Quantum source of the back reaction on a classical field

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Pair creation in an external electric field is presented in terms of the localized events which contribute to the mean current that is produced. Each of these localized contributions is given by a nondiagonal matrix element of the current operator. The physical relevance of this matrix element is displayed. It can also be interpreted as the result of a weak measurement in the sense of Aharonov *et al.*

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I. INTRODUCTION

The problem of the back reaction on a classical field due to a quantum process is to a great extent an unsolved problem in modern physics. Ignorance of the solution is (at the least) one of the barriers towards the solution of the so-called unitarity problem posed by black hole physics [1]; i.e., how does the black hole evaporation engendered by gravitational collapse convey all the information that has gone into the initial conditions describing the quantum state prior to the collapse?

The state of the art of black hole reaction physics is at present rather primitive. For example in the black hole problem one calculates the expectation value of the energy-momentum due to Hawking radiation [2] (for a review see Ref. [3]), i.e., the thermal radiation that occurs at large Schwarzchild times where the collapsing body hugs the horizon infinitely closely. This expectation value then serves as a source to the classical Einstein's equations [4–6]. Clearly the fluidlike description characteristic of this semiclassical approach begs the question of the details of how the gravitational field, better that part of the wave function which describes gravity, reacts to the quantum process, i.e., tunneling [7], which results in the emission of a single (or a few) Schwarzschild quantum (quanta). At best the semiclassical theory describes some coarse-grained mean evolution. So it is very difficult to imagine that such an approach to the unitary issue can reveal the information, phases among other things, being sought.

The considerations of this paper are conceived as a (very small) step towards the solution of this problem. We study the source of the back reaction on the electric field due to production of pairs induced by its presence. As pointed out by several authors [8–11], this problem, owing to the presence of horizons (caused by the constant acceleration induced by the field on charged quanta), bears many analogies to black hole evaporation. Indeed, the amplitudes for production in the two cases have essentially the same mathematical structure. But there are profound differences which cause the black hole problem to be more difficult. For example the "member" of the pair that is not measured (or measurable) by the Schwarzschild observer is hidden from him by the horizon (except if it turns out in the end that the complete back reaction destroys the horizon; but then the problem is even more complicated). Of course in the electric case both members of the produced pair are accessible to observation and subtle problems of loss of information do not arise (provided one allows for ubiquitous measurement). Nevertheless interesting nontrivial problems occur even in this case when one delves into the quantum mechanics of the production of a single pair.

The problem of how to deal with electroproduction in the mean has been dealt with by Cooper *et al.* [12] in a semiclassical approach similar to that brought to bear in black hole physics. Their approach is valid in the presence of large density of pairs. Our purpose here is to ask more detailed questions appropriate to the opposite case where pairs are rare. How does one isolate the counter electric field carried by a localized pair? And how does this counter electric field modify the probability of find-

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ing the next pair? We recall that a pair is created by tunneling [13,9,10] out of a region of spatial dimension $|\Delta x|$, which is of order a^{-1} where a is the acceleration $(= E/m)$, E being the electric field with the charge of the field quanta absorbed into its definition, and m their mass. For $|\Delta x| > a^{-1}$ the produced quanta propagate on mass shell in opposite directions, and one expects in a wave packet description that this asymptotic region can be adequately described in conventional terms. But what happens *as* the pair is created in the tunneling region?

In our effort to answer these questions we found difficulty for the following reason. If in the distant past one has vacuum, then in the future, pairs are produced everywhere and all the time. Since the final state is *not* a single pair state we must address the problem of how to isolate the source of the change of the field E produced by a single pair. The solution to this problem is to consider only those outcomes wherein a specific localized pair is produced. By considering only this restricted set of final states one will learn information about the pair, isolated from all the other pairs.

In order to isolate the current carried by the pair and to prove its physical relevance we ask specific questions, such as how is the probability of creating this pair modified by a modification of the external electric field? It is seen that, in first order of perturbation theory, the answer to all such questions depends on a certain nondiagonal matrix element of the current operator. Thus this matrix element controls the back reaction in that particular channel exactly as the mean value of the current controls the back reaction when no further specification of the final state of the system is required. It is these matrix elements which we study.

These nondiagonal matrix elements were first analyzed in detail by Aharonov *et al.* [14] in the context of measurement theory. These authors fix both the initial and the final state of a system. They then inquire into the response of a measuring device which interacts "weakly" with the system at an intermediate time. The answer is given in terms of these nondiagonal matrix elements. The repeated appearance of these matrix elements in a large variety of physical problems is what motivates our detailed analysis of them.

In brief what we obtain is a description of the current density and charge density of the pair *as* it is produced. When the two members of the pair are on shell, the matrix elements are real and are localized around the classical trajectories. There is also a production region of dimension $\Delta x \times \Delta t \simeq 2a^{-1} \times 2a^{-1}$. In this region the nondiagonal matrix elements of the charge and current are complex and oscillate. This reflects the highly quantum nature of what is taking place in this region: a vacuum fluctuation is being converted into a pair. The complexity has a natural interpretation in the context of the questions we ask about the pair (for instance, a small modification of the external electric field modifies the probability of finding the pair only if the electric field is different where the charge and current are complex, i.e., in the production region). These results are depicted in Figs. 3(a)–3(e).

Some of the paper is concerned with technical matters which rely on previously published material. To keep the paper self-contained, relevant previous work is presented in Sec. II and Appendix A. Section II contains a resume of the mode decomposition in the presence of an E field, Sec. III the description of particle production in terms of wave packets, Sec. IV our most interesting new results, an analysis of the current carried by a localized produced pair given in terms of nondiagonal matrix elements. Appendixes A and B contain, respectively, a summary of the weak value theory of Aharonov *et al.* and an analysis of the Hadamard subtraction procedure adapted to the present problem.

II. QUANTIZATION IN AN E FIELD

We summarize, in this section, quantization in the presence of an E field. To facilitate the presentation we first proceed formally by giving the mode decomposition in the presence of an E field which is constant in both space and time. The relevance of this idealization to the case where the E field is switched on at some time and whose effects are investigated at some later time is then discussed. It will then be seen that temporal effects arise through a proper mode counting procedure. For the sake of clarity we shall develop the formalism by treating two different gauges which we shall call the x gauge $(A_0 = -Ex, A_x = 0)$ and t gauge $(A_0 = 0, A_x = Et)$. Wave packets are constructed in Sec. III and are gauge independent up to an irrelevant phase factor.

We shall work with the simplest example, a complex scalar field in 1 + 1 dimensions. The Klein-Gordon (KG) equation is

$$(\mathcal{D}_\mu \mathcal{D}^\mu + m^2)\phi = 0, \tag{1}$$

where $\mathcal{D}_\mu = \partial_\mu + iA_\mu$ where the electric charge e has been englobed in the definition of A_μ. In the x gauge the KG operator commutes with ∂_t and the modes are of the form

$$\varphi_\omega(t,x) = Ce^{i\omega t}\chi_\omega(x) = Ce^{i\omega t}\chi(x - \omega/E), \tag{2}$$

where C is chosen to norm φ to unit charge $(i\int dx|C|^2[\varphi^*_\omega(\mathcal{D}_0\varphi_\omega) - (\mathcal{D}^*_0\varphi^*_\omega)\varphi_\omega] = 1)$. $\chi_\omega(x)$ obeys the equation

$$\left[\frac{\partial^2}{\partial x^2} + E^2(x - \omega/E)^2\right]\chi_\omega(x) = m^2\chi_\omega(x). \tag{3}$$

Upon dividing by $-2m$ one comes upon a Schrödinger equation describing tunneling of a particle of mass m in an upside down oscillator potential centered at $x = \omega/E$ whose curvature is $-E^2/2m$. The energy of the particle is $-m/2$ so that its classical turning points are at $(x - \omega/E) = \pm a^{-1}$ where a is the acceleration $(= E/m)$. In this effective Schrödinger problem an incoming particle (in a wave packet) gives rise to reflected and transmitted waves of flux R and T, respectively, and for unit incident flux we have the unitarity condition

$$|R|^2 + |T|^2 = 1. \tag{4}$$

We record here the well-known result [11]

$$|T/R| = e^{-\pi m^2/2E}. \tag{5}$$

Since these two branches accelerate in opposite directions we surmise that in the KG context they correspond to particles of opposite charge and that the reflected and incident wave have the same charge.

This identification is borne out by an analysis of the motion of localized wave packets of the modes $\varphi_\omega(t,x)$, that is, linear combinations of the form $\int d\omega f(\omega-\omega_0)\varphi_\omega(t,x)$ (rather than the usual sum over energy eigenfunctions of the Schrödinger equation). This construction is carried out explicitly in Sec. III. For example, if E points to the right and if an incoming packet from the right carries unit positive charge one establishes that the transmitted wave having an amplitude β carries a negative charge (with flux $|\beta|^2$) and the reflected wave with amplitude α carries a positive charge (with flux $-|\alpha|^2$). Charge conservation yields

$$|\alpha|^2 - |\beta|^2 = 1. \tag{6}$$

As explained in Refs. [10,11], the relation between Eqs. (4) and (6) is established through the swap of incident and reflected waves necessitated by the movements of the wave packets. Thus one obtains Eq. (6) from Eq. (4) by dividing the latter by $|R|^{-2}$ with $\alpha = 1/R$, $\beta = T/R$. In this way β is identified with the amplitude for pair production.

The modes $\varphi_\omega(t,x)$ satisfying the above initial conditions (incident flux in the direction of E) provide basis functions for the in-quantization scheme since they correspond to the propagation of a single particle in the past and a particle plus a pair in the future. Its parity conjugate obtained by $(x-\omega/E) \to -(x-\omega/E)$ then corresponds to the presence of an antiparticle in the past, once more yielding an additional pair in the future. Introducing labels p and a for particle (antiparticle) and following the standard convention for Whittaker functions [15] we thus have the in-basis functions

$$\begin{aligned}\varphi^{\rm in}_{\omega,p}(x,t) &= \frac{e^{-3\pi\mu^2/4}}{(2E)^{1/4}} e^{i\omega t} D_{i\mu^2-1/2}[e^{-3i\pi/4}\sqrt{2E}(x-\omega/E)],\\ \varphi^{\rm in}_{\omega,a}(x,t) &= \varphi^{\rm in}_{-\omega,p}(-x,t)\ ,\end{aligned} \tag{7}$$

where $\mu^2 = m^2/2E$. The second-quantized field is then written, following the usual rules, as

$$\phi = \sum_\omega (a^{\rm in}_\omega \varphi^{\rm in}_{\omega,p} + b^{\rm in\dagger}_\omega \varphi^{\rm in*}_{\omega,a}). \tag{8}$$

The operators $a^{\rm in}_\omega$ and $b^{\rm in}_\omega$ define the in vacuum by

$$a^{\rm in}_\omega|0\rangle_{\rm in} = 0,\ \ b^{\rm in}_\omega|0\rangle_{\rm in} = 0. \tag{9}$$

Since the in-basis functions contain a particle plus a pair in the future they are not useful to describe quantization in terms of single quanta in the future (i.e., those which would be registered in counters). Their time-reversed versions are clearly the right set and these are given simply by

$$\begin{aligned}\varphi^{\rm out}_{\omega,p}(x,t) &= \varphi^{\rm in*}_{\omega,p}(x,-t),\\ \varphi^{\rm out}_{\omega,a}(x,t) &= \varphi^{\rm in*}_{-\omega,p}(-x,-t).\end{aligned} \tag{10}$$

Since for each ω the set $\varphi^{\rm in}_{\omega,p}$ and $\varphi^{\rm in}_{\omega,a}$ is complete one may express the $\varphi^{\rm out}_\omega$'s as linear combinations of $\varphi^{\rm in}_\omega$'s. One finds [10]

$$\begin{aligned}\varphi^{\rm out}_{\omega,p} &= \alpha\varphi^{\rm in}_{\omega,p} + \beta\varphi^{\rm in*}_{\omega,a},\\ \varphi^{\rm out}_{\omega,a} &= \beta\varphi^{\rm in*}_{\omega,p} + \alpha\varphi^{\rm in}_{\omega,a},\end{aligned} \tag{11}$$

where

$$\alpha = \frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\mu^2/2}}{\Gamma(1/2+i\mu^2)},\quad \beta = e^{i\pi/2}e^{-\pi\mu^2}. \tag{12}$$

Inserting Eq. (11) into the expansion

$$\phi = \sum_\omega (a^{\rm out}_\omega \varphi^{\rm out}_{\omega,p} + b^{\rm out\dagger}_\omega \varphi^{\rm out*}_{\omega,a}), \tag{13}$$

one finds the Bogoliubov transformation

$$\begin{aligned}a^{\rm out}_\omega &= \alpha^* a^{\rm in}_\omega - \beta^* b^{\rm in\dagger}_\omega,\\ b^{\rm out}_\omega &= \alpha^* b^{\rm in}_\omega - \beta^* a^{\rm in\dagger}_\omega.\end{aligned} \tag{14}$$

As in Eq. (9) the operators $a^{\rm out}_\omega$ and $b^{\rm out}_\omega$ define the out vacuum by

$$a^{\rm out}_\omega|0\rangle_{\rm out} = 0,\ \ b^{\rm out}_\omega|0\rangle_{\rm out} = 0. \tag{15}$$

It follows from Eq. (14) that the in vacuum is a superposition of out states given in terms of produced pairs:

$$|0\rangle_{\rm in} = \prod_\omega \frac{1}{\alpha}\exp\left[-\frac{\beta^*}{\alpha}a^{\rm out\dagger}_\omega b^{\rm out\dagger}_\omega\right]|0\rangle_{\rm out}. \tag{16}$$

The probability to find no out particles in the future is then found to be [16]

$$\begin{aligned}|{}_{\rm out}\langle 0|0\rangle_{\rm in}|^2 = \prod_\omega \frac{1}{|\alpha|^2} &= \exp\left[-\sum_\omega \ln(1+|\beta|^2)\right]\\ &= \exp\left[-\frac{LTE}{2\pi}\ln(1+|\beta|^2)\right],\end{aligned} \tag{17}$$

where LT is the volume of the spacetime box over which E is nonvanishing. See the end of this section for the proof and discussion of $\sum_\omega = (2\pi)^{-1}LTE$.

We now sketch the corresponding analysis in the gauge $A_x = Et$, $A_0 = 0$. It is in this gauge that we have carried out our computations. Here the modes are of the form $Ce^{ikx}\xi_k(t)$, where $\xi_k(t)$ obeys the same type of equation as $\chi_\omega(x)$. This is obtained by replacing x by t and ω by k. The term $m^2\chi_\omega$ on the right-hand side (RHS) of Eq. (3) becomes $-m^2\xi_k$. Thus the effective Schrödinger equation is the same but for the sign of the energy ($+m/2$ rather than $-m/2$). Tunneling does not exist in these modes, rather it is replaced by back-scattering in time.

A packet moving forward in time in the distant past gives rise to a transmitted wave propagating forward in time and a reflected wave moving backward in time. Following the analytic procedures of [15] one readily confirms that the amplitude of this backward wave is equal to the amplitude of the transmitted wave in the corresponding tunneling problem. Furthermore, analysis of wave packets confirms one's expectations that the role of the transmitted wave in x gauge is played by that of the reflected wave in t gauge. Therefore up to a phase the coefficients α and β remain the same, and the whole previous analysis of quantization in x gauge is applicable as such. One simply changes to the basis functions $e^{ikx}\xi_k(t)$ which in terms of Whittaker functions are

$$\varphi^{\rm in}_{k,p}(t,x) = \frac{e^{-\pi\mu^2/4}}{(2E)^{1/4}} e^{ikx} D_{i\mu^2-1/2}[e^{3i\pi/4}\sqrt{2E}(t+k/E)]. \tag{18}$$

With this choice of function and using the same definition for the antiparticle functions and the out-basis functions as in Eq. (7) and Eq. (10) the α and β coefficients are equal in both gauges and given by Eq. (12). In this gauge the sum $\sum_\omega$ in Eq. (17), becomes the $\sum_k$ which will now be shown to be equal to $LTE/2\pi$, as well.

Until now we have proceeded formally with $E = \text{const}$. In consequence one has come upon the undetermined expressions $\sum_\omega$ or $\sum_k$. (Note that the coefficients α and β are independent of the mode indices in these formal expressions.) We also remark that in the formal method of functional integration of Schwinger [16] one comes upon a volume factor LT with L the length of the system over which E is nonvanishing and T the duration of time from switch on of E to the observation (which may or may not correspond to the switch off of E).

The origin of a temporal description in this formal mode analysis has been the source of some confusion to several of our colleagues so we shall now dwell at length on this point. We discuss the physics in the two different gauges. Before the field is switched on, the vacuum fluctuations move to the left and to the right with equal weight. After the field is turned on (say in the left direction) a certain number of right- (left-) moving particle (antiparticle) fluctuations reverse their direction (in the sense of a wave packet) during time T. It is only the fluctuations in this class which result in particle production. To make this point clear we note that we are working in a specific frame: the field is produced by a pair of condenser plates at rest, separated by a length L. The coefficient β, which encodes pair production, arises from tunneling (in the x gauge) or back scattering (t gauge), these phenomena occur only near turning points, and the space position of these turning points are calculated in this frame. Therefore it is necessary to qualify all the formulas presented so far since, as written, they are valid only for $E = \text{const}$. Clearly the modes we have derived are good approximations to the physical modes (i.e., those that respond to a field switched on at time zero by producing pairs in the interval T) provided they have turning points well within the space-time volume: $0 < x < L$ and $0 < t < T$. Quantitatively, "well within" means removed from the edges by $O(a^{-1})$ since the production event occurs on this scale. (This is also discussed at the end of Appendix B.)

Thus the sums $\sum_\omega$ or $\sum_k$ are evaluated by counting the number of modes whose turning points lie within $0 < x < L$ and $0 < t < T$. One then gets an asymptotic formula valid for $L \gg a^{-1}$ and $T \gg a^{-1}$. To count these is an easy matter. In x gauge $\sum_\omega = \frac{T}{2\pi}\int d\omega$ and since the turning points occur at $x_t = \omega/E$ and $0 < x_t < L$ we have $\int d\omega = EL$ so that $\sum_\omega = ELT/2\pi$. Similarly in t gauge one has $\sum_k = \frac{L}{2\pi}\int dk = ELT/2\pi$ since the interval of k for which there are turning points in the interval $0 \le t \le T$ is $-Et \le k \le 0$.

More amusing is the following physical argument. Using classical mechanics to describe the trajectory of the incoming wave packet, one has, in t gauge, $p_x = mdx/d\tau - A_x = mdx/d\tau - Et$ where τ is the proper time. Since p_x is constant, the action being independent of x, we have $mdx/d\tau - Et = k$. The constant k is thus identified with the velocity times the mass $mdx/d\tau$ when the field is turned on at $t = 0$. Turning points $(dx/d\tau = 0)$ which occur in the interval $0 \le t \le T$ thus correspond to $-ET \le k \le 0$ as stated. In this variety of ways one recovers Schwinger's formula [16] for the case of 1+1 dimensions.

[An interesting aside is that this last counting procedure is precisely what one can use to get the axial anomaly in 0+2 dimensions. Here one first computes the final velocity of the vacuum of fermions in 1+1 dimensions induced by an E field turned on during a time T over a length L. Since this is $\int \bar\psi\gamma_0\gamma_1\psi dx$ and $\gamma_0\gamma_1$ is the Minkowski analogue of $\gamma_1\gamma_2$ one has by analytic continuation the anomaly in the presence of a magnetic field perpendicular to the (12) plane.]

We note that our Bogoliubov coefficients may be unfamiliar to some readers in the sense that our asymptotic waves are not plane waves, but rather accelerating waves. This, however, is irrelevant for the definition of the Bogoliubov transformation. For our purposes the necessary condition is the existence of in and out single-particle asymptotic states defined in a gauge independent way by constructing wave packets that carry unit charge of a given sign in the asymptotic regions. The Bogoliubov transformation is the (unitary) mapping of one set into the other.

Finally, it will have been noticed that we have used modes in the sense of broad wave packets, a time-honored procedure in scattering theory. However, in the next section, we shall be more precise and reformulate things entirely in terms of well-localized packets.

III. WAVE PACKETS

In terms of the basis functions of Eq. (7) we make the packet construction to describe an in-particle

$$\psi^{\rm in}_{k_0x_0,p}(x,t) = \int dk f(k-k_0) e^{-ikx_0}\varphi^{\rm in}_{k,p}(x,t). \tag{19}$$

The wave packet is centered at $x = x_0$ and $t_0 = k_0/E$

[since $\varphi^{\rm in}_{k,p}(x,t)$ is a function of $t+k/E$, and the quadratic potential of the Schrödinger problem is centered at $t = k/E$]. Without loss of generality we take a wave packet centered at the origin, i.e., $x_0 = 0$, $k_0 = 0$, and henceforth we drop the indices x_0, k_0.

The notion of a minimal wave packet emerges from the asymptotic behavior of the modes $\varphi^{\rm in}_{k,p}$ as $t \to \pm\infty$ where the WKB approximation prevails:

$$\lim_{t\to-\infty} \varphi^{\rm in}_{k,p} \simeq e^{ikx} e^{iE(t+k/E)^2/2},$$

$$\lim_{t\to+\infty} \varphi^{\rm in}_{k,p} \simeq \alpha^* e^{ikx} e^{-iE(t+k/E)^2/2} - \beta e^{ikx} e^{iE(t+k/E)^2/2}, \tag{20}$$

where we have kept only the leading exponential behavior of φ and α, β are the Bogoliubov coefficients Eq. (12). The analysis is greatly facilitated by taking f to be Gaussian, whereupon in the asymptotic regions the packet (always taking $x_0 = 0$, $k_0 = 0$) is

$$\begin{aligned}\lim_{t\to-\infty} \psi^{\rm in}_p &\simeq \int dk e^{-k^2/2\sigma^2} e^{ikx} e^{iE(t+k/E)^2/2} \\ &\simeq e^{iEt^2/2} e^{-(x+t)^2/2\Sigma_-^2}\,, \\ \lim_{t\to+\infty} \psi^{\rm in}_p &\simeq \int dk e^{-k^2/2\sigma^2} (\alpha^* e^{ikx} e^{-iE(t+k/E)^2/2} \\ &\qquad -\beta e^{ikx} e^{iE(t+k/E)^2/2}) \\ &\simeq \alpha^* e^{-iEt^2/2} e^{-(x-t)^2/2\Sigma_+^2} \\ &\qquad -\beta e^{iEt^2/2} e^{-(x+t)^2/2\Sigma_-^2}\,,\end{aligned} \tag{21}$$

where

$$\Sigma_+^2 = \left(\frac{1}{\sigma^2} + \frac{i}{E}\right), \quad \Sigma_-^2 = \left(\frac{1}{\sigma^2} - \frac{i}{E}\right). \tag{22}$$

As announced in Sec. II the wave packet $\psi^{\rm in}_p$ in the far past $t \to -\infty$ carries only positive charge in a localized region of space time (verified directly by acting on $\psi^{\rm in}_p$ with the charge operator $i\overleftrightarrow{\mathcal{D}}_0$). In the distant future $t \to +\infty$ it carries positive and negative charge in distinct regions of space time with weights $|\alpha|^2$ and $|\beta|^2$, respectively.

The asymptotic width of the wave packets is given by $[{\rm Re}(1/\Sigma_\pm^2)]^{-1/2}$. If ${\rm Im}(\sigma) \neq 0$, then the two branches of $\psi^{\rm in}_p$ as $t \to +\infty$ have unequal widths. This implies an asymmetric treatment of the particles and antiparticles. As nothing warrants such partiality we take ${\rm Im}(\sigma) = 0$. In this case the asymptotic width of the wave packets is $[{\rm Re}(1/\Sigma_+^2)]^{-1/2} = [{\rm Re}(1/\Sigma_-^2)]^{-1/2} = [(E^2 + \sigma^4)/E^2\sigma^2]^{1/2}$ and is minimized by the choice $\sigma^2 = \sigma^2_{\rm min} = E = ma$ corresponding to wave packets of width $[{\rm Re}(1/\Sigma^2_{\rm min})]^{-1/2} = [2/E]^{1/2}$.

To illustrate these properties we have plotted in Figs. 1(a)–1(c) and 2 the norm and the current carried by localized wave packets. These figures are plots of an exact solution of the wave Eq. (1) built as a Gaussian superposition of modes $\varphi^{\rm in}_{k,p}$. We now construct this solution. To this end we introduce an integral representation of the Whittaker functions. It is given in terms of eigenfunctions of the operator UV where $U = (-\frac{1}{\sqrt{E}}i\partial_t - t\sqrt{E})/\sqrt{2}$ and $V = (-i\frac{1}{\sqrt{E}}\partial_t + t\sqrt{E})/\sqrt{2}$ as explained in [17,11] in physical terms. It is also this integral representation which serves as a basic tool in revealing interesting properties of these functions. It now has the added luster that the integral over k in Eq. (19) is Gaussian thereby leading to a simple integral representation for Gaussian packets which is nothing more than another Whittaker

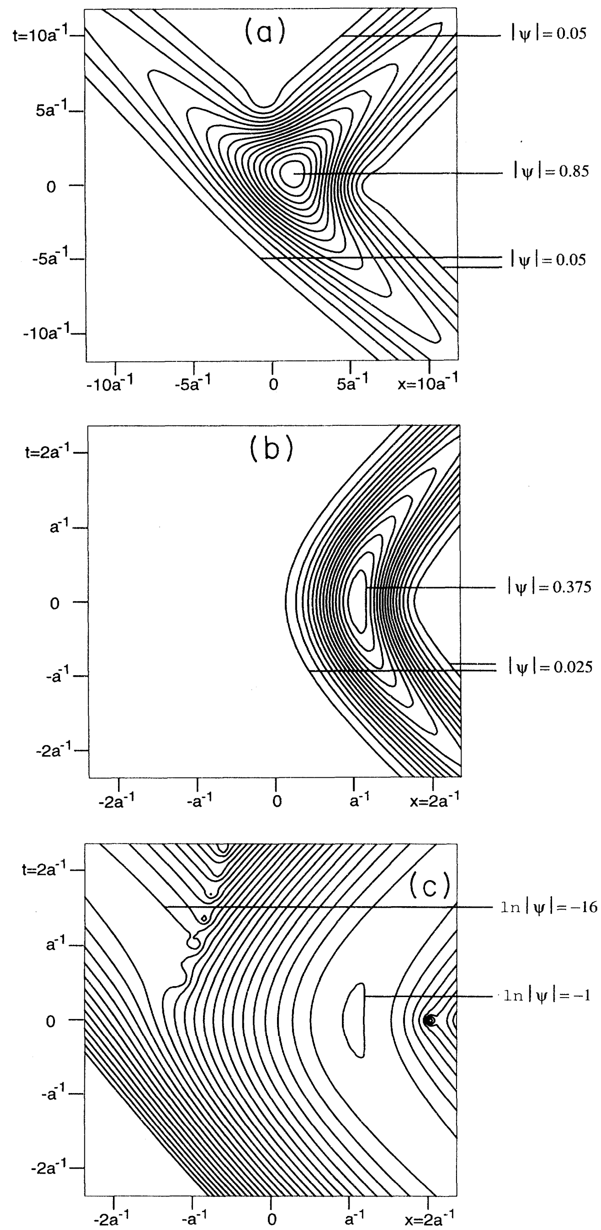


FIG. 1. Representation by equally spaced contour lines of the absolute value of wave packets $|\psi^{\rm in}_p|$ [Eq. (24) with $\sigma = \sigma_{\rm min}$] centered on the origin of space time. We have illustrated $|\psi^{\rm in}_p|$ (a) for $m/a = 1/4$ as typical for $m/a < 1$ and (b) for $m/a = 9$ as typical for $m/a \gg 1$. In the latter case the created antiparticle does not appear since its amplitude is too small to see. (c) is a drawing of $\ln|\psi^{\rm in}_p|$ for $m/a = 9$ which permits the display of the created antiparticle. In all figures the length unit is the inverse acceleration a^{-1}.

function. This we now display, where we work in the temporal gauge and choose for simplicity units where $E = 1$. The integral representation of $\varphi^{\rm in}_{k,p}(x,t)$ we use is (with $\mu^2 \equiv m^2/2E$ in conventional units)

$$\varphi^{\rm in}_{k,p}(x,t) = \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}} e^{ikx} D_{i\mu^2-1/2}[e^{3i\pi/4}\sqrt{2}(t+k)]$$
$$= \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}\Gamma(1/2-i\mu^2)} e^{ikx} e^{i(k+t)^2/2} \times \int_0^{+\infty} du e^{(1-i)(k+t)u-u^2/2} u^{-i\mu^2-1/2}, \tag{23}$$

whereupon the Gaussian wave packet centered at $t_0 = 0$ and $x_0 = 0$ is

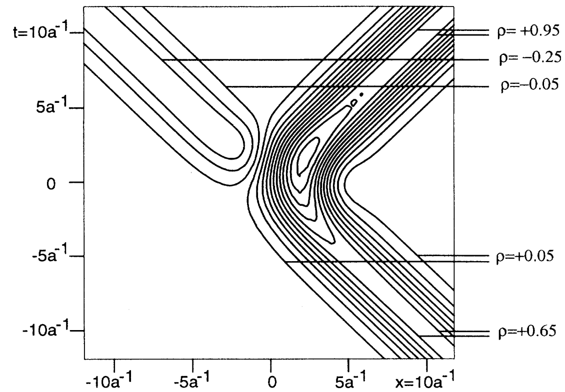


FIG. 2. Representation by equally spaced contour lines of the charge density $\rho(t,x)$ carried by the wave packet drawn in Fig. 1(a). The created antiparticle (on the left) carries negative charge.

$$\psi^{\rm in}_p(x,t) = \int_{-\infty}^{+\infty} dk(2\pi)^{-1/2}\sigma^{-1}e^{-k^2/2\sigma^2}\varphi^{\rm in}_{k,p}(x,t)$$
$$= \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}\Gamma(1/2-i\mu^2)} \int_{-\infty}^{+\infty} dk \frac{e^{-k^2/\sigma^2}}{\sqrt{2\pi}\sigma} \int_0^{+\infty} du\, u^{-i\mu^2-1/2} \exp\left(ikx + i\frac{(k+t)^2}{2} + (1-i)(k+t)u - \frac{u^2}{2}\right)$$
$$= \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}} \frac{1}{\Sigma_-\sigma} \xi_-^{1/2-i\mu^2} e^{it^2/2} e^{-(t+x)^2/2\Sigma_-^2} e^{z^2/4} D_{i\mu^2-1/2}(z), \tag{24}$$

where $\Sigma_-^2 = (1/\sigma^2 - 1)$, $\xi_-^2 = (1-i\sigma^2)/(1+i\sigma^2)$, and $z = -\xi_-(1-i)(t+i\sigma^2 x)/(1-i\sigma^2)$ with $|\arg\Sigma_-| < \pi/4$ and $|\arg\xi_-| < \pi/2$. We now present the demonstration of Eq. (24).

To prepare for the integral over k we first obtain a slightly more general integral representation of $\varphi^{\rm in}_{k,p}$ than Eq. (23) by applying Cauchy's theorem to the contour $(0, \infty, \infty e^{i\arg\lambda}, 0)$:

$$\varphi^{\rm in}_{k,p}(x,t) = \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}} e^{ikx} D_{i\mu^2-1/2}[e^{3i\pi/4}\sqrt{2}(t+k)]$$
$$= \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}\Gamma(1/2-i\mu^2)} \lambda^{-i\mu^2+1/2} e^{ikx} e^{i(k+t)^2/2} \int_0^{+\infty} du\, e^{(1-i)(k+t)\lambda\mu-\lambda^2u^2/2} u^{-i\mu^2-1/2}, \tag{25}$$

where $|\arg\lambda| < \pi/4$. A Gaussian wave packet centered on $x_0 = 0$, $t_0 = 0$ is

$$\psi^{\rm in}_p(x,t) = \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}\Gamma(1/2-i\mu^2)} \lambda^{-i\mu^2+1/2} \times \int_{-\infty}^{+\infty} dk \frac{e^{-k^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \int_0^{+\infty} du \exp\left(ikx + i\frac{(k+t)^2}{2} + (1-i)(k+t)\lambda u - \frac{\lambda^2u^2}{2}\right) u^{-i\mu^2-1/2}$$
$$= \frac{e^{-\pi\mu^2/4}}{(2)^{1/4}\Gamma(1/2-i\mu^2)} \lambda^{-i\mu^2+1/2} \frac{1}{\Sigma_-\sigma} e^{it^2/2} e^{-(t+x)^2/2\Sigma_-^2} \times \int_0^{+\infty} du \exp\left(-\lambda^2 \frac{1+i\sigma^2}{1-i\sigma^2}\frac{u^2}{2}\right) e^{\lambda u(1-i)[t+i(t+x)\Sigma_-^{-2}]} u^{-i\mu^2-1/2}, \tag{26}$$

where Σ_- is defined previously. In order that the permutation of the integrals carried out above be meaningful, λ must be taken such that $\mathrm{Re}\lambda^2 > 0$ and $\mathrm{Re}(\lambda^2 \frac{1+i\sigma^2}{1-i\sigma^2}) > 0$. The integral over u in Eq. (26) is of the same form as Eq. (25) and, therefore is a representation of a Whittaker function. By identification one obtains the second line of Eq. (24).

We note that the asymptotic behavior discussed in Eq. (21) and following can be recovered by using the exact wave packets and the asymptotic expansion for Whittaker functions [15].

The three other interesting wave packets centered at $k = 0$ and $x = 0$ are

$$\psi^{\rm in}_a(x,t) = \psi^{\rm in}_p(-x,t),$$
$$\psi^{\rm out}_p(x,t) = \psi^{\rm in*}_p(x,-t),$$
$$\psi^{\rm out}_a(x,t) = \psi^{\rm in*}_p(-x,-t). \tag{27}$$

Since the Bogoliubov coefficients Eq. (12) are independent of k, the functions $\psi_{p(a)}^{\rm in(out)}$ are related one to another by the same Bogoliubov transformation. Note that this simplification will not occur when considering wave packets in the analogue black hole problem.

Of course the wave packets under gauge transformation simply differ by the gauge phase e^{iExt}. More interesting is that for Gaussian packets the two constructions run completely in parallel. Indeed, in the x gauge, one obtains

$$\begin{aligned}\psi_p'^{\rm in}(x,t) &= \int_{-\infty}^{+\infty} d\omega \frac{e^{-\omega^2/2\sigma^2}}{\sqrt{2\pi}\sigma}\varphi_{\omega,p}^{\rm in}(x,t)\\ &= \frac{e^{-3\pi\mu^2/4}}{(2)^{1/4}}\frac{1}{\Sigma_+\sigma}\xi_+^{1/2-i\mu^2}e^{-ix^2/2}e^{-(t+x)^2/2\Sigma_+^2}\\ &\quad\times e^{z^2/4}D_{i\mu^2-1/2}(z),\end{aligned} \tag{28}$$

where $\Sigma_+^2=(1/\sigma^2+i)$, $\xi_+^2=(1+i\sigma^2)/(1-i\sigma^2)$, and $z=-\xi_+(1+i)(x-i\sigma^2t)/(1+i\sigma^2)$ with $|\arg\Sigma_+|<\pi/4$ and $|\arg\xi_+|<\pi/2$. For minimal wave packets ($\sigma^2=1$) these two wave packets Eqs. (24) and (28) coincide up to the gauge phase e^{iExt}. Thus a minimal Gaussian packet of k is equivalent to a minimal Gaussian packet of ω.

IV. THE CURRENT DUE TO A SINGLE PAIR

In this section we shall consider the production of one pair in the wave packet ψ introduced above. More precisely we consider the state

$$|1_p1_a\rangle_{\rm out} = \int dk f(k)a_k^{\rm out\dagger}\int dk' f^*(k')b_{k'}^{\rm out\dagger}|0\rangle_{\rm out}, \tag{29}$$

where we have fixed f to be $f(k)=(2\pi)^{-1/2}\sigma^{-1}e^{-k^2/2\sigma^2}$ as in Eq. (26). The probability to find this state starting from $|0\rangle_{\rm in}$ is given by the overlap

$$\begin{aligned}P_E &= |{}_{\rm out}\langle 1_p1_a|0\rangle_{\rm in}|^2\\ &= \left(\int dk|f(k)|^2|\beta/\alpha|^2\right)\left(\prod_\omega\frac{1}{|\alpha|^2}\right)\\ &= (|\beta/\alpha|^2)\left(\prod_\omega\frac{1}{|\alpha|^2}\right),\end{aligned} \tag{30}$$

where we have used Eq. (16) and developed the exponential to first order in β/α.

The second factor in Eq. (30) is the probability to create no pairs in all modes including $\psi^{\rm out}$ [see Eq. (17)]. To grasp the content of Eq. (30) it is useful to envisage that the field ϕ is quantized in terms of a complete orthogonal set of localized wave packets labeled by i, one of which ($i=0$) is taken to be ψ. We then rewrite Eq. (30) so as to isolate the contribution from the pair $i=0$ which is created from all the other modes:

$$P_E = \frac{|\beta|^2}{|\alpha|^4}\left(\prod_{i\neq 0}\frac{1}{|\alpha|^2}\right). \tag{31}$$

It is now clear that the probability to create a pair in the mode ψ is $|\beta|^2/|\alpha|^4$ times the probability not to create any pairs ($=|\alpha|^{-2}$) in each of the other modes. (Similarly the probability of creating n pairs in the mode ψ is $|\beta|^n/|\alpha|^{n+1}$ times the probability not to create any pairs in each of the other modes. Summing over all numbers of pairs and over all modes one finds that the total probability is 1 as it should be.) The factorization of the probabilities which we have just encountered follows from neglecting interactions among pairs.

In order to introduce the concept of the current carried by this particular pair Eq. (29) we consider the following problem. Suppose that the electric field is slightly modified $E\to E+\delta E(x,t)$ where $\delta E(x,t)$ is a small and slowly varying function thereby justifying a first-order treatment. The probability of finding the pair described in Eq. (29) is modified by δE. Since the coupling is $\delta S=\int dt\int dx J_\mu\delta A^\mu$, the new probability is given by

$$\begin{aligned}P_{E+\delta E} &= |{}_{\rm out}\langle 1_p1_a|e^{i\delta S}|0\rangle_{\rm in}|^2\\ &\simeq |{}_{\rm out}\langle 1_p1_a|(1+i\delta S)|0\rangle_{\rm in}|^2\\ &= P_E\left(1+\int dx\int dt\delta A^\mu(x,t)\right.\\ &\quad\left.\times 2{\rm Im}\left[\frac{{}_{\rm out}\langle 1_p1_a|J_\mu(x,t)|0\rangle_{\rm in}}{{}_{\rm out}\langle 1_p1_a|0\rangle_{\rm in}}\right]\right),\end{aligned} \tag{32}$$

where we work in interaction representation. [More precisely the states $|0\rangle_{\rm in}$ and $|1_p1_a\rangle_{\rm out}$ are given by Eqs. (9) and (29); i.e., they are defined by the modes in the unperturbed electric field E. The operator J_μ is $J_\mu(x,t)=(i/2)[\phi^\dagger(x,t)\overleftrightarrow{\mathcal{D}}_\mu\phi(x,t)+\phi(x,t)\overleftrightarrow{\mathcal{D}}_\mu\phi^\dagger(x,t)]$, where $\phi(x,t)$ is given by Eq. (8) or (13), i.e., once more in terms of the modes in the presence of the unperturbed electric field.] It is thus the imaginary part of the matrix element

$$\frac{{}_{\rm out}\langle 1_p1_a|J_\mu|0\rangle_{\rm in}}{{}_{\rm out}\langle 1_p1_a|0\rangle_{\rm in}} \tag{33}$$

which controls the modification of the probability of finding the pair (and not the mean current ${}_{\rm in}\langle 0|J_\mu|0\rangle_{\rm in}$ as in the semiclassical back reaction).

This matrix element Eq. (33) will also come up when considering the effect of the current on an additional system if one requires that the final state be Eq. (29). It plays exactly the same role as the mean current ${}_{\rm in}\langle 0|J_\mu|0\rangle_{\rm in}$ if no restriction is imposed on the final state (as in the semiclassical treatment [12]). This is readily understood from the identity:

$$\begin{aligned}{}_{\rm in}\langle 0|J_\mu|0\rangle_{\rm in} &= \sum_m {}_{\rm in}\langle 0|m\rangle_{\rm out}\,{}_{\rm out}\langle m|J_\mu|0\rangle_{\rm in}\\ &= \sum_m |{}_{\rm in}\langle 0|m\rangle_{\rm out}|^2\left(\frac{{}_{\rm out}\langle m|J_\mu|0\rangle_{\rm in}}{{}_{\rm out}\langle m|0\rangle_{\rm in}}\right),\end{aligned} \tag{34}$$

where $|m\rangle_{\rm out}$ constitute a complete set of out states one of which can be taken to be Eq. (29). Thus the mean current is the sum of matrix elements of the form Eq. (33) weighted by the probability of finding the final state $|m\rangle_{\rm out}$. Therefore to first order the corresponding weighted sum of back reactions induced by ${}_{\rm out}\langle m|J_\mu|0\rangle_{\rm in}/{}_{\rm out}\langle m|0\rangle_{\rm in}$ is indeed given by the mean current.

We anticipate subsequent developments and mention that Eq. (34) can be greatly simplified for the ideal situation treated in this paper wherein interactions among the pairs (or a pair with itself) has been neglected. Referring to Appendix B, upon differentiating Eq. (B18) one finds

$$ {}_{\rm in}\langle 0|J_\mu|0\rangle_{\rm in} = |\beta|^2 \sum_k \frac{-1}{\beta^*\alpha}\varphi^{\rm in*}_{p,k}(x) i\overleftrightarrow{\mathcal{D}}_\mu \varphi^{\rm in*}_{a,k}(x), \qquad (35) $$

where we have used the result ${}_{\rm out}\langle 0|J_\mu|0\rangle_{\rm in} = 0$ (see Appendix B). The sum on modes in Eq. (35) can also be rewritten as a sum over a complete orthogonal set of wave packets. Once more the sum is over the relevant set, those which have turning points within LT. As we shall see [cf. Eq. (41)], each term in the sum Eq. (35) is the matrix element Eq. (33). The prefactor, $|\beta|^2$, is the mean number of pairs created in a given mode (independent of the mode number). We emphasize that for our noninteracting model Eqs. (34) and (35) say the same thing. However most of the interesting quantum effects present in each term on the right-hand side (RHS) of Eq. (35) wash out in the sum. Indeed J_μ being Hermitian, the sum of Eq. (35) is manifestly real whereas each term on the RHS is complex. Moreover the mean current is the sum of the currents carried by the produced particles. It is therefore proportional to the number of produced particles and grows linearly in time. The only place where the detailed structure of the RHS is felt is in edge effects where they have not had time to wash out.

A general analysis of matrix elements such as Eq. (33) was carried out by Aharonov *et al.* [14] in the context of measurement theory. For the sake of writing a self-contained paper we give in Appendix A a brief review of the theory of Aharonov *et al.* These authors showed that matrix elements of the form Eq. (33) could be measured by certain weak measurements and in this context are interpreted as the measured value of the current if the final state is Eq. (29). The occurrence of the same matrix elements in the modification of the probability [Eq. (32)] and in measurement theory comes from the universal character of the first-order perturbation in quantum mechanics (Born term). Such nondiagonal matrix elements [correctly normalized as in Eq. (33)] are called weak values. The restriction to a particular outcome is called postselection. We shall in the sequel conform to this vocabulary.

However, as we have just shown, the physical significance of these nondiagonal matrix elements goes beyond their detectability through a measuring device external to the system. They will indeed govern the modification of multipair production due to current-current interactions among the pairs produced. For example in lowest order in the pair-pair interaction is

$$ H_{\rm int} = J_\mu(x) D_{\mu\nu}(x-x') J_\nu(x'), \qquad (36) $$

where $D_{\mu\nu}(x-x')$ is the photon propagator. Thus there exists a matrix element ${}_{\rm out}\langle {\rm pair}, {\rm pair}'|H_{\rm int}|0\rangle_{\rm in}$ which contains a term proportional to

$$ {}_{\rm out}\langle {\rm pair}|J_\mu(x)|0\rangle_{\rm in} D_{\mu\nu}(x-x')\,{}_{\rm out}\langle {\rm pair}'|J_\nu(x')|0\rangle_{\rm in} \quad (37) $$

which indicates that the second pair lives in a reduced E field owing to the counter electric field carried by the former one, in the case where pair and pair$'$ are here taken nonoverlapping, and the second one is created in the causal future of the first. More generally these matrix elements enter into the perturbative S matrix at the tree level. Thus, this is a part only of the more difficult program of treating the self-interaction among pairs. The difficulty arises from the necessity of considering loops in addition to trees. This is currently under study and we intend to report on it in a forthcoming paper [18]. It should be noted that the interaction written in Eq. (37) has two physical effects: one will affect the production acts localized within regions of $O(a^{-2})$ in space-time and the other will be "final state interactions" among the pairs which have been created. It will be the job of the S-matrix formulation to sort out these effects and to express the out states with the final state interactions included. In the present paper we shall restrict ourselves to a detailed analysis of the properties of the weak value of J_μ without back reaction [Eq. (33)].

In preparation for the evaluation of Eq. (33) it is fitting first to calculate matrix elements of $\phi^\dagger(x)\phi(x')$ between states of interest. Matrix elements of operators constructed from bilinear forms of ϕ such as the current j_μ or the energy momentum tensor may then be obtained by differentiation and going to the coincidence limit. In the present case we then first calculate ${}_{\rm out}\langle 1_p 1_a|\phi^\dagger(x)\phi(x')|0\rangle_{\rm in}$ for which one finds the equality

$$ \frac{{}_{\rm out}\langle 1_p 1_a|\phi^\dagger(x)\phi(x')|0\rangle_{\rm in}}{{}_{\rm out}\langle 1_p 1_a|0\rangle_{\rm in}} = -\frac{1}{\alpha\beta^*}\psi^{\rm in*}_p(x)\psi^{\rm in*}_d(x') + G_F(x,x'), \qquad (38) $$

where G_F is the familiar in-out propagator:

$$ G_F(x,x') = \frac{{}_{\rm out}\langle 0|\phi^\dagger(x)\phi(x')|0\rangle_{\rm in}}{{}_{\rm out}\langle 0|0\rangle_{\rm in}}. \qquad (39) $$

We prove Eq. (38) by availing ourselves of the identity

$$ a^{{\rm out}\dagger}_k b^{{\rm out}\dagger}_{k'}|0\rangle_{\rm out} = \frac{1}{\alpha^{*2}} a^{{\rm in}\dagger}_k b^{{\rm in}\dagger}_{k'}|0\rangle_{\rm out} - \frac{\beta}{\alpha^*}\delta(k-k')|0\rangle_{\rm out} \qquad (40) $$

which follows from the definition of the Bogoliubov transformation Eq. (14) $a^{{\rm in}\dagger}_k = \alpha^* a^{{\rm out}*}_k + \beta b^{\rm out}_k$ and similarly for $b^{{\rm in}\dagger}_k$. Use of the definition of $|1_p 1_a\rangle_{\rm out}$, Eq. (29), together with (40) and ϕ expanded in the in-basis gives the required relation.

The first term of Eq. (38) refers to the postselected pair with $\psi^{\rm in}_p$ and $\psi^{\rm in}_a$ being the associated particle and antiparticle wave functions, respectively. The second term is equal to the weak value of $\phi^\dagger(x)\phi(x')$ if the postselected state is $|0\rangle_{\rm out}$, i.e., if no pairs are produced.

The second term is formally infinite, and therefore to be handled carefully. Furthermore it is of interest in its own right, in that is contains a physically meaningful finite part. Therefore we have written an extensive Appendix (Appendix B) devoted to this point and in particular to how we adapt the Hadamard subtraction scheme to our problem. We find that upon renormalization the current carried by the noise term vanishes (as it should: if the postselected state is out vacuum, no one-mass shell pairs are created and the current vanishes). Nevertheless for other operators quadratic in ϕ (the energy momentum, etc.) it is nonvanishing.

We have in hand the relevant matrix element obtained (after subtracting the noise) by operating on Eq. (38) with $\mathcal{D}_\mu$ as follows

$$j_\mu(x) = \frac{{}_{\text{out}}\langle 1_p 1_a | J_\mu(x)|0\rangle_{\text{in}}}{{}_{\text{out}}\langle 1_p 1_a |0\rangle_{\text{in}}} = -\frac{1}{\alpha\beta^*}[\psi_a^{\text{in}*}(-i\mathcal{D}_\mu^*\psi_p^{\text{in}*}) + \psi_p^{\text{in}*}(i\mathcal{D}_\mu\psi_a^{\text{in}*})]. \quad (41)$$

Asymptotically $j_\mu(x)$ behaves as it should, classically. It vanishes in the past (since ψ_a^{in} and ψ_p^{in} being centered on the past trajectories $x = \pm t$ have no overlap for $t \to -\infty$). In the future it behaves as a classical pair, each member of which carries unit charge. This is due to the remarkable formula

$$\lim_{t\to+\infty}\left[-\frac{1}{\alpha\beta^*}\psi_a^{\text{in}*}\psi_p^{\text{in}*}\right] = \psi_p^{\text{out}}\psi_p^{\text{out}*} + \psi_a^{\text{out}*}\psi_a^{\text{out}} \quad (42)$$

obtainable from the Bogoliubov transformation between in and out modes Eq. (11) plus the fact that the overlap of ψ_a^{out} and ψ_p^{out} vanishes in the distant future.

Figures 3(a)–3(e) display the real and imaginary part of $j_\mu(x)$ with the ψ's constructed from minimal wave packets [Eq. (24) with $\sigma = \sigma_{\min}$]. Unlike Figs. 1(a)–1(c) and 2 there is now no advantage in displaying pictures for $m/a \sim O(1)$ since the normalization factor in Eq. (33) brings the produced current up to order of magnitude unity (recall we are working with "conditional" amplitudes). We have chosen $m/a = 9$. The WKB approximation is then almost always valid, therefore the tunneling interpretation *de rigueur* (as well as the whole theoretical framework). We now discuss some interesting features of these drawings.

In Sec. III we constructed packets, say $\psi_p^{\text{in}}(x)$, whose branches are of fixed width $[= (2/E)^{1/2}]$ straddling the corresponding classical trajectories. It may also be shown that for $m/a \gg 1$, in the tunneling region, $|x| < a^{-1}$, the tunneling bridge also has width $O(E^{-1/2})$. This situation however changes radically when one studies products such as $\psi_a^{\text{in}*}(x)\psi_p^{\text{in}*}(x)$ or derivatives thereof such as in $j_\mu(x)$. Here the bridge thickens in time to give a symmetric space-time tunneling region of dimensions $a^{-1}\times a^{-1}$. This is understood both physically and mathematically as follows. Physically it has been explained in [10] that one needs the space separation of a pair in a vacuum fluctuation to be $O(a^{-1})$ in order for the negative electric energy $(= -E|\Delta x|)$ to overcome the rest mass threshold. Alternatively the virtual particles must be accelerated in a time interval $\Delta t = O(a^{-1})$ to pick up the energy necessary to overcome the threshold. Mathematically, the problem may be posed in either of the gauges discussed in this paper. In space gauge, tunneling is in the region between turning points $(|\Delta x| = 2/a)$ whereas in time gauge backscattering occurs during a time interval $|\Delta t| = O(a^{-1})$. We also may point out that when WKB approximation applies to the tunneling region, there is a Euclidean classical path $x^2 + (\text{Im}t)^2 = a^{-2}$ which is used to get the tunneling action. Since this is the result of a steepest descent calculation wherein a contour has been distorted so as to give imaginary values of t, it is not unexpected that as a function of x and Ret, the relevant production region is spread throughout the circle of radius a^{-1}.

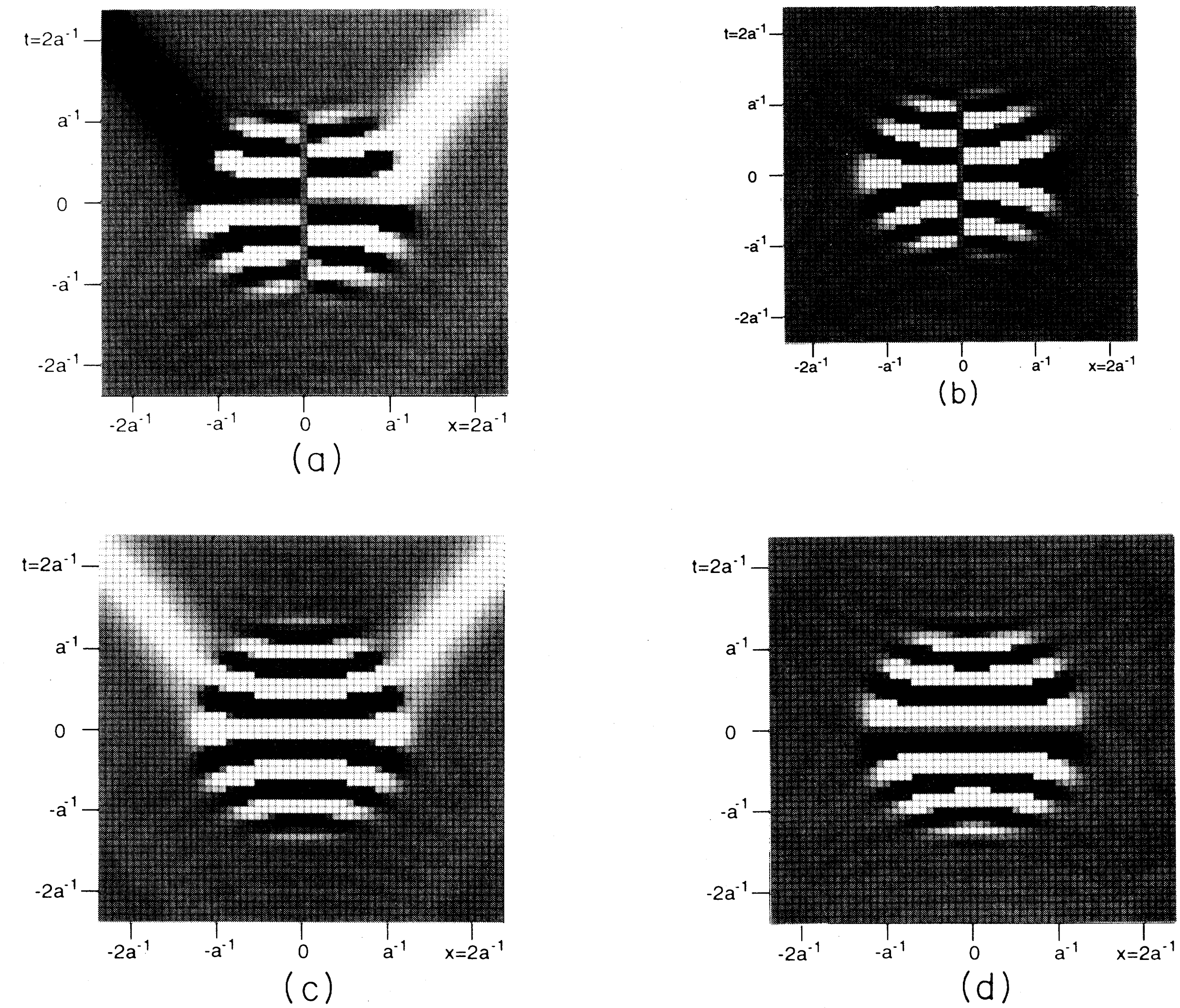
It should be noted that the description of production in terms of minimal packets is a very precise representation of the physics in localized terms. This is because these minimal packets serve as a starting point for the rigorous construction of complete orthogonal basis functions for the quantization of the field $\phi(x)$ [19]. More precisely, by complete set we mean the set necessary and sufficient to describe the modes which lead to pair production in the space-time region $\mathcal{R}$, for which E is nonvanishing: $-\frac{L}{2} \le x \le +\frac{L}{2}$, $0 \le t \le T$. As discussed in Sec. II, these are the modes that scatter off potentials whose centers lie within this region. Their number is $ELT/2\pi$. Since the minimal wave packet has width $E^{-1/2}$ about the classical orbit in space-time, we see that if the packets are separated one from the other by $O(E^{-1/2})$, the number of them that can be fitted into $\mathcal{R}$ is $O(ELT)$. Thus with some tinkering on their size and shape they will constitute a complete set, in the above sense, and orthogonal because they are nonoverlapping.

One then comes upon the physical picture of production of quanta of minimal size $O(E^{-1/2})$ where the production zone is within a space-time cell of size $O(a^{-1}\times a^{-1})$. For $m/a \gg 1$, one may think of production as a set of shots emerging from cells. The number of packets which contribute to a given cell is $[(a^{-1}\times a^{-1})/E] = m/a$. We have displayed in Figs. 3(a)–3(e) the production of a single pair of these quanta.

Another noteworthy feature is the existence of oscillations in time that occur within the "circle" of production, i.e., bounded by radius a^{-1} wherein the particles are still virtual. In Sec. I it was stated that we were getting "inside the golden rule," and indeed this is what we are now seeing inside the circle of production. To put these oscillations into evidence in this region it is most simple to examine the modes in temporal gauge [Eq. (23)]. The function $D_{i\mu^2-1/2}[e^{3i\pi/4}\sqrt{2}(t+k)]$ is the solution of

$$[\partial_t^2 + (Et+k)^2 + m^2]D_{i\mu^2-1/2}[e^{3i\pi/4}\sqrt{2}(t+k)] = 0, \quad (43)$$

whereupon it is seen that for small t $(|t| \ll m/E = a^{-1})$ the modes oscillate with frequency $\omega_k = \sqrt{k^2+m^2}$. For $t \le a^{-1}$, one sees from Eq. (43) that the frequency of oscillation is still bounded by $O(m)$. These oscillations of the modes are reflected in the oscillation of j_μ formed from wave packets. For $x = 0$ the frequency is



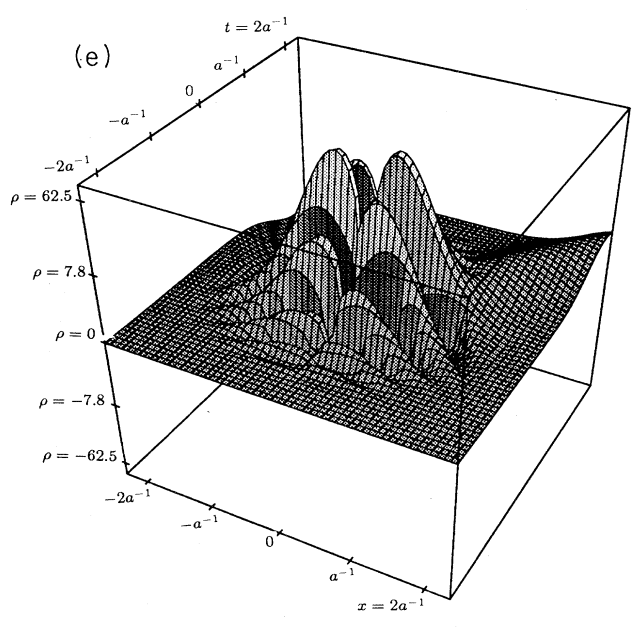


FIG. 3. Plots of the charge and current due to a pair created in a minimal wave packet for $m/a = 9$: (a) is $\mathrm{Re}[\rho(t,x)]$; (b) is $\mathrm{Im}[\rho(t,x)]$; (c) is $\mathrm{Re}[j(t,x)]$; (d) is $\mathrm{Im}[j(t,x)]$. These four plots do not describe the true amplitude of the oscillations. Rather, black and white simply designate the signals while gray shows the transition between them. The gray background means zero. (e) is a "three-dimensional" version of (a) which gives an idea of the amplitude of the charge oscillations. Note the nonlinearity of the vertical scale. As mentioned in the text the variation of the amplitude as a function of x and t is very large. It is seen upon inspection of (a)–(d) that the region of production is confined to the expected circular cell of radius $\simeq O(a^{-1})$. Outside this region $\mathrm{Re}[j_\mu(t,x)]$ is due to particles moving in classical orbits and $\mathrm{Im}[j_\mu(t,x)] \simeq 0$. Within the tunneling region $\mathrm{Im}[j_\mu(t,x)]$ is of the same order of magnitude as $\mathrm{Re}[j_\mu(t,x)]$ and both oscillate.

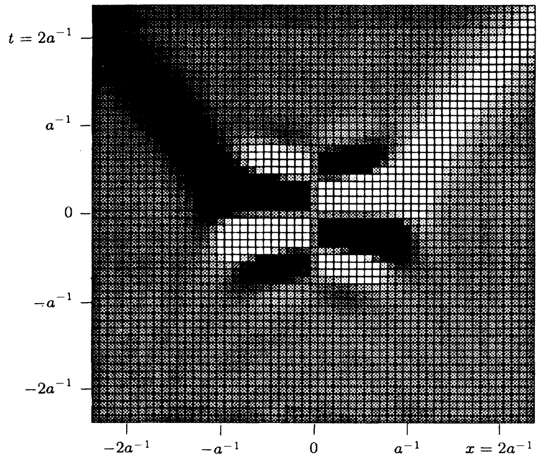


FIG. 4. The real part of the charge density $\rho(t,x)$ due to a created pair in a Gaussian wave packet with $m/a = 9$, $\sigma = \sqrt{2}\sigma_{\min}$ is plotted using the same conventions as in Fig. 3. Compared to it the amplitude and frequency of the oscillations have decreased.

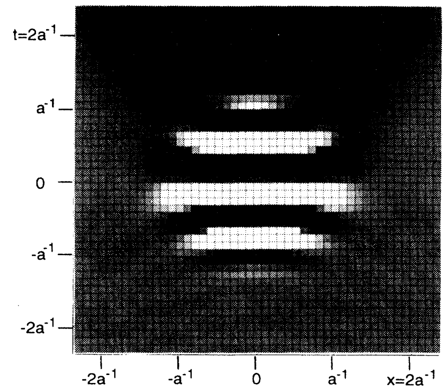


FIG. 5. A picture of the real part of the electric field obtained by integrating Gauss's law $\nabla \cdot E = \rho$ when the charge density is that of the pair drawn in Fig. 3(a). The electric field oscillates in the region where the pair is created whereas for late times when the particles are on mass shell the back reaction electric field becomes constant between the particles.

then $m/(\pi\sqrt{1+\sigma^4})$ obtainable from Eq. (24). In Fig. 4 we have plotted $\mathrm{Re}[\rho]$ for $\sigma = \sqrt{2}$ whereupon one sees the corresponding decrease in the frequency. The picture emerges, then, that near the origin the modes are those of free particles, plane wave of momentum k. The pair creation begins as a vacuum fluctuation of free particles which subsequently upon separation is converted into a real pair. The oscillations are thus the manifestation of the "time-energy" uncertainty relation (in the sense of Fourier transform) during the phenomenon of creation. They disappear upon completion of the creation act. It is remarkable that the quantum number k gradually changes its significance from a momentum to the time which dates the origin of the creation of the given pair.

We call attention to the asymmetry in space and time even though the production region is symmetric. The uncertainty principle implies oscillations in time and not in space, in preparation for particles which propagate casually along the forward light cone. We also mention that the amplitude of these oscillations varies in space-time, approximately according to the law $\exp[E(a^{-2}-t^2-x^2)]$. This extreme sensitivity to the wave packet construction will probably be washed out in most physical applications. For instance in Eq. (32) if δE is a constant independent of x, t the result $(P_{E+\delta E} - P_E)/P_E \simeq \pi\delta E/a^2$ is obtained directly from Eq. (30) and is not exponentially large. Another case where this occurs is in the mean Eq. (35): J_μ being Hermitian, the sum in Eq. (35) is manifestly real and the imaginary piece arising in the production region of each pair cancels out in the sum. Moreover the piece of Eq. (35) which grows linearly in time is due to the asymptotic current arising from each pair, obtained from Eq. (42). So only at the edges of the interval $0 \le t \le T$ and of the box $0 \le x \le L$ does the detailed structure of the RHS of Eq. (35) manifest itself.

In this paper we have given a description of the source which is responsible for the back reaction at the tree level upon postselecting a particular outcome. It oscillates, it is complex, and outside the production region one's classical expectations are satisfied. In Fig. 5 we have integrated Gauss's law $\nabla \cdot E = \mathrm{Re}\rho$ to get the weak value of the electric field (now treated as an operator) due to the postselected pair.

Inspired by the present work, we have carried out in [20,21] an analysis of the individual emission events which give rise to Hawking radiation. The result is given in terms of nondiagonal matrix elements of $T_{\mu\nu}$. In response to these complex energy-momentum tensors the back reaction to Hawking radiation will display very specific quantum effects encoded in the wave function of the metric. What the incidence of this will be on the unitary issue remains to be seen.

APPENDIX A: POSTSELECTION AND WEAK MEASUREMENT

For the sake of writing a self-contained paper we give a brief review of the postselection weak measurement formalism of Aharonov *et al.* (This formalism has been slightly generalized in [21] to include incomplete postselections—a necessary development in the black hole case owing to the inaccessibility of the region beyond the horizon—and a more quantum mechanical treatment of postselection.) In quantum mechanics one usually starts with an ensemble of identically prepared systems, subjects them to different experiments, and analyzes the probability distribution of the results. In such a situation one deals with a "preselected" ensemble: the preparation of the ensemble took place before the experiments to which it was subjected. A "pre- and postselected" ensemble involves a supplementary step, a final measurement. According to the result of this final measurement the original ensemble is split into subensembles; each of these subensembles is called a pre- and postselected ensemble. In each of these subensembles the distribution of the results of the intermediate measurements is different from that in the original, preselected only, ensemble, and depends on both the pre- and postselections. As shown in [14] the analysis of these distributions reveals both surprising physical effects and interesting features of quantum mechanics.

The most surprising effects occur when, between the pre- and postselections, the system interacts only weakly with test particles. Consider the following scenario in which a variable A is "weakly" measured. A system prepared at time t_1 in the state $|\psi_1\rangle$ evolves according to a Hamiltonian H_0 until a time t_0 when it interacts with a test particle (which can be viewed as the pointer of a measuring device). The interaction Hamiltonian is

$$H_{\rm int} = \delta(t - t_0)Ap, \tag{A1}$$

where A is a variable of the system and p is the canonical momentum of the test particle, conjugate to a position variable q. The above interaction Hamiltonian is the one used by von Neumann to model the standard measurement of A. Indeed, during the interaction the effect of H_0, the proper Hamiltonian of the system can be neglected, and the time evolution is

$$U(t_0 + \epsilon, t_0 - \epsilon) = \int_{t_0-\epsilon}^{t_0+\epsilon} e^{-iH_{\rm int}(t)}dt = e^{-iAp}. \tag{A2}$$

If the state of the system at $t_0 - \epsilon$ is an eigenstate of A, say $|A = a\rangle$, and the initial position of the pointer precisely defined, say in the state $|q = 0\rangle$, the effect of the interaction would be

$$U(t_0 + \epsilon, t_0 - \epsilon)|A = a\rangle|q = 0\rangle = |A = a\rangle|q = a\rangle; \tag{A3}$$

that is, the pointer is moved from $q = 0$ to $q = a$, showing the value of A. Furthermore, if at $t_0 - \epsilon$ the system is in a superposition of eigenstates of A, say $\sum_i c_i|A = a_i\rangle$, the system will get correlated with the pointer

$$U(t_0 + \epsilon, t_0 - \epsilon)\sum_i c_i|A = a_i\rangle|q = 0\rangle = \sum_i c_i|A = a_i\rangle|q = a_i\rangle. \tag{A4}$$

When "reading" the pointer we obtain different values $q = a_i$ with probability $|c_i|^2$, as prescribed by the postulates of quantum mechanics.

Suppose, however, as opposed to von Neumann, that the initial position of the pointer is not accurately defined; say it is represented by a broad Gaussian $\exp(-q^2/\Delta^2)$. As a result the measurement is imprecise (even if one reads very precisely the final position of the pointer, one still does not know how much the pointer moved, as one does not know its initial position exactly). On the other hand the system is only slightly disturbed by the measurement. Indeed, the effective magnitude of the interaction depends on the values of p in Eq. (A2), which in this case are essentially bounded by $1/\Delta$, supposed to be small. Alternatively $H_{\rm int}$ may be multiplied by a small coupling constant. After this "weak measurement" the system evolves again according to H_0 until a final time t_2 when a final measurement takes place and the system is found to be in a state $|\psi_2\rangle$. What is the final state of the measuring device corresponding to this sequence of events? Putting all this together we get

$$\Phi_{\rm fin} = \langle\psi_2|U(t_2, t_0 + \epsilon)e^{-iAp}U(t_0 - \epsilon, t_1)|\psi_1\rangle e^{-q^2/\Delta^2}, \tag{A5}$$

where $\Phi_{\rm fin}$ is the final state of the pointer, and we have supposed for simplicity that the mass of the pointer is so large that its evolution under its own unperturbed Hamiltonian is negligible. Since Δ is large we can use a first-order approximation to obtain

$$\begin{aligned}\Phi_{\rm fin} &\approx \langle\psi_2|U(t_2, t_0 + \epsilon)(1 - iAp)U(t_0 - \epsilon, t_1)|\psi_1\rangle e^{-q^2/\Delta^2} \\ &= \langle\psi_2|U(t_2, t_0 + \epsilon)U(t_0 - \epsilon, t_1)|\psi_1\rangle(1 - iA_wp)e^{-q^2/\Delta^2},\end{aligned} \tag{A6}$$

where

$$A_w = \frac{\langle\psi_2|U(t_2, t_0 + \epsilon)AU(t_0 - \epsilon, t_1)|\psi_1\rangle}{\langle\psi_2|U(t_2, t_0 + \epsilon)U(t_0 - \epsilon, t_1)|\psi_1\rangle}. \tag{A7}$$

The important part here is that one has added an increment to the detector wave function that owing to the complexity of A_w, leads to unexpected phase-dependent effects. One may see this in a picturesque way by reexponentiation to obtain

$$\begin{aligned}\Phi_{\rm fin} &\approx \langle\psi_2|U(t_2, t_0 + \epsilon)U(t_0 - \epsilon, t_1)|\psi_1\rangle e^{-iA_wp}e^{-q^2/\Delta^2} \\ &= Ce^{-(q-A_w)^2/\Delta^2} \\ &\approx Ce^{-(q-{\rm Re}A_w)^2/\Delta^2}e^{iq{\rm Im}Aw},\end{aligned} \tag{A8}$$

bringing into view the real and imaginary parts of A_w. Therefore the test particle, which weakly interacted with the pre- and postselected systems, behaves as if the variable A of the system has the complex value A_w. The real part of A_w causes a shift in the position of the test particle while the imaginary part of A_w produces a shift in its momentum.

The above discussion was done in Schrödinger representation. In Heisenberg representation A_w becomes

$$A_w(t_0) = \frac{{}_{\rm out}\langle\psi_2|A(t_0)|\psi_1\rangle_{\rm in}}{{}_{\rm out}\langle\psi_2|\psi_1\rangle_{\rm in}}, \tag{A9}$$

where $A_w(t_0)$ is the Heisenberg operator A evaluated at time t_0 and where $|\psi_1\rangle_{\rm in}$ and $|\psi_2\rangle_{\rm out}$ represent incoming and outgoing states, respectively.

APPENDIX B: HADAMARD'S SUBTRACTION

In order to compute finite quantities quadratic in the field it is necessary to regularize the matrix elements of $\phi^\dagger(x_0)\phi(x_0)$. The regularization scheme we adopt is to subtract the Hadamard function; i.e., we subtract the solution of the field equations that most resembles the Minkowskian propagator in the neighborhood of a point x_0 [22,23]. The subtraction is universal insofar as it is independent of the matrix element to be calculated (it is constructed using only the field and its derivatives at x_0). To express this universality of the subtraction we introduce the following expression for the renormalized product of two field operators:

$$\phi^\dagger(x)\phi(x_0)|_{\rm ren} = \phi^\dagger(x)\phi(x_0) - G_H(x,x_0)I, \qquad \text{(B1)}$$

where $G_H(x,x_0)$ is the Hadamard function, and it is understood that the limit $x \to x_0$ is to be taken at the end of any computation. I is the identity matrix in Hilbert space so that the Hadamard function cancels in the difference of matrix elements discussed in Sec. IV.

Upon renormalizing according to Eq. (B1) the matrix elements appearing in Sec. IV always contain a finite term plus a noise term equal to $G_F(x,x_0) - G_H(x,x_0)$ (where G_F is the in-out propagator). In Sec. IV the finite term was discussed at length. In this appendix we consider the noise term. To this end we first obtain closed forms for G_H and G.

To construct the Hadamard function it is of conceptual interest to carry through the calculation for a general space-time varying electromagnetic field $F_{\mu\nu}$. The informed reader will then recognize the more familiar analogous construction used in the presence of gravity. One first chooses a gauge such that the wave equation coincides at x_0 with the free field equation, i.e., by taking $A^\mu(x_0) = 0$ and $\partial_\mu A^\mu(x) = 0$; in addition one requires that the derivatives of A^μ at x_0 depend only on $F_{\mu\nu}$ and its derivatives at x_0 (this choice of gauge—Poincaré's gauge—is the analogue of the Riemann normal coordinates in curved space-time). Explicitly this is obtained by taking

$$A_\mu(x^\beta) = \int_0^1 t(x^\alpha - x_0^\alpha)F_{\alpha\mu}(x_0^\beta + t(x^\beta - x_0^\beta))dt. \qquad \text{(B2)}$$

(For a constant electric field this gauge is $A_0 = -Ex/2$, $A_x = Et/2$.) The Hadamard function is then fixed by requiring that its short distance behavior in this gauge be that of a free field

$$G_H(x,x_0) \underset{\lambda\to 0}{\approx} \frac{i}{2\pi}\left(\ln\frac{\sqrt{m^2\lambda}}{2} + \gamma\right), \qquad \text{(B3)}$$

where $\lambda = (t-t_0)^2 - (x-x_0)^2$.

The Hadamard function can be expressed for finite λ as

$$G_H(x,x_0) = e^{-iE(xt - x_0t_0)/2}H_{\mu^2}(E\lambda), \qquad \text{(B4)}$$

where $H_{\mu^2}(z)$ (with $\mu^2 = m/2a$) is a solution of

$$\left(z\frac{d^2}{dz^2} + \frac{d}{dz} + \frac{z}{16} + 2\mu^2\right)H_{\mu^2}(z) = 0, \qquad \text{(B5)}$$

and, therefore, of the form

$$H_{\mu^2}(z) = e^{-iE\lambda/4}\left[aU\left(\frac{1}{2} + i\mu^2, 1, i\frac{E\lambda}{2}\right)\right.$$
$$\left. + bM\left(\frac{1}{2} + i\mu^2, 1, i\frac{E\lambda}{2}\right)\right], \qquad \text{(B6)}$$

where U and M are confluent hypergeometric functions (Kummers' functions). In order to satisfy condition Eq. (B3) the constants a and b have to be equal to

$$a = -\frac{i}{2\pi}\Gamma\left(\frac{1}{2} + i\mu^2\right),$$
$$b = \frac{i}{2\pi}\left[\ln\mu^2 - \frac{i\pi}{2} - \psi\left(\frac{1}{2} + i\mu^2\right)\right]. \qquad \text{(B7)}$$

A closed form for the in-out propagator can be obtained from its representation as an integral over Schwinger fifth time derived in Refs. [16,9,10]. The equivalence of this representation with the sum over modes given at the end of this appendix has been proven in [24]. Using formula [1IV.8] of Ref. [25] it is straightforward to verify that, in the gauge Eq. (B2),

$$G_F(x,x_0) = e^{-iE(xt - x_0t_0)/2}\Delta_{\mu^2}(E\lambda), \qquad \text{(B8)}$$

$$\Delta_{\mu^2}(z) = -\frac{i}{2\pi}\Gamma\left(\frac{1}{2} + i\mu^2\right)U\left(\frac{1}{2} + i\mu^2, 1, i\frac{E\lambda}{2}\right). \qquad \text{(B9)}$$

We are now in a position to discuss the physical content of the noise term. The charge and current carried by this term are zero as is seen by acting on $G_F(x,x_0) - G_H(x,x_0)$ with the differential operator $i(\mathcal{D}_{\mu x_0} - \mathcal{D}^*_{\mu x})$. For short distances we have, in Poincaré's gauge,

$$G_F(x,x_0) - G_H \underset{\lambda\to 0}{\approx} -b\exp[-iE(xt - x_0t_0)/2]$$
$$\times\left[1 + \left(\frac{1}{2} + i\mu^2\right)i\frac{E\lambda}{2} + O(\lambda^2)\right]. \qquad \text{(B10)}$$

As the charge and the current are obtained from the action of a first-order differential operator, only the first phase factor of this expression will contribute in the coincidence limit and we obtain

$$\begin{aligned}
\rho_{\mathrm{in,out}} &= \lim_{(t,x)\to(t_0,x_0)} i[(\partial_{t_0} - iEx_0/2) - (\partial_t - iEx/2)]\exp[-iE(xt - x_0t_0)/2] \\
&= 0, \\
j_{\mathrm{in,out}} &= \lim_{(t,x)\to(t_0,x_0)} i[(\partial_{x_0} + iEt_0/2) - (\partial_x - iEt/2)]\exp[-iE(xt - x_0t_0)/2] \\
&= 0.
\end{aligned} \tag{B11}$$

These results can also be obtained by calculating the current carried by the in-out propagator when it is expressed as a sum over modes whereupon it is seen to vanish identically. As this confirms that the noise term in Eq. (38) vanishes when considering the charge we now give this alternative proof.

The vanishing of the charge is most conveniently exhibited by working in temporal gauge wherein it occurs mode by mode. Explicitly using the first equality of Eq. (B18) one obtains

$$\begin{aligned}
\rho_{\mathrm{in,out}}(k) &= \varphi^{\mathrm{out}}_{p,k}(x,t)(i\overleftrightarrow{\mathcal{D}}_0)\varphi^{\mathrm{in}*}_{p,k}(x,t) + \varphi^{\mathrm{out}}_{a,k}(x,t)(i\overleftrightarrow{\mathcal{D}}_0)^*\varphi^{\mathrm{in}*}_{a,k}(x,t) \\
&= e^{ikx}\chi^*(-t-k)(i\overleftrightarrow{\partial}_t)e^{-ikx}\chi^*(t+k) + e^{-ikx}\chi^*(-t-k)(-i\overleftrightarrow{\partial}_t)e^{ikx}\chi^*(t+k) \\
&= 0,
\end{aligned} \tag{B12}$$

where in passing from the first to the second line we have expressed all the modes in terms of $\chi(t+k)$ as in Eqs. (7) and (10). To exhibit the vanishing of j_x it is convenient to pass to the spatial gauge whereupon j_x vanishes mode by mode for the particles and antiparticles separately,

$$\begin{aligned}
j_{\mathrm{in,out}}(\omega) &= \varphi^{\mathrm{out}}_{p,\omega}(i\overleftrightarrow{\mathcal{D}}_x)\varphi^{\mathrm{in}*}_{p,\omega} \\
&= e^{i\omega t}\chi^*(x-\omega)(i\overleftrightarrow{\partial}_x)e^{-i\omega t}\chi^*(x-\omega) \\
&= 0,
\end{aligned} \tag{B13}$$

since the second line is the Wronskian of two identical functions and thus is zero.

Another interesting property of the noise term to calculate is the expectation value of $\phi^\dagger(x_0)\phi(x_0)$ [i.e., the coincidence point of $G_F(x,x_0) - G_H(x,x_0)$] since the imaginary part of $G_F(x_0,x_0)$ is related to the rate Γ of pair creation per unit time and unit length by

$$\Gamma LT = \mathrm{Im}\int d^2x \int_{m^2}^{+\infty} dm^2 (G_F(x,x)) \tag{B14}$$

(where LT is the space-time dimension of the region wherein $E \neq 0$). It is therefore a natural propriety of the formalism (easily verified from the explicit expression for G_H) that in the coincidence limit the imaginary part of $G_F(x_0,x_0)$ is unchanged by the renormalization. The real part of $G_F(x_0,x_0) - G_H(x_0,x_0)$ is related in a similar manner to the renormalized energy of the vacuum.

We have just shown in Eq. (B11) that the current density ${}_{\mathrm{out}}\langle 0|J_\mu|0\rangle_{\mathrm{in}}$ vanishes within the box LT, wherein E is constant. Nevertheless there exist residual edge effects which can be seen by direct differentiation of the in-out matrix element of e^{iS} where S is the action. For example the Schwinger formula, Eq. (17), gives the imaginary part of this quantity. Indeed differentiation of this latter with respect to E gives

$$\begin{aligned}
\frac{d}{dE}&\ln|{}_{\mathrm{out}}\langle 0|0\rangle_{\mathrm{in}}|^2 \\
&= \int dx\,dt \frac{\partial A_\mu}{\partial E} 2\mathrm{Im}\left[\frac{{}_{\mathrm{out}}\langle 0|J_\mu|0\rangle_{\mathrm{in}}}{{}_{\mathrm{out}}\langle 0|0\rangle_{\mathrm{in}}}\right] \\
&= \frac{d}{dE}\left[-\frac{LTE}{2\pi}\ln(1+|\beta|^2)\right] \\
&= -\frac{LT}{2\pi}\left[\ln(1+|\beta|^2) - \frac{\pi m^2}{E}e^{-\pi m^2/E}(1+|\beta|^2)^{-1}\right].
\end{aligned} \tag{B15}$$

This looks like a volume term, but it is due to the accumulation of surface effects related to the behavior of modes (or packets) near the surface. (We note that the same type of accumulation of surface effects occurs within the radiation emitted by an accelerated box (Unruh effect), see [21].) For example in the gauge $A_0 = -Ex$, there is a contribution from $J_0(x)$ equal to $J_0(x)x$. At the edge $x = L$, this contributes a term proportional to L for all t hence a term proportional to LT in the above.

It is instructive to rewrite Eq. (B14) as a sum over the members of a complete set of modes.

$$\Gamma TL = \mathrm{Im}\int d^2x \sum_k \int_{m^2}^{+\infty} dm^2 \left[|\beta|^2 \frac{1}{\beta^*\alpha}\varphi^{\mathrm{in}*}_{a,k}(x)\varphi^{\mathrm{in}*}_{p,k}(x)\right]. \tag{B16}$$

This last equality is obtained by first expressing G_F as

$$\begin{aligned}
G_F(x,x') &= \frac{{}_{\mathrm{out}}\langle 0|\phi^\dagger(x)\phi(x')|0\rangle_{\mathrm{in}}}{{}_{\mathrm{out}}\langle 0|0\rangle_{\mathrm{in}}} \\
&= \sum_{k,k'} \varphi^{\mathrm{out}}_{a,k}(x)\varphi^{\mathrm{in}*}_{a,k'}(x') \frac{{}_{\mathrm{out}}\langle 0|a^{\mathrm{out}}_k a^{\mathrm{in}\dagger}_{k'}|0\rangle_{\mathrm{in}}}{{}_{\mathrm{out}}\langle 0|0\rangle_{\mathrm{in}}},
\end{aligned} \tag{B17}$$

expressing $a_k^{\rm out}$ in terms of $a_k^{\rm in}$ and $b_k^{\rm in\dagger}$ yields

$$G_F(x,x') = \sum_k \frac{1}{\alpha}\varphi^{\rm out}_{a,k}(x)\varphi^{\rm in*}_{a,k}(x') = {}_{\rm in}\langle 0|\phi^\dagger(x)\phi(x')|0\rangle_{\rm in} + \sum_k \frac{\beta}{\alpha}\varphi^{\rm in*}_{p,k}(x)\varphi^{\rm in*}_{a,k}(x'), \quad \text{(B18)}$$

where ${}_{\rm in}\langle 0|\phi^\dagger(x)\phi(x')|0\rangle_{\rm in} = \sum_k \varphi^{\rm in}_{a,k}(x)\varphi^{\rm in*}_{a,k}(x')$ is the propagator in in vacuum. In the coincidence point as limit the propagator is real and does not contribute to Eq. (B16). One may then rewrite Eq. (B16) as a sum over a complete orthogonal set of localized wave packets [$\psi_{a,i}$ and $\psi_{p,i}$ labeled by i—see the discussion after Eq. (30)] as

$$\Gamma T L = {\rm Im}\int d^2x \sum_i \int_{m^2}^{+\infty} dm^2 \left[|\beta|^2 \frac{1}{\beta^*\alpha}\psi^{\rm in*}_{a,i}(x)\psi^{\rm in*}_{p,i}(x)\right]. \quad \text{(B19)}$$

The probability of not creating pairs is expressed as the sum over all possible pairs of the imaginary part of $\int dm^2\phi^\dagger\phi$ if that pair was created [i.e., the same term as Eq. (41) times the mean number of pairs ($=|\beta|^2$)]. Since for a fixed pair the imaginary part of $\psi^{\rm in*}_{i,a}\psi^{\rm in*}_{i,p}$ is localized within the corresponding cell, the volume dependence is as a sum over all cells times the probability that a pair be created in that cell, thereby giving a local meaning to the Schwinger formula.

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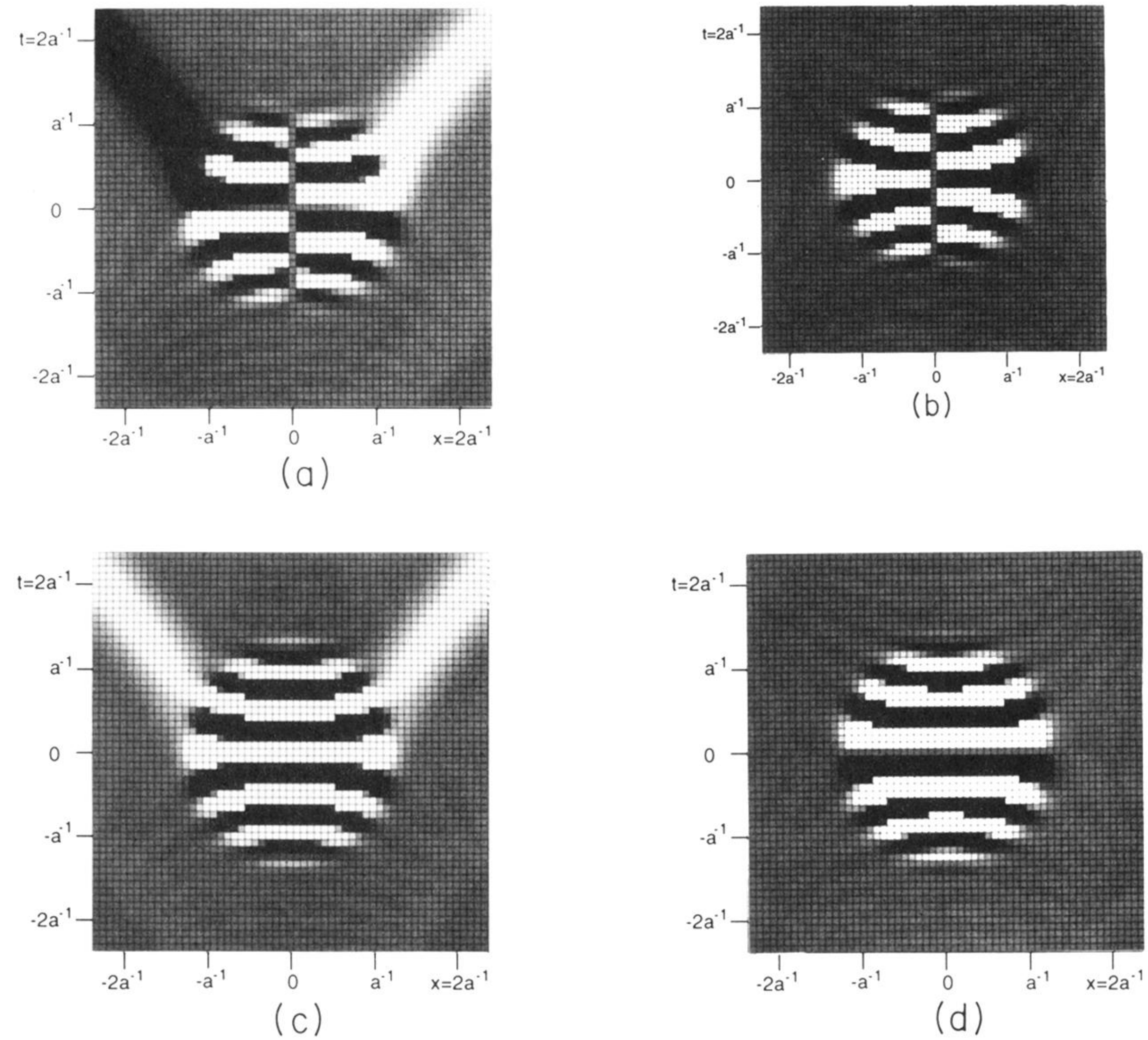

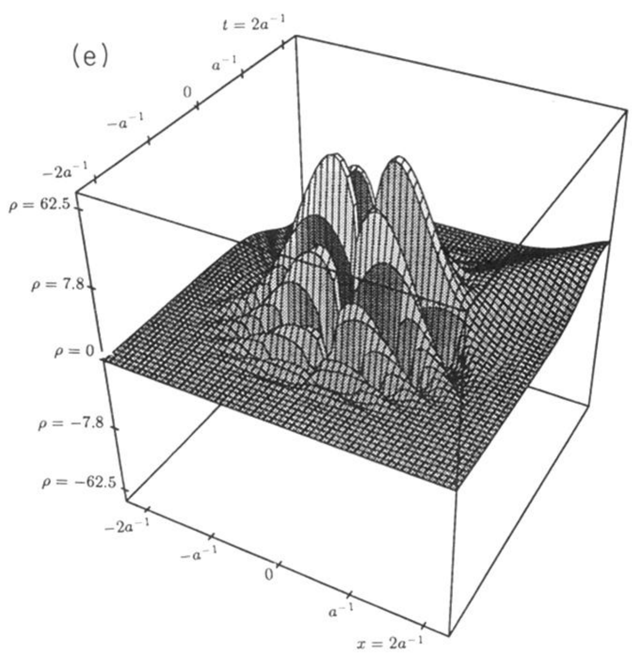

FIG. 3. Plots of the charge and current due to a pair created in a minimal wave packet for $m/a = 9$: (a) is $\mathrm{Re}[\rho(t,x)]$; (b) is $\mathrm{Im}[\rho(t,x)]$; (c) is $\mathrm{Re}[j(t,x)]$; (d) is $\mathrm{Im}[j(t,x)]$. These four plots do not describe the true amplitude of the oscillations. Rather, black and white simply designate the signals while gray shows the transition between them. The gray background means zero. (e) is a "three-dimensional" version of (a) which gives an idea of the amplitude of the charge oscillations. Note the nonlinearity of the vertical scale. As mentioned in the text the variation of the amplitude as a function of x and t is very large. It is seen upon inspection of (a)–(d) that the region of production is confined to the expected circular cell of radius $\simeq O(a^{-1})$. Outside this region $\mathrm{Re}[j_\mu(t,x)]$ is due to particles moving in classical orbits and $\mathrm{Im}[j_\mu(t,x)] \simeq 0$. Within the tunneling region $\mathrm{Im}[j_\mu(t,x)]$ is of the same order of magnitude as $\mathrm{Re}[j_\mu(t,x)]$ and both oscillate.

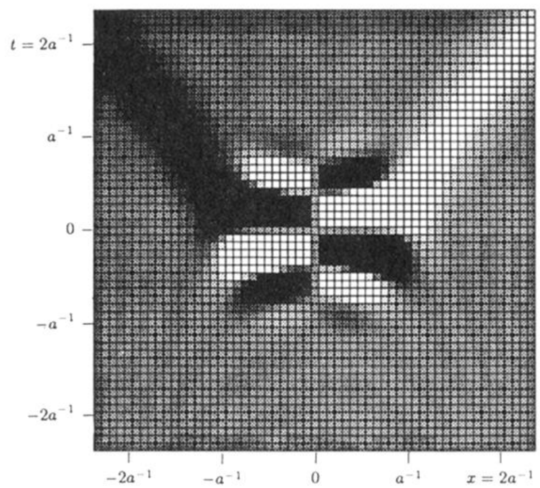

FIG. 4. The real part of the charge density $\rho(t,x)$ due to a created pair in a Gaussian wave packet with $m/a = 9$, $\sigma = \sqrt{2}\sigma_{\min}$ is plotted using the same conventions as in Fig. 3. Compared to it the amplitude and frequency of the oscillations have decreased.

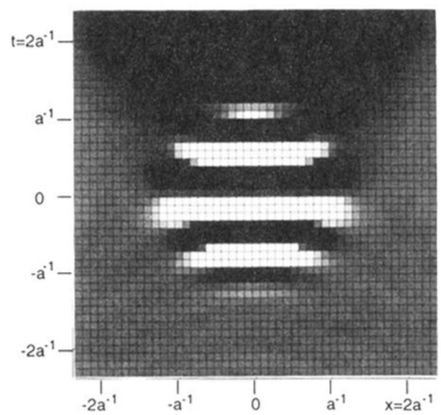

FIG. 5. A picture of the real part of the electric field obtained by integrating Gauss's law $\nabla \cdot E = \rho$ when the charge density is that of the pair drawn in Fig. 3(a). The electric field oscillates in the region where the pair is created whereas for late times when the particles are on mass shell the back reaction electric field becomes constant between the particles.