/p^2)L$; in Eq. (3.11b) the third term with ln (p^2/p'^2) should have a $\frac{1}{2}$ factor in front; in their equation (3.11c) the third coefficient I_0, $(p^2-p'^2)^2/8$ should be $(p^2+p'^2)^2/8$; in Eq. (3.12) there is an overall factor of 2 missing; in Eq. (3.14) factor $\frac{1}{6}$ should be $\frac{1}{12}$; in their equation (A18) the coefficient of the first term should not have a factor of $\frac{1}{2}$; in Eq. (A19) first coefficient $p\cdot p'/2$ should be $p\cdot p'$, where the equation numbers refer to those in [4].

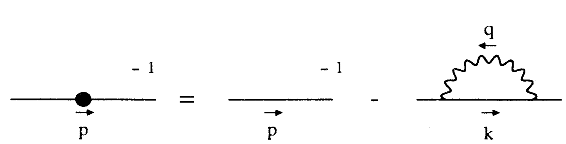

FIG. 2. The inverse fermion propagator to one-loop order in perturbation theory.

IV. FREEDOM FROM KINEMATIC SINGULARITIES

Clearly the full vertex, $\Gamma^\mu(k,p)$, is free of kinematic singularities. The Ball-Chiu construction ensures that the longitudinal part is free, so the transverse part must be. However, after decomposing this transverse part into eight components, it is not necessary that the individual components will each be free of kinematic singularities. Ball and Chiu showed that with their choice of eight basis vectors, the transverse vertex in the Feynman gauge possessed this property of being singularity-free. Here we explicitly consider whether this is true in an arbitrary covariant gauge. Indeed such checks are far longer than the initial calculation reported above. We consider several limits in turn.

A. $\Delta^2 \to 0$

The proof depends crucially on the behavior of the combination of Spence functions forming the integral J_0 that appears in every τ_i. Thus, for instance, when we consider the limit $\Delta^2 \to 0$, i.e., $(k\cdot p)^2 \to k^2p^2$, we can deduce from Eqs. (26), (A15)–(A19) that J_0 can be expanded in powers of Δ^2 as

$$J_0 = J_0^0 + J_0^1\Delta^2 + O(\Delta^4)\ , \tag{75}$$

where

$$J_0^0 = -\frac{1}{m^2+\sqrt{k^2p^2}}\left[4S - 2\frac{k^2+\sqrt{k^2p^2}}{k^2-p^2}L' + 2\frac{p^2+\sqrt{k^2p^2}}{k^2-p^2}L\right]\ , \tag{76}$$

$$J_0^1 = \left[\frac{2}{3q_0^2(m^2+\sqrt{k^2p^2})\sqrt{k^2p^2}} - Y_1(k^2,p^2)L' - Y_1(p^2,k^2)L - Z_1(k^2,p^2)S\right]\ , \tag{77}$$

and Y_1 and Z_1 are defined as

$$Y_1(k^2,p^2) = \frac{k^2-m^2}{3(\sqrt{k^2}-\sqrt{p^2})^3(m^2+\sqrt{k^2p^2})^3\sqrt{k^2p^2}} \times\left[3\sqrt{k^2}(m^2-p^2)+\sqrt{p^2}(k^2-m^2)\right]\ , \tag{78}$$

$$Z_1(k^2,p^2) = -\frac{1}{q_0^2(m^2+\sqrt{k^2p^2})^3(q_0^2-4m^2)\sqrt{k^2p^2}} \times\left\{8m^6 - 8m^4\left(k^2+p^2-\frac{4}{3}\sqrt{k^2p^2}\right) + m^2\left[2q_0^4+\frac{8}{3}\sqrt{k^2p^2}(k^2+p^2+\sqrt{k^2p^2})\right] + \frac{2}{3}\sqrt{k^2p^2}q_0^4\right\}\ , \tag{79}$$

$$q_0^2 = k^2+p^2-2\sqrt{k^2p^2}\ . \tag{80}$$

Together with the known behavior of all the other functions, such as L, L', and S, it is a lengthy but straightforward calculation to deduce that each τ_i is finite in the limit $\Delta^2 \to 0$, despite the appearance of explicit $1/\Delta^2$ and $1/\Delta^4$ terms.

B. $\chi \to 0$

As seen from Eq. (20) the full vertex, and hence the transverse part, has no pole singularities when $\chi \to 0$. However, the expressions for τ_2, τ_3, τ_4, τ_5, τ_6, and τ_7, Eqs. (68)–(73), have explicit factors of $1/\chi$ in all but the Feynman gauge. As can be seen from Eq. (55) χ only vanishes if *both* p^2 and k^2 tend to m^2, i.e., when both of the fermion legs, Fig. 1, are on the mass shell, then when $k^2 \to p^2$, $\chi = (q^2-4m^2)(p^2-m^2)^2$. In this limit the full vertex only has logarithmic singularities, such as $\ln(1-m^2/p^2)$. Consequently, an acceptable basis for the transverse vertex is one in which only these logarithmic singularities occur. Explicit calculation shows that τ_2, τ_3, τ_5, and τ_6, given by Eqs. (68), (69), (71), and (72), do have only these logarithmic terms when $\chi \to 0$. However, both τ_4 and τ_7 have poles in $1/(p^2-m^2)$ term. These singularities are readily removed by choosing a new basis for the transverse vertex, the S_i^μ $(i = 1,\ldots,8)$. Clearly this only involves changes to T_4^μ and T_7^μ. Note that these singularities do not arise in the Feynman gauge ($\xi = 1$), and so Ball and Chiu were not aware of this constraint.

We write

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8}\sigma^i S_i^\mu\ , \tag{81}$$

where

$$S_i^\mu \equiv T_i^\mu \quad \text{for } i = 1,2,3,5,6,7,8 \tag{82}$$

and

$$S_4^\mu = q^2[\gamma^\mu(\not p+\not k) - p^\mu - k^\mu] + 2(p-k)^\mu k^\lambda p^\nu \sigma_{\lambda\nu}\ , \tag{83}$$

then

$$\sigma_i \equiv \tau_i \quad \text{for } i = 1,2,3,5,6,8 \tag{84}$$

and

$$\sigma_4 = \frac{k^2-p^2}{4}\tau_4\ , \tag{85}$$

$$\sigma_7 = \tau_7 + \frac{q^2}{2}\tau_4\ . \tag{86}$$

σ_7 is then given explicitly as

$$\sigma_7 = \frac{\alpha m}{8\pi\Delta^2}(\xi-1)\left\{\left(-q^2-\frac{3q^4}{4\Delta^2}(m^2+k\cdot p)\right)J_0 - \frac{q^2}{p^2k^2}k\cdot p - 2\frac{\Delta^2}{p^2k^2} + \frac{1}{\chi}\left[\Delta^2\left\{-2\frac{m^4q^2}{(p^2-k^2)}\left(\frac{L}{p^2}-\frac{L'}{k^2}\right) - 2m^2(p^2-k^2)\left(\frac{L}{p^2}-\frac{L'}{k^2}\right)\right\}\right.\right.$$

$$
\begin{aligned}
&-\frac{m^4q^4}{2(p^2-k^2)}[k^2+2(k\cdot p)+p^2]\left(\frac{L}{p^2}-\frac{L'}{k^2}\right)\\
&-\frac{q^4}{2}\left\{p^2\left(\frac{m^4}{p^4}-1\right)L+k^2\left(\frac{m^4}{k^4}-1\right)L'\right\}\\
&+\frac{3}{2}\frac{m^2q^4}{\Delta^2}(p^2-k^2)\{(m^2+p^2)L-(m^2+k^2)L'\}-\frac{3q^4}{4\Delta^2}(p^2-k^2)(m^4-p^2k^2)(L-L')\\
&-\frac{3q^6}{4\Delta^2}(m^4-p^2k^2)(L+L')+\frac{3m^2q^2}{2}(p^2-k^2)(L-L')\\
&-\frac{3m^2q^6}{2\Delta^2}[(p^2-m^2)L+(k^2-m^2)L']+q^2k\cdot p[(p^2-m^2)L+(k^2-m^2)L']\\
&-q^2[p^2(k^2-m^2)L+k^2(p^2-m^2)L']\\
&+\left\{-10m^2q^4-q^4(p^2+k^2)-\frac{6m^2q^6}{\Delta^2}(m^2+k\cdot p)+\frac{3q^6}{\Delta^2}(m^4-p^2k^2)\right.\\
&\left.\left.\left.-2q^4(m^2+k\cdot p)-4q^2\Delta^2\right\}S\right]\right\} . \qquad (87)
\end{aligned}
$$

In this new basis, all the σ_i's $(i=1,\ldots,8)$ have no singularities other than the expected logarithmic ones. Note that in this new basis, the C-parity operation of Eq. (15) requires

$$\sigma_4(k^2,p^2,q^2)=-\sigma_4(p^2,k^2,q^2)\ , \qquad (88)$$

which Eq. (83) ensures.

C. Asymptotic limit

It is convenient to give here the simple asymptotic limit for the transverse vertex. In the limit that either of the fermion momenta are large, e.g., $k^2 \gg k\cdot p \gg p^2 \gg m^2$,

$$
\begin{aligned}
J_0=\frac{2}{k^2}&\left[\left(1+\frac{(k\cdot p)}{k^2}-\frac{p^2}{3k^2}+\frac{4(k\cdot p)^2}{3k^4}\right)\ln\frac{k^2}{p^2}\right.\\
&\left.+\left(2+\frac{(k\cdot p)}{k^2}-\frac{2p^2}{9k^2}+\frac{8(k\cdot p)^2}{9k^4}\right)\right]\\
&+O(1/k^5)\ . \qquad (89)
\end{aligned}
$$

Consequently, the transverse vertex has the well-known limit

$$\Gamma_T^\mu=\frac{\alpha\xi}{8\pi}\left[\frac{k^\mu \not k}{k^2}-\gamma^\mu\right]\ln\left(\frac{k^2}{p^2}\right)\ . \qquad (90)$$

Equations (67)–(74) and (85)–(87) give our results for the transverse components. Their forms in specific limits are given in Ref. [13]. However, it is worth noting that they greatly simplify if the fermion mass m is zero. Then not only do σ_1, σ_4, σ_5, and σ_7 explicitly vanish, but the other four σ_i's $(i=2,3,6,8)$ have far shorter expressions [13].

V. CONCLUSIONS

This paper presents the complete one-loop calculation of the fermion-boson vertex in QED in an arbitrary covariant gauge. This calculation has, in fact, been performed independently by the present authors. The authors have joined forces only to compare and check their answer and to write this paper. The coupling of two spin-$\frac{1}{2}$ particles with a vector boson involves 12 independent spin and Lorentz vectors. Each of these vectors has a coefficient that is an analytic function of the three Lorentz scalars, k^2, p^2, and q^2, that can be formed from the two independent four-momenta flowing through the vertex. These 12 components are given as functions of the covariant gauge parameter. They have been previously calculated by Ball and Chiu [4] in the Feynman gauge. Our results correct some typographical errors in their publication. The vertex has only logarithmic singularities: these arise either when the external legs are on-shell or when the internal fermions can be real.

Four of the 12 components define what is called the longitudinal vertex. This is related by the Ward-Takahashi identity to the fermion propagator. This fact allows three of these components to be expressed in terms of the fermion wave function renormalization $F(p^2)$, and its mass function $M(p^2)$ and forces a fourth to be zero. Ball and Chiu have shown how to construct this longitudinal vertex in a way free of kinematic singularities. This freedom is essential in ensuring the Ward identity is the $q\to 0$ limit of the Ward-Takahashi identity. Subtraction of this longitudinal vertex from our one-loop answer leaves the transverse vertex $O(\alpha)$. This can be represented in terms of a basis of eight vectors orthogonal to the boson momentum, each unconstrained by the Ward-Takahashi identity.

We propose a new transverse basis S_i^μ $(i=1,\ldots,8)$, Eqs. (82) and (83), which has components with only the logarithmic singularities of the full vertex. This basis modifies the T_i^μ $(i=1,\ldots,8)$ of Eq. (13) proposed by Ball and Chiu [4]. Though their basis has no additional singularities in the Feynman gauge, this is not the case in any other gauge. Equations (19), (66)–(74), and (81)–(87) constitute our new result in QED to one loop. That any potential singularities at $\Delta^2=0$ or $\chi=0$ cancel in the full vertex is in fact a sensitive test of the correctness of our results. The same and/or related integrals arise in QCD, and so this calculation could, in principle, be extended to non-Abelian theories in any covariant gauge too.

Though our calculations are self-evidently only true to $O(\alpha)$, our aim is wider. The hope is that the coefficients of each of the transverse vectors, S_i^μ, like those of the longitudinal component, are free of kinematic singularities at all orders in perturbation theory and even nonperturbatively. Just as use of the Ward-Takahashi identity specifies nonperturbatively the longitudinal vertex in terms of the fermion propagator, Eq. (9), multiplicative renormalizability too imposes relationships between the vertex and the fermion propagator. These constrain the transverse vertex. A start has been made in analyzing these powerful conditions. Ignoring such requirements and use of, for instance, a bare vertex (*the rainbow approximation*) in studies of chiral symmetry breaking leads to the generation of highly gauge-dependent masses. In contrast nonperturbative enforcement of the Ward-Takahashi identity and the constraints of multiplicative renormalizability dramatically reduces or even eliminates [5,6] this unphysical gauge dependence. Indeed, knowing the vertex in any covariant gauge may give us an understanding of how the essential gauge dependence of the vertex demanded by its Landau-Khalatnikov transformation [14] is satisfied nonperturbatively. Moreover, having a basis for the transverse vertex with coefficients free of nondynamical singularities is a key step in further investigations of a meaningful nonperturbative truncation of Schwinger-Dyson equations. For instance, studying the behavior of just the propagators one must have an *Ansatz* for the three-point vertex that embodies as completely as possible the constraints of gauge invariance on all the higher n-point functions. Satisfying the Ward-Takahashi identity and multiplicative renormalizability are essential in constructing such an *Ansatz.* Moreover, in the weak-coupling limit this vertex must agree with perturbation theory—hence this one-loop calculation.

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APPENDIX A

Here we collect the results of evaluating all the integrals necessary for our calculation of Sec. II B in $d = 4 + \epsilon$ dimensions:

$$Q_1 = \int_M d^dk \frac{1}{(k-p)^2[k^2-m^2]} = i\pi^2\mu^\epsilon \left\{ C - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} , \tag{A1}$$

$$Q_2^\nu = \int_M d^dk \frac{k^\nu}{(k-p)^2[k^2-m^2]} = i\pi^2\mu^\epsilon \left\{ \frac{p^\nu}{2} \left[C + 2 + \frac{m^2}{p^2} - \left(1 - \frac{m^4}{p^4}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right] \right\} , \tag{A2}$$

$$Q_3 = \int_M d^dk \frac{1}{(k-p)^4[k^2-m^2]} = \mu^\epsilon \frac{i\pi^2}{m^2 - p^2} \left\{ C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} , \tag{A3}$$

$$Q_4^\nu = \int_M d^dk \frac{k^\nu}{(k-p)^4[k^2-m^2]} = \mu^\epsilon \frac{i\pi^2 p^\nu}{m^2 - p^2} \left\{ C + \left(1 - \frac{m^2}{p^2}\right) - \left(1 + \frac{m^4}{p^4}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} , \tag{A4}$$

$$Q_5 = \int_M d^dk \frac{k^2}{(k-p)^4[k^2-m^2]} = m^2 Q_3 , \tag{A5}$$

$$Q_6^\nu = \int_M d^dk \frac{k^2 k^\nu}{(k-p)^4[k^2-m^2]} = m^2 Q_4 , \tag{A6}$$

$$Q_7 = \int_M d^dw \frac{1}{[(k-w)^2 - m^2][(p-w)^2 - m^2]} = i\pi^2\mu^\epsilon [C + 2 - 2S] , \tag{A7}$$

$$Q_8 = \int_M d^dw \frac{1}{[(p-w)^2 - m^2]w^2} = i\pi^2\mu^\epsilon [C + 2 - L] , \tag{A8}$$

$$Q_9^\nu = \int_M d^dw \frac{w^\nu}{[(k-w)^2 - m^2][(p-w)^2 - m^2]} = \frac{i\pi^2}{2}\mu^\epsilon (p^\nu + k^\nu)[C + 2 - 2S] , \tag{A9}$$

$$Q_{10}^\nu = \int_M d^dw \frac{w^\nu}{[(p-w)^2 - m^2]w^2} = \frac{i\pi^2}{2}\mu^\epsilon p^\nu \left\{ C + 2 - \frac{m^2}{p^2} - \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} , \tag{A10}$$

$$Q_{11} = \int_M d^dw \frac{1}{[(p-w)^2 - m^2]w^4} = \frac{i\pi^2}{m^2 - p^2}\mu^\epsilon \left\{ C - \left(1 + \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) \right\} , \tag{A11}$$

$$Q_{12}^\nu = \int_M d^dw \frac{w^\nu}{[(p-w)^2 - m^2]w^4} = i\pi^2 \frac{p^\nu}{p^2} \left[1 + \frac{m^2}{p^2} \ln\left(1 - \frac{p^2}{m^2}\right) \right] , \tag{A12}$$

$$Q_{13} = \int_M d^d w \frac{1}{[(p-w)^2 - m^2][(k-w)^2 - m^2]w^4} = I^{(0)}$$
$$= i\pi^2 \mu^\epsilon \left\{ \frac{1}{\chi} \left[-2q^2 S + p^2 \frac{(p^2 - m^2)q^2 + 2m^2(k^2 - p^2)}{(p^2 - m^2)^2} L \right. \right.$$
$$\left. \left. + k^2 \frac{(k^2 - m^2)q^2 - 2m^2(k^2 - p^2)}{(k^2 - m^2)^2} L' \right] - \frac{C}{(p^2 - m^2)(k^2 - m^2)} \right\} , \qquad \text{(A13)}$$

where we recall

$$C = -\frac{2}{\epsilon} - \gamma - \ln(\pi) - \ln\frac{m^2}{\mu^2} , \quad L = \left(1 - \frac{m^2}{p^2}\right) \ln\left(1 - \frac{p^2}{m^2}\right) , \quad L' = L(p \leftrightarrow k) ,$$
$$S = \frac{1}{2}\left(1 - 4\frac{m^2}{q^2}\right)^{1/2} \ln\frac{[(1 - 4m^2/q^2)^{1/2} + 1]}{[(1 - 4m^2/q^2)^{1/2} - 1]} , \quad Q_{14} = \int_M d^d w \frac{1}{[(p-w)^2 - m^2][(k-2)^2 - m^2]w^2} = J^{(0)} . \qquad \text{(A14)}$$

$J^{(0)}$ is naturally expressed in terms of the Spence function $\mathrm{Sp}(x)$,

$$\mathrm{Sp}(x) = -\int_0^x dy \frac{\ln(1-y)}{y} , \qquad \text{(A15)}$$

so that

$$J^{(0)} = \frac{i\pi^2}{-2\Delta} \left\{ \mathrm{Sp}\left(\frac{y_1}{y_1 - 1}\right) + \mathrm{Sp}\left(\frac{y_1}{y_1 - \frac{m^2}{p^2}}\right) - \mathrm{Sp}\left(\frac{y_1 - 1}{y_1 - \frac{m^2}{p^2}}\right) \right.$$
$$- \mathrm{Sp}\left(\frac{y_2}{y_2 - 1}\right) - \mathrm{Sp}\left(\frac{y_2}{y_2 - \frac{m^2}{k^2}}\right) + \mathrm{Sp}\left(\frac{y_2 - 1}{y_2 - \frac{m^2}{k^2}}\right)$$
$$\left. + \mathrm{Sp}\left(\frac{y_3}{y_3 - q_1}\right) - \mathrm{Sp}\left(\frac{y_3 - 1}{y_3 - q_1}\right) + \mathrm{Sp}\left(\frac{y_3}{y_3 - q_2}\right) - \mathrm{Sp}\left(\frac{y_3 - 1}{y_3 - q_2}\right) \right\} , \qquad \text{(A16)}$$

where

$$\alpha = 1 + \frac{-(k \cdot p) + \Delta}{p^2}, \quad y_1 = y_0 + \alpha, \quad y_2 = \frac{y_0}{(1-\alpha)}, \quad y_3 = -\frac{y_0}{\alpha} ,$$
$$y_0 = \frac{1}{2p^2\Delta}[k^2p^2 - 2(k \cdot p)^2 + 2(k \cdot p)\Delta - p^2\Delta + p^2(k \cdot p) - m^2(k \cdot p - \Delta)] ,$$
$$q_1 = \frac{1 + \sqrt{1 - 4m^2/q^2}}{2}, \quad q_2 = \frac{1 - \sqrt{1 - 4m^2/q^2}}{2} . \qquad \text{(A17)}$$

In the massless case, J_0 simplifies to

$$J_0 = \frac{2}{\Delta}\left[\mathrm{Sp}\left(\frac{p^2 - k \cdot p + \Delta}{p^2}\right) - \mathrm{Sp}\left(\frac{p^2 - k \cdot p - \Delta}{p^2}\right) + \frac{1}{2}\ln\left(\frac{k \cdot p - \Delta}{k \cdot p + \Delta}\right)\ln\left(\frac{q^2}{p^2}\right)\right] . \qquad \text{(A18)}$$

APPENDIX B

In this appendix the coefficients, Eq. (65), of the 12 vectors V_i^μ in Eq. (6) are explicitly tabulated:

$$a_1^{(1)} = 3p^2(k^2 - m^2) - 2k \cdot p(p^2 - m^2) + 4\Delta^2 ,$$
$$a_2^{(1)} = k \cdot p(p^2 + m^2) - \frac{3m^2p^2}{2} - \frac{p^2k^2}{2} ,$$
$$a_1^{(2)} = k^2(k^2 - m^2) ,$$
$$a_2^{(2)} = k^2 k \cdot p - \frac{k^2}{2}(m^2 + k^2) ,$$
$$a_1^{(3)} = k^2(p^2 - m^2) - 2k \cdot p(k^2 - m^2) + 4\Delta^2 ,$$
$$a_2^{(3)} = m^2 k \cdot p + \Delta^2 - \frac{k^2}{2}(m^2 + p^2) ,$$
$$a_1^{(4)} = k^2(p^2 - m^2) - 2k \cdot p(k^2 - m^2) ,$$
$$a_2^{(4)} = -\frac{3k^2p^2}{2} - \frac{m^2k^2}{2} + k \cdot p(m^2 + k^2) - 2\Delta^2 ,$$
$$a_1^{(5)} = -2\Delta^2 ,$$
$$a_2^{(5)} = -\frac{\Delta^2}{2} ,$$
$$a_1^{(6)} = -(k^2 + m^2)\Delta^2 ,$$
$$a_2^{(6)} = -\left(\frac{m^2}{2} + k^2\right)\Delta^2 ,$$

$$a_1^{(7)} = -8m\Delta^2 ,$$
$$a_2^{(7)} = m[-3(k\cdot p)^2 + k^2k\cdot p + 2p^2k\cdot p - 2\Delta^2] ,$$
$$a_2^{(8)} = m\left[2k^2k\cdot p - \frac{k^2p^2}{2} - (k\cdot p)^2 - \frac{k^4}{2}\right] ,$$
$$a_2^{(9)} = m[-k^2 + k\cdot p] ,$$
$$a_2^{(10)} = m\left[-\frac{3p^2}{2} - \frac{k^2}{2} + 2k\cdot p\right] ,$$
$$a_2^{(11)} = -m\frac{q^2}{2}k\cdot p ,$$
$$a_2^{(12)} = m\frac{q^2}{2}k^2 ,$$
$$a_1^{(i)} = 0,\quad i = 8,9,10,11,12 ; \tag{B1}$$

$$b_1^{(1)} = a_1^{(2)}(k\leftrightarrow p),\quad b_1^{(2)} = a_1^{(1)}(k\leftrightarrow p) ,$$
$$b_1^{(i)} = a_1^{(i)}(k\leftrightarrow p),\quad i = 3,4,5,6 ,$$
$$b_1^{(7)} = a_1^{(8)}(k\leftrightarrow p),\quad b_1^{(8)} = a_1^{(7)}(k\leftrightarrow p) ,$$
$$b_1^{(i)} = 0,\quad i = 9,10,11,12 ,$$
$$b_2^{(1)} = a_2^{(2)}(k\leftrightarrow p),\quad b_2^{(2)} = a_2^{(1)}(k\leftrightarrow p) ,$$
$$b_2^{(i)} = a_2^{(i)}(k\leftrightarrow p),\quad i = 5,6 ,$$
$$b_2^{(9)} = a_2^{(10)}(k\leftrightarrow p),\quad b_2^{(10)} = a_2^{(9)}(k\leftrightarrow p) ,$$
$$b_2^{(11)} = -a_2^{(12)}(k\leftrightarrow p),\quad b_2^{(12)} = -a_2^{(11)}(k\leftrightarrow p) ,$$
$$b_2^{(3)} = (m^2+p^2)(k\cdot p) - (k\cdot p)^2 - \frac{p^2}{2}(m^2+k^2) ,$$
$$b_2^{(4)} = m^2(k\cdot p) - \frac{p^2}{2}(m^2+k^2) ,$$
$$b_2^{(7)} = m\left[-\Delta^2 + \frac{p^2}{2}(p^2-k^2)\right] ,$$
$$b_2^{(8)} = m[k^2(k\cdot p) - (k\cdot p)^2 - 2\Delta^2] ; \tag{B2}$$

$$c_1^{(i)} = 0,\quad i = 1,\ldots,12 ,$$
$$c_2^{(1)} = -\frac{p^4k^2}{2} + \frac{3m^4p^2}{2} - m^2p^2k^2 + p^2k^2k\cdot p - m^4k\cdot p ,$$
$$c_2^{(2)} = -\frac{k^4p^2}{2} - m^2k^2k\cdot p + \frac{m^4k^2}{2} + k^4k\cdot p ,$$
$$c_2^{(3)} = (m^2-k^2)\left(\Delta^2 + \frac{k^2}{2}(k^2+m^2) - m^2k\cdot p\right) ,$$
$$c_2^{(4)} = \frac{m^4k^2}{2} + k^2p^2k\cdot p - m^4k\cdot p - \frac{3k^4p^2}{2} + m^2p^2k^2 ,$$
$$c_2^{(5)} = \frac{k^2-m^2}{2}\Delta^2 ,$$
$$c_2^{(6)} = \frac{m^2(m^2-k^2)}{2}\Delta^2 ,$$
$$c_2^{(7)} = m\Bigg[[3(k\cdot p)^2 - 2p^2k\cdot p - k^2k\cdot p]m^2 + \frac{p^4k^2}{2} + 3p^2k^2k\cdot p - 2p^2(k\cdot p)^2 - \frac{p^2k^4}{2} - k^2(k\cdot p)^2\Bigg] ,$$
$$c_2^{(8)} = m\Bigg[\left((k\cdot p)^2 - 2k^2k\cdot p + \frac{k^2p^2}{2} + \frac{k^4}{2}\right)m^2 - p^2k^4 + p^2k^2k\cdot p - k^2(k\cdot p)^2 + k^4k\cdot p\Bigg] ,$$
$$c_2^{(9)} = m\left[(k^2-k\cdot p)m^2 - \frac{q^2}{2}k^2\right] ,$$
$$c_2^{(10)} = m\left[\left(\frac{3p^2}{2} - 2k\cdot p + \frac{k^2}{2}\right)m^2 + q^2k\cdot p + 2\Delta^2\right] ,$$
$$c_2^{(11)} = m\left(m^2q^2\frac{k\cdot p}{2} + \frac{k^2p^2}{2}(k^2-p^2) + p^2k\cdot p(k\cdot p - k^2)\right) ,$$
$$c_2^{(12)} = m\left[-m^2\frac{q^2}{2}k^2 - \frac{k^2k\cdot p}{2}(p^2+k^2)\right] ; \tag{B3}$$

$$d_1^{(i)} = 0,\quad i = 1,\ldots,12 ,$$
$$d_2^{(1)} = c_2^{(2)}(k\leftrightarrow p),\quad d_2^{(2)} = c_2^{(1)}(k\leftrightarrow p) ,$$
$$d_2^{(i)} = c_2^{(i)},\quad i = 5,6 ,$$
$$d_2^{(9)} = c_2^{(10)}(k\leftrightarrow p),\quad d_2^{(10)} = c_2^{(9)}(k\leftrightarrow p) ,$$
$$d_2^{(11)} = -c_2^{(12)}(k\leftrightarrow p),\quad d_2^{(12)} = -c_2^{(11)}(k\leftrightarrow p) ,$$
$$d_2^{(3)} = m^2(k\cdot p)^2 - \frac{p^4k^2}{2} + \frac{m^2p^2}{2} - p^2(k\cdot p)^2 + p^2k^2k\cdot p - m^4k\cdot p ,$$
$$d_2^{(4)} = \frac{m^4p^2}{2} + m^2p^2k\cdot p - m^4k\cdot p - \frac{k^2p^4}{2} ,$$
$$d_2^{(7)} = m\left[\left(\Delta^2 + \frac{p^2}{2}(k^2-p^2)\right)m^2 - p^2\Delta^2\right] ,$$
$$d_2^{(8)} = m\Bigg[[-k^2k\cdot p + (k\cdot p)^2]m^2 - p^2\Delta^2 - \frac{q^2}{2}k^2p^2\Bigg] ; \tag{B4}$$

$$e_1^{(1)} = (m^4-p^4)\Delta^2 ,$$
$$e_2^{(1)} = (m^4-p^4)\Delta^2 + m^2p^4(k^2-p^2) ,$$
$$e_1^{(2)} = -2(p^2-m^2)^2(k^2-p^2)k\cdot p - (p^4-m^4)\Delta^2 ,$$
$$e_2^{(2)} = (m^4-p^4)\Delta^2 + m^2p^2k^2(k^2-p^2) - 2m^4k\cdot p(k^2-p^2) ,$$
$$e_1^{(3)} = -(p^4-m^4)\Delta^2 + p^2(k^2-p^2)(p^2-m^2)^2 ,$$
$$e_2^{(3)} = (m^4-p^4)\Delta^2 - 2m^2p^2k\cdot p(k^2-p^2) + m^4p^2(k^2-p^2) ,$$
$$e_1^{(4)} = -(p^4-m^4)\Delta^2 + p^2(k^2-p^2)(p^2-m^2)^2 ,$$
$$e_2^{(4)} = (m^4-p^4)\Delta^2 + m^4p^2(k^2-p^2) ,$$
$$e_2^{(5)} = 0 ,$$
$$e_1^{(6)} = (p^4-m^4)(k^2-p^2)\Delta^2 ,$$

$$
\begin{aligned}
e_2^{(6)} &= (p^4 - m^4)(k^2 - p^2)\Delta^2 \ , \\
e_1^{(7)} &= 8mp^2(p^2 - m^2)\Delta^2 \ , \\
e_2^{(7)} &= m[2p^2(p^2 - m^2)\Delta^2 - m^2p^4(k^2 - p^2)] \ , \\
e_1^{(8)} &= 8mp^2(p^2 - m^2)\Delta^2 \ , \\
e_2^{(8)} &= m[2p^2(p^2 - m^2)\Delta^2 + m^2p^2k^2(k^2 - p^2)] \ , \\
e_2^{(9)} &= m^3(k^2 - p^2)[p^2 - 2k\cdot p] \ , \\
e_2^{(10)} &= m^3p^2(k^2 - p^2) \ , \\
e_2^{(11)} &= m^3p^2(k^2 - p^2)[p^2 - k\cdot p] \ , \\
e_2^{(12)} &= m^3(k^2 - p^2)[2(k\cdot p)^2 - p^2(k\cdot p) - p^2k^2] \ , \\
e_1^{i} &= 0, \quad i = 5,9,10,11,12 \ ;
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
f_1^{(1)} &= -e_1^{(2)}(k \leftrightarrow p), \quad f_1^{(2)} = -e_1^{(1)}(k \leftrightarrow p) \ , \\
f_1^{(3)} &= -e_1^{(4)}(k \leftrightarrow p), \quad f_1^{(4)} = -e_1^{(3)}(k \leftrightarrow p) \ , \\
f_1^{(i)} &= 0, \quad i = 5,9,10,11,12 \ , \\
f_1^{(i)} &= -e_1^{(i)}(k \leftrightarrow p), \quad i = 6,7,8 \ , \\
f_2^{(1)} &= -e_2^{(2)}(k \leftrightarrow p), \quad f_2^{(2)} = -e_2^{(1)}(k \leftrightarrow p) \ , \\
f_2^{(3)} &= -e_2^{(4)}(k \leftrightarrow p), \quad f_1^{(4)} = -e_2^{(3)}(k \leftrightarrow p) \ , \\
f_2^{(9)} &= -e_2^{(10)}(k \leftrightarrow p), \quad f_2^{(10)} = -e_2^{(9)}(k \leftrightarrow p) \ , \\
f_2^{(11)} &= e_2^{(12)}(k \leftrightarrow p), \quad f_2^{(12)} = e_2^{(11)}(k \leftrightarrow p) \ , \\
f_2^{(i)} &= -e_2^{(i)}(k \leftrightarrow p), \quad i = 5,6 \ ,
\end{aligned}
$$

$$
\begin{aligned}
f_2^{(7)} &= m\left\{(k^2 - p^2)[-k^2p^2 - 2k^2k\cdot p + 4(k\cdot p)^2]m^2 \right. \\
&\qquad \left. -2k^2(k^2 - m^2)\Delta^2\right\} \ , \\
f_2^{(8)} &= m\left\{[-2k^2m^2(k^2 - p^2)k\cdot p - k^4m^2(k^2 - p^2)] \right. \\
&\qquad \left. -2k^2(k^2 - m^2)\Delta^2\right\} \ ;
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
g_1^{(i)} &= 0, \quad i = 5,7,8,9,10,11,12 \ , \\
g_1^{(i)} &= 2k\cdot p, \quad i = 1,2 \ , \\
g_1^{(i)} &= -(k^2 + p^2), \quad i = 3,4 \ , \\
g_1^{(6)} &= -2\Delta^2 \ , \\
g_2^{(i)} &= 0, \quad i = 1,\ldots,12 \ ;
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
h_1^{(i)} &= 0, \quad i = 1,2,4,9,10,11,12 \ , \\
h_1^{(3)} &= -4\Delta^2 \ , \\
h_1^{(5)} &= 2\Delta^2 \ , \\
h_1^{(6)} &= -2m\Delta^2 \ , \\
h_1^{(i)} &= 4m\Delta^2, \quad i = 7,8 \ , \\
h_2^{(i)} &= 0, \quad i = 5,6,11,12 \ , \\
h_2^{(1)} &= h_2^{(2)}(k \leftrightarrow p) = p^2[m^2 - k\cdot p]\Delta^2 \ , \\
h_2^{(3)} &= h_2^{(4)}(k \leftrightarrow p) = k^2p^2 - m^2k\cdot p \ , \\
h_2^{(7)} &= h_2^{(8)}(k \leftrightarrow p) = m[-p^2k\cdot p + \Delta^2 + (k\cdot p)^2] \ , \\
h_2^{(9)} &= h_2^{(10)}(k \leftrightarrow p) = m[k^2 - k\cdot p] \ ;
\end{aligned} \tag{B8}
$$

$$
\begin{aligned}
l_1^{(1)} &= l_1^{(2)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2k^2p^2} - \left(m^2\frac{(k\cdot p)}{k^2} + p^2\right) \ , \\
l_1^{(3)} &= l_1^{(4)}(k \leftrightarrow p) = (m^2 + k\cdot p) + \frac{m^2\Delta^2}{2k^2p^2} \ , \\
l_1^{(i)} &= 0, \quad i = 5,7,8,9,10,11,12 \ , \\
l_1^{(6)} &= \left[-1 - \frac{m^2}{2}\left(\frac{1}{p^2} + \frac{1}{k^2}\right)\right]\Delta^2 \ , \\
l_2^{(1)} &= l_2^{(2)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2k^2p^2} - m^2\frac{k\cdot p}{k^2} + p^2 \ , \\
l_2^{(3)} &= l_2^{(4)}(k \leftrightarrow p) = \frac{m^2\Delta^2}{2k^2p^2} + m^2 - k\cdot p \ , \\
l_2^{(5)} &= 0 \ , \\
l_2^{(6)} &= \left[1 - \frac{m^2}{2}\left(\frac{1}{p^2} + \frac{1}{k^2}\right)\right]\Delta^2 \ , \\
l_2^{(7)} &= m\left[-k\cdot p + 2\frac{(k\cdot p)^2}{k^2} - p^2\right] \ , \\
l_2^{(8)} &= m\left[-k\cdot p + k^2\right] \ , \\
l_2^{(9)} &= l_2^{(10)}(k \leftrightarrow p) = m\left[-\frac{k\cdot p}{p^2} + 1\right] \ , \\
l_2^{(11)} &= -l_2^{(12)}(k \leftrightarrow p) = m\left[\frac{(k\cdot p)^2}{k^2} - p^2\right] \ .
\end{aligned} \tag{B9}
$$

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