Large deformations of relativistic membranes: A generalization of the Raychaudhuri equations

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A coupled system of nonlinear partial differential equations is presented which describes nonperturbatively the evolution of deformations of a relativistic membrane of arbitrary dimension D in an arbitrary background spacetime. These equations can be considered from a formal point of view as higher dimensional analogues of the Raychaudhuri equations for point particles to which they are shown to reduce when D=1. For D=1 or D=2 (a string), there are no constraints on the initial data. If $D>2$, however, there will be constraints with a corresponding complication of the evolution problem. The consistent evolution of the constraints is guaranteed by an integrability condition which is satisfied when the equations of motion are satisfied. Explicit calculations are performed for membranes described by the Nambu action.

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I. INTRODUCTION

A surprising variety of physical systems can be modeled as relativistic membranes of an appropriate dimension propagating in a fixed background spacetime. The phenomenological action describing the dynamics of the membrane is a sum of various relevant scalars associated with the geometry of its trajectory (world sheet). At lowest order this action is proportional to the area of the world sheet, the Nambu action [1]. If the approximation stops here, the classical trajectory of the membrane is an extremal surface of the background spacetime [2]. At a higher order, one can consider rigidity corrections quadratic in the extrinsic curvature of the world sheet [3].

The dynamics of a Nambu membrane is reasonably well understood. A large body of information has been accumulated on the dynamics of geometrically symmetrical extremal configurations [1]. What is less complete is a satisfactory description of the deformations of relativistic membranes. How does a variation in the symmetry of the membrane evolve? If the world sheet develops a singularity, such as a cusp, how will this singularity evolve? Will it be smoothed out or will it grow?

It is clear from the outset what criteria a description of deformations should satisfy. First, it should be covariant not only with respect to world-sheet diffeomorphisms, but also with respect to local rotations of the normals to the world sheet itself when the codimension of the world sheet is greater than one. This is crucial because once we deviate from a symmetrical configuration the choice of normals will no longer be obvious. A second requirement is that, ideally, the description should be independent of the specific dynamics of the membrane. Apart from obvious economical considerations, this permits one to isolate the kinematical features of the deformation, common to all membrane theories, from those which depend on the dynamics. Since the dynamics under consideration is an approximation in the first place, one would like to know what features of the evolution of deformations are influenced by a change in the dynamics.

Various aspects of this problem have been addressed recently by several authors. Garriga and Vilenkin described the evolution of small disturbances propagating on planar and spherically symmetrical Nambu membranes in background Minkowski and de Sitter spacetimes [4]. Following this work, Guven [5] and, almost simultaneously, Carter as well as Frolov and Larson [6] approached the problem of small deformations of a Nambu membrane in a manifestly covariant way, independent of the particular symmetry of the defect, and of the background spacetime. In [5], the role of the twist potential in ensuring manifest covariance under normal rotations was made explicit. The deformation is described by a set of massive scalar fields, which satisfy a coupled system of linear wave equations. The scalar fields are the projection of the infinitesimal deformation in the embedding function describing the world sheet onto each normal direction. The effective mass matrix is the sum of a term quadratic in the extrinsic curvature, and a matrix of curvature projections. This framework was subsequently generalized by the authors to permit the stability anal-

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ysis of any membrane described by a local action constructed from any world-sheet scalars [7]. This was done by shifting the focus from the particular dynamics of an extremal surface to a systematic kinematical description of the deformation of the world-sheet intrinsic and extrinsic geometries.

The progress we have described has been made entirely in the examination of small deformations. If the membrane is described by the Nambu action there is no energy penalty prohibiting the formation of kinks or cusps—we should therefore not be surprised to find that structures such as these form in the course of the evolution. Unfortunately, the analysis of such structures lies outside the scope of perturbation theory. A formalism permitting us to examine large deformations is required.

From a formal point of view, the equations describing the evolution of small deformations of the world sheet are higher-dimensional analogues of the Jacobi equations describing the infinitesimal separation of neighboring timelike geodesics [8]. These higher-dimensional analogues possess, however, an interpretation without any one-dimensional analogue in that they describe a physical stretching of the membrane itself.

In the prototype of a geodesic curve, an alternative nonperturbative framework for describing the focusing of trajectories is provided by the Raychaudhuri equations [8,9]. These are a system of coupled nonlinear ordinary differential equations describing the evolution along a curve of the orthogonal deformations of its velocity vector. By examining the trace of these equations over the normal directions, which describes the expansion in the volume occupied by a given pencil of geodesics, it is possible to show that under very reasonable assumptions on the material sources, spacetime will generally be geodesically incomplete. As such, the Raychaudhuri equations constitute one of the cornerstones of the classical singularity theorems in general relativity.

If the Raychaudhuri equations are any guideline, we should shift our focus from the deformation in the embedding function describing the world sheet of the membrane to the deformation in the tangent vectors to this surface [10]. In this way we derive an analogous nonperturbative, and nonlinear, coupled system of partial differential equations to describe the deformation of higher-dimensional surfaces. Unlike the Jacobi equation, where the formal generalization is in some sense the obvious one, the generalization of the Raychaudhuri equations is extremely nontrivial.

The particular value of this set of equations is that they provide us with an analytically tractable procedure for examining various peculiarities of the dynamics of relativistic membranes whose description is beyond the scope of perturbation theory. Physical applications will be considered elsewhere [11].

The content of this article is as follows. To establish our notation, we begin in Sec. II by summarizing the classical kinematical description of an embedded timelike surface (world sheet) of dimension D, in a fixed background spacetime of dimension N, in terms of its intrinsic and extrinsic geometry. The latter is characterized completely by the extrinsic curvature and the extrinsic twist potential. This world sheet will be generated from some initial configuration of the membrane appropriate to the truncation of the action describing its dynamics.

We now want to consider the evolution of a deformation of this world sheet. We begin by providing a purely kinematical description of the deformation of the world sheet, a nonperturbative analogue of the analysis performed in [7]. In this way we identify the structures that characterize the deformation. When $D=1$, these are the world-sheet scalars J^{ij} constructed by taking the projection onto the jth normal of the gradient of the tangent vector to the curve along the ith normal. When $D>1$, there will be one object of this kind for each tangent vector to the surface, $J_a{}^{ij}$ $(a,b,\ldots = 0,1,\ldots,D-1)$. In addition to such straightforward generalizations from one to higher dimensions there will be a new structure without any one-dimensional analogue, which we interpret geometrically as the deformation of the world-sheet connection preserving manifest covariance under tangent frame rotations after deformation.

In Sec. III, a "naive" generalization of the Raychaudhuri equations which reduces to the familiar prototype in one dimension is provided. In Sec. IV, we point out the inadequacies of this simpleminded generalization. If our system of equations is to possess any predictive power, it is important that it possess a Cauchy formulation, modulo the membrane dynamics, so that the evolution of independent initial deformations can be tracked. We encounter various obstacles to the implementation of such a formulation.

First, the source for the deformation in the equation of motion must not involve unknown elements. To obtain a consistent system of equations, we need then to form suitable linear combinations of the naive generalization of the Raychaudhuri equations, such that these elements are eliminated from the source. One set of linear combinations is an antisymmetric sum with respect to world-sheet indices. This linear combination eliminates these unknowns no matter what the background dynamics is. In Sec. V, we consider extremal surfaces, and for this case the remaining linear combination is a trace.

We are still not out of the woods, however. This is because when $D>2$ not all of our equations are dynamical. The nondynamical equations must be considered as constraints on the initial data, i.e., the deformations on some spacelike hypersurface of the world sheet. These data are not freely specifiable. The existence of constraints complicates the implementation of a Cauchy solution, for we need to ensure that they be preserved by the evolution. This could require us to impose a nontrivial integrability condition on the solution, thereby further complicating the solution of the Cauchy problem. We demonstrate, however, that the integrability conditions we require are trivially satisfied, modulo the differential Bianchi identities on the curvature associated with the twist potential introduced in Sec. II. Once the constraints are satisfied by the initial data, the equations of motion will ensure that they continue to be satisfied at all subsequent times. We point out three important dimensional exceptions: the cases of a point particle, $D=1$, and of a string, $D=2$, where there are no constraints on the initial data, and that of a hypersurface, $D = N-1$, where the constraint reduces to a condition on the rotationality of the spatial initial data.

Now that we have a consistent generalization of the Raychaudhuri equations to higher dimensions, we focus on the generalized expansion, given by tracing J_{aij} over normal indices, i.e., $J_a{}^i{}_i = \Theta_a$. The antisymmetric linear equation implies that $\Theta_a = \partial_a \Upsilon$. For extremal surfaces, inserting this in the traced evolution equation for Θ_a gives Eq. (5.15) in the text:

$$\Delta\Upsilon + \frac{1}{N-D}\partial_a\Upsilon\partial^a\Upsilon - \Lambda^2 + \Sigma^2 + M^2 = 0 \ ,$$

where Λ^2, Σ^2, M^2 are world-sheet scalars defined in the text. This equation generalizes the familiar first-order ordinary differential equation for the expansion of neighboring geodesics.

In Sec. VI, we compare our results to perturbation theory. For an extremal surface, the "naive" truncation consisting only of the traced equations (or the appropriate set of equations if the dynamics is not extremal) permits us to recover the coupled linear scalar equations of motion describing the evolution of an infinitesimal deformation of the extremal world sheet derived in Refs. [5,7]. The antisymmetric Raychaudhuri equations play no role in perturbation theory.

Finally, we conclude in Sec. VII, with a discussion of our results. For simplicity, we confine our attention to closed surfaces without physical boundaries. All considerations are local.

II. MATHEMATICAL PRELIMINARIES

A. Geometry of the world sheet

We consider an oriented world sheet m of dimension D described by the embedding

$$x^\mu = X^\mu(\xi^a) \qquad (2.1)$$

($\mu, \nu, \ldots = 0, 1, \ldots, N-1$, and $a, b, \ldots = 0, 1, \ldots, D-1$) in a spacetime $\{M, g_{\mu\nu}\}$ of dimension N. We adopt the signature $\{-, +, \ldots, +\}$ for $g_{\mu\nu}$. The D vectors

$$e_a = X^\mu_{,a}\partial_\mu \qquad (2.2)$$

form a basis of tangent vectors to m at each point of m. The metric induced on the world sheet is then given by

$$\gamma_{ab} = X^\mu_{,a}X^\nu_{,b}g_{\mu\nu} = g(e_a, e_b) \ . \qquad (2.3)$$

To facilitate comparison with the one-dimensional Raychaudhuri prototype, where the single tangent vector is the unit velocity vector along the particle world line, it is useful, if not essential, to replace the coordinate tangent basis $\{e_a\}$ by an orthonormal basis of tangent vectors $\{E_a\}$:

$$g(E_a, E_b) = \eta_{ab} \ . \qquad (2.4)$$

In order to avoid cluttering our equations, we continue to use Latin letters for orthonormal indices.

We assume that the world sheet is timelike everywhere, and that we can always consistently choose one timelike tangent vector field, E_0. That the vectors $\{E_a\}$ form a surface is encoded in an integrability condition, the closure of their commutator algebra, $[E_a, E_b]$, by Frobenius theorem [2].

The ith unit normal to the world sheet ($i, j, \ldots = 1, 2, \ldots, N-D$) is denoted with n^i, and is defined up to a local $O(N-D)$ rotation by

$$g(n^i, n^j) = \delta^{ij}, \quad g(n^i, E_a) = 0 \ . \qquad (2.5)$$

Normal vielbein indices are raised and lowered with δ^{ij} and δ_{ij}, respectively, whereas tangential indices are raised and lowered with η^{ab} and η_{ab}, respectively.

We define the world-sheet projections of the spacetime covariant derivatives, $D_a = E^\mu{}_a D_\mu$, where D_μ denotes the (torsionless) covariant derivative compatible with the spacetime metric $g_{\mu\nu}$. Let us consider the world-sheet gradients of the basis vectors $\{E_a, n^i\}$. Since they are spacetime vectors, they can always be decomposed with respect to the orthonormal basis $\{E_a, n^i\}$ [2], as

$$D_a E_b = \gamma_{ab}{}^c E_c - K_{ab}{}^i n_i \ , \qquad (2.6a)$$

$$D_a n^i = K_{ab}{}^i E^b + \omega_a{}^{ij} n_j \ . \qquad (2.6b)$$

These kinematical expressions, which describe the extrinsic geometry of the world sheet, are generalizations of the classical Gauss-Weingarten equations. The $\gamma_{ab}{}^c$ are the world-sheet Ricci rotation coefficients

$$\gamma_{abc} \equiv g(D_a E_b, E_c) = -\gamma_{acb} \ . \qquad (2.7)$$

The quantity $K_{ab}{}^i$ is the ith extrinsic curvature of the world sheet:

$$K_{ab}{}^i \equiv -g(D_a E_b, n^i) = K_{ba}{}^i \ . \qquad (2.8)$$

Its symmetry in the tangential indices is a consequence of the integrability of the $\{E_a\}$.

The twist potential of the world sheet is defined by

$$\omega_a{}^{ij} \equiv g(D_a n^i, n^j) = -\omega_a{}^{ji} \ . \qquad (2.9)$$

With respect to a normal frame rotation, $n^i \to O^i{}_j n^j$, $\omega_a{}^{ij}$ transforms as a connection, $\omega_a \to O\omega_a O^{-1} + O_{,a}O^{-1}$. As discussed, e.g., in [5], it is therefore associated with the gauging of normal frame rotations. It is desirable to implement normal frame covariance in a manifest way. We therefore introduce a world-sheet covariant derivative, defined on fields transforming as tensors under normal frame rotations as

$$\tilde{\nabla}_a \Phi^{i_1\cdots i_n} = \nabla_a \Phi^{i_1\cdots i_n} - \omega_a^{i_1 j}\Phi_j^{i_2\cdots i_n} - \cdots - \omega_a^{i_n j}\Phi_j^{i_1\cdots i_{n-1}} \ , \qquad (2.10)$$

where ∇_a is the intrinsic world-sheet covariant derivative.

The embedding $X^\mu(\xi)$ is overspecified by Eqs. (2.6).

There are integrability conditions: the intrinsic and extrinsic geometry must satisfy the Gauss-Codazzi, Codazzi-Mainardi, and Ricci integrability conditions, given, respectively, by

$$g(R(E_b,E_a)E_c,E_d) = \mathcal{R}_{abcd} - K_{ac}{}^i K_{bdi} + K_{ad}{}^i K_{bci} \ , \tag{2.11a}$$

$$g(R(E_b,E_a)E_c,n^i) = \tilde{\nabla}_a K_{bc}{}^i - \tilde{\nabla}_b K_{ac}{}^i \ , \tag{2.11b}$$

$$g(R(E_b,E_a)n^i,n^j) = \Omega_{ab}{}^{ij} - K_{ac}{}^i K_b{}^{cj} + K_{bc}{}^i K_a{}^{cj} \ . \tag{2.11c}$$

We use the notation $g(R(Y_1,Y_2)Y_3,Y_4) = R_{\alpha\beta\mu\nu} Y_2^\alpha Y_1^\beta Y_3^\mu Y_4^\nu$. $R^\alpha{}_{\beta\mu\nu}$ is the Riemann tensor of the spacetime covariant derivative D_μ, whereas $\mathcal{R}^a{}_{bcd}$ is the Riemann tensor of the world-sheet covariant derivative ∇_a. $\Omega_{ab}{}^{ij}$ is the curvature associated with the twist potential $\omega_a{}^{ij}$, defined by,

$$\Omega_{ab}{}^{ij} = \nabla_b \omega_a{}^{ij} - \nabla_a \omega_b{}^{ij} + \omega_a{}^{ik}\omega_{bk}{}^j - \omega_b{}^{ik}\omega_{ak}{}^j \ . \tag{2.12}$$

Since it is a curvature, it satisfies the differential Bianchi identity

$$\tilde{\nabla}_{[c}\Omega_{ab]}{}^{ij} = 0 \ . \tag{2.13}$$

The gauge-invariant measure of the twist is completely determined by the extrinsic curvature, and the spacetime Riemann tensor. The twist potential can be gauged away locally if and only if $\Omega_{ab}{}^{ij}=0$. Setting $\Omega_{ab}{}^{ij}=0$ in Eq. (2.11c), therefore, provides the necessary and sufficient condition that the twist potential can be gauged away, in terms of an equality between a projection of the spacetime Riemann tensor, and an antisymmetric sum of the square of the extrinsic curvature.

B. Geometry of a deformed world sheet

The Gauss-Weingarten equations (2.6) describe a single surface embedded in spacetime. Let us consider now a one-parameter family of neighboring surfaces, $x^\mu = X^\mu(\xi^a, s)$. To provide a measure of the relative displacement of such surfaces, we consider the gradients of the spacetime basis $\{E_a, n^i\}$ along the directions orthogonal to the world sheet. Let $\delta = \partial_s$. We define $\tilde{\delta}^\mu = (g^{\mu\nu} - E^\mu{}_a E^{\nu a})\delta_\nu$, and $D_{\tilde{\delta}} = \tilde{\delta}^\mu D_\mu$. Now, the two measures of the orthogonal deformation of m are $g(n_i, D_{\tilde{\delta}} E_a)$, and $g(n_i, D_{\tilde{\delta}} n_j)$. In particular, suppose that $\tilde{\delta} = n_i$. Then, with $D_i = n^\mu{}_i D_\mu$, the gradients of the spacetime basis $\{E_a, n^i\}$ along the directions orthogonal to the world sheet, can be expressed as

$$D_i E_a = S_{abi} E^b + J_{aij} n^j \ , \tag{2.14a}$$

$$D_i n_j = -J_{aij} E^a + \gamma_{ij}^k n_k \ . \tag{2.14b}$$

These are the analogues of the Gauss-Weingarten equations, Eqs. (2.6), along the distribution spanned by the $\{n^i\}$. Note that one can obtain integrability conditions analogous to Eqs. (2.11), but we will not need them here.

The quantity $J_a{}^{ij}$, which plays a central role in the description of the deformations of the world sheet, is defined by

$$J_a{}^{ij} \equiv g(D^i E_a, n^j) \ . \tag{2.15}$$

In general, it does not possess any symmetry under interchange of the normal indices i and j, reflecting the fact that the $\{n^i\}$ (unlike the $\{E_a\}$) do not generally form an integrable distribution. It is the analogue of $K_{ab}{}^i$ in the Gauss-Weingarten equations.

The quantity $S_{ab}{}^i$ is defined by

$$S_{ab}{}^i \equiv g(D^i E_a, E_b) = -S_{ba}{}^i \ . \tag{2.16}$$

For an embedded curve, $S_{ab}{}^i = 0$, identically. It is the analogue of the extrinsic twist potential, $\omega_a{}^{ij}$, in Eqs. (2.6). Note that under a tangent frame orthonormal rotation, $E^a \to O^a{}_b E^b$, $S_{ab}{}^i$ transforms as a connection, $S_i \to O S_i O^{-1} + O_{,i} O^{-1}$. $S_{ab}{}^i$ is the deformation of the world-sheet connection associated with the gauging of local tangent frame rotations. We can introduce an associated covariant derivative $\tilde{\nabla}_i$ in a manner directly analogous to Eq. (2.10) by

$$\tilde{\nabla}_i \Phi_{a_1\cdots a_n} = \nabla_i \Phi_{a_1\cdots a_n} - S_{a_1 bi}\Phi^b{}_{a_2\cdots a_n} - \cdots - S_{a_n bi}\Phi_{a_1\cdots a_{n-1}}{}^b \ , \tag{2.17}$$

where

$$\nabla_i \Phi_j = D_i \Phi_j - \gamma_{ijk}\Phi^k \tag{2.18}$$

is the normal covariant derivative, and

$$\gamma_{ijk} \equiv g(D_i n_j, n_k) = -\gamma_{ikj} \tag{2.19}$$

are the Ricci rotation coefficients associated with the normal basis.

III. NATURAL GENERALIZATION OF THE RAYCHAUDHURI EQUATIONS

For a geodesic curve, the Raychaudhuri equations describe the evolution of $J^{ij} \equiv J_0{}^{ij}$, connecting neighboring geodesics along the curve, given specified values for J^{ij} at some initial instant. These are ordinary differential equations. Their generalization to higher-dimensional surfaces will clearly involve partial differential equations.

The natural generalization of the proper time derivative along the trajectory, $d_s J_{ij}$, in the one-dimensional context is given by the covariant world-sheet derivatives $\tilde{\nabla}_b J_a{}^{ij}$. Therefore, to generalize the Raychaudhuri equations, we evaluate this quantity. We find

$$\tilde{\nabla}_b J_a{}^{ij} = -\tilde{\nabla}^i K_{ab}{}^j - J_b{}^i{}_k J_a{}^{kj} - K_{bc}{}^i K_a{}^{cj} + g(R(E_b, n^i)E_a, n^j) \ . \tag{3.1}$$

The details of the evaluation of $\tilde{\nabla}_b J_a{}^{ij}$ are contained in Appendix A. We emphasize that the evaluation does not depend on the membrane equations of motion.

When D=1, we reproduce the Raychaudhuri equations for a curve. For then there is only one tangent vector, the unit timelike vector $E_0 = V$. Now $K_{00}{}^i = -K^i = g(n^i, D_V V)$. In addition, $S_{00}{}^i$=0 by antisymmetry, and we can always orient the normals along the curve such that the extrinsic twist potential vanishes, $\omega_0{}^{ij}$=0 [5]. (In [8], this is accomplished by Fermi-Walker transporting the normals.) With respect to proper time s along a physical trajectory, Eq. (3.1) assumes the form

$$\frac{dJ^{ij}}{ds} + \nabla^i K^j + J^i{}_k J^{kj} + K^i K^j = g(R(V,n^i)V,n^j) \ , \tag{3.2}$$

which agrees, before symmetrization, with Eq. (4.25) in [8].

To establish contact with the literature on the Raychaudhuri equations, we note that the analogs of the quantities which more frequently appear in the literature are the spacetime tensors

$$J_{\mu\nu a} = n_\mu{}^i n_\nu{}^j J_{aij} = H_\mu{}^\alpha H_\nu{}^\beta D_\alpha E_{\beta a} \ , \tag{3.3}$$

where

$$H_{\mu\nu} = n^i_\mu n_{i\nu} = g_{\mu\nu} - E_\mu{}^a E_{\nu a} \tag{3.4}$$

is the projection orthogonal to the world sheet.

IV. DIFFICULTIES WITH THE GENERALIZATION

Given some initial conditions on an initial spacelike hypersurface, the membrane equations of motion, together with the integrability conditions, Eqs. (2.11), determine uniquely the embedded world sheet described by Eq. (2.1). This will in turn determine both the intrinsic geometry γ_{ab}, or $\{E_a\}$, and the extrinsic geometry, as characterized by $K_{ab}{}^i$ and $\omega_a{}^{ij}$.

Suppose we also prescribe some initial values for $J_a{}^{ij}$ on this spacelike hypersurface. It is then possible to determine $J_a{}^{ij}$ at subsequent times using Eq. (3.1)? Unfortunately, there are at least two obstructions to the implementation of a Cauchy solution of Eq. (3.1).

The first problem is one which must be confronted even when D=1. The term $\tilde{\nabla}^i K_{ab}{}^j$, which is not determined on the world sheet by the equations of motion, appears as a source for the world-sheet derivative of $J_a{}^{ij}$ in Eq. (3.1). In the traditional one-dimensional application, this does not present any problem because one is concerned there only with geodesics satisfying K^i=0. Therefore this term vanishes, and along with it the dependence on the normal frame Ricci rotation coefficients, $\gamma_{ij}{}^k$. If, however, the curve satisfies some other equation of motion the source is just as much an unknown as is J^{ij}, and the equations useless in practice as they stand. In general, when $D > 1$, we will not only have the normal gradients of $K_{ab}{}^i$ to contend with in the source term, but also the connection $S_{ab}{}^i$.

We need therefore to find appropriate linear combinations of Eq. (3.1) such that the troublesome source term is eliminated modulo the equations of motion of the membrane. This is unfortunate since our ideal is a description of deformations of the world sheet which is independent of the dynamics, in a way analogous to the perturbative analysis of [7]. The presence of the source term does not leave us any choice, but to exploit the equations of motion to eliminate it. We focus on the case of a membrane satisfying the Nambu dynamics, i.e., extremal membranes. The generalization to membranes described by more complicated dynamical rules will be briefly sketched in the conclusions.

V. RAYCHAUDHURI EQUATIONS FOR EXTREMAL MEMBRANES

In this section, we consider membranes which satisfy the Nambu dynamics. The Nambu action is proportional to the world-sheet area

$$S[X^\mu, X^\mu{}_{,a}] = -\sigma \int_m d^D\xi \sqrt{-\gamma} \ , \tag{5.1}$$

where the constant σ is the membrane tension. The equations of motion are given by [2]

$$K^i \equiv \gamma^{ab} K_{ab}{}^i = 0 \ . \tag{5.2}$$

We want to eliminate the term in Eqs. (3.1) involving normal derivatives of $K_{ab}{}^i$, using Eqs. (5.2). The obvious thing to do is to consider the linear combination of Eqs. (3.1) obtained by tracing with γ^{ab} over world-sheet indices,

$$\tilde{\nabla}_a J^{aij} + J_a{}^i{}_k J^{akj} + K_{ac}{}^i K^{acj} = g(R(E_a, n^i)E^a, n^j) \ , \tag{5.3}$$

where we have used Eq. (5.2) to eliminate the source term, which now takes the form $\tilde{\nabla}^i K^j$.

When D=1, this equation reduces to the familiar Raychaudhuri equations Eq. (3.2). When $D > 1$, Eqs. (5.3) assume the form of a system of coupled continuity-type equations involving the time derivative of $J_0{}^{ij}$ (a gradient along the timelike tangent vector, E_0), and world-sheet spatial gradients of the remaining $J_a{}^{ij}$.

Now, to evolve initial data, we need one equation to evolve each variable $J_a{}^{ij}$. The traced equation (5.3) does this for $J_0{}^{ij}$. To complete the description of the evolution, we need corresponding evolution equations for the remaining $J_A{}^{ij}$, where capital Latin letters denote world-sheet *spatial* indices $(A, B, \ldots = 1, 2, \ldots, D-1)$.

An important observation towards this goal is the recognition that the source term in Eqs. (3.1) is symmetric in the world sheet indices a and b. If we now anti-symmetrize Eq. (3.1) with respect to its world sheet indices, we get

$$\tilde{\nabla}_a J_b{}^{ij} - \tilde{\nabla}_b J_a{}^{ij} = G_{ab}{}^{ij} \ , \tag{5.4}$$

where we have defined

$$G_{ab}{}^{ij} \equiv -J_a{}^i{}_k J_b{}^{kj} - K_{ac}{}^i K_b{}^{cj} + g(R(E_a, n^i)E_b, n^j) - (a \leftrightarrow b) \ . \qquad (5.5)$$

The source term has been canceled out independent of the background dynamics. We note that when D=1, Eqs. (5.4) are vacuous. When $D = N-1$, for a hypersurface, $G_{ab}{}^{ij}$ and the extrinsic twist potential $\omega_a{}^{ij}$ vanish identically, and Eq. (5.4) reads

$$\partial_a J_b - \partial_b J_a = 0 \ , \qquad (5.4')$$

with $J_a \equiv J_a{}^{11}$. J_a is rotationless.

The right-hand side of Eq. (5.4) can be put in a simpler form. Using the Ricci integrability conditions, Eq. (2.11c), the antisymmetric sum of quadratics in the extrinsic curvature can be identified with $\Omega_{ab}{}^{ij}$, modulo a projection of the spacetime Riemann tensor. This gives

$$G_{ab}{}^{ij} = -J_a{}^i{}_k J_b{}^{kj} + J_b{}^i{}_k J_a{}^{kj} - \Omega_{ab}{}^{ij} \ , \qquad (5.6)$$

where the spacetime Riemann tensor projections vanish because of the spacetime cyclic Bianchi identities, $R_{\mu[\nu\rho\sigma]}$=0.

At this point, to facilitate our counting of the evolution equations, let us separate the timelike component of Eq. (5.3), from the spatial components:

$$\tilde\nabla_0 J_B{}^{ij} - \tilde\nabla_B J_0{}^{ij} = G_{0B}{}^{ij} \ , \qquad (5.7a)$$

$$\tilde\nabla_B J_A{}^{ij} - \tilde\nabla_B J_A{}^{ij} = G_{BA}{}^{ij} \ . \qquad (5.7b)$$

For each pair (ij), Eq. (5.7a) give the required $D-1$ evolution equations for the $J_A{}^{ij}$. With the trace equations (5.3), they provide the desired number of equations of evolution.

When $D > 1$, for each pair (ij), Eq. (5.7b) are $(D-1)(D-2)/2$ equations, that involve only spatial gradients, and, as such, do not evolve initial data. If these equations are to be interpreted as constraints on the initial data, it is essential that they be preserved by the evolution. In the worst possible scenario, this could require us to impose integrability conditions on the solution, somewhat analogous to the second-class constraints in Dirac's classification of constrained dynamical systems [12], thereby further complicating the solution of the Cauchy problem. What is remarkable is that the integrability conditions are, in fact, trivially satisfied. Let us examine how this happens.

Formally, Eqs. (5.3) and (5.4) represent an overdetermined system. Ignoring the initial value interpretation of this system for the moment, a solution will exist if and only if the following integrability conditions are satisfied:

$$2\tilde\nabla_{[a}\tilde\nabla_b J_{c]}{}^{ij} = \tilde\nabla_{[a} G_{bc]}{}^{ij} \ . \qquad (5.8)$$

When D=2, i.e., for a string, these integrability conditions are vacuus, and Eqs. (5.3) and (5.4) therefore form a consistent system of partial differential equations for $J_a{}^{ij}$.

When $D > 2$, these integrability conditions are identically satisfied. To see this, note that the left-hand side gives

$$2\tilde\nabla_{[a}\tilde\nabla_b J_{c]}{}^{ij} = \Omega_{[ab}{}^{ik} J_{c]k}{}^j + \Omega_{[ab}{}^{jk} J_{c]}{}^i{}_k \ ,$$

and that the right-hand side gives the same:

$$\begin{aligned}\tilde\nabla_{[a} G_{bc]}{}^{ij} &= -\tilde\nabla_{[a}\Omega_{cb]}{}^{ij} - 2(\tilde\nabla_{[a} J_b{}^{ik}) J_{c]k}{}^j - 2J_{[b}{}^{ik}(\tilde\nabla_a J_{c]k}{}^j) \\ &= -G_{[ab}{}^{ik} J_{c]k}{}^j - J_{[b}{}^{ik} G_{ac]k}{}^j \\ &= \Omega_{[ba}{}^{ik} J_{c]k}{}^j - J_{[b}{}^{ik}\Omega_{ca]k}{}^j \\ &= \Omega_{[ab}{}^{ik} J_{c]k}{}^j + \Omega_{[ab}{}^{jk} J_{c]}{}^i{}_k \ .\end{aligned}$$

We have used the Bianchi differential identity Eq. (2.12) in the first line, and in the third line a combination cubic in $J_a{}^{ij}$ vanishes identically.

The integrability conditions that might have obstructed the implementation of the Cauchy problem are then trivial. It is therefore true that if the constraints are satisfied on an initial spacelike hypersurface, and the equations of evolution are satisfied, then the constraints will continue to be satisfied at all subsequent times. In Appendix B, we illustrate this, using a simple example in which the intrinsic geometry of the world sheet is flat, and the extrinsic twist potential vanishes.

To summarize, for an extremal membrane, the generalization of the Raychaudhuri equations for a curve is given by Eqs. (5.3) and (5.4). These equations describe the evolution of the deformation of the world sheet, $J_a{}^{ij}$.

The quantity $J_a{}^{ij}$ is difficult to work with. By analogy with continuum mechanics, we can decompose $J_a{}^{ij}$ into its symmetric and antisymmetric parts with respect to the normal indices $\Theta_a{}^{ij}$ and $\Lambda_a{}^{ij}$, respectively:

$$J_a{}^{ij} = \Theta_a{}^{ij} + \Lambda_a{}^{ij} \ . \qquad (5.9)$$

We further decompose $\Theta_a{}^{ij}$ into its trace-free and trace parts:

$$\Theta_a{}^{ij} = \Sigma_a{}^{ij} + \frac{1}{N-D}\delta^{ij}\Theta_a \ . \qquad (5.10)$$

In one dimension Θ, Σ^{ij}, and Λ^{ij} describe, respectively, the expansion, the shear, and the vorticity of a trajectory with respect to neighboring trajectories. No such clear interpretation appears to be available in higher dimensions.

We consider now the trace, trace-free, and antisymmetric parts of the Raychaudhuri equations for an extremal membrane, Eqs. (5.3) and (5.4), in order to obtain "equations of motion" for Θ_a, $\Theta_a{}^{ij}$, and $\Lambda_a{}^{ij}$.

From the traced Raychaudhuri equation, Eq. (5.3), one finds

$$\tilde\nabla_a\Lambda^{aij} + \Lambda^{ak[i}\Lambda_a{}^{j]}{}_k + \Sigma^{ak[i}\Sigma_a{}^{j]}{}_k - 2\Lambda^{ak[i}\Sigma_a{}^{j]}{}_k + \frac{2}{N-D}\Lambda_a{}^{ij}\Theta^a = 0 \ , \qquad (5.11a)$$

$$\nabla_a\Theta^a - \Lambda^{aij}\Lambda_{aij} + \Sigma^{aij}\Sigma_{aij} + \frac{1}{N-D}\Theta_a\Theta^a - (M^2)^i{}_i = 0 \;, \quad (5.11b)$$

$$\tilde\nabla_a\Sigma^{aij} + (\Lambda^{aik}\Lambda_{ak}{}^j + \Sigma^{aik}\Sigma_{ak}{}^j)^{\rm str} + \frac{2}{N-D}\Sigma_a{}^{ij}\Theta^a - [(M^2)^{ij}]^{\rm str} = 0 \;, \quad (5.11c)$$

where the symbol $(\cdots)^{\rm str}$ denotes the symmetric traceless part of the matrix under parenthesis, and we have defined the "effective mass" matrix

$$(M^2)^{ij} = -K_{ab}{}^i K^{abj} g(R(E_a, n^i)E^a, n^j) = (M^2)^{ji} \;. \quad (5.12)$$

This terminology is borrowed from the perturbative analysis of [5–7], where it appears as a variable mass in the world-sheet wave equation that describes the evolution of small perturbations.

The antisymmetric Raychaudhuri equations, Eq. (5.4), give

$$2\tilde\nabla_{[a}\Lambda_{b]}{}^{ij} = -2\Lambda_{[a}{}^{k[i}\Lambda_{b]}{}^{j]}{}_k - 2\Sigma_{[a}{}^{k[i}\Sigma_{b]}{}^{j]}{}_k - \Omega_{ab}{}^{ij} \;, \quad (5.13a)$$

$$2\partial_{[a}\Theta_{b]} = 0 \;, \quad (5.13b)$$

$$2\tilde\nabla_{[a}\Sigma_{b]}{}^{ij} = -2(\Lambda_{[a}{}^{ik}\Lambda_{b]k}{}^j + \Sigma_{[a}{}^{ik}\Sigma_{b]k}{}^j)^{\rm str} + 4\Lambda_{[a}{}^{k(i}\Lambda_{b]k}{}^{j)} \;. \quad (5.13c)$$

First of all, note that if we set $\Lambda_a{}^{ij}, \Sigma_a{}^{ij}$ equal to zero initially, $\Lambda_a{}^{ij}$ will continue to vanish only if the curvature of the extrinsic twist, $\Omega_{ab}{}^{ij}$, vanishes. The generalized shear $\Sigma_a{}^{ij}$ is picked up if the matrix $(M^2)^{ij}$ has a nonvanishing traceless part.

Let us now focus our attention on the equations which describe the evolution of the generalized expansion, Θ_a. Equation (5.13b) implies that

$$\Theta_a = \partial_a\Upsilon \;, \quad (5.14)$$

at least locally, for some potential function Υ. [However, recall that Eqs. (5.4),and thus Eqs. (5.13), are vacuus for a curve.] Inserting this expansion in Eq. (5.11b), we find

$$\Delta\Upsilon + \frac{1}{N-D}\partial_a\Upsilon\partial^a\Upsilon - \Lambda^2 + \Sigma^2 - M^2 = 0 \;, \quad (5.15)$$

where we have defined the world-sheet scalar quantities $\Lambda^2 \equiv \Lambda^{aij}\Lambda_{aij}$, $\Sigma^2 \equiv \Sigma^{aij}\Sigma_{aij}$, and $M^2 \equiv (M^2)^i{}_i$. This equation describes the evolution of the expansion of the world sheet. It is a quasilinear, second-order hyperbolic partial differential equation. Note the nonlinear term which depends quadratically in world-sheet derivatives of Υ. Neither this term nor the other world-sheet scalars which follow it have a positive sign, since the world-sheet metric has indefinite signature.

It is useful to compare Eq. (5.11b) for Θ_a [and its development through Eq. (5.14) to Eq. (5.15)] with its one-dimensional analogue. For a geodesic curve with tangent vector V, the Raychaudhuri equation describing the evolution of the expansion Θ is given by [8]

$$\dot\Theta + \tfrac{1}{3}\Theta^2 + \Sigma^2 - \Lambda^2 + R_{\mu\nu}V^\mu V^\nu = 0 \;. \quad (5.16)$$

This is a first-order ordinary differential equation. It can always be converted into a second-order equation by simply redefining $\Theta = \dot\Upsilon$. Note, however, that the nonlinear term in Θ in Eq. (5.16) is positive definite. In general relativity, when the vorticity is set to zero, and the weak energy condition holds, $R_{\mu\nu}V^\mu V^\nu \geq 0$, one comes to the conclusion that $\dot\Theta$ is negative, i.e., geodesics focus, and diverges within a finite proper length [8]. It would be nice to be able to apply a similar argument to Eq. (5.15), but it is not obvious to us how to do this.

We can consider Υ as a generalized relative volume expansion potential. If l represents the characteristic length of the expansion, we can set

$$\Upsilon = (N-D)\ln l \;. \quad (5.17)$$

With this elementary change of variables, Eq. (5.15) translates into the linear equation

$$\Delta l + \frac{1}{N-D}[\Sigma^2 - \Lambda^2 + M^2]l = 0 \;. \quad (5.18)$$

This is a wave equation on the world sheet for a massive *positive definite* scalar field l with an effective mass term, $\mu^2 = [\Sigma^2 - \Lambda^2 + M^2]/(N-D)$. Superficially, we have reduced the analysis of Θ_a to the solution of a linear wave equation. However, we need to remember that μ^2 involves Σ^{ij}_a and Λ^{ij}_a explicitly, as a result, depends implicitly also on Θ_a. We note that just as M^2 does not have a definite sign neither does μ^2.

Note that for a geodesic curve, with $\Theta = 3(d/ds)(\ln l)$, the geodesic Raychaudhuri equation Eq. (5.16) reduces to

$$\ddot l + \tfrac{1}{3}(\Sigma^2 - \Lambda^2 + R_{\mu\nu}V^\mu V^\nu)l = 0 \;. \quad (5.19)$$

In general relativity, using the same argument sketched above, one finds that $\ddot l$ is negative, implying that l cannot have a local minimum.

To conclude this section, we note that, for a hypersurface, $D = N-1$, there is only one normal vector so that $J_{aij} = J_{a11} \equiv \Theta_a$. The only degree of freedom describing the deformation of the hypersurface is therefore the breathing mode or dilation of the world sheet. The extrinsic twist potential $\omega_a{}^{11}$ vanishes identically, by antisymmetry, and the normal frame rotation coefficients vanish as well. Is it also possible to orient the tangent vectors along the normal direction so that $S_{ab}{}^1 = 0$, in a way analogous to a curve.

Therefore, the Raychaudhuri equations for this special case reduce to Eq. (5.14), and

$$\Delta\Upsilon + \partial_a\Upsilon\partial^a\Upsilon - M^2 = 0 \;. \quad (5.20)$$

The corresponding linear equation (5.18) is genuinely linear.

VI. JACOBI EQUATIONS

In this section, we show how the nonperturbative generalized Raychaudhuri equations describing arbitrary deformations of extremal membranes can be linearized to reproduce the perturbative Jacobi equations describing small deformations derived in [5–7].

We demonstrate that, for an extremal membrane, the traced Raychaudhuri equation, Eq. (5.3), alone completely encodes perturbation theory. In fact, it is a simple matter to obtain the Jacobi equations derived in [5,7] directly from Eq. (5.3). The antisymmetric Raychaudhuri equations, Eqs. (5.4), turn out to have a vacuous perturbative limit.

In [7], the infinitesimal deformations are described by a multiplet of scalar fields, Φ^i, living on the world sheet. The perturbations are characterized by the normal projections of the infinitesimal perturbations of the world-sheet basis:

$$J_a{}^i \equiv \Phi^j g(D_j E_a, n^i) \ . \tag{6.1}$$

For infinitesimal perturbations, it was found that

$$J_a{}^i = \tilde{\nabla}_a \Phi^i \ . \tag{6.2}$$

Therefore, the reduction of the generalized Raychaudhuri equations to their perturbative limit consists simply in the case of the relation

$$J_a{}^{ji}\Phi_j = J_a{}^i = \tilde{\nabla}_a \Phi^i \ . \tag{6.3}$$

Let us consider now the (infinitesimal) normal linear combination of Eq. (5.3) obtained by contracting with the scalar fields Φ_j:

$$[\tilde{\nabla}_a J^{aji} + J_a{}^j{}_k J^{aki} - (M^2)^{ji}]\Phi_j = 0 \ , \tag{6.4}$$

where we use the effective mass $(M^2)^{ij}$ defined in Eq. (5.12).

This expression may be rewritten as

$$\tilde{\nabla}_a(J^{aji}\Phi_j) - J^{aji}(\tilde{\nabla}_a\Phi_j) + J_a{}^j{}_k J^{aki}\Phi_j - (M^2)^{ji}\Phi_j = 0 \ .$$

Using Eq. (7.3), the second and third terms cancel, and one recovers Eq. (5.3) of [7]:

$$\tilde{\nabla}^a\tilde{\nabla}_a\Phi^i - (M^2)^i{}_j\Phi^j = 0 \ . \tag{6.5}$$

We show now that the antisymmetric combination (5.4) is vacuous. To see this, note that with the use of relation (6.3), one has

$$[\tilde{\nabla}_b J_a{}^{ij} - \tilde{\nabla}_a J_b{}^{ij} + J_b{}^i{}_k J_a{}^{kj} - J_a{}^i{}_k J_b{}^{kj}]\Phi_i = \Omega_{ab}{}^{ij}\Phi_i \ . \tag{6.6}$$

However, the left-hand side of this equation gives

$$\tilde{\nabla}_b(J_a{}^{ij}\Phi_i) - J_a{}^{ij}(\tilde{\nabla}_b\Phi_i) - \tilde{\nabla}_a(J_a{}^{ij}\Phi_i) + J_a{}^{ij}(\tilde{\nabla}_b\Phi_i) + J_b{}^i{}_k J_a{}^{kj}\Phi_i - J_a{}^i{}_k J_b{}^{kj}\Phi_i = 2\tilde{\nabla}_{[b}\tilde{\nabla}_{a]}\Phi^j = \Omega_{ba}{}^{ji}\Phi_i = \Omega_{ab}{}^{ij}\Phi_i \ ,$$

which is identical to the right hand side of Eq. (6.6).

VII. CONCLUSIONS

We have provided a nonperturbative framework involving a coupled system of nonlinear partial differential equations to examine the evolution of deformations of relativistic membranes of an arbitrary dimension propagating in a background spacetime of arbitrary codimension. The construction of this system of equations was motivated by the Raychaudhuri equations describing point particles to which they reduce when D=1. Despite the complexity of these equations, they do share many features of the one-dimensional prototype. Clearly, however, work remains to be done to sharpen our understanding of this system. Work is in progress on the examination of the dynamics of large deformations about various simple symmetrical configurations [11,13].

To conclude, we comment briefly on the derivation of the appropriate generalizations of Raychaudhuri equations when the dynamics is not of the Nambu type. Recall that to yield a useful system of equations from the simple-minded generalization of the Raychaudhuri equations, Eq. (3.1), we need to eliminate the source term $\tilde{\nabla}_i K_{ab}{}^j$ in terms of quantities determined on the world sheet by the equations of motion. For the case of an extremal membrane, this was achieved by forming appropriate linear combinations of Eq. (3.1), i.e., by considering its trace over the world-sheet indices, Eq. (5.3). We note, however, that the other linear combinations we considered, the antisymmetric combination Eq. (5.4), are in fact independent of the dynamics, since the source term is eliminated. Thus we continue to use these equations. For a nonextremal surface, therefore, we need to find the appropriate linear combination to replace Eq. (5.3). The generalization of the procedure we followed for extremal membranes is to consider linear combinations of Eq. (3.1) such that any source term which is not determined on the world sheet is proportional to the equations of motion. If, for example, rigidity corrections are incorporated into the action, the dynamical equations will involve two world-sheet derivatives ($\tilde{\nabla}_a$) of K^i_{ab} (four derivatives of the embedding functions) [14]. This will require us to strike Eq. (3.1) twice with $\tilde{\nabla}_a$. The source term will be removed in favor of quantities which are determined completely on the world sheet by commuting these derivatives through the normal gradient operating on K^i_{ab}. Details will be presented elsewhere [15].

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APPENDIX A

In this appendix we give the details of the derivation of the generalized Raychaudhuri equations (3.1). This involves the evaluation of the quantity $\tilde{\nabla}_b J_{aij}$. We emphasize that we are not making any assumptions about the equations of motion of the world sheet.

The definition of J_{aij} in Eq. (2.13) implies

$$\begin{aligned} D_b J_{aij} + D_b g(n_j, D_i E_a) \\ = g(D_b n_j, D_i E_a) + g(n_j, D_b D_i E_a) \ . \end{aligned} \quad \text{(A1)}$$

Using Eqs. (2.4) and (2.11), the first term gives us

$$\begin{aligned} g(D_b n_j, D_i E_a) &= g(K_{bcj} E^c + \omega_{bjk} n^k, S_{adi} E^d + J_{ail} n^l) \\ &= S_{aci} K_b{}^c{}_j + \omega_{bj}{}^k J_{aik} \ . \end{aligned} \quad \text{(A2)}$$

We now apply the Ricci identity to the last term on the second line of Eq. (A1):

$$\begin{aligned} D_b D_i E_a = D_i D_b E_a + R(E_b, n_i) E_a \\ + (D_b n_i^\mu - D_i E^\mu{}_b) D_\mu E_a \ . \end{aligned} \quad \text{(A3)}$$

We note that

$$\begin{aligned} D_i(D_b E_a) &= D_i[\gamma_{ba}{}^c E_c - K_{ba}{}^k n_k] \\ &= (D_i \gamma_{ba}{}^c) E_c + \gamma_{ba}{}^c D_i E_c - (D_i K_{ba}{}^k) n_k \\ &\quad - K^k_{ba} D_i n_k \ . \end{aligned}$$

Thus

$$g(n_j, D_i(D_b E_a)) = \gamma_{ba}{}^c J_{cij} - D_i K_{abj} - \gamma_{ikj} K_{ab}{}^k \ . \quad \text{(A4)}$$

The first term appearing on the right-hand side (RHS) of Eq. (A4) combines with the LHS of Eq. (A1) to provide a world-sheet covariant derivative of J_{aij} defined by

$$\nabla_b J_{aij} = D_b J_{aij} - \gamma_{ba}{}^c J_{cij} \ . \quad \text{(A5)}$$

The latter two terms add, to yield a normal covariant derivative acting on $K_{ab}{}^j$,

$$\nabla_i K_{ba}{}^j = D_i K_{ba}{}^j - \gamma_i{}^{jk} K_{bak} \ . \quad \text{(A6)}$$

The term which remains to be evaluated is

$$g(n_j, (D_b n^\mu{}_i - D_i E^\mu{}_b) D_\mu E_a) \ .$$

To evaluate it we insert unity to write

$$\begin{aligned} g(n_j, (D_b n^\mu{}_i - D_i E^\mu{}_b) D_\mu E_a) &= g(n_j, (D_b n_{\mu i} - D_i E_{\mu b})(E^\mu{}_c E^{\nu c} + n^\mu{}_k n^{\nu k}) D_\nu E_a) \\ &= g(n_j, (D_b n_{\mu i} - D_i E_{\mu b})(E^{\mu c} D_c E_a + n^{\mu k} D_k E_a)) \\ &= \omega_{bi}{}^k J_{akj} - K_{bc}{}^i K_a{}^{cj} + S_{bc}{}^i K_a{}^{cj} - J_b{}^{ik} J_{ak}{}^j \ . \end{aligned} \quad \text{(A7)}$$

Summing terms, we obtain the sought for expression in the form

$$\begin{aligned} \tilde{\nabla}_b J_{aij} = \tilde{\nabla}_i K_{abj} - J_{bi}{}^k J_{akj} - K_{bc}{}^i K_a{}^{cj} \\ + g(R(E_b, n_i) E_a, n_j) \ , \end{aligned} \quad \text{(A8)}$$

where we have exploited the definitions of $\tilde{\nabla}_b$ and $\tilde{\nabla}_i$, Eqs. (2.9) and (2.17), respectively, to write

$$\tilde{\nabla}_b J_{aij} = \nabla_b J_{aij} - \omega_{bj}{}^k J_{aki} - \omega_{bi}{}^k J_{akj} \quad \text{(A9)}$$

and

$$\tilde{\nabla}_i K_{ab}{}^j = \nabla_i K^j_{ab} - K_b{}^{cj} S_{aci} - K_a{}^{cj} S_{bci} \ . \quad \text{(A10)}$$

APPENDIX B

In this appendix, we illustrate how the integrability conditions Eq. (5.7) imply that the spatial constraints (5.6b) are preserved by the evolution. We use a simpler analogous system of equations, assuming that the world-sheet geometry is flat, and that the extrinsic twist potential, $\omega_a{}^{ij}$, vanishes. Under these assumptions, Eqs. (5.4) simplify to

$$\partial_a J_b - \partial_b J_a = G_{ab}(J_c) \ . \quad \text{(B1)}$$

These equations fall into two categories. The first consists of dynamical equations for the world sheet spatial components J_A:

$$\partial_0 J_A - \partial_A J_0 = G_{0A} \ . \quad \text{(B2)}$$

The latter is a set of constraints on these variables:

$$\mathcal{C}_{AB} \equiv \partial_A J_B - \partial_B J_A - G_{AB} = 0 \ . \qquad \text{(B3)}$$

The integrability conditions for (B1) can be written as

$$\partial_c G_{ab} + \partial_a G_{bc} + \partial_b G_{ca} = 0 \ . \qquad \text{(B4)}$$

These equations are vacuous if two or more of the indices are equal. To propagate the spatial constraint, we require only the equation

$$\partial_0 G_{AB} + \partial_A G_{B0} - \partial_B G_{A0} = 0 \ . \qquad \text{(B5)}$$

For now,

$$\partial_0 \mathcal{C}_{AB} = \partial_A(\partial_0 J_B) - \partial_B(\partial_0 J_A) - \partial_0 G_{AB} \ , \qquad \text{(B6)}$$

the right-hand side of which is zero modulo Eqs. (B2) and (B4). Thus $\mathcal{C}_{AB}=0$ is preserved by the evolution.

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