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Chern-Simons vortices in an open system

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A gauge-invariant quantum field theory with a spacetime-dependent Chern-Simons coefficient is studied. Using a constraint formalism together with the Schwinger action principle it is shown that nonzero gradients in the coefficient induce magnetic-moment corrections to the Hall current and transform vortex singularities into nonlocal objects. The fundamental commutator for the density fluctuations is obtained from the action principle and the Hamiltonian of the Chern-Simons field is shown to vanish only under the restricted class of variations which satisfy the gauge invariance constraint.

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The study of Chern-Simons theories is motivated principally by two observations: namely, that important aspects of the quantum Hall phenomenon are described efficiently by a Chern-Simons theory, and that a viable theory of high temperature superconductivity should be characterized by a parity-violating antiferromagnetic state [1]. Symmetry considerations alone suggest that such an interaction should be present in these systems.

In the case of the superconductor, where electrons are effectively two dimensional by virtue of the layered symmetry, neighboring planes can be expected to play a nontrivial role on the dynamics of the two-dimensional system. In particular, donor sites and irregularities in neighboring two-dimensional systems could have a sufficiently coherent influence on a two-dimensional system that the physical properties in the two-dimensional superconductor are modulated by the presence of their neighbors. This would suggest an *effective* field theory with position-dependent couplings. In a similar vein, it was suggested by Jacobs [2] that certain desirable features might be achieved if the Chern-Simons term was coupled, not by a coupling constant, but through an "axion" field—i.e., a spacetime-dependent coupling. In a continuum theory of the quantum Hall effect, a stepping Chern-Simons coefficient is also natural in the vicinity of the edges of the Hall sample where the statistics parameter passes through a sequence of values dictated by the Landau level structure. Recent work by the author [3,4] has lead to a formalism for dealing with the apparent inconsistencies in the interpretation of such a theory. Although originally motivated on other grounds, the formalism is easily adapted to the problem of Chern-Simons particles (the anyon system [1]) which has been investigated in Refs. [2,5].

The apparent difficulty with a variable Chern-Simons coefficient is that the resulting theory is not explicitly gauge invariant. One might argue that this is because one starts with the action S which is not a physical object. One could, after all, simply start with the field equations and make the Chern-Simons coefficient spacetime dependent. However, in present day quantum field theory the action is increasingly regarded as being a physical object, not only its variation. The Chern-Simons term is a case in point. It is therefore important to secure a formalism which guarantees consistency between variations of the action and the dynamical structure of the theory at all levels. Such a formalism was recently constructed and the physical meaning of the procedure identified as being that of closing an open physical system through the use of a constraint. The formalism is easily adapted to the quantized theory by adopting Schwinger's action principle [6]. Let us therefore begin by examining the formalism.

The fundamental relation in Schwinger's quantum action principle is

$$\delta\langle t'|t\rangle = i\left\langle t'\left|\int_t^{t'} L(q)\,dt\right|t\right\rangle. \tag{1}$$

From this relation one infers both the operator equations of motion $\frac{\delta S}{\delta q}=0$, for dynamical variables q and the generator of infinitesimal unitary transformations G which is obtained from the total time derivative in δS. S is an action symmetrized with respect to the kinematical derivatives of the dynamical variables. From this, one obtains the variation of any operator A on the basis $|t\rangle$:

$$\delta A = -i[A,G]. \tag{2}$$

Consider first the usual Chern-Simons theory for constant μ. This will serve as a point of reference for the remainder of the paper. It can be noted that the present formalism bears a certain resemblance to the Schrödinger quantization examined by Dunne, Jackiw, and Trugenberger [7] and reproduces the relevant results. The pure Chern-Simons theory is described by the action

$$S=\int dtd^2x\left(\frac{1}{2}\mu\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - J^\mu A_\mu\right) \tag{3}$$

where J^μ is a gauge-invariant current operator and μ is constant. The variation of this action operator with respect to A_μ leads to the operator equation of motion

$$\frac{1}{2}\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = J^\mu \tag{4}$$

and the generator of infinitesimal unitary transformations on the field variables [8]:

$$G(\sigma) = \int d\sigma_\mu \; \mu\epsilon^{\mu\nu\lambda}A_\mu \delta A_\lambda. \tag{5}$$

Taking σ to be a spacelike hypersurface, with unit normal parallel to the time t, one obtains the fundamental commutator for A_μ trivially by considering δA_μ in (2):

$$[A_i(x), A_j(x')]\Big|_{t=t'} = i\mu^{-1}\epsilon_{ij}\delta(\mathbf{x}, \mathbf{x}'). \tag{6}$$

No restrictions are placed on the A_0 component which is therefore not a true canonical variable, rather it should be understood as a Lagrange multiplier which enforces the relation $\mu B = -\rho$.

The generator $G(\sigma)$ is not obviously gauge invariant but, if one ignores the source J_μ for a moment, it is clear that the constraint $B = 0$ can be satisfied by $A_i = \partial_i \xi$, for some scalar field ξ. If one uses this in the generator, it is evident that there is no dynamical evolution unless $[\partial_1, \partial_2]\xi \neq 0$. This indicates that vortex singularities play a special role in this theory and that a nontrivial generator with $B = 0$ could only be satisfied by a pointlike source J^μ, as in the flux line singularities of anyon theory.

More generally, if one solves the field equations giving

$$A_i(x) = \frac{\epsilon_{ji}\partial_j \rho(x)}{\mu \nabla^2} \tag{7}$$

and uses this to express the gauge field purely in terms of gauge-invariant operators, one obtains an implicit equation for the commutator of the density operator, thus identifying density fluctuations as the basic excitations:

$$4\pi\mu i \delta(\mathbf{x}, \mathbf{x}') = \int d^2x'' d^2x''' [\rho(x''), \rho(x''')] \epsilon_{lj} \frac{(\mathbf{x} - \mathbf{x}'')^l}{|\mathbf{x} - \mathbf{x}''|^2} \frac{(\mathbf{x}' - \mathbf{x}''')^j}{|\mathbf{x}' - \mathbf{x}'''|^2}\Big|_{t'=t} \tag{8}$$

or

$$[\rho(\mathbf{x}'', t), \rho(\mathbf{x}''', t)] = 4\pi^2 \mu \Omega^{-1} i \delta(\mathbf{x}'', \mathbf{x}''') \epsilon_{lj} \frac{(\mathbf{x} - \mathbf{x}'')^l}{|\mathbf{x} - \mathbf{x}''|^2} \frac{(\mathbf{x} - \mathbf{x}''')^j}{|\mathbf{x} - \mathbf{x}'''|^2} \tag{9}$$

where Ω has the dimensions of volume. Since the Chern-Simons action is linear in the time derivative, it possesses no dynamics independently of J^μ and thus its sole effect is to induce certain symmetry relations on the field operators, a fact which is manifest in the above expression. In deriving (9), a number of relations concerning vortex fluxline singularities have been used. It is convenient to state these for the record:

$$\tan\theta(\mathbf{x} - \mathbf{x}') = \frac{(\mathbf{x} - \mathbf{x}')^2}{(\mathbf{x} - \mathbf{x}')^1} \; , \tag{10}$$

$$-\frac{1}{2\pi}(\partial_l \theta) = \epsilon_{lj}\partial_j g(\mathbf{x} - \mathbf{x}') \; , \tag{11}$$

$$\nabla^2 g(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \; , \tag{12}$$

$$g(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi}\ln|\mathbf{x} - \mathbf{x}'| \; . \tag{13}$$

θ is formally the winding angle between two flux singularities and satisfies the curious relation

$$[\partial_1, \partial_2]\theta(\mathbf{x} - \mathbf{x}') = 2\pi\delta(\mathbf{x} - \mathbf{x}'). \tag{14}$$

These relations will be a useful reference later when interpreting the equations of motion for the field operators. (Note also the discussion in [9] concerning these relations.)

Let us now turn to the case in which the coefficient $\mu(x)$ is an arbitrary function. As shown in Ref. [3], this necessitates an additional variable coupling to the source in order to satisfy a suitable gauge-invariance constraint:

$$S = \int dV_x \left(\frac{1}{2}\mu(x)\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda - f(x) J^\mu A_\mu \right). \tag{15}$$

Since both couplings are position dependent, this represents a phenomenological system rather than a fundamental one. In order to proceed, one needs to apply a physical boundary condition to the source. As explained earlier [3], the consistency of this theory then requires that the source be adjusted in such a way that gauge invariance is maintained and energy is conserved. Since we do not want the source coupling to vanish when μ is constant, the natural boundary condition in this instance is $f(x) = \mu(x)/\alpha$, for some constant mass scale α. Thus, after a convenient rescaling, one may write

$$S = \int dV_x \mu(x) \left(\frac{1}{2}\alpha\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda - J^\mu A_\mu \right) \tag{16}$$

where $\mu(x)$ is now a dimensionless field. The role of $\mu(x)$ is to present the system through a "distorting glass." The physical picture is that of a two-dimensional gas of particles influenced microscopically but smoothly by sites in neighboring planar systems. The special form of the action together with the constraint results in the preservation of gauge invariance.

The allowed class of variations of the action is determined from the consideration of an infinitesimal gauge transformation $A_\mu \to A_\mu + \partial_\mu \xi$, which provides us with an operator constraint. We shall assume that the current J^μ is conserved and that the variation of ξ commutes with the field. On varying the action with respect to $\delta\xi$, one obtains the constraint

$$\frac{1}{2}\alpha\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = J^\mu \tag{17}$$

and the generator of infinitesimal gauge transformations

$$G_\xi = \int d\sigma \left(\frac{1}{2}\alpha\mu\epsilon^{ij}\partial_j A_i - \mu J^0 \right) \delta\xi. \quad (18)$$

These are gauge invariant, indeed one sees how the formalism which includes the physical boundary condition repairs the canonical structure of the theory in the presence of variable $\mu(x)$. The solutions to (17) determine now the class of variations under which the quantum theory will be gauge invariant. Choosing the Coulomb gauge to eliminate the unphysical degrees of freedom from the field operators, one may solve (17) to get

$$A_\sigma = 2\alpha^{-1} \int d^2\mathbf{x}' \epsilon_{\sigma\rho\lambda}\partial^\rho J^\lambda(\mathbf{x}') g(\mathbf{x}, \mathbf{x}'). \quad (19)$$

The variation of this result now yields the allowed values for δA_μ. Returning to (16) one may thus vary with respect to the dynamical variable A_μ to obtain the gauge-invariant equations of motion for the field operators:

$$J^i = \alpha\epsilon_{ij}E^j + \alpha\epsilon^{ij}(\partial_j\mu)\mu^{-1}\frac{\epsilon_{lm}\partial^l J^m}{\nabla^2} - \alpha\epsilon^{ij}(\partial_0\mu)\mu^{-1}\frac{\epsilon_{lj}\partial^l\rho}{\nabla^2} , \quad (20)$$

$$\rho = -\alpha B - \alpha\epsilon^{ij}(\partial_j\mu)\mu^{-1}\frac{\epsilon_{li}\partial^l\rho}{\nabla^2}. \quad (21)$$

The first of these equations clearly describes a modification to the Hall current of the system. The spatial gradient of μ makes the current dependent on its own curl in precisely the manner of a magnetic-moment interaction [10,4,11]. It is interesting to compare this form to the parallel theory [11] in which the gauge field couples directly to the source through a parity-violating term. The same magnetic current loop interaction appears in both cases. The time gradient term leads to an additional induction effect.

To ascertain the meaning of the second equation, it is useful to define a field θ by analogy with Eq. (11). Now, integrating by parts and assuming only weakly varying μ, one obtains

$$[\partial_1, \partial_2]\theta(\mathbf{x}) \sim 2\pi B \sim 2\pi\delta(\mathbf{x}) \quad (22)$$

since $\rho \sim -\alpha B$. The translational invariance of the field θ has also been assumed. This "rough and ready" last step serves mainly as a guide to physical intuition and shows that (21) predicts a nonlocal generalization of the vortex lines in the theory with constant μ.

Extracting the generator of infinitesimal unitary transformations from the variation of the action operator, one easily determines that the commutator analogous to (8) is given by the implicit equation

$$i\alpha\mu^{-1}\pi^2\delta(\mathbf{x}, \mathbf{x}') = \int d^2\mathbf{x}'' d^2\mathbf{x}''' [\rho(\mathbf{x}''), \rho(\mathbf{x}''')] \times \epsilon_{lj}\frac{(\mathbf{x} - \mathbf{x}'')^l}{|\mathbf{x} - \mathbf{x}''|^2}\frac{(\mathbf{x}' - \mathbf{x}''')^j}{|\mathbf{x}' - \mathbf{x}'''|^2}. \quad (23)$$

Finally, since the Chern-Simons term imparts no dynamics to the system, the Hamiltonian must be expected to vanish. The Hamiltonian for the Chern-Simons action can be computed from $H = -\frac{\delta S}{\delta t}$ and is indeed found to vanish under the restricted class of variations in (19). Under general variations, it is nonvanishing when $\mu(x)$ is spacetime dependent. The time variation may be defined by

$$\delta S = \int_t^{t+\delta t} L(A_\mu, J_\nu) = \int dt\, \delta L, \quad (24)$$

where, to first order,

$$\delta\mu(x) = \frac{\partial\mu}{\partial t}\delta t , \quad (25)$$

$$\delta A_\mu = F_\mu{}^\sigma \delta x_\sigma. \quad (26)$$

The latter gauge invariant transformation is required to generate the symmetrical, conserved energy-momentum tensor for the theory [12]. The Hamiltonian operator is therefore

$$H = -\int d^2\mathbf{x}\, (\partial_t\mu)\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \quad (27)$$

On using the solution of the operator equations of motion (19) this is seen to vanish as required. The reason has already been described in earlier work: the interpretation of the naive unconstrained theory is that of an open system and the energy is therefore not automatically conserved. One would therefore encounter a nonvanishing Hamiltonian.

An interesting feature of the present vortex system is that the gauge invariance constraint (17) does not involve the spacetime-dependent field $\mu(x)$ unlike the Maxwell-Chern-Simons theory in Ref. [3,4]. This has an important implication—namely, that, in the absence of external magnetic fields, the flux lines can form arbitrary stable gradients in μ without violating gauge invariance. This must be understood as a topological phenomenon since the relations provide no dynamical reason for such behavior. It might be possible in certain cases to identify these with spin textures. The obvious information we are missing which decides these gradients is the details of the neighboring system(s). One would expect, on the basis of experience with the Maxwell-Chern-Simons system, that when the coupling to the external system is removed, the Chern-Simons coefficient would have to decay to a constant value. This is indeed the case. If one relaxes the imposed boundary condition and takes $f(x) \to$ const, then the gauge invariance condition leads to the familiar equation [3]

$$(\partial_0\mu)B + (\partial_i\mu)\epsilon^{ij}E_j = 0 , \quad (28)$$

which has decaying solutions in the manner of the Langevin equation. Thus the interpretation of the system is fully self-consistent.

To summarize, a Chern-Simons field theory coupled to a gauge invariant current J^μ through the field $\mu(x)$ is only gauge invariant and unitary under a restricted class of operator variations. This can be understood as arising from an interaction with an external system. The restricted theory can be explored with the help of a constraint formalism applied to the Schwinger action principle. The corrections to regular Chern-Simons theory indicate a modification of the Hall current for vortex lines in a manner which resembles a magnetic moment interaction term and an induction term. The sharp nature of the vortices is distorted by the gradients in $\mu(x)$ but the basic excitations are of a similar nature.

It should be possible, by supplementing the source terms with extra impulsive sources, to compute the many-body Green functions for this theory directly from the Schwinger action principle. These may then be used to determine the corrections to the thermodynamical and transport properties of this model, particularly the effect of the gradients in $\mu(x)$ on the conductivity in a model for a superconductor. The present results are model independent, but agree well with the specific model presented in Ref. [2] and back up the work of Ref. [4].

The present model, motivated essentially by symmetry considerations and its connection with the widely discussed anyon model, has been simplified as far as possible for the sake of illustration. A more realistic model would be more specific about the origin of the source terms and must provide some empirical estimate of the strength of the coupling, perhaps using data for the observed magnetic moment interactions in high-T_c superconductors. These points turn out to involve some subtle issues and will be pursued elsewhere.

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