

Towards solving QCD: The transverse zero modes in light-cone quantization

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We formulate QCD in $d+1$ dimensions using Dirac's front form with periodic boundary conditions, that is, within discretized light-cone quantization. The formalism is worked out in detail for $SU(2)$ pure glue theory in 2+1 dimensions which is approximated by restriction to the lowest *transverse* momentum gluons. The dimensionally reduced theory turns out to be an $SU(2)$ gauge theory coupled to adjoint scalar matter in 1+1 dimensions. The scalar field is the remnant of the transverse gluon. This field has modes of both nonzero and zero *longitudinal* momentum. We categorize the types of zero modes that occur into three classes, dynamical, topological, and constrained, each well known in separate contexts. The equation for the constrained mode is explicitly worked out. The Gauss law is rather simply resolved to extract physical, namely, color singlet states. The topological gauge mode is treated according to two alternative scenarios related to the elimination of the cutoff. In one, a spectrum is found consistent with pure $SU(2)$ gluons in 1+1 dimensions. In the other, the gauge mode excitations are estimated and their role in the spectrum with genuine Fock excitations is explored. A color singlet state is given which satisfies Gauss' law. Its invariant mass is estimated and discussed in the physical limit.

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I. INTRODUCTION

It remains notoriously difficult to understand the low-energy regime of quantum chromodynamics (QCD) in terms of the simplistic but otherwise successful constituent quark picture. In line with the formulation of Feynman's parton model [1] in the infinite momentum frame [2,3], a promising approach could be that of Pauli and Brodsky [4,5] which adapts Dirac's "front form" Hamiltonian dynamics [6] for nonperturbative quantum field theory. Perhaps a misnomer, the method carries the name discretized light-cone quantization (DLCQ). Numerous applications have been carried out with reasonable success in extracting bound state spectra and wave functions for both 1+1 dimensions in Abelian [7,8] and non-Abelian gauge theories [9–13] as well as for higher space-time dimensions [14–18]. DLCQ combines the aspects of a simple vacuum [2] with a careful treatment of the infrared degrees of freedom. The latter are controlled by the finite volume regularization of the method. There is legitimate hope that one can thus get both a manageable treatment of the "vacuum problem" and explicit invariant mass spectra and wave functions for physical particles.

In this paper we concern ourselves with more than just the vacuum problem, which in this context manifests itself as the so-called "zero mode problem." Contrary to the original expectations of the framers of DLCQ, the zero momentum modes of the Lagrangian field operators have proved far more than just a "set of measure zero." Indeed, in the ϕ^4 theory in 1+1 dimensions they are crucial in reproducing the vacuum properties of the theory,

namely, spontaneous symmetry breaking and a vacuum condensate [19–25]. This is achieved in the DLCQ framework by the property of the zero mode of ϕ being, not dynamical, but satisfying a constraint [26]. This preserves the simplicity of the vacuum and thus the partonic picture of light-cone field theory. The desired symmetry is *explicitly* broken upon solution of the constraint by nonperturbative approximation and substitution in the Hamiltonian [23–25]. Now the four-point coupling of *gluons* suggests at least some aspect of this feature will be present in (3+1)-dimensional QCD (QCD_{3+1}). We thus set out in the sequence of papers [27–30] to *disentangle* the constraint problem from that of the gauge symmetry of the non-Abelian theory. Constrained zero modes occur even in Abelian theory [27,28] for which one requires gauge fixing to solve [29] and that gauge choice may *not* in general be that on the nonzero modes. But not all zero modes in a gauge theory are constrained. Less significant in QED_{3+1} , some of these dynamical modes are intimately related to the nontrivial topology of both the hypertorus implicit in DLCQ and become important in the presence of non-Abelian gauge groups [30]. We thus find a diverse range of zero mode types, all of which will evidently be present in QCD_{3+1} . Treating these types *together* in a single theory, albeit a still simplified one, is the subject of this paper.

The essential principle we shall use for getting to simpler theories from QCD_{3+1} was espoused in [30]. Lower dimensional "regimes" of a higher dimensional theory can be systematically explored because of the finite volume regularization: zero and nonzero momentum modes can be cleanly distinguished and so one can, for exam-

ple, excite one and not the other. One thus obtains effective theories in lower space-time dimensions that are not identical to the original theory defined *a priori* in lower dimensions. This is essentially “dimensional reduction” [31,32]. A similar idea for the instant form was recently suggested by [33]. In [30] we examined (1+1)-dimensional pure SU(2) gauge theory coupled to external sources in DLCQ. We suppressed *all* momentum excitations and obtained a (0+1)-dimensional, quantum mechanical, theory of a single gluon zero mode whose dynamics depended on external sources coupled to the gluons. This mode corresponds to the quantized flux loop around the circle defining space. The mode is purely of topological origin, and thus the theory was manifestly isomorphic to a quantization in the instant form defined on the analogous topology [34,35]. This is one of the types of zero modes discussed above.

The “next step,” taken in this paper, is to begin with (2+1)-dimensional SU(2) theory and looking at the nested (1+1)-dimensional theory by suppressing transverse gluon momentum excitations. The topological mode appears here again, but now coupled to true dynamical field modes that are the Fock modes of the transverse gluon component. As well, other types of zero modes appear. This is now the simplest type of non-Abelian gauge theory we can construct in which all the types of zero modes encountered previously couple together into a nontrivial dynamical problem. Many of the structures we unveil here were already foreseen in 1981 by Franke *et al.* [36,37] in 3+1 dimensions. It will become clear in the course of the present work that these types of models, first discussed in DLCQ by [11,12] but without zero modes and assuming only color singlet string states, enable insight into how to overcome the obstacles that impeded Franke.

The aim of this work then is twofold. The first is a formal aim: to show how a treatment of the non-Abelian gauge theory can be achieved which keeps the advantages

of the front-form approach while controlling the infrared problem. This we succeed in doing insofar as we can give a Hamiltonian in which the nature of all modes, zero and nonzero, is clarified and their means of solution at least understood. Sections II–IV deal with these formal aspects. In particular, in Sec. III we make the restriction in the gauge fixing to the so-called “fundamental modular domain” [38]. The second aim is to gain insight into the physical spectrum of the aforementioned Hamiltonian of the pure glue theory by at least semianalytic methods. Here we relax rigor and make several simplifying assumptions within the context of a cutoff regularization of the large momentum region of the theory. A point-splitting treatment will be presented elsewhere [39]. Several insights into the spectrum are obtained. The purely contraction parts of the Hamiltonian lead to the potential for the Schrödinger equation in the gauge zero mode sector. This is the analogue for the problem we solved in [30]. Two alternative methods are described for dealing with the singular structure of the potential, either keeping the cutoff or “renormalizing” the potential. In both scenarios, we are able to simultaneously diagonalize the energy and momentum operators. This is discussed in Sec. V. However, in the absence of a definite counterterm for the renormalization approach we use the solution to the gauge zero mode keeping the cutoff and implement it in the particle sector of the theory. We give a color singlet state which is an eigenstate to part of the Hamiltonian. The invariant mass squared of this state is seen to diverge when all cutoffs are taken to their physical values. We comment on this in the discussion.

II. FORMULATION FOR PURE SU(N) GAUGE THEORY

Consider an SU(N) gauge theory without fermions in $d + 1$ dimensions defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) \quad \text{with} \quad \mathbf{F}^{\mu\nu} \equiv \partial^\mu\mathbf{A}^\nu - \partial^\nu\mathbf{A}^\mu + ig[\mathbf{A}^\mu, \mathbf{A}^\nu] \equiv \partial^\mu\mathbf{A}^\nu - \mathbf{D}^\nu\mathbf{A}^\mu. \quad (1)$$

The \mathbf{A}^μ are the SU(N) *vector potentials*. We shall reserve the term “gauge potential” for something else, discussed below. The energy-momentum tensor is derived from Eq. (1) in the usual way [40] by $\Theta^{\mu\nu} = 2\text{Tr}(\mathbf{F}^{\mu\kappa}\mathbf{F}_{\kappa}^\nu) - g^{\mu\nu}\mathcal{L}$. This and the notation are explained in more detail in Appendix A. But at this stage we keep the discussion general for SU(N) for arbitrary number of colors N . In the *front form*, it is convenient to separate the Lorentz indices $\mu(\nu)$ into *longitudinal* values $\alpha(\beta) = +, -$ and *transversal* values $j(i) = 2, 3, \dots, d$. The indicial sums in the Lagrangian and the (light-cone) energy density Θ^{+-} then disentangle cleanly:

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(\mathbf{F}^{\alpha\beta}\mathbf{F}_{\alpha\beta} + \mathbf{F}^{ij}\mathbf{F}_{ij} + 2\mathbf{F}^{\alpha j}\mathbf{F}_{\alpha j})$$

and

$$\Theta^{+-} = \frac{1}{2}\text{Tr}(\mathbf{F}^{\alpha\beta}\mathbf{F}_{\beta\alpha} + \mathbf{F}^{ij}\mathbf{F}_{ij}), \quad (2)$$

respectively. The dimensionality of the problem is not manifest but resides in the dimensions of the fields. Working in dimensions of length l , we have for the field and coupling the usual $\dim[\mathbf{A}^\mu] \equiv l^{(1-d)/2}$ and $\dim[g] \equiv l^{(d-3)/2}$. The energy density in 3+1 and 2+1 dimensions has the simple structure

$$\Theta_{3+1}^{+-} = \text{Tr}(\mathbf{F}^{-+}\mathbf{F}^{+-} + \mathbf{F}^{23}\mathbf{F}^{23})$$

and

$$\Theta_{2+1}^{+-} = \text{Tr}(\mathbf{F}^{-+}\mathbf{F}^{+-}). \quad (3)$$

As in previous work [27,28] it is convenient to disentangle such an expression into “zero modes” and “normal modes.” A *zero mode* of some function $f(x^-, x_\perp)$ with respect to any one of the space coordinates, say y with interval length L_y , is defined by

$$\langle f(\bar{x}) \rangle_0 \equiv \frac{1}{2L_y} \int_{-L_y}^{L_y} dy f(\bar{x}, y), \quad (4)$$

where \bar{x} are the remaining spatial coordinates not being integrated over. When y is the longitudinal direction x^- we shall denote the zero mode by \bar{f} . The *normal mode* is, in general, the complement $\bar{f}^n = f - \langle f \rangle_0$. We see then that since P^- involves an integration of the density with respect to the spatial coordinates x^i , it is like evaluating the transversal “zero mode” of the energy density. Taking the zero mode with respect to any spacelike coordinate, one realizes that the Hamiltonian is *additive* in the zero and normal mode contributions: i.e.,

$$\begin{aligned} \text{Tr}(\mathbf{F}^{-+} \mathbf{F}^{-+})_0 &\equiv \text{Tr}(\langle \mathbf{F}^{-+} \rangle_0 \langle \mathbf{F}^{-+} \rangle_0) \\ &+ \text{Tr}(\bar{\mathbf{F}}^{-+} \bar{\mathbf{F}}^{-+})_0. \end{aligned} \quad (5)$$

Of course one should not take this too far, since zero and normal modes of the individual quantum fields can (and do) reside in both terms of this expression. However the separation allows for a conceptual simplification: By lack of insight, the original formulation of discretized light-cone quantization (DLCQ) was formulated in terms of *only* the normal modes. It should be useful and even complementary to analyze the theory in terms of only the transversal zero modes.

We therefore consider a model which only has *transversal zero modes* by requiring

$$\partial_i \mathbf{A}^\mu = 0 \quad \text{for all } \mu. \quad (6)$$

This is to be regarded as a genuine dynamical restriction on the full theory. Of course this theory will now involve both zero and normal mode longitudinal gluon excitations. Since the lengths L_\perp and L are now decoupled in scale, it is convenient to readjust units by scaling out the transverse length L_\perp :

$$\mathbf{A}^\mu \longrightarrow \tilde{\mathbf{A}}^\mu = \mathbf{A}^\mu (2L_\perp)^{(d-1)/2}$$

and

$$g \longrightarrow \tilde{g} = g(2L_\perp)^{(1-d)/2}. \quad (7)$$

In the sequel the tilde is dropped. The dimensionality resides then only in the Lorentz indices. For simplicity we now restrict ourselves to consider the original theory as having been formulated in 2+1 dimensions. The result of the assumption Eq. (6) is to have *dimensionally reduced* the 2+1 theory to 1+1 dimensions. However a reminder of the original 2+1 structure resides in the continued presence of the transverse gluon component \mathbf{A}^\perp . In the spirit of Siegel [31,32], who introduced dimensional reduction for regularizing supersymmetric theories, we identify this gluon as a *scalar* field Φ transforming under the adjoint representation of the color group. To completely avoid playing with Lorentz indices we go further and introduce the notation

$$\mathbf{A}^\mu = (\mathbf{A}^+, \mathbf{A}^-, \mathbf{A}^\perp) \equiv (\mathbf{V}, \mathbf{A}, \Phi). \quad (8)$$

The model theory then takes the form of a (1+1)-dimensional non-Abelian gauge theory covariantly coupled to scalar adjoint matter:

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{2} \mathbf{F}^{\alpha\beta} \mathbf{F}_{\alpha\beta} + \mathbf{D}^\alpha \Phi \mathbf{D}_\alpha \Phi \right). \quad (9)$$

The covariant derivative \mathbf{D}_α is implicitly defined in Eq. (1). A similar treatment of this theory in DLCQ was recently given by [11,12]. The equations of motion in the two parts of the theory can be deduced from the sourceless color Maxwell equations and are

$$\begin{aligned} \mathbf{D}_\beta \mathbf{F}^{\beta\alpha} &= g \mathbf{J}_M^\alpha, \quad \text{with } \mathbf{J}_M^\alpha = -i[\Phi, \mathbf{D}^\alpha \Phi] \quad \text{and} \\ \mathbf{D}^\alpha \mathbf{D}_\alpha \Phi &= 0. \end{aligned} \quad (10)$$

Note that the “matter current” \mathbf{J}_M^α is not conserved, $\partial_\alpha \mathbf{J}_M^\alpha \neq 0$, whereas the total “gluon current” $\mathbf{J}_G^\alpha = \mathbf{J}_M^\alpha - i[\mathbf{F}^{\alpha\beta}, \mathbf{A}_\beta]$ is conserved. To quantize the theory in terms of as few redundant degrees of freedom as possible it is essential to fix the gauge. We follow the procedure given in [30] and find that A^+ only has a zero mode, i.e., $\partial_- \mathbf{A}^+ = 0$. At this point we specialize to SU(2). Then a single rotation in color space suffices to diagonalize the SU(2) color matrix \mathbf{A}^+ . The simple way to see this mode cannot be gauged away is that it is related to the Wilson loop for a contour (line) along the x^- space: a gauge invariant quantity that cannot be set to a fixed value by gauge choice. There remain a set of “large” gauge transformations which generate shifts in \mathbf{V} known as Gribov copies [41,42]. These matters are discussed further later and in Appendix B. In the instant form this gauge has been used by a range of authors, [34,35,43,44] to list a few. In a context related to the front form it has also been used by [45]. Finally, the diagonal zero mode of $\mathbf{A}^-(x_0^+)$ can be gauged away [30] at some fixed light-cone time x_0^+ . For writing the Hamiltonian later, it is convenient to choose this time as $x_0^+ = 0$, the null-plane initial value surface on which we specify the independent fields.

In this gauge then, $\mathbf{F}^{-+} = \partial_+ \mathbf{V} - \mathbf{D}_- \mathbf{A}$. The first of our three equations of motion, $\alpha = +$, is simply Gauss’ law, $\mathbf{D}_- \mathbf{F}^{-+} = -\mathbf{D}_-^2 \mathbf{A} = g \mathbf{J}_M^+$, realized here as a *second class* constraint in the nomenclature of Dirac [46–48]. In the absence of gauge fixing these are first-class constraints. They are a consequence of, and generate, gauge symmetry. With the gauge fixing, these can be realized as quantum operator constraints with an exception which we discuss below. This aside, we can be cavalier and solve this equation strongly yielding

$$\mathbf{A} = -g \frac{1}{\mathbf{D}_-^2} \mathbf{J}_M^+. \quad (11)$$

The exception is a remnant first-class constraint, namely, the zero mode diagonal part of Gauss’ law. The reason for this is that in selecting out the color-three direction we have factored the group $\text{SO}(2) \otimes \text{SO}(2)$ from SU(2) [locally isomorphic to SO(3)], leaving a subgroup of SO(2) rotations which leave the preferred axis invariant. This

itself is isomorphic to $U(1)$. In other words we have a residual symmetry with respect to global Abelian gauge transformations. It means in the quantum theory there are redundant states in the Hilbert space corresponding to the “fibers” or “orbits” in gauge configuration space. Unique representatives on these orbits must be selected by projecting a “reduced Hilbert space” out of the larger space by requiring the selected, so-called “physical” [46], state vectors have vanishing “charge” $Q_0|\text{phys}\rangle = 0$. The charge here is the zero mode of the matter density $\mathbf{J}_{M,\text{diag}}^+$. Although this does not necessarily imply that $|\text{phys}\rangle$ is an actual state in the physical *spectrum*, states outside this subspace render the theory *a priori* inconsistent. Analogous constraints can be found in many other contexts, for example, in [49].

The second Yang-Mills equation of motion, with $\alpha = -$, is a genuine dynamical equation for \mathbf{V} , i.e.,

$$\partial_+ \partial_+ \mathbf{V} + 2gi[\mathbf{A}, \partial_+ \mathbf{V}] - \partial_+ \partial_- \mathbf{A} - ig[\mathbf{A}, \partial_- \mathbf{A}] - ig[\mathbf{V}, \partial_+ \mathbf{A}] + g^2[\mathbf{A}, [\mathbf{V}, \mathbf{A}]] = -g\mathbf{J}_M^- . \quad (12)$$

Since we are ultimately interested in a Hamiltonian treatment of the dynamics of \mathbf{V} , we do not address ourselves to solving dynamical Euler-Lagrange equations.

Next we discuss the equations in the scalar field sector, $\mathbf{D}^\alpha \mathbf{D}_\alpha \Phi = 0$, which reads explicitly

$$\partial_- (\partial_+ + \mathbf{D}_+) \Phi + ig(\partial_+ [\mathbf{V}, \Phi] + [\mathbf{A}^\alpha, \mathbf{D}_\alpha \Phi]) = 0 . \quad (13)$$

The zero mode of its color-diagonal part will turn out to be a true operator constraint equation, occurring as second class in the Dirac procedure.

We complete this overview of the theory with the Hamiltonian and light-cone momentum operator. One calculates them to be, respectively,

$$P^- = \int_{-L}^{+L} dx^- \text{Tr} (\partial_+ \mathbf{V} - \mathbf{D}_- \mathbf{A})^2 = \int_{-L}^{+L} dx^- \text{Tr} \left(\partial_+ \mathbf{V} \partial_+ \mathbf{V} - g^2 \mathbf{J}_M^+ \frac{1}{\mathbf{D}_-^2} \mathbf{J}_M^+ \right) , \quad (14)$$

$$P^+ = 2 \int_{-L}^{+L} dx^- \text{Tr} (\mathbf{D}_- \Phi)^2 \equiv \frac{\pi}{L} \hat{K} . \quad (15)$$

The quantity \hat{K} is independent of the two-dimensional parameters L and g . The Hamiltonian describes the interaction of two scalar matter currents via an instantaneous gluonlike interaction [11,12]. The instantaneous gluon is “dressed” by the zero mode of \mathbf{A}^+ . This zero mode, a color singlet object as shown in Appendix B, acts like a “screening mass.” Above all, $1/\mathbf{D}_-^2$ is never singular. The system has what we will call a set of pseudo-ground-states generated by the zero mode operator \mathbf{V} . All these states have zero P^+ but only one of them is

the true ground state having zero P^- upon subtraction of the zero point energy. On all these pseudovacua there will be a rich structure of matter states that are allowed by Gauss’ law. For example, they include, but are not exhausted by the string states discussed in [11,12].

III. QUANTIZATION AND MATTER CURRENTS FOR $SU(2)$ GAUGE THEORY

The construction of the field Φ is quite complicated in $SU(N)$ for any $N > 2$. This is the primary stumbling block to formulating a large N analysis in the presence of the zero mode of \mathbf{A}^+ . The extension to $SU(N)$ in the Chevalley basis [50,51] is easier and will be presented in a future work. In the sequel we will thus proceed to analyze the model for $SU(2)$. We will use a color helicity basis of the form

$$\Phi = \tau^3 \varphi_3 + \tau^+ \varphi_+ + \tau^- \varphi_- \quad (16)$$

for all field matrices. This is explained in more detail in Appendix A.

By gauge choice, the zero mode matrix \mathbf{V} is diagonal; thus, $\mathbf{V} = v\tau_3$. The component $v \equiv v(x^+)$ is a quantum-mechanical operator which we treat in the manner of [52]. We have previously encountered it in DLCQ in [30] where, we showed that in the absence of dynamical quanta it is the basis of a topological field theory isomorphic to equal-time quantization. The quantum v has a conjugate momentum $p \equiv \delta L/\delta v = 2L\partial_+ v$ and satisfies the commutation relation

$$[v, p] = [v, 2L\partial_+ v] = i . \quad (17)$$

In the following it will be useful to invoke the dimensionless combination

$$z \equiv \frac{gvL}{\pi} . \quad (18)$$

Gribov copies then correspond to $z \rightarrow z + n_0$ for n_0 some even integer. The odd integers are “copies” generated by the group of center conjugations of $SU(2)$, namely, Z_2 symmetry [53]. The finite interval $0 < z < 1$ is called the *fundamental modular domain*, see for example [38]. We emphasize the otherwise trivial fact that z is an operator, better denoted \hat{z} . In the subsequent analysis it is understood that we work in a representation which diagonalizes that operator. Thus $\hat{z}|z'\rangle = z'|z'\rangle$ and $\langle z'|\hat{p}|z''\rangle = -i\frac{\partial}{\partial z'}\delta(z' - z'')$. In the sequel we shall drop the δ functions, that is, we shall work in Schrödinger representation.

The diagonal components of the Hermitian color matrix Φ are φ_3 . The quantization, with the exception of the zero mode $\hat{\varphi}_3 = a_0/\sqrt{4\pi}$, is canonical. Any real-valued boson field subject to periodic boundary conditions can be represented by

$$\varphi_3(x^+, x^-) = \frac{a_0(x^+)}{\sqrt{4\pi}} + \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} [a_n(x^+) w_n e^{-in\frac{\pi}{L}x^-} + a_n^\dagger(x^+) w_n e^{+in\frac{\pi}{L}x^-}] . \quad (19)$$

The factor $1/\sqrt{2\pi}$ is of course arbitrary but will turn out to be convenient. The notation $a_n(x^+)$ should indicate that the creation and destruction operators depend on the (light-cone) time. The momentum field conjugate to φ_3 is pure normal mode since $\pi^3 = \partial_- \varphi_3$. The quantum commutation relation at equal x^+ for the normal modes is

$$\frac{1}{i} [\varphi_3(x), \pi^3(y)]_{x^+=y^+} = \frac{1}{2} \left[\delta(x^- - y^-) - \frac{1}{2L} \right], \quad (20)$$

where the last term ensures consistency for the commutator restricted to normal mode fields [27]. The Fock modes must consequently satisfy $[a_n, a_m^\dagger] = \delta_n^m$ ($n, m = 1, \dots, \infty$) and the coefficients must be $w_n = 1/\sqrt{n}$. The Kronecker δ_n^m is equivalent to $\delta_{n,m}$. The commutation relations of the zero mode $a_0 = a_0^\dagger$ cannot be determined by any of these relations: the mode obeys the “constraint equation” given below, with a commutator $[a_0, a_n] \neq 0$ that must be solved for via the constraint.

The off-diagonal components of Φ are complex valued operators with $\varphi_+(x^+, x^-) = \varphi_-^\dagger(x^+, x^-)$. With $\pi^- = (\partial_- + igv)\varphi_+$ as the momentum field conjugate to φ_- , and $\pi^+ = (\partial_- - igv)\varphi_-$ conjugate to φ_+ , they obey the canonical commutation relations

$$\begin{aligned} \frac{1}{i} [\varphi_-(x), \pi^-(y)]_{x^+=y^+} &= \frac{1}{i} [\varphi_+(x), \pi^+(y)]_{x^+=y^+} \\ &= \frac{1}{2} \delta(x^- - y^-). \end{aligned} \quad (21)$$

$$m_0(z) = \text{st}(z) - \frac{1}{2}, \quad \zeta(z) = z - m_0(z)$$

Since $\text{st}(z) + \text{st}(-z) = 1$, we can derive the fundamental relations

$$\begin{aligned} m_0(z+1) &= m_0(z) + 1, & m_0(-z) &= -m_0(z), \\ \zeta(z+1) &= \zeta(z), & \zeta(-z) &= -\zeta(z). \end{aligned} \quad (25)$$

Important is that $-\frac{1}{2} < \zeta(z) < \frac{1}{2}$ for all values of z because this allows one to rewrite Eq. (23) as

$$\begin{aligned} \varphi_-(x) &= \frac{e^{+im_0 \frac{\pi}{L} x^-}}{\sqrt{4\pi}} \sum_{m=\frac{1}{2}}^{\infty} \left(b_m u_m e^{-im \frac{\pi}{L} x^-} \right. \\ &\quad \left. + d_m^\dagger v_m e^{+im \frac{\pi}{L} x^-} \right), \end{aligned} \quad (26)$$

i.e., in terms of particle and antiparticle operators, b_m and d_m , respectively. The analogy with the (complex) Dirac spinor components [7] is intentional, but the present particles obey boson commutation relations

$$[b_n, b_m^\dagger] = [d_n, d_m^\dagger] = \delta_n^m$$

and

$$[b_n, d_m] = [b_n, d_m^\dagger] = 0. \quad (27)$$

Following Franke [36,37], this is achieved by the expansion over the momentum modes

$$\varphi_-(x^+, x^-) = \sum_{n \in Z} \frac{\tilde{C}_n(x^+)}{\sqrt{4\pi|n+z|}} e^{-in \frac{\pi}{L} x^-}$$

with

$$[\tilde{C}_n(x^+), \tilde{C}_{n'}^\dagger(x^+)] = \delta_n^{n'} \text{sgn}(n+z), \quad (22)$$

where the set of all integers is denoted by $Z \equiv \{0, \pm 1, \pm 2, \dots, \pm \infty\}$, as opposed to set of all half-integers $H \equiv \{\pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm \infty\}$, to be used below. We want to go beyond Franke by explicitly introducing particle and hole operators. But this is met with difficulties, because $\text{sgn}(n+z)$ is a very asymmetric function of n for arbitrary values of z . It is however always possible to shift the summation index in Eq. (22) by $n = m - m_0$, with an arbitrary but given value of $m_0 \equiv m_0(z)$. Relabeling the operators $\tilde{C}_{m-m_0} \equiv C_m$ by matter of convention gives identically

$$\varphi_-(x) = \frac{e^{+im_0 \frac{\pi}{L} x^-}}{\sqrt{4\pi}} \sum_m \frac{C_m(x^+)}{\sqrt{|m+z-m_0|}} e^{-im \frac{\pi}{L} x^-}$$

with

$$[C_m, C_{m'}^\dagger] = \delta_m^{m'} \text{sgn}(m+z-m_0). \quad (23)$$

Now we need only define the shift constant $m_0(z) \in H$ in terms of the “stair-function” $\text{st}(z)$

$$\text{with } \text{st}(z) = \begin{cases} [z] + 1 & \text{for } z \geq 0, \\ [z] & \text{for } z < 0. \end{cases} \quad (24)$$

The real coefficients $u_m(z) = 1/\sqrt{m+\zeta}$ and $v_m(z) = 1/\sqrt{m-\zeta}$ depend on z through ζ . Finally one notes that a large gauge transformation $z \rightarrow z+1$ produces only $m_0 \rightarrow m_0+1$ and thus only a change of the overall phase in Eq. (26). Most importantly it does *not* change the particle-hole assignment and thus the Fock vacuum defined with respect to b_m and d_m is *invariant* under these transformations.

The Gauss law can be rewritten in terms of its explicit components, namely,

$$-\partial_-^2 A_3 = gJ_3^+, \quad -(\partial_- + igv)^2 A_+ = gJ_+^+, \quad (28)$$

and the Hermitian conjugate of the latter with $(J_+^+)^\dagger \equiv J_-^+$. One would like to invert them to express A_3 and A_\pm in terms of the currents $J \equiv J_M$, which according to Eq. (10) are defined as

$$J_3^+ = \frac{1}{i} (\varphi_+ \pi_- - \varphi_- \pi_+)_s$$

and

$$J_+^+ = \frac{1}{i} (\varphi_3 \pi_+ - \varphi_+ \pi_3)_s. \quad (29)$$

The index s indicates that noncommuting operators in

this product and in general must be symmetrized in order to preserve Hermiticity. Before the inversion of Eqs. (28) one should investigate the zero mode structure of the currents. An effective way to do so is to consider their discrete Fourier transforms \tilde{J} , defined for convenience by

$$J_3^+(x^-) \equiv -\frac{1}{4L} \sum_{k \in Z} e^{-ik\frac{x^-}{L}} \tilde{J}_3^+(k)$$

and

$$J_{\pm}^+(x^-) \equiv -\frac{1}{4L} \sum_{k \in H} e^{-ik\frac{x^-}{L}} \tilde{J}_{\pm}^+(k). \quad (30)$$

One verifies that $[\tilde{J}_3^+(k)]^\dagger = \tilde{J}_3^+(-k)$ and $[\tilde{J}_{\pm}^+(k)]^\dagger = \tilde{J}_{\pm}^+(-k)$. In order to notationally disentangle a_0 from the dynamic modes we introduce composite ‘‘charge operators’’ Q which are independent of a_0 and the symmetrized remainders $B_k \equiv (a_0 b_k + b_k a_0)/2$ and $D_k \equiv (a_0 d_k + d_k a_0)/2$: i.e.,

$$\tilde{J}_3^+(k) \equiv Q_3(k), \quad \tilde{J}_+^+(k) \equiv Q_+(k) + \frac{D_k}{v_k},$$

and

$$-\tilde{J}_-^+(k) \equiv Q_+(k) + \frac{B_k}{u_k}. \quad (31)$$

The explicit expressions for the operators Q can be found in Appendix C.

Because of the boundary conditions, the first of the Gauss equations (28) can be solved only if the zero mode $\langle J_3^+ \rangle_0 \equiv Q_0$ on the right-hand side (RHS) vanishes. This cannot be satisfied as an operator, but must be used to select out physical states, i.e., $Q_0|\text{phys}\rangle \equiv 0$. In second-quantized form this gives

$$\sum_{k=\frac{1}{2}}^{\infty} \left[u_k^2 \left(B_k^\dagger b_k + B_k b_k^\dagger \right)_s + v_k^2 \left(D_k^\dagger d_k + D_k d_k^\dagger \right)_s \right] = - \sum_{k=\frac{1}{2}}^{\infty} \{ u_k^3 [Q_-^\dagger(k) b_k + b_k^\dagger Q_-(k)] + v_k^3 [Q_+^\dagger(k) d_k + d_k^\dagger Q_+(k)] \}. \quad (35)$$

This is the most compact expression for the constraint. It is clearly linear in a_0 and therefore quite different in structure from the constraint equation of ϕ_{1+1}^4 . It is not clear then how it could give rise to spontaneous symmetry breaking in the scenario of [19–25]. Despite its linearity, Eq. (35) is still complicated to solve as a quantum operator constraint. At this point, we therefore isolate this part of the overall problem and return to it in a future treatment. For example, under active consideration now is a solution via Fock space truncation methods as employed in [23–25].

We end this section by summarizing the zero mode ‘‘zoo’’ of this theory. The complex field φ_+ actually has no true zero mode in the sense of vanishing eigenvalue of P^+ . Second, there is the topological zero mode z which must be treated by diagonalizing its Hamiltonian. Third, there is the constrained zero mode, a_0 . It must be solved

$$Q_0|\text{phys}\rangle = \sum_{m=\frac{1}{2}}^{\infty} \left(b_m^\dagger b_m - d_m^\dagger d_m \right) |\text{phys}\rangle = 0. \quad (32)$$

It is thus simple to find states satisfying this: they must have the same total number of b and d particles. The resemblance to the electric-charge neutrality condition is because the residual global gauge symmetry we are factoring out of the Hilbert space is, as mentioned earlier, Abelian.

To complete the specification of the Hilbert space we give the momentum operator in second quantized form. Evaluating the trivial integrals in Eq. (15), one gets the dimensionless operator

$$\hat{K} = \sum_{n=1}^{\infty} n a_n^\dagger a_n + \sum_{m=\frac{1}{2}}^{\infty} \left[(m + \zeta) b_m^\dagger b_m + (m - \zeta) d_m^\dagger d_m \right]. \quad (33)$$

We finally turn to the constraint equation. Taking the zero mode of the matter field equation (13) explicitly yields three equations. Two of them give true dynamical equations for the zero modes $\tilde{\varphi}_{\pm}$, which for reasons given above we do not wish to solve explicitly so we do not give them here. The third equation, with $c \equiv 8\pi \text{Tr} \langle \tau^3 \mathbf{D}^\alpha \mathbf{D}_\alpha \Phi \rangle_0 / g^2$, becomes, after some algebra,

$$c = \left\langle \varphi_+ \frac{1}{(\partial_- - igv)} J_-^+ - \varphi_- \frac{1}{(\partial_- + igv)} J_+^+ \right\rangle_{0,s} = 0, \quad (34)$$

and is the constraint equation. Insertion of the above yields

at the level of the constraint equation (35) and its presence in the Hamiltonian eliminated in favor of the true degrees of freedom.

IV. THE HAMILTONIAN FOR PURE SU(2) GAUGE THEORY

The front-form Hamiltonian, Eq. (14), rewritten in terms of the components becomes

$$P^- = L(\partial_+ v)^2 + \frac{1}{2} \int_{-L}^{+L} dx^- \left[\partial_- A_3 \partial_- A_3 + (\partial_- + igv) A_+ (\partial_- - igv) A_- + (\partial_- - igv) A_- (\partial_- + igv) A_+ \right]. \quad (36)$$

Note that the zero mode of A_3 does not occur since this

field always appears here acted upon by a space derivative. Because of periodic boundary conditions one can integrate by parts and eliminate A in favor of $J \equiv J_M^+$ using Eq. (28): i.e.,

$$P^- = \frac{p^2}{4L} - \frac{g^2}{2} \int_{-L}^{+L} dx \left[\tilde{J}_3 \frac{1}{\partial_-^2} \tilde{J}_3 + J_+ \frac{1}{(\partial_- - igv)^2} J_- + J_- \frac{1}{(\partial_- + igv)^2} J_+ \right]. \quad (37)$$

Notice that the operator has no ill-defined singularities. This expression is best written in discrete Fourier space. We can factor out all dimensionful parameters by $P^- = 2L(g/4\pi)^2 H$, which defines the dimensionless Hamiltonian H . The discrete Fourier transforms Eq. (30) of the currents can be used and the subsequent momentum sums expressed over positive values of k by $\tilde{J}_3(-k) = \tilde{J}_3^\dagger(k)$, $\tilde{J}_+(-k) = \tilde{J}_-^\dagger(k)$, and $\tilde{J}_-(-k) = \tilde{J}_+^\dagger(k)$. In terms of the coefficients u and v the result is

$$H = -4 \frac{d^2}{dz^2} + \sum_{k=1}^{\infty} w_k^4 [\tilde{J}_3^\dagger(k) \tilde{J}_3(k) + \tilde{J}_3(k) \tilde{J}_3^\dagger(k)] + \sum_{k=\frac{1}{2}}^{\infty} v_k^4 [\tilde{J}_+^\dagger(k) \tilde{J}_+(k) + \tilde{J}_+(k) \tilde{J}_+^\dagger(k)] + \sum_{k=\frac{1}{2}}^{\infty} u_k^4 [\tilde{J}_-^\dagger(k) \tilde{J}_-(k) + \tilde{J}_-(k) \tilde{J}_-^\dagger(k)]. \quad (38)$$

This result resembles in many respects the structure found in treatments of gauge theory on a ‘‘cylinder’’ in standard instant form Hamiltonian quantization [43]. Insofar as [11,12] omit zero modes, it disagrees with their expression for the Hamiltonian. We now look in more detail at the separate contributions to this expression from, respectively, the gauge mode z , the Fock operators, and the constrained zero mode. In particular, in the next section we shall be especially interested in the vacuum expectation value (VEV) of the Hamiltonian. This will guide our present analysis.

First we consider the pieces which do not involve the constrained zero mode a_0 . This is simply achieved by replacing, in the above Hamiltonian, the currents \tilde{J} with the operators Q via the definition (31). Inspecting the expressions for Q given in Appendix C one observes they all annihilate the Fock vacuum $|0\rangle$, while $Q^\dagger|0\rangle \neq 0$. So the full Hamiltonian has a VEV, henceforward denoted by $V_0(z)$. Since it depends on the *quantum operator* z , V_0 cannot be removed by the usual trivial vacuum normalization. Rather the VEV plays the role of a ‘‘potential energy’’ in what we call the ‘‘gauge part’’ of the Hamiltonian which also includes the kinetic term for z : namely,

$$H_{\text{gauge}} \equiv -4 \frac{d^2}{dz^2} + \langle 0|H|0\rangle \equiv -4 \frac{d^2}{dz^2} + V_0(z). \quad (39)$$

We extract the expression for $V_0(z)$ by bringing the

Hamiltonian to normal ordered form. Commuting the operators Q^\dagger with Q in the substitute of Eq. (38) leaves us with the contribution $H_{\text{Fock}}^{(1)}$ to the Fock space part of the Hamiltonian $H_{\text{Fock}} = H_{\text{Fock}}^{(1)} + H_{\text{Fock}}^{(2)}$: i.e.,

$$H_{\text{Fock}}^{(1)} \equiv 2 \sum_{k=1}^{\infty} w_k^4 Q_3^\dagger(k) Q_3(k) + 2 \sum_{k=\frac{1}{2}}^{\infty} v_k^4 Q_+^\dagger(k) Q_+(k) + 2 \sum_{k=\frac{1}{2}}^{\infty} u_k^4 Q_-^\dagger(k) Q_-(k). \quad (40)$$

The VEV of the leftover commutator term generates the gauge mode potential:

$$V_0 \equiv \sum_{k=1}^{\infty} w_k^4 \left\langle 0 \left| \left[Q_3(k), Q_3^\dagger(k) \right] \right| 0 \right\rangle + \sum_{k=\frac{1}{2}}^{\infty} v_k^4 \left\langle 0 \left| \left[Q_+(k), Q_+^\dagger(k) \right] \right| 0 \right\rangle + \sum_{k=\frac{1}{2}}^{\infty} u_k^4 \left\langle 0 \left| \left[Q_-(k), Q_-^\dagger(k) \right] \right| 0 \right\rangle. \quad (41)$$

This is analyzed in detail in Appendix D and represented in Fig. 1. Subtracting Eq. (41) from the commutator expression gives the other contribution to the Fock-space Hamiltonian:

$$H_{\text{Fock}}^{(2)} \equiv \sum_{k=1}^{\infty} w_k^4 \left[Q_3(k), Q_3^\dagger(k) \right] + \sum_{k=\frac{1}{2}}^{\infty} v_k^4 \left[Q_+(k), Q_+^\dagger(k) \right] + \sum_{k=\frac{1}{2}}^{\infty} u_k^4 \left[Q_-(k), Q_-^\dagger(k) \right] - V_0(z). \quad (42)$$

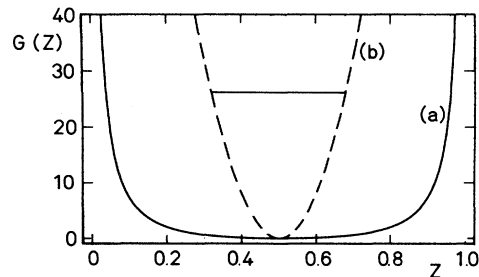


FIG. 1. The gauge potential $G(z) \equiv V_0(z) - V_0(\frac{1}{2})$ in the fundamental modular domain depicted for the two scenarios: (a) (solid curve) with the cutoff dependence removed; (b) (dashed curve) with the cutoff dependence kept. In the latter case the cutoff $s = 131$ leading to the potential $V_0(z - \frac{1}{2})^2$. In this oscillator well we give the lowest energy eigenvalue $E_0 = 25.66$.

Thus far it would appear to suffice to consider the Q operators and z for the determination of the spectrum.

This changes if one addresses the constrained mode a_0 as the constraint cannot be expressed purely in terms of Q operators. From Eqs. (38) and (31) one finds a_0 makes its appearance in the Hamiltonian both linearly and quadratically which we separate into $H_{\text{constr}} = H_{\text{constr}}^{(1)} + H_{\text{constr}}^{(2)}$. The term linear in a_0 is

$$H_{\text{constr}}^{(1)} = 2 \sum_{k=\frac{1}{2}}^{\infty} \left\{ u_k^3 \left[B_k^\dagger Q_-(k) + Q_-^\dagger(k) B_k \right]_s + v_k^3 \left[D_k^\dagger Q_+(k) + Q_+^\dagger(k) D_k \right]_s \right\}. \quad (43)$$

The term quadratic in a_0 is

$$H_{\text{constr}}^{(2)} = \sum_{k=\frac{1}{2}}^{\infty} \left[u_k^2 (B_k B_k^\dagger + B_k^\dagger B_k) + v_k^2 (D_k D_k^\dagger + D_k^\dagger D_k) \right]. \quad (44)$$

H_{constr} could have a nonzero VEV that could contribute to the potential of the gauge mode. There are two possibilities where such nonvanishing contributions could arise. The first is when an a_0 appears either to the extreme left or extreme right in H_{constr} . However, preliminary studies of the structure of the constraint suggest that no nonzero VEV of a_0 or $(a_0)^2$ can arise. Thus, the role played by this mode is quite different from that of its counterpart in the (1+1)-dimensional ϕ^4 [$(\phi^4)_{1+1}$] theory. The second possibility of nonzero contribution to the gauge potential is when the zero mode lies between an annihilation operator on the left and a creation operator on the right, such as $\langle 0|b_k a_0 a_0 b_k^\dagger|0\rangle$. A nonzero VEV could potentially arise from such contributions if $[a_0, b_k]$ and $[a_0, b_k^\dagger]$ are nonzero.

This completes the analysis of the light-cone Hamiltonian. In summary though, we have reexpressed the Hamiltonian as a sum of three contributions, i.e.,

$$H = H_{\text{gauge}} + H_{\text{Fock}} + H_{\text{constr}}. \quad (45)$$

In the next section we will see how far analytic methods can take us in diagonalizing parts of the Hamiltonian.

Finally, we should emphasize that we have refrained thus far from *ad hoc* approximations to define a consistent Hamiltonian and Hilbert space. As we are unable to give here exact analytical solutions we consider their numerical simulation as a challenge for the future. In the sequel we therefore will continue with several simplifying assumptions which will allow us to solve part of the Hamiltonian analytically. The most drastic among them is the omission of the constraint part. We thus will concentrate efforts in the following on solving approximately H_{gauge} and H_{Fock} subject to the omission of H_{constr} .

V. APPROXIMATE SOLUTIONS IN THE GAUGE SECTOR

The potential energy $V_0(z)$ which appears in H_{gauge} via Eq. (39), i.e., $H_{\text{gauge}} \equiv -d^2/dz^2 + V_0(z)$, is what we shall loosely refer to as the “gauge potential.” As analyzed in Appendix D it is *invariant* under large gauge transformations. One therefore can restrict oneself to calculate it only for the fundamental modular domain $0 < z < 1$. In Fig. 1 it is displayed for the two cases explained below. In either case, because of the singular behavior at $z = 0$ and $z = 1$, there will be a discrete spectrum of excitation energies in this potential. These will be labeled by a quantum number N . Generalizing the representation in [30] we use wave functions $\Psi_N(z) \equiv \langle z|N\rangle$ so that we address ourselves to solving the Schrödinger equation:

$$H_{\text{gauge}} \Psi_N(z) = \tilde{E}_N \Psi_N(z). \quad (46)$$

The symbol E_N will be reserved for the vacuum normalized eigenenergy, namely, $\tilde{E}_N - \tilde{E}_0$.

In the fundamental modular domain the potential is symmetric about $z = \frac{1}{2}$. As mentioned, it is singular at $z = 0$ and 1 . It has a minimum at $z = \frac{1}{2}$, however the value of the function there is actually divergent. Moreover its curvature diverges at that point, that is, $V_0''(\frac{1}{2}) \propto \omega_0^2 \ln s$, $s \rightarrow \infty$, where s is the value of a dimensionless regulator truncating sums. The precise meaning of this divergent behavior, namely, whether it is physical or formal, is unclear to us. Lacking a definite answer, we pursue below the two alternative scenarios, labeled (a) and (b).

Scenario (a) – Cutoff-Independent Approach. Let us take the point of view that the divergent $\ln s$ behavior of the potential should be canceled by a counterterm in the Lagrangian. In fact we do not know what this counterterm should be and the answer may come from the, as-yet unavailable, solution to the constrained zero mode a_0 . We therefore proceed by simply numerically subtracting the function $\omega_0^2 \ln s (z - \frac{1}{2})^2$ from the explicit expression in Appendix D. We plot the so-obtained function in Fig. 1. Upon inspection, one notes the very flat base in the vicinity of $z = 1/2$. A good approximation might therefore be the infinite square well. Restoring units, the problem is expressed by $-L(g/2\pi)^2 (d^2/dz^2) \Psi_N = \tilde{E}_N \Psi_N$. After a vacuum energy subtraction, the eigenenergy *densities* $\epsilon_N \equiv E_N/(2L)$ are $g^2(n^2 - 1)/8$. We thus recover the spectrum for SU(2) pure zero mode glue in 1+1 dimensions as obtained in [34,35], and verified by us on the light cone in [30]. Wave functions respecting boundary conditions are then $\sin(N\pi z)$ up to normalization. As for subtleties concerning this choice over the cosine we refer the reader to [30]. Thus we have succeeded in diagonalizing the gauge part of P^- . As the vacuum part of P^+ has no dependence on z these wave functions are also eigenfunctions of P^+ with momentum zero.

Scenario (b) – Cutoff-Dependent Approach. For reasonably large values of the cutoff s , the $\ln s$ term dominates, as mentioned, over the s -independent part and

thus here the latter can be omitted. As discussed in Appendix D, with the cutoff finite but large the gauge potential can be approximated by

$$V_0(z) = V_0(\frac{1}{2}) + 4\omega^2(z - \frac{1}{2})^2 \quad \text{with} \quad \omega^2 = \omega_0^2 \ln s. \quad (47)$$

The numerical value of ω_0^2 is given in Appendix D, while $V_0(\frac{1}{2})$ is an unspecified constant. The eigenvalues and eigenfunctions are now those of the harmonic oscillator:

$$\tilde{E}_N = V_0(\frac{1}{2}) + 4\omega(2N + 1)$$

and

$$\Psi_N(z) = \mathcal{N}_N H_N[\sqrt{\omega}(z - \frac{1}{2})] e^{-\frac{\omega}{2}(z - \frac{1}{2})^2}, \quad (48)$$

respectively, where the H_N are the Hermite polynomials of order N , and where the \mathcal{N}_N normalize all eigenfunctions to unity. The state with lowest energy has $N = 0$, by inspection, and has to be identified with the true “vacuum.” The vacuum-renormalized eigenvalues and lowest energy eigenfunctions are thus

$$E_N \equiv \tilde{E}_N - \tilde{E}_0 = 8N\omega_0\sqrt{\ln s} \quad (49)$$

and

$$\Psi_0(z) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\omega}{2}(z - \frac{1}{2})^2} \sim \sqrt{\delta(z - \frac{1}{2})}. \quad (50)$$

In the limit $s \rightarrow \infty$ the eigenfunctions degenerate into, roughly speaking, δ “functions,” namely, the wave functions strongly localize about $z = \frac{1}{2}$. This extremely sharp peaking implies that the *operator* z can be replaced by the *number* $\frac{1}{2}$ everywhere but in the gauge Hamiltonian, a significant simplification. It is what makes this scenario radically different from the former. As a consequence, the gauge excitations $\Psi_N(z)$, the “gaugeons,” have for finite but sufficiently large s a finite and nondegenerate (light-cone) energy but strictly zero mass.

VI. APPROXIMATE SOLUTIONS TO THE GAUGE PLUS FOCK SECTOR

In the remainder we address ourselves to finding eigenstates to $H_{\text{gauge}} + H_{\text{Fock}}$. In the absence of a complete picture for the renormalization in scenario (a), we implement here the conclusions of scenario (b) for the gauge mode: we substitute $z = \frac{1}{2}$ where appropriate. The Hilbert space will be spanned by the product states of gauge eigenfunctions $\Psi_N(z)$ and “Fock states.” The product states $\Psi_N(z)|0\rangle_{\text{Fock}}$ are what we loosely call *pseudovacua*. The true vacuum is $|0\rangle = \Psi_0|0\rangle_{\text{Fock}} \sim \sqrt{\delta(z - \frac{1}{2})}|0\rangle_{\text{Fock}}$. On top of it we now build Fock space excitations.

Let us consider three types of two-particle Fock-space excitations, the toy states

$$\begin{aligned} |k; 3\rangle &\equiv Q_3^\dagger(k)|0\rangle, \\ |k; +\rangle &\equiv Q_+^\dagger(k)|0\rangle, \end{aligned}$$

and

$$|k; -\rangle \equiv Q_-^\dagger(k)|0\rangle, \quad (51)$$

to which we shall refer collectively as $|k; a\rangle$. We mention already here that only one of these, $|k; 3\rangle$, satisfies Gauss’ law. Using the approximate δ -function behavior of the gauge wave functions, the toy states are seen to be eigenstates of P^+ : i.e.,

$$\hat{K}|k; 3\rangle = k|k; 3\rangle, \quad \hat{K}|k; +\rangle = k|k; +\rangle,$$

and

$$\hat{K}|k; -\rangle = k|k; -\rangle. \quad (52)$$

Are they also eigenstates to the full P^- ?

In general, the commutators of charge operators $Q_a(k)$ acting on the vacuum can be separated into c -number and operator parts:

$$[Q_a(p), Q_b^\dagger(k)]|0\rangle = [\delta_{ab}S_a(k)\delta_p^k + \mathcal{O}_{ab}(p, k)]|0\rangle, \quad (53)$$

with the c -number coefficients S_a defined in Appendix D. The operator part $\mathcal{O}_{ab}(p, k)$ is complicated to write down in full and is generally nonzero. However, for $p = k$, $\mathcal{O}_{ab} = 0$. Taking this as a hint, we shall *assume* that the effects of \mathcal{O}_{ab} are in some sense “small” and set the entire operator to zero by hand.

With these simplifications now, the commutator $H_{\text{Fock}}^{(2)}$ vanishes and all three toy states $|k; a\rangle$ become eigenstates of $H_{\text{Fock}}^{(1)}$, such as, for example,

$$\begin{aligned} H_{\text{Fock}}^{(1)}|k; 3\rangle &= 2 \sum_{p=\frac{1}{2}}^{\infty} v_p^4 Q_+^\dagger(p) [Q_+(p), Q_3(k)^\dagger]|0\rangle \\ &\quad + 2 \sum_{p=1}^{\infty} w_p^4 Q_3^\dagger(p) [Q_3(p), Q_3(k)^\dagger]|0\rangle \\ &\quad + 2 \sum_{p=\frac{1}{2}}^{\infty} u_p^4 Q_-^\dagger(p) [Q_-(p), Q_3(k)^\dagger]|0\rangle \\ &\simeq \frac{4}{k} \ln k |k; 3\rangle, \end{aligned} \quad (54)$$

for sufficiently large values of k , according to Eqs. (D8). Recall that the continuum limit is reached [4,5] by the limit $k \rightarrow \infty$. The combined action of the energy and momentum operators $\hat{H} \equiv \hat{K}(H_{\text{gauge}} + H_{\text{Fock}})$ becomes thus $\hat{H}|k; a\rangle = 4 \ln k |k; a\rangle$. The action of the mass-squared operator $2P^+P^-$ on the toy states gives finally, after restoring units according to Eq. (7):

$$\begin{aligned} 2P^+P^-|k; a; N\rangle \\ = \frac{g^2}{4\pi L_\perp} \left(\ln k + 2kN\omega_0\sqrt{\ln s} \right) |k; a; N\rangle, \end{aligned} \quad (55)$$

with

$$|k; a; N\rangle = Q_3^\dagger(k) \Psi_N(z) |0\rangle_{\text{Fock}}, \quad (56)$$

independent of the longitudinal interval length L . This then is an approximate mass spectrum of the model in the two-particle sector with all cutoffs large but finite. How does it behave as cutoffs are removed? We take the necessary limits as follows. (1) There is no meaning to the transversal continuum limit in the present model so we consider L_\perp as arbitrary but fixed. (2) Since it is meaningful to consider physics in a finite volume or interval, s should be taken to its physical limit before L . This removes from the spectrum all the “gaugeon” excitations N . (3) As mentioned, the longitudinal continuum limit is defined by $k \rightarrow \infty, L \rightarrow \infty$, but $p^+ = \pi k/L$ fixed. Since the longitudinal length does not appear one has to take the isolated limit $k \rightarrow \infty$. Thus the degenerate triplet of states with $N = 0$ also diverge in the continuum limit and do not survive in the bare spectrum.

VII. DISCUSSION AND PERSPECTIVES

Let us briefly restate the approach we took here. Beginning with SU(2) gauge theory in 2+1 dimensions in the front form we suppressed transverse momenta of the gluons and obtained a (1+1)-dimensional gauge theory coupled to adjoint scalar matter. Gauge fixing of this theory revealed for the content many dynamical normal modes of the scalar field, a topological gauge zero mode, and a constrained zero mode. The constrained mode satisfies a linear but nonetheless complicated operator constraint. The gauge fixing involved a space-time-independent color rotation that reduced the remnant Gauss law to be implemented to an Abelian global symmetry generator. We succeeded in specifying the space of states which would be annihilated by the Gauss operator, namely, that of color singlet states built from the Fock or parton operators. Not performing the gauge rotation would not even permit one to easily solve Gauss’ law. We succeeded in diagonalizing both P^+ and P^- in the gauge mode sector in two separate approaches which, respectively, involved keeping or removing an ultraviolet cutoff in the calculation. The cutoff independent approach lead to gauge mode wave functions that were unlocalized. With the cutoff, the solution of the gauge mode problem in the Fock space sector reduced to substituting in the Fock sector the minimum value of its potential by the value $\zeta = 0$. We approximately and analytically solved for the invariant mass of three composite states, one of which satisfied Gauss’ law. Even for this one, the energy diverged in the continuum limit.

We now interpret the meaning of this result. It might be that some aspect of the nontrivial renormalization required in full 2+1 dimensions manifests itself even in this (1+1)-dimensional subregime. While the theory is superrenormalizable by virtue of dimensionality the structure of the (finite) number of divergences is not of the usual two-dimensional QCD-type but reflect some substructure of the higher dimensional theory and its renor-

malization and scaling properties. Recalling that every Lagrangian field theory has an open scale, only *mass ratios* can be meaningful quantities. If our toy states reflect correctly the behavior of the lowest energy singlet state, namely, running as $\frac{g^2}{2\pi} \ln k$, then renormalization of the spectrum is achieved by “renormalizing the coupling constant.” This would then read explicitly

$$g^2 = g_{\text{phys}}^2 / \ln k \quad (57)$$

leading to a lowest excitation of mass “1” in arbitrary units.

We now return to the question of the diagonalization of the gauge part of the Hamiltonian in scenario (a). It leads to a more complicated treatment of the z mode when including the Fock space excitations. There is nothing in principle hindering a full solution of this but we leave it for future work. What remains to be understood is what type of counterterm could remove this divergence. It is possible that the omitted constrained mode contributions assist in the renormalization of the potential. We are presently exploring this in the context of Fock space truncation approximations [23–25] for solving the constraint. That the constrained zero mode can provide renormalization counterterms is not without precedent, for example, this has already been seen in perturbative QED in [29]. On the other hand, the picture emerging in scenario (b), of a special role for the value $\zeta = 0$, has also been seen via point-splitting regularization of Gauss’ law and the momentum operator. This will be reported elsewhere [39].

More can be done analytically: evidently some nontrivial linear combination of the $Q_a(k)$ operators could build a color singlet for which an approximate eigenvalue might be obtainable. Nevertheless, the treatment of the first four sections has prepared the way for treating the theory with the full power of *standard* DLCQ numerical techniques, now including zero modes. Such numerical work is underway.

Going beyond the present theory would mean addressing dimensionally reduced QCD₃₊₁ where all the features discussed here will continue. The hope that DLCQ can allow us to understand QCD in an intuitively simple way but with its full richness remains undiminished.

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APPENDIX A: NOTATION AND CONVENTIONS

Equations and Constants of Motion for QCD. In quantum chromodynamics the gauge fields are traceless Hermitian 3×3 matrices \mathbf{A}^μ . More generally for $SU(N)$, they are $N \times N$ matrices parametrized in terms of “color vector potentials” A_a^μ , i.e., $\mathbf{A}^\mu \equiv T^a A_a^\mu$. The glue index

a (or r, s, t) is implicitly summed with no attention paid to the lowering or raising and runs from 1 to $N^2 - 1$. The quark field Ψ is a color triplet spinor, i.e., $\Psi_{\alpha,c}$, but the Dirac index α and the color index $c = 1, \dots, N$ are usually suppressed. The color matrices $T_{cc'}^a$ obey $[T^a, T^b]_{cc'} = if^{abr} T_{cc'}^r$ and $\text{Tr}(T^a T^b) = \frac{1}{2} \delta_{ab}$. They are related to the Gell-Mann matrices λ^a by $T^a = \frac{1}{\sqrt{2}} \lambda^a$. For $SU(2)$ the λ^a are the Pauli matrices σ^a , and for $SU(3)$ one has, e.g.,

$$(\mathbf{A}^\mu)_{cc'} = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{3}} A_8^\mu + A_3^\mu & A_1^\mu - iA_2^\mu & A_4^\mu - iA_5^\mu \\ A_1^\mu + iA_2^\mu & \frac{1}{\sqrt{3}} A_8^\mu - A_3^\mu & A_6^\mu - iA_7^\mu \\ A_4^\mu + iA_5^\mu & A_6^\mu + iA_7^\mu & -\frac{2}{\sqrt{3}} A_8^\mu \end{pmatrix}. \quad (\text{A1})$$

The Lagrangian density for QCD can thus be written in two equivalent conventions:

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}) + \frac{1}{2} [\bar{\Psi}(i\gamma^\mu \mathcal{D} m_\mu - m)\Psi + \text{H.c.}] \quad \text{with } \mathbf{F}^{\mu\nu} \equiv \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + ig[\mathbf{A}^\mu, \mathbf{A}^\nu], \quad (\text{A2})$$

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \frac{1}{2} [\bar{\Psi}(i\gamma^\mu \mathcal{D} m_\mu - m)\Psi + \text{H.c.}] \quad \text{with } F_a^{\mu\nu} \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gf^{ars} A_r^\mu A_s^\nu. \quad (\text{A3})$$

The covariant derivative matrix is $(\mathcal{D}^\mu)_{cc'} \equiv \delta_{cc'} \partial^\mu + ig(\mathbf{A}^\mu)_{cc'}$. The variational derivatives are

$$\frac{\delta \mathcal{L}}{\delta(\partial^\kappa A_\lambda^r)} = -F_{\kappa\lambda}^r \quad \text{and} \quad \frac{\delta \mathcal{L}}{\delta A_\lambda^r} = -gJ_\lambda^r \quad \text{with} \quad J_\lambda^r \equiv f^{ars} F_r^{\nu\kappa} A_\kappa^s + \bar{\Psi} \gamma^\nu T^a \Psi. \quad (\text{A4})$$

Canonical field theory yields straightforwardly to the *color Dirac equations* $(i\gamma^\mu \mathcal{D}_\mu - m)\Psi = 0$. It also gives the *color Maxwell equations*, which are given here in two conventions: i.e.,

$$\partial_\mu \mathbf{F}^{\mu\nu} = g\mathbf{J}^\nu \quad \text{with} \quad \mathbf{J}^\nu \equiv -i[\mathbf{F}^{\nu\kappa}, \mathbf{A}_\kappa] + \bar{\Psi} \gamma^\nu T^a \Psi T^a, \quad (\text{A5})$$

$$\mathbf{D}_\mu \mathbf{F}^{\mu\nu} = g\mathbf{J}_Q^\nu \quad \text{with} \quad \mathbf{J}_Q^\nu \equiv \bar{\Psi} \gamma^\nu T^a \Psi T^a \quad \text{and} \quad \mathbf{D}_\mu \equiv \partial_\mu + ig[\mathbf{A}_\mu, *]. \quad (\text{A6})$$

The *color Maxwell current* \mathbf{J}^μ and the *quark matter current* \mathbf{J}_Q^μ have different conservation laws. In particular $\partial_\mu \mathbf{J}^\mu = 0$. The stress tensor $\Theta^{\mu\nu} = F_a^{\mu\kappa} \partial^\nu A_\kappa^a + \frac{1}{2} [\bar{\Psi} i\gamma^\mu \partial^\nu \Psi + \text{H.c.}] - g^{\mu\nu} \mathcal{L}$ is, at first, is *not* manifestly gauge invariant. But with the Maxwell equations one derives $F_a^{\mu\kappa} \partial^\nu A_\kappa^a = F_a^{\mu\kappa} F_{\nu\kappa}^a + gJ_a^\mu A_\nu^a + gf^{ars} F_a^{\mu\kappa} A_r^\nu A_\kappa^s + \partial_\kappa (F_a^{\mu\kappa} A_\nu^a)$, and thus

$$\Theta^{\mu\nu} = 2 \text{Tr}(\mathbf{F}^{\mu\kappa} \mathbf{F}_\kappa^\nu) + \frac{1}{2} [\bar{\Psi} i\gamma^\mu \mathcal{D}^\nu \Psi + \text{H.c.}] - g^{\mu\nu} \mathcal{L} - 2\partial_\kappa \text{Tr}(\mathbf{F}^{\mu\kappa} \mathbf{A}^\nu). \quad (\text{A7})$$

All explicit gauge dependence resides in the last term. For periodic boundary conditions it vanishes upon integration. The generalized momenta “on the light cone” become then *manifestly gauge invariant*: i.e.,

$$\begin{aligned} P^\nu &\equiv \int_\Omega d\omega \Theta^{+\nu} \\ &= \int_\Omega d\omega [2 \text{Tr}(\mathbf{F}^{+\kappa} \mathbf{F}_\kappa^\nu) - g^{+\nu} \mathcal{L} \\ &\quad + \frac{1}{2} (\bar{\Psi} i\gamma^+ \mathcal{D}^\nu \Psi + \text{H.c.})]. \end{aligned} \quad (\text{A8})$$

Integration goes over all spacelike coordinates ($d\omega$) and Ω denotes the integration volume. This was first shown in [40]. Note that all this holds rigorously for $SU(N)$ in $d+1$ dimensions.

Light-Cone Coordinates. We follow the convention of Kogut and Soper [54], in particular with $x^\pm \equiv (x^0 \pm x^1)/\sqrt{2}$.

Color Helicity Basis. We define the color helicity basis for $SU(2)$ by the Pauli matrices σ^a :

$$\tau^3 = \frac{1}{2} \sigma^3, \quad \tau^\pm \equiv \frac{1}{2\sqrt{2}} (\sigma^1 \pm i\sigma^2). \quad (\text{A9})$$

We can turn this into a vector space by introducing elements x^a such that tilde quantities are defined with respect to the helicity basis, and untilde the usual Cartesian basis:

$$x^a = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \quad \text{and} \quad \tilde{x}^a = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{pmatrix} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}. \quad (\text{A10})$$

The relation between the tilde and untilde basis can be written $\tilde{x}^a = \Lambda_b^a x^b$ and $x^a = \tilde{\Lambda}_b^a \tilde{x}^b$ where $\tilde{\Lambda} = \Lambda^\dagger$. With these elements we can construct the metric in terms of the tilde basis. Essentially we must demand the invariance of the inner product of any two vector space elements, $x^a y_a = \tilde{x}^a \tilde{y}_a$. Using the fact that the metric in the $a = 1, 2$ basis is just the Kronecker delta δ_{ab} and the transformed metric is $\tilde{\mathcal{G}}_{ab} = \tilde{\Lambda}_a^c \delta_{cd} \Lambda_b^d$. Thus

$$\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{G}}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A11})$$

The metric to raise and lower indices in the helicity basis becomes $x_\pm = x^\mp$. The color algebra looks formally like the Lorentzian structure in light-cone coordinates.

APPENDIX B: GRIBOV COPIES, CENTER CONJUGATIONS, AND FOCK SPACE

Gribov Copies. Because of the torus geometry of our space and the non-Abelian structure of the gauge group, there remain large gauge transformations which are still symmetries of the theory [41,42] despite our complete fixing of the theory with respect to small gauge transformations. These are generated by local SU(2) elements

$$V(x^-) = \exp\left(-in_0\pi\frac{x^-}{L}\tau_3\right),$$

n_0 an even integer, (B1)

which satisfy periodic boundary conditions. Another symmetry of the theory is Z_2 center symmetry which here means allowing for antiperiodic V or alternately n_0 odd. In both cases one preserves the periodic boundary conditions on the gauge potentials. On the diagonal component of A^+ it generates shifts that are best expressed in terms of the dimensionless z , namely, $z \rightarrow z' = z + n_0$. On the scalar adjoint field and its momenta the effect of the transformation is

$$\varphi_3 \rightarrow \varphi_3 \quad \text{and} \quad \varphi_{\pm} \rightarrow \varphi_{\pm} \exp\left(\mp in_0\frac{\pi}{L}x^-\right), \quad (\text{B2})$$

$$\pi^3 \rightarrow \pi^3 \quad \text{and} \quad \pi^{\pm} \rightarrow \pi^{\pm} \exp\left(\pm in_0\frac{\pi}{L}x^-\right). \quad (\text{B3})$$

Color Property of z . We now show that the gauge mode z can be written in terms of an explicitly color singlet object, thus demonstrating that it itself is a color singlet and a viable physical degree of freedom. We construct the Wilson line by a contour C along the x direction from $-L$ to L :

$$\begin{aligned} W &= \text{Tr}P \exp\left(ig \int_C dx_{\mu} \mathbf{A}^{\mu}\right) \\ &= \text{Tr}P \exp\left(ig \int_{-L}^{+L} dx \mathbf{A}^+\right). \end{aligned} \quad (\text{B4})$$

In the gauge employed in this paper, this is simply $W = \text{Tr} \exp(2iz\pi\tau^3) = 2 \cos(2\pi z)$, and one can relate z to W modulo the integers, $z = \frac{1}{2\pi} \arccos\left(\frac{W}{2}\right)$. The integer shifts are nothing but the Gribov copies discussed earlier. Observe that the dynamical quantity W attains its minimum value at $z = \frac{1}{2}$ matching with the minimum in the Fock vacuum potential. Since W is explicitly constructed in terms of a color trace, z is a color singlet.

APPENDIX C: THE CHARGE OPERATORS

In the text we introduced the operators which are the discrete Fourier transforms of the scalar current components with the constrained zero mode *removed*. The explicit expressions for these in terms of the various Fock operators are

$$\begin{aligned} Q_3(k) &= - \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} b_m d_n \left(\frac{u_m}{v_n} - \frac{v_n}{u_m}\right) \delta_{n+m}^k \\ &\quad + \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} b_n^{\dagger} b_m \left(\frac{u_n}{u_m} + \frac{u_m}{u_n}\right) \delta_{n+k}^m \\ &\quad - \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} d_n^{\dagger} d_m \left(\frac{v_n}{v_m} + \frac{v_m}{v_n}\right) \delta_{n+k}^m, \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} Q_+(k) &= + \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n d_m \left(\frac{w_n}{v_m} - \frac{v_m}{w_n}\right) \delta_{n+m}^k \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n^{\dagger} d_m \left(\frac{w_n}{v_m} + \frac{v_m}{w_n}\right) \delta_{n+k}^m \\ &\quad - \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n b_m^{\dagger} \left(\frac{w_n}{u_m} + \frac{u_m}{w_n}\right) \delta_{m+k}^n, \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} Q_-(k) &= + \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n b_m \left(\frac{w_n}{u_m} - \frac{u_m}{w_n}\right) \delta_{n+m}^k \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n^{\dagger} b_m \left(\frac{w_n}{u_m} + \frac{u_m}{w_n}\right) \delta_{n+k}^m \\ &\quad - \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} a_n d_m^{\dagger} \left(\frac{w_n}{v_m} + \frac{v_m}{w_n}\right) \delta_{m+k}^n. \end{aligned} \quad (\text{C3})$$

APPENDIX D: ANALYSIS OF THE GAUGE POTENTIAL

The gauge potential $V_0(z)$ was defined in Eq. (41). We express it here conveniently

$$\begin{aligned} V_0(z) &= \sum_{k=1}^{\infty} S_3(k) w_k^4 \\ &\quad + \sum_{k=\frac{1}{2}}^{\infty} \left[S_+(k) v_k^4 + S_-(k) u_k^4 \right], \end{aligned} \quad (\text{D1})$$

with

$$\begin{aligned} S_3(k, z) &\equiv \langle 0 | [Q_3(k), Q_3^{\dagger}(k)] | 0 \rangle \\ &= \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\frac{u_m}{v_n} - \frac{v_n}{u_m}\right)^2 \delta_{m+n}^k, \end{aligned} \quad (\text{D2})$$

$$\begin{aligned} S_+(k, z) &\equiv \langle 0 | [Q_+(k), Q_+^{\dagger}(k)] | 0 \rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\frac{w_n}{v_m} - \frac{v_m}{w_n}\right)^2 \delta_{m+n}^k, \end{aligned} \quad (\text{D3})$$

$$\begin{aligned} S_-(k, z) &\equiv \langle 0 | [Q_-(k), Q_-^{\dagger}(k)] | 0 \rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\frac{w_n}{u_m} - \frac{u_m}{w_n}\right)^2 \delta_{m+n}^k, \end{aligned} \quad (\text{D4})$$

in terms of the commutator functions $S(k)$. Thus

$$\begin{aligned} V_0(z) &= \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\frac{u_m}{v_n} - \frac{v_n}{u_m} \right)^2 w_{m+n}^4 \\ &+ \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\frac{w_n}{v_m} - \frac{v_m}{w_n} \right)^2 v_{m+n}^4 \\ &+ \left(\frac{w_n}{u_m} - \frac{u_m}{w_n} \right)^2 u_{m+n}^4. \end{aligned} \quad (\text{D5})$$

All these functions depend on z through $\zeta(z)$ in the coefficients $u_n(z) = 1/\sqrt{n+\zeta}$, $v_n(z) = 1/\sqrt{n-\zeta}$, and $w_n =$

$1/\sqrt{n}$. The gauge potential is thus *manifestly invariant* under large gauge transformations. By inspection, it has *singularities* at all values $z \in Z$, particularly at $z = 0$ and $z = 1$. Noting that the coefficients have the symmetry $u_m(-\zeta) = v_m(\zeta)$ and $v_m(-\zeta) = u_m(\zeta)$, one observes another symmetry, namely, $V_0(-\zeta) = V_0(\zeta)$. This implies that the gauge potential is symmetric around $z = \frac{1}{2}$ in the fundamental modular domain, having there a *minimum*. It is thus reasonable to renormalize the gauge potential by $G(z) = V_0(z) - V_0(\frac{1}{2})$. One can expand $V_0(z)$ around its minimum at $z = \frac{1}{2}$. The first derivative vanishes there, and the second can be cast into the form

$$\begin{aligned} V_0''(z) &= \sum_{n=\frac{1}{2}}^s \sum_{m=\frac{1}{2}}^{\infty} \left[\frac{2}{(n-\zeta)(m+\zeta)^3} + \frac{2}{(n+\zeta)(m-\zeta)^3} - \frac{2}{(n-\zeta)^2(m+\zeta)^2} \right] \\ &+ \sum_{n=1}^s \sum_{m=\frac{1}{2}}^{\infty} \left[\frac{2}{n(m-\zeta)^3} + \frac{2}{n(m+\zeta)^3} - \frac{24}{(n+m-\zeta)^4} - \frac{24}{(n+m+\zeta)^4} \right]. \end{aligned} \quad (\text{D6})$$

The first two terms in each line diverge as $s \rightarrow \infty$. A measure for the curvature at $\zeta(\frac{1}{2}) = 0$ is obtained by means of Riemann's zeta function ζ [not to be confused with $\zeta(z)$]: namely,

$$\begin{aligned} \omega^2 &\equiv \frac{1}{8} V_0''(\frac{1}{2}) = \ln s \sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})^3} \\ &= 7\zeta(3) \ln s \\ &\simeq 8.4144 \ln s \\ &\equiv \omega_0^2 \ln s. \end{aligned} \quad (\text{D7})$$

We have thus analytically obtained the divergent s -dependent part of the gauge potential.

Finally, we analyze the commutator functions at $\zeta = 0$.

Direct evaluation of Eqs. (D2)–(D4) yields the positive values

$$S_3(k) = \sum_{n=\frac{1}{2}}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\sqrt{\frac{n}{m}} - \sqrt{\frac{m}{n}} \right)^2 \delta_{m+n}^k$$

and

$$S_{\pm}(k) = \sum_{n=1}^{\infty} \sum_{m=\frac{1}{2}}^{\infty} \left(\sqrt{\frac{n}{m}} - \sqrt{\frac{m}{n}} \right)^2 \delta_{m+n}^k. \quad (\text{D8})$$

For sufficiently large values all of them approach $S(k) \simeq 2k \ln k$.

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