# Superconducting domain walls from a supersymmetric action

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(Received 16 March 1995)

A model constructed from chiral superfields is considered, and a simple, but exact, domain wall solution is found which interpolates between supersymmetric vacuum states. The domain wall supports fermionic zero modes, for which approximate analytic solutions are found. It is also found that bosonic bound states localized within or near the core of the wall can exist. The fermionic superconducting domain wall can therefore be endowed with both fermionic and bosonic charges and currents.

PACS number(s): 11.27.+d, 12.60.Jv

## I. INTRODUCTION

A domain wall [1,2] can arise from a situation wherein a discrete symmetry is spontaneously broken. The disconnected vacuum manifold then contains distinct vacuum states, and the field associated with the broken discrete symmetry can settle into different energy minimizing vacuum states in different spatial domains, with the formation of a domain wall between these regions. The field giving rise to the domain wall thus interpolates between the distinct vacuum states. It is also known that domain-wall solutions can exist in supersymmetric theories [3] with interesting gravitational properties [4,5] that differ from those of domain walls occurring in nonsupersymmetric theories [1,2,6,7]. A domain wall may also be superconducting [8,9] in that it may support a complex scalar field condensate which forms by the Witten mechanism [10] or fermionic zero modes which propagate along the wall. If these fields are associated with a U(1) gauge group, then the wall can acquire an "electromagnetic" charge and current, giving rise to long-range gauge field interactions.

Here, attention is focused upon the field-theoretic aspects of a domain wall which arises from a supersymmetric action. In particular, an $N = 1$ supersymmetric action constructed from two chiral superfields is considered. A superpotential is chosen which reflects a discrete $Z_2$ symmetry of one of the superfields. The scalar potential of the model also contains this $Z_2$ symmetry, and the vacuum manifold contains two discrete, but degenerate, vacuum states which allow a spontaneous breaking of this $Z_2$ symmetry, along with the formation of a domain wall. However, the vacuum states have vanishing energy, so that supersymmetry remains unbroken in the vacuum. Because of the nontrivial interactions between the component fields of the theory, it is found that the domain wall supports fermionic zero modes which propagate at the speed of light. The domain wall is therefore superconducting and can carry fermionic charge and current. It is also found that, instead of a bosonic condensate, domain-wall-bosonic particle bound states can exist, with the bosonic field being concentrated within or near the core of the wall, contributing to a bosonic charge and current. The domain wall therefore displays a fermionic superconductivity and can be endowed with both fermionic and bosonic charges and currents. The interactions of the wall with fermionic and bosonic fields can allow transitions to occur among the domain-wall-boson particle states.

In Sec. II, the supersymmetric model is presented and the vacuum states are found. The bosonic and fermionic mass matrices are calculated, and the positive mass spinor eigenstates are identified. The field equations for the boson and fermion fields are derived, and a simple, but exact, solution of the field equations describing a topologically stable domain wall is given in Sec. III, where it is seen that the Majorana and Dirac positive mass spinor eigenstates in two different domains are related by a relative factor of $\gamma_5$. It can also be noted that the effective fermion masses decrease near the domain wall, implying an attraction of fermions toward the wall. Analytic solutions are found for the fermionic zero modes in Sec. IV, and the existence of domain-wall-boson particle bound states is examined in Sec. V. A brief summary of the results forms Sec. VI.

## II. THE MODEL

### A. Action and Lagrangian

Consider a supersymmetric model constructed from two chiral superfields $\Phi_i$, $i = 1, 2$, with component fields $(\phi_i, \psi_i, F_i)$, with the $F_i$ representing the auxiliary boson fields. The boson fields $\phi_i$ and $F_i$ are complex scalar fields and each of the fermion fields $\psi_i$ is a Weyl two-spinor. The superfields have a superspace representation [11] given by

$$\Phi_i(z) = \Phi_i(y, \theta) = \phi_i(y) + \sqrt{2}\theta\psi_i(y) + \theta^2 F_i(y) \ , \quad (1)$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\theta^2 = \theta\theta = \theta^\alpha\theta_\alpha$, $\alpha = 1, 2$.

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A metric $g_{\mu\nu}$ with signature $(+,-,-,-)$ is used. (See the Appendix for a brief description of the conventions and $\gamma$ matrices.) The supersymmetry transformations are given by

$$\delta\phi_i(x) = -\sqrt{2}\xi\psi_i(x) \ , \tag{2}$$

$$\delta\psi_{i\alpha}(x) = -\sqrt{2}[\xi_\alpha F_i(x) + i(\sigma^\mu\bar{\xi})_\alpha\partial_\mu\phi_i(x)] \ , \tag{3}$$

$$\delta F_i(x) = i\sqrt{2}\partial_\mu\psi_i(x)\sigma^\mu\bar{\xi} \ , \tag{4}$$

where $\xi$ is a constant Weyl spinor.

The supersymmetric action is

$$I = \int d^4x L = \int d^4x\, d^2\theta d^2\bar{\theta}\Phi_i^*\Phi_i + \int d^4x\, d^2\theta W(\Phi) + \int d^4x\, d^2\bar{\theta} W^*(\Phi^*) \ , \tag{5}$$

where a summation over the index $i$ is implied, and $W(\Phi)$ is the superpotential. The superpotential is chosen to be

$$W = \tfrac{1}{2}\lambda\Phi_2(\Phi_1^2 - v^2) \tag{6}$$

with $\lambda$ and $v$ being positive real-valued constants.

The auxiliary fields can be eliminated so that the Lagrangian can be written as [11,12]

$$L = L_{\rm KE} + L_Y - V \ , \tag{7}$$

where $L_{\rm KE}$ contains the kinetic terms, $L_Y$ contains the Yukawa interactions, and $V$ is the scalar potential. The complex scalar field $\phi_i(x)$ can be displayed in terms of real scalar fields $A_i(x)$ and $B_i(x)$ as

$$\phi_i(x) = \frac{1}{\sqrt{2}}[A_i(x) + iB_i(x)] \ , \tag{8}$$

and Majorana four-spinors $\Psi_1(x)$ and $\Psi_2(x)$ are defined by

$$\Psi_i = \begin{pmatrix} \psi_{i\alpha} \\ \bar{\psi}_i^{\dot{\alpha}} \end{pmatrix}, \quad i=1,2, \ \alpha=1,2, \ \dot{\alpha}=1,2 \ . \tag{9}$$

The kinetic term is

$$L_{\rm KE} = \partial^\mu\phi_i^*\partial_\mu\phi_i + \frac{i}{2}[(\partial_\mu\psi_i)\sigma^\mu\bar{\psi}_i - \psi_i\sigma^\mu\partial_\mu\bar{\psi}_i] \ , \tag{10}$$

with a sum over $i$. The Yukawa part takes the form

$$\begin{aligned} L_Y &= -\tfrac{1}{2}\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)\psi_i\psi_j - \tfrac{1}{2}\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)^*\bar{\psi}_i\bar{\psi}_j \\ &= -\tfrac{1}{2}(Y_{ij}\psi_i\psi_j + Y_{ij}^*\bar{\psi}_i\bar{\psi}_j) \\ &= -\tfrac{1}{2}\lambda(\phi_2\psi_1\psi_1 + 2\phi_1\psi_1\psi_2 + \phi_2^*\bar{\psi}_1\bar{\psi}_1 + 2\phi_1^*\bar{\psi}_1\bar{\psi}_2) \end{aligned} \tag{11}$$

with a sum over $i$ and $j$, and

$$Y_{ij} \equiv \frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \ . \tag{12}$$

The scalar potential $V$, obtained from the superpotential $W$, is

$$\begin{aligned} V &= \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2 = \lambda^2\{\tfrac{1}{4}(\phi_1^{*^2} - v^2)(\phi_1^2 - v^2) + |\phi_1\phi_2|^2\} \\ &= \tfrac{1}{4}\lambda^2\left\{\left[\tfrac{1}{2}\left(A_1^2 - B_1^2\right) - v^2\right]^2\right\} \\ &\quad + \tfrac{1}{4}\lambda^2\{A_1^2B_1^2 + A_1^2A_2^2 + A_1^2B_2^2 + A_2^2B_1^2 + B_1^2B_2^2\} \ . \end{aligned} \tag{13}$$

By (9)–(11) the terms $L_{\rm KE}$ and $L_Y$, in terms of the Majorana spinors $\Psi_1$ and $\Psi_2$, become

$$L_{\rm KE} = L_{\rm KE}^B + \frac{i}{2}\left(\bar{\Psi}_i\gamma^\mu\partial_\mu\Psi_1 + \bar{\Psi}_2\gamma^\mu\partial_\mu\Psi_2\right) \ , \tag{14}$$

$$\begin{aligned} L_Y &= \frac{i\lambda}{\sqrt{2}}\left[\bar{\Psi}_1(A_1 + \gamma_5 B_1)\Psi_2 + \tfrac{1}{2}\bar{\Psi}_1(A_2 + \gamma_5 B_2)\Psi_1\right] \\ &= \frac{i\lambda}{2}\left[2\phi_1(\bar{\Psi}_1 P_L\Psi_2) + 2\phi_1^*(\bar{\Psi}_1 P_R\Psi_2)\right] \\ &\quad + \frac{i\lambda}{2}\left[\phi_2(\bar{\Psi}_1 P_L\Psi_1) + \phi_2^*(\bar{\Psi}_1 P_R\Psi_1)\right] \ , \end{aligned} \tag{15}$$

where

$$\begin{aligned} L_{\rm KE}^B &= \partial^\mu\phi_1^*\partial_\mu\phi_1 + \partial^\mu\phi_2^*\partial_\mu\phi_2 \\ &= \tfrac{1}{2}[(\partial_\mu A_1)^2 + (\partial_\mu B_1)^2 + (\partial_\mu A_2)^2 + (\partial_\mu B_2)^2] \ , \end{aligned} \tag{16}$$

and $P_{L,R}$ are the chiral projectors defined by

$$\begin{aligned} P_L &= \tfrac{1}{2}(1 - i\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ P_R &= \tfrac{1}{2}(1 + i\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \end{aligned} \tag{17}$$

with the properties $(P_{L,R})^2 = P_{L,R}$ and $P_L + P_R = 1$.

### B. Vacuum states

The vacuum states of the theory, obtained from the scalar potential $V$, satisfy the conditions

$$\begin{aligned} \left(\frac{\partial V}{\partial\phi_i}\right)_{\phi_{iv}} &= 0 \ , \\ \left(\frac{\partial^2 V}{\partial\phi_1^*\partial\phi_1}\right)_{\phi_{1v}} &> 0 \ , \\ \left(\frac{\partial^2 V}{\partial\phi_2^*\partial\phi_2}\right)_{\phi_{2v}} &> 0 \ . \end{aligned} \tag{18}$$

From (13) and (18), the vacuum states are therefore given by

$$\phi_{1v} = \pm v, \quad \phi_{2v} = 0 \ , \tag{19}$$

or, equivalently,

$$A_{1v} = \pm\sqrt{2}v, \quad B_{1v} = A_{2v} = B_{2v} = 0 \ . \tag{20}$$

The discrete $Z_2$ reflection symmetry (associated with the field $\phi_1$) of the scalar potential $V$ is therefore spontaneously broken by the vacuum. Spatial regions that are separated by more than a coherence length $\xi \sim (m_{\phi_1})^{-1}$ can accommodate different vacuum configurations of the field $\phi_1$, so that $\phi_1$ can settle into a $\phi_1 = +v$ configuration in some domains and into a $\phi_1 = -v$ configuration in other domains, with some region between different domains where $\phi_1 = 0$, which will locate a domain wall. However, from (13) and (19) it is seen that in the vacuum states $V = 0$, indicating that the vacuum states respect supersymmetry, so that the supersymmetry of the theory is unbroken in the vacuum region outside of a domain wall.

### C. Mass matrices

In terms of the complex scalar fields, the boson mass matrix can be written in the form

$$M_B^2 = \begin{pmatrix} M_{11}^2 & M_{12}^2 \\ M_{21}^2 & M_{22}^2 \end{pmatrix}, \quad M_{ij}^2 = \begin{pmatrix} X_{ij} & Z_{ij} \\ Z_{ij}^\dagger & X_{ij} \end{pmatrix} , \tag{21}$$

where

$$X_{ij} = \left( \frac{\partial^2 V}{\partial\phi_i \partial\phi_j^*} \right)_{\text{vac}} , \quad Z_{ij} = \left( \frac{\partial^2 V}{\partial\phi_i^* \partial\phi_j^*} \right)_{\text{vac}} , \tag{22}$$

with $()_{\text{vac}}$ indicating evaluation in vacuum. From (13) and (19) it is found that $M_B^2$ is diagonal with $M_{ij}^2 = m^2\delta_{ij}$, where $m \equiv \lambda v$ is the mass of each boson field. The mass for each of the real scalar fields $A_i$ and $B_i$ is $m$.

The fermion mass matrix which is associated with the spinors $\psi_i$ is $M_{Fij} = (Y_{ij})_{\text{vac}}$, which gives, by (11) and (19),

$$M_F = \begin{pmatrix} 0 & \lambda\phi_1 \\ \lambda\phi_1 & 0 \end{pmatrix}_{\text{vac}} = \begin{pmatrix} 0 & \pm m \\ \pm m & 0 \end{pmatrix} \quad \text{for } \phi_{1v} = \pm v \ . \tag{23}$$

From (23) we have $(M_F^2)_{ij} = m^2\delta_{ij}$ and the supertrace relation is satisfied, with $\mathrm{Tr}(M_B^2) = 2\,\mathrm{Tr}(M_F^2) = 4m^2$ . We can notice that the spinors $\psi_1, \psi_2$ (or $\Psi_1, \Psi_2$) are not the mass eigenstates since they do not diagonalize $M_F$. Also, we will see that the positive mass Weyl spinor states in two different domains are related by a phase rotation.

### D. Spinor mass eigenstates

From the Yukawa part of the Lagrangian $L_Y$ given by (11), along with the expression for the fermion mass matrix in (23), it can be seen that different spinor states are required in different $\phi_{1v}$ domains to yield positive fermion masses.

#### 1. The vacuum state with $\phi_{1v} = +v$

For $\phi_{1v} = +v$, $\phi_2 = 0$, $L_Y$ gives rise to the fermion mass term

$$L_{\text{mass}}^F = -m(\psi_1\psi_2 + \bar{\psi}_1\bar{\psi}_2) \ , \tag{24}$$

and we are therefore led to define a mass eigenstate Dirac spinor

$$\Psi \equiv \begin{pmatrix} \omega_\alpha \\ \bar{\omega}_c^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \end{pmatrix} = P_L\Psi_1 + P_R\Psi_2,$$
$$\bar{\Psi} = i(\omega_c^\alpha, \bar{\omega}_{\dot{\alpha}}) \ , \tag{25}$$

so that (24) becomes

$$L_{\text{mass}}^F = im\bar{\Psi}\Psi \ , \tag{26}$$

which describes a Dirac fermion of mass $m$.

We can also define the Weyl spinors $\alpha$ and $\beta$ by

$$\alpha = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \beta = \frac{-i}{\sqrt{2}}(\psi_1 - \psi_2) \ , \tag{27}$$

and Majorana spinors $M_1$ and $M_2$ by

$$M_1 = \begin{pmatrix} \alpha_\alpha \\ \bar{\alpha}^{\dot{\alpha}} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \beta_\alpha \\ \bar{\beta}^{\dot{\alpha}} \end{pmatrix} , \tag{28}$$

so that (24) can also be written as

$$\begin{aligned} L_{\text{mass}}^F &= -\tfrac{1}{2}m[(\alpha\alpha + \bar{\alpha}\bar{\alpha}) + (\beta\beta + \bar{\beta}\bar{\beta})] \\ &= \tfrac{1}{2}im(\bar{M}_1M_1 + \bar{M}_2M_2) \ , \end{aligned} \tag{29}$$

which describes two Majorana fermions, each of mass $m$.

The Majorana spinors $M_{1,2}$ are related to the Majorana spinors $\Psi_{1,2}$ by

$$\begin{aligned} M_1 &= \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2), \quad M_2 = \frac{1}{\sqrt{2}}\gamma_5(\Psi_2 - \Psi_1), \\ \Psi_1 &= \frac{1}{\sqrt{2}}(M_1 + \gamma_5 M_2), \quad \Psi_2 = \frac{1}{\sqrt{2}}(M_1 - \gamma_5 M_2) \ . \end{aligned} \tag{30}$$

The Majorana spinors $M_{1,2}$ are related to the Dirac spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}$$

and its conjugate

$$\Psi_c = \begin{pmatrix} \psi_2 \\ \bar{\psi}_1 \end{pmatrix}$$

by

$$\begin{aligned} M_1 &= \frac{1}{\sqrt{2}}(\Psi + \Psi_c), \quad M_2 = \frac{-i}{\sqrt{2}}(\Psi - \Psi_c), \\ \Psi &= \frac{1}{\sqrt{2}}(M_1 + iM_2), \quad \Psi_c = \frac{1}{\sqrt{2}}(M_1 - iM_2). \end{aligned} \tag{31}$$

#### 2. *The vacuum state with $\phi_{1v} = -v$*

For the vacuum state labeled by $\phi_{1v} = -v$, $\phi_{2v} = 0$, $L_Y$ gives rise to the fermion mass term

$$L'^{F}_{\rm mass} = +m(\psi_1\psi_2 + \bar\psi_1\bar\psi_2) \equiv -m(\psi'_1\psi'_2 + \bar\psi'_1\bar\psi'_2) \; . \tag{32}$$

We can therefore identify

$$\psi'_1 = i\psi_1, \;\; \psi'_2 = i\psi_2, \;\; \bar\psi'_1 = -i\bar\psi_1, \;\; \bar\psi'_2 = -i\bar\psi_2 \; . \tag{33}$$

Then the positive mass four-spinor eigenstates in this vacuum sector are

$$\Psi' = \gamma_5\Psi, \;\; \Psi'_c = \gamma_5\Psi_c, \;\; M'_{1,2} = \gamma_5 M_{1,2} \; , \tag{34}$$

with $\Psi'_{1,2} = \gamma_5\Psi_{1,2}$.

In each vacuum sector there is a Dirac particle of mass $m$, although the associated Weyl spinors differ by a constant phase and the four-spinors differ by a factor of $\gamma_5$. (The kinetic term $L_{\rm KE}$ is invariant under the rescaling $M_{1,2} \to \gamma_5 M_{1,2}$.)

## III. THE DOMAIN WALL

### A. Field equations

The field equations for the complex scalar fields $\phi_i$ and the Majorana spinors $\Psi_i$, obtained from the Lagrangian given by (13)–(16), are given by

$$\Box\phi_1 + \lambda^2[\tfrac{1}{2}\phi_1^*(\phi_1^2 - v^2) + |\phi_2|^2\phi_1] - i\lambda\bar\Psi_1 P_R\Psi_2 = 0 \; , \tag{35}$$

$$\Box\phi_2 + \lambda^2|\phi_1|^2\phi_2 - \frac{i\lambda}{2}\bar\Psi_1 P_R\Psi_1 = 0 \; , \tag{36}$$

$$\gamma^\mu\partial_\mu\Psi_1 + \lambda[2(\phi_1 P_L + \phi_1^* P_R)\Psi_2 + (\phi_2 P_L + \phi_2^* P_R)\Psi_1] = 0 \; , \tag{37}$$

$$\gamma^\mu\partial_\mu\Psi_2 + 2\lambda(\phi_1 P_L + \phi_1^* P_R)\Psi_1 = 0 \; . \tag{38}$$

where $\Box = \partial^\mu\partial_\mu = \partial_0^2 - \nabla^2$ .

### B. Domain-wall solution

An exact solution of the field equations which describes a domain wall can be obtained by setting $\phi_2 = 0$, and $\Psi_1 = \Psi_2 = 0$. By (35) the equations of motion for the real scalar fields $A_1$ and $B_1$ are

$$\Box A_1 + \frac{\lambda^2}{2}A_1\left[\frac{1}{2}(A_1^2 - B_1^2) - v^2 + B_1^2\right] = 0 \; , \tag{39}$$

$$\Box B_1 + \frac{\lambda^2}{2}B_1\left[A_1^2 - \frac{1}{2}(A_1^2 - B_1^2) + v^2\right] = 0 \; . \tag{40}$$

Upon setting $B_1 = 0$, (39) collapses to

$$\Box A_1 + \tfrac{1}{4}\lambda^2 A_1(A_1^2 - 2v^2) = 0 \; . \tag{41}$$

The static solution is just the $\phi^4$ kink solution

$$A_w(x) = \sqrt{2}v\tanh\left(\frac{x}{w}\right), \;\; w = \frac{2}{\lambda v} = \frac{2}{m} \; , \tag{42}$$

which describes a domain wall of thickness $w = 2/m$ located at $x = 0$, interpolating between the supersymmetric vacuum states with $\phi_{1v} = +v$ at $x = +\infty$ and $\phi_{1v} = -v$ at $x = -\infty$.

Upon passing through the domain wall from a position $(x,y,z)$ to a position $(-x,y,z)$ we have $\phi_w(x) \to \phi_w(-x) = -\phi_w(x)$, where $\phi_w(x) \equiv (1/\sqrt{2})A_w(x)$, and for the physical spinor states

$$\Psi(x) \to \Psi(-x) = \Psi'(x) = \gamma_5\Psi(x), M_i(x) \to \gamma_5 M_i(x) \; .$$

Also, from (15) and (25) we notice that the Dirac fermion mass vanishes in the core of the domain wall at $x = 0$, from which we can infer that upon collision with the domain wall, fermions will be attracted inward toward the core.

## IV. FERMIONIC ZERO MODES

The fermion fields become effectively massless in the core of the domain wall, allowing fermionic zero modes to form. Consider the fermion fields in the domain-wall background with $\phi_2 = 0$ and $\phi_1 = \phi_w(x)$. From the fermionic field equations given by (37) and (38) we obtain

$$\gamma^\mu\partial_\mu\Psi_1 = -2\lambda\phi_w\Psi_2, \;\; \gamma^\mu\partial_\mu\Psi_2 = -2\lambda\phi_w\Psi_1 \; . \tag{43}$$

Let us first look for static solutions $\Psi_i = \Psi_i^0(x)$. Multiplying (43) by $\gamma^1$ [with $(x^0,x^1,x^2,x^3) = (t,x,y,z)$] yields

$$\partial_1\Psi_1^0 = -2\lambda\phi_w\gamma^1\Psi_2^0, \;\; \partial_1\Psi_2^0 = -2\lambda\phi_w\gamma^1\Psi_1^0 \; . \tag{44}$$

A solution is obtained for $\Psi_2^0 = \gamma^1\Psi_1^0$ (and therefore $\Psi_1^0 = \gamma^1\Psi_2^0$) so that (44) becomes

$$\partial_1\Psi_1^0 = -2\lambda\phi_w\Psi_1^0 \; . \tag{45}$$

The solutions are

$$\begin{aligned}\Psi_1^0(x) &= \tau\,\exp[-2\lambda\int_0^x \phi_w(x')dx'] \\ &= \tau\left[\cosh\left(\frac{x}{w}\right)\right]^{-2\lambda v w} = \tau\left[\cosh\left(\frac{x}{w}\right)\right]^{-4} \; , \\ \Psi_2^0(x) &= \gamma^1\Psi_1^0(x) \; ,\end{aligned} \tag{46}$$

where $\tau$ is an arbitrary constant Majorana spinor. These solutions decrease exponentially away from the core of the wall and therefore describe fermion fields concentrated within the wall. As $x \to 0$, then $\gamma^\mu\partial_\mu\Psi_i \to 0$, so that the fermions are effectively massless in the core.

Traveling wave solutions can be constructed by using $\Psi_2 = \gamma^1\Psi_1$ with $\Psi_i(x,z,t) = \alpha(z,t)\Psi_i^0(x)$, from which we obtain, by using (43) and (44),

$$(\partial_0 - \gamma^0\gamma^3\partial_3)\alpha\tau = 0 \ . \qquad (47)$$

Defining the spinor eigenvectors of $\gamma^0\gamma^3$ by $\gamma^0\gamma^3\tau_\pm = \pm\tau_\pm$, (47) reduces to

$$(\partial_0 \mp \partial_3)\alpha\tau_\pm = 0 \ , \qquad (48)$$

which is solved by

$$\alpha(z,t) = \begin{cases} \alpha_+(z+t) & \text{for } \tau = \tau_+ , \\ \alpha_-(z-t) & \text{for } \tau = \tau_- . \end{cases} \qquad (49)$$

Then (47) and (48) are solved by choosing either $\tau = \tau_+$ or $\tau = \tau_-$. The traveling wave solutions are then

$$\Psi_1(x,z,t) = \alpha(z,t)\Psi_1^0(x), \quad \Psi_2(x,z,t) = \gamma^1\Psi_1(x,z,t) \ . \qquad (50)$$

These solutions describe effectively massless fermions trapped within the core of the domain wall, traveling in either the $+z$ direction or in the $-z$ direction. The domain wall is therefore superconducting in that it supports nondissipative fermionic currents which are associated with the spinor traveling waves.

From the fermion current densities $J_i^\mu = \bar\Psi_i\gamma^\mu\Psi_i = \alpha^2(z,t)\bar\Psi_i^0(x)\gamma^\mu\Psi_i^0(x)$ (with $i=1,2$), we find $J_2^1 = -J_1^1$ and $J_2^{0,3} = J_1^{0,3}$, so that for the total fermion current $J^\mu = J_1^\mu + J_2^\mu$ we find $J^1 = 0$, $J^{0,3} = 2J_1^{0,3}$. The associated charge per unit area of the domain wall is $Q_i = \int_{-\infty}^{\infty} J_i^0 dx \approx \int_{-w/2}^{w/2} J_i^0 dx$ and the linear current density (amount of current per unit width of domain wall) is $I_i = \int_{-\infty}^{\infty} J_i^3 dx \approx \int_{-w/2}^{w/2} J_i^3 dx$.

## V. $\phi_2$ PARTICLE BOUND STATES

At the same level of approximation, let us examine the $\phi_2$ field in the domain-wall background by setting $\phi_1 = \phi_w(x)$ and $\Psi_i = 0$; i.e., as a first approximation we examine the fields $\Psi_i$ and $\phi_2$ in the absence of one another in the domain-wall background. Then (36) reduces to

$$\Box\phi_2 + \lambda^2\phi_w^2(x)\phi_2 = 0 \ . \qquad (51)$$

Writing $\phi_2(x,z,t) = \varphi(x)\exp[i(kz - wt)]$ then leads to

$$-\partial_1^2\varphi + \lambda^2\phi_w^2(x)\varphi = \mu^2\varphi, \quad \mu^2 \equiv \omega^2 - k^2 \ , \qquad (52)$$

which is a time-independent Schrödinger equation for a particle of one-half unit of "mass" and "energy" $\mu^2$ in the presence of a potential well $U(x) = m^2\tanh^2(x/w)$. This attractive potential can accommodate one or more normalizable bound states, depending upon the values of the parameters, with $0 < \mu < m$. Since the field $\phi_2$ can be associated with normalizable stationary bound states, we infer that $\phi_2$ particles can be localized within or near the core of the domain wall in the form of domain-wall, $\phi_2$ particle bound states, rather than in the form of a boson condensate. [Note that we do not expect a *condensate* to form, at least at the level of approximation being used, since from (51) the only *constant* value of $\phi_2$ in the core of the wall is given by $\phi_2 = 0$, so that there is no nonvanishing value of $\phi_2$ which minimizes the scalar potential in the absence of fermion fields or $\phi_1$ excitations.] The spectrum described by $\{\mu\}$ also includes scattering states, and we expect, in general, transitions between states to be possible due to interactions of the $\phi_2$ field with the $\Psi_1$ or $\phi_1$ fields.

From the current density $j_\mu = i(\phi_2^*\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_2^*)$ we find $j^1 = 0$, $j^0 = 2\omega\varphi^2(x)$, and $j^3 = 2k\varphi^2(x)$, yielding a surface charge density $q = \int_{-\infty}^{\infty} j^0 dx$ and a linear current density $i = \int_{-\infty}^{\infty} j^3 dx$, with the charge and current associated with the normalizable bound states being effectively confined to the core of the domain wall.

## VI. SUMMARY

The field-theoretic structures of topological and nontopological defects, along with possible interactions and physical consequences involving such entities, have been studied in a variety of settings [2]. Here, attention has been focused upon some of the field-theoretic aspects of a simple topological defect, a domain wall, that arises from a supersymmetric action. The supersymmetric action dictates the forms of the interactions between the component boson and fermion fields of the theory, and such a theory can possess an interesting nonperturbative sector. The Lagrangian for the model presented here has been constructed from chiral superfields, and a superpotential has been selected that allows the scalar potential to exhibit a spontaneous breaking of a discrete $Z_2$ reflection symmetry, without a spontaneous breaking of supersymmetry in the vacuum sectors. The spontaneously broken $Z_2$ symmetry allows the existence of a domain-wall solution which smoothly interpolates between the distinct supersymmetric vacuum states of the theory. A simple, but exact, solution describing the domain wall has been found, boson and fermion mass matrices have been calculated, and positive mass Dirac and Majorana spinor eigenstates have been found for each spatial domain.

In the domain-wall background, approximate analytic solutions have been found that describe the fermionic zero modes entrapped by the domain wall. Since these zero modes have nondissipative currents, the domain wall has a fermionic superconductivity, and can support fermionic charge and current. The $\phi_2$ boson field can interact with the domain wall, not by forming a condensate within it, but rather by the formation of one or more sets of domain wall, $\phi_2$ particle bound states describing a boson concentration within or near the core of the wall. Scattering states describing wall, $\phi_2$ interactions also exist. The bound states contribute a bosonic charge and current to the domain wall, and transitions between states can occur due to $\phi_2$ interactions with the $\Psi_1$ and $\phi_1$ excitations. The domain wall therefore has a fermionic superconductivity, and can carry both fermionic and bosonic charges and currents.

## APPENDIX: CONVENTIONS

Some of the notation and conventions are briefly listed here. A metric $g_{\mu\nu}$ is used with signature $(+,-,-,-)$. Aside from the metric, the notation, conventions, and $\gamma$ matrices used conform to those of [11]. The $\gamma$ matrices can be written in the form

$$\gamma^\mu = i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A1}$$

with

$$\sigma^\mu = (1,\vec{\sigma}), \quad \bar{\sigma}^\mu = (1,-\vec{\sigma}) \ , \tag{A2}$$

where $\vec{\sigma}$ represents the Pauli matrices. Then

$$\gamma^0 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = i\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3 \ , \tag{A3}$$

and $\gamma_5$ is given by

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A4}$$

The $\gamma$ matrices have the properties

$$\begin{gathered}\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu}, \quad \{\gamma^\mu,\gamma_5\} = 0, \\ \gamma_5^\dagger = -\gamma_5, \quad (\gamma_5)^2 = -1 \ .\end{gathered} \tag{A5}$$

A Majorana four-spinor $\Psi$ is expressed in terms of the Weyl two-spinors $\psi$ and $\bar{\psi}$ by

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

and we use the summation conventions for Weyl spinors [with $\bar{\psi}^{\dot{\alpha}} = (\psi^\alpha)^*$]

$$\xi\psi \equiv \xi^\alpha\psi_\alpha, \quad \bar{\xi}\bar{\psi} \equiv \bar{\xi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, \quad \alpha = 1,2, \quad \dot{\alpha} = 1,2 \ , \tag{A6}$$

with $\varepsilon$ metric tensors (for raising and lowering Weyl spinor indices)

$$\begin{gathered}(\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot{\alpha}\dot{\beta}}) = i\sigma_2, \quad (\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot{\alpha}\dot{\beta}}) = -i\sigma_2, \\ \varepsilon^{12} = 1 = \varepsilon^{\dot{1}\dot{2}} \ .\end{gathered} \tag{A7}$$


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