

Mass spectrum of strings in anti-de Sitter spacetime

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We perform string quantization in anti-de Sitter (AdS) spacetime. The string motion is stable, oscillatory in time with real frequencies $\omega_n = \sqrt{n^2 + m^2 \alpha'^2 H^2}$, and the string size and energy are bounded. The string fluctuations around the center of mass are well behaved. We find the mass formula which is also well behaved in all regimes. There is an *infinite* number of states with arbitrarily high mass in AdS spacetime [in de Sitter (dS) spacetime there is a *finite* number of states only]. The critical dimension at which the graviton appears is $D = 25$, as in de Sitter space. A cosmological constant $\Lambda \neq 0$ (whatever its sign) introduces a fine structure effect (splitting of levels) in the mass spectrum at all states beyond the graviton. The high mass spectrum changes drastically with respect to flat Minkowski spacetime. For $\Lambda < 0$, we find $\langle m^2 \rangle \sim |\Lambda| N^2$, independent of α' , and the level spacing *grows* with the eigenvalue of the number operator N . The density of states $\rho(m)$ grows like $\exp[(m/\sqrt{|\Lambda|})^{1/2}]$ [instead of $\rho(m) \sim \exp(m\sqrt{\alpha'})$ as in Minkowski space], thus *discarding* the existence of a critical string temperature. For the sake of completeness, we also study the quantum strings in the black string background, where strings behave, in many respects, as in ordinary black hole backgrounds. The mass spectrum is equal to the mass spectrum in flat Minkowski space.

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I. INTRODUCTION AND RESULTS

String dynamics in curved spacetime and its associated physical phenomena started to be systematically studied in Refs. [1,2]. Since then, this subject has received systematic and increasing attention. String propagation in curved spacetime reveals new insight and new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetime). The results of this program are relevant both for fundamental (quantum) strings and for cosmic strings, which behave essentially in a classical way. Approximate [1–5] and exact [5–10] string methods have been developed. Classical and quantum string dynamics have been investigated in black hole backgrounds [5, 11–14], cosmological spacetimes [15], cosmic string spacetimes [16], gravitational wave backgrounds [17], supergravity backgrounds [18] (which are necessary for non-trivial propagation of fermionic strings), and near spacetime singularities [19]. Physical phenomena such as the Hawking-Unruh effect in string theory [2,20], horizon string stretching [2,20], particle transmutation [11,21], string scattering [11,16], mass spectrum and critical dimension [1,11,16], string instabilities with nonoscillatory motion in time [1,5–9,15], and multistring solutions [7–9] have been found. The physically rich dynamics of strings in $D > 2$ curved spacetimes is lost in $D = 2$, in which most of the interesting stringy effects referred to above disappear. String propagation in the $D = 2$ stringy black hole [22] reduces to the quantization of a massless scalar particle (the dilaton) with a new peculiar effect: the quantum renormalization of the speed of light [23], which appears restricted to strings in two-dimensional

curved spacetimes.

Recently [5], classical string dynamics was studied in the 2+1 black hole anti-de Sitter spacetime as well as in its dual, the black string background, within the string perturbation series approach of de Vega and Sanchez [1]. First- and second-order string fluctuations around the center of mass were obtained. A comparison was made to the ordinary ($D \geq 4$) black hole backgrounds (with cosmological constant). The circular string motion was exactly and completely solved (in terms of either elementary or elliptic functions) in all these backgrounds.

In the present paper we go further in the investigation of the physical properties of strings in these backgrounds, by performing string quantization. Since the 2+1 black hole spacetime is asymptotically anti-de Sitter (AdS) spacetime, the results of Ref. [5] can be extended to classical strings in D -dimensional anti-de Sitter spacetime as well. For the sake of completeness, we also investigate string quantization in the black string background, for which, in many respects, strings behave as in the ordinary black hole backgrounds. In AdS spacetime, the string motion is oscillatory in time and is *stable*; all fluctuations around the string center of mass are well behaved and bounded. Local gravity of AdS spacetime is always negative and string instabilities do not develop. The string perturbation series approach, considering fluctuations around the center of mass, is particularly appropriate in AdS spacetime, the natural dimensionless expansion parameter being $\lambda = \alpha'/l^2 > 0$, where α' is the string tension and the “Hubble constant” $H = 1/l$. The negative cosmological constant of AdS spacetime is related to the Hubble constant H by $\Lambda = -(D-1)(D-2)H^2/2$.

All the spatial ($\mu = 1, \dots, D-1$) modes in D -

dimensional AdS spacetime oscillate with frequency $\omega_n = \sqrt{n^2 + m^2 \alpha'^2 H^2} = \sqrt{n^2 + m^2 \alpha' \lambda}$, which are real for all n (m being the string mass). In this paper, we perform a canonical quantization procedure. From the conformal generators L_n , \tilde{L}_n and the constraints $L_0 = \tilde{L}_0 = 0$, imposed at the quantum level, we obtain the mass formula

$$m^2 \alpha' = (D-1) \sum_{n>0} \Omega_n(\lambda) + \sum_{n>0} \Omega_n(\lambda) \sum_{R=1}^{D-1} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R], \quad (1.1)$$

where

$$\Omega_n(\lambda) = \frac{2n^2 + m^2 \alpha' \lambda}{\sqrt{n^2 + m^2 \alpha' \lambda}}, \quad (1.2)$$

and we have applied symmetric ordering of the operators. The operators α_n^R , $\tilde{\alpha}_n^R$ satisfy

$$[\alpha_n^R, (\alpha_n^R)^\dagger] = [\tilde{\alpha}_n^R, (\tilde{\alpha}_n^R)^\dagger] = 1, \quad \text{for all } n \geq 0, \quad (1.3)$$

and we have eliminated the zero modes. To the first term in the mass formula (the zero point energy) we apply ζ -function regularization; see Eqs. (3.14), (3.15). For $\lambda \ll 1$, which is clearly satisfied in most interesting cases, we find the lower mass states $m^2 \alpha' \lambda \ll 1$ and the quantum mass spectrum. Physical states are characterized by the eigenvalue of the number operator

$$N = \frac{1}{2} \sum_{n>0} n \sum_{R=1}^{D-1} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R], \quad (1.4)$$

and the ground state is defined by

$$\alpha_n^R | 0 \rangle = \tilde{\alpha}_n^R | 0 \rangle = 0 \quad \text{for all } n > 0. \quad (1.5)$$

We find that $m^2 \alpha' = 0$ is an *exact* solution of the mass formula in $D = 25$ and that there is a graviton at $D = 25$, which indicates, as in de Sitter space [1], that the critical dimension of AdS space is 25 (although it should be stressed that the question whether de Sitter space is a solution to the full β -function equations remains open). The ground state is a tachyon; its mass is given by Eq. (3.18). Remarkably enough, for $N \geq 2$ we find that a generic feature of all excited states beyond the graviton is the presence of a *fine structure* effect: For a given eigenvalue $N \geq 2$, the corresponding states have different masses. For the lower mass states the expectation value of the mass operator in the corresponding states (generically labeled $| j \rangle$) turns out to have the form [Eq. (3.32)]

$$\langle j | m^2 \alpha' | j \rangle_{\text{AdS}} = a_j + b_j \lambda^2 + c_j \lambda^3 + \mathcal{O}(\lambda^4). \quad (1.6)$$

The collective index j generically labels the state $| j \rangle$ and the coefficients a_j , b_j , c_j are all well-computed numbers, different for each state. The corrections to the mass in Minkowski spacetime appear to order λ^2 . Therefore, the leading Regge trajectory for the lower mass states is

$$J = 2 + \frac{1}{2} m^2 \alpha' + \mathcal{O}(\lambda^2). \quad (1.7)$$

In Minkowski spacetime the mass and number operators of the string are related by $m^2 \alpha' = -4 + 4N$. In AdS [as well as in de Sitter (dS)], spacetime there is no such simple relation between the mass and the number operators; the splitting of levels increases considerably for very large N . The *fine structure* effect we find here is also present in de Sitter space. Up to order λ^2 , the lower mass states in dS and AdS space are the same; the differences appear to order λ^3 . The lower mass states in de Sitter spacetime are given by Eq. (1.6) but with the λ^3 term getting an opposite sign ($-c_j \lambda^3$).

For the very high mass spectrum, we find more drastic effects. States with very large eigenvalue N , namely, $N \gg 1/\lambda$, have masses [Eq. (3.33)]

$$\langle j | m^2 \alpha' | j \rangle_{\text{AdS}} \approx d_j \lambda N^2 \quad (1.8)$$

and angular momentum

$$J^2 \approx \frac{1}{\lambda} m^2 \alpha', \quad (1.9)$$

where d_j are well-computed numbers different for each state. Since $\lambda = \alpha'/l^2$, we see from Eq. (1.8) that the masses of the high mass states are *independent* of α' . In Minkowski spacetime, very large N states all have the same mass $m^2 \alpha' \approx 4N$, but here in AdS spacetime the masses of the high mass states with the same eigenvalue N are *different* by factors d_j . In addition, because of fine structure effect, states with different N can get mixed up. For high mass states, the level spacing *grows* with N (instead of being constant as in Minkowski spacetime). As a consequence, the density of states $\rho(m)$ as a function of mass grows like $\exp[(m/\sqrt{|\Lambda|})^{1/2}]$ [instead of $\exp(m\sqrt{\alpha'})$ as in Minkowski spacetime], and *independently* of α' . The partition function for a gas of strings at a temperature β^{-1} in AdS spacetime is well defined for all finite temperatures β^{-1} , discarding the existence of the Hagedorn temperature and the possibility of a phase transition (as occurs in Minkowski spacetime and in other curved spacetimes).

In the black string background, we calculate explicitly the first- and second-order string fluctuations around the center of mass. We then determine the world-sheet energy-momentum tensor and we derive the mass formula in the asymptotic region. The mass spectrum is equal to the mass spectrum in flat Minkowski spacetime. Therefore, for a gas of strings at temperature β^{-1} in the asymptotic region of the black string background, the partition function can only be defined for $\beta > \sqrt{\alpha'}$; i.e., there is a Hagedorn temperature, Eqs. (4.51), (4.52).

This paper is organized as follows: In Sec. II we review the classical string propagation in the 2 + 1 black hole anti-de Sitter spacetime [5] and we generalize the results to ordinary D -dimensional anti-de Sitter spacetime. In Sec. III we perform a canonical quantization of strings in D -dimensional anti-de Sitter spacetime and we analyze the string spectrum. In Sec. IV we consider classical and quantum string propagation in the black string

background. A summary of our results and conclusions is presented in Sec. V and in Table I.

II. CLASSICAL STRING PROPAGATION

The classical propagation of strings in the $(2+1)$ -dimensional black hole anti-de Sitter spacetime was considered in detail in Ref. [5] using both approximative and exact methods. In this section we give a short review of the results of the perturbation series approach and we give the generalization to ordinary D -dimensional anti-de Sitter space.

The $(2+1)$ -dimensional black hole anti-de Sitter spacetime, recently found by Banados, Teitelboim, and Zanelli [24], is a two-parameter family (mass M and angular momentum J) of solutions to the Einstein equations with a cosmological constant $\Lambda = -1/l^2$:

$$ds^2 = \left(M - \frac{r^2}{l^2} \right) dt^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2 - J dt d\phi + r^2 d\phi^2. \quad (2.1)$$

The solution can be obtained by a discrete identification of points in anti-de Sitter space [25] and thus has the local geometry of anti-de Sitter space. Globally, however, it is very different. It has two horizons r_{\pm} and a static limit r_{erg} defining an ergosphere:

$$r_{\pm} = \sqrt{\frac{Ml^2}{2} \pm \frac{l}{2} \sqrt{M^2 l^2 - J^2}}, \quad r_{\text{erg}} = \sqrt{M}l. \quad (2.2)$$

Ordinary D -dimensional anti-de Sitter space can formally be obtained by taking $M = -1$ and $J = 0$ in the line element (2.1), and by adding $D - 3$ extra angular coordinates. In that case, of course, there are no horizons and no static limit.

The string equations of motion and constraints, in the conformal gauge, take the form

$$\ddot{x}^{\mu} - x''^{\mu} + \Gamma_{\rho\sigma}^{\mu}(\dot{x}^{\rho}\dot{x}^{\sigma} - x'^{\rho}x'^{\sigma}) = 0, \quad (2.3)$$

$$g_{\mu\nu}\dot{x}^{\mu}x'^{\nu} = g_{\mu\nu}(\dot{x}^{\mu}\dot{x}^{\nu} + x'^{\mu}x'^{\nu}) = 0, \quad (2.4)$$

where the overdot and prime stand for derivatives with respect to τ and σ , respectively. In the string perturbation series approach [1], solutions to this set of equations are obtained by expanding around an exact solution, typically taken as the string center of mass:

$$x^{\mu}(\tau, \sigma) = q^{\mu}(\tau) + \eta^{\mu}(\tau, \sigma) + \xi^{\mu}(\tau, \sigma) + \dots \quad (2.5)$$

where

$$\ddot{q}^{\mu} + \Gamma_{\rho\sigma}^{\mu}(q)\dot{q}^{\rho}\dot{q}^{\sigma} = 0, \quad g_{\mu\nu}(q)\dot{q}^{\mu}\dot{q}^{\nu} = -m^2\alpha'^2. \quad (2.6)$$

For the $(2+1)$ -dimensional black hole anti-de Sitter spacetime these equations can be easily separated and integrated in terms of elementary functions [5]. Concerning the first-order perturbations $\eta^{\mu}(\tau, \sigma)$ it is convenient to consider from the beginning only those perturbations

transverse to the geodesic of the center of mass [5]. We therefore introduce $D - 1$ normal vectors n_R^{μ} ,

$$g_{\mu\nu}(q)n_R^{\mu}\dot{q}^{\nu} = 0, \quad g_{\mu\nu}(q)n_R^{\mu}n_S^{\nu} = \delta_{RS}, \quad (2.7)$$

and consider first order perturbations in the form

$$\eta^{\mu} = \delta x^R n_R^{\mu}, \quad (2.8)$$

where δx^R are the comoving perturbations, i.e., the perturbations as seen by an observer traveling with the center of mass of the string. In the case of the $(2+1)$ -dimensional black hole anti-de Sitter spacetime it has been shown that [5]

$$\delta x^R(\tau, \sigma) = \sum_n [A_n^R e^{-i(n\sigma + \omega_n \tau)} + \tilde{A}_n^R e^{-i(n\sigma - \omega_n \tau)}], \quad (2.9)$$

where

$$\omega_n = \sqrt{n^2 + \frac{m^2\alpha'^2}{l^2}} \quad (2.10)$$

and

$$A_n^R = (\tilde{A}_{-n}^R)^{\dagger}. \quad (2.11)$$

Notice that in de Sitter spacetime [1] the perturbations satisfy Eq. (2.9) but with frequency $\omega_n = \sqrt{n^2 - m^2\alpha'^2/l^2}$; thus, unstable modes (for $|n| < m\alpha'/l$) appear and the perturbations blow up. On the contrary, in the present case the first-order perturbations are completely regular and finite trigonometric functions oscillating with real frequencies ω_n for all n .

The second-order perturbations $\xi^{\mu}(\tau, \sigma)$ are somewhat more complicated since they couple to the first-order perturbations. The explicit expressions can be found in Ref. [5], and so we shall not go into any detail here. From the center-of-mass solution and the first- and second-order perturbations we can finally calculate the worldsheet energy-momentum tensor

$$T_{\pm\pm} = g_{\mu\nu}\partial_{\pm}x^{\mu}\partial_{\pm}x^{\nu}, \quad (2.12)$$

where $\partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial_{\sigma})$, as well as the conformal generators L_n, \tilde{L}_n ,

$$T_{--} = \frac{1}{2\pi} \sum_n \tilde{L}_n e^{-in(\sigma-\tau)}, \quad (2.13)$$

$$T_{++} = \frac{1}{2\pi} \sum_n L_n e^{-in(\sigma+\tau)}.$$

For the $(2+1)$ -dimensional black hole anti-de Sitter spacetime, L_0, \tilde{L}_0 are given by [5]

$$L_0 = \pi \sum_n (\omega_n + n)^2 \sum_{R=1}^2 A_n^R \tilde{A}_{-n}^R - \frac{\pi}{2} m^2 \alpha'^2, \quad (2.14)$$

$$\tilde{L}_0 = \pi \sum_n (\omega_n - n)^2 \sum_{R=1}^2 A_n^R \tilde{A}_{-n}^R - \frac{\pi}{2} m^2 \alpha'^2. \quad (2.15)$$

The classical constraints $L_0 = \tilde{L}_0 = 0$ then provide a formula for the mass:

$$m^2 \alpha'^2 = 2 \sum_n \left(2n^2 + \frac{m^2 \alpha'^2}{l^2} \right) \sum_{R=1}^2 A_n^R \tilde{A}_{-n}^R, \quad (2.16)$$

where the constants A_n^R and \tilde{A}_{-n}^R are subject to the restriction

$$\sum_n n \omega_n \sum_{R=1}^2 A_n^R \tilde{A}_{-n}^R = 0. \quad (2.17)$$

Notice that in our normalization, Eq. (2.9), the constants A_n^R and \tilde{A}_{-n}^R have the dimension of length and that Eq. (2.17) is just the usual closed string restriction, implying that there must be an equal amount of left and right movers.

The conformal generators for $n \neq 0$ involve only the free oscillators introduced in the second-order perturbations [5] and therefore do not lead to any further restrictions on A_n^R and \tilde{A}_{-n}^R . This is in agreement with the result found in de Sitter space [1] but is very different from Minkowski space where there is still one unused constraint that in principle eliminates a pair of oscillators A_n^R and \tilde{A}_{-n}^R (cf. light-cone gauge quantization).

This concludes the discussion of the classical picture of string propagation in the $(2+1)$ -dimensional black hole anti-de Sitter spacetime using the string perturbation approach. These results can be easily generalized to D -dimensional anti-de Sitter space. The first-order perturbations (2.9)–(2.11) and the conformal generators (2.14) and (2.15) are independent of the black hole mass M , and so the results in D -dimensional anti-de Sitter space are obtained by simply changing the range of the index R from 1, 2 to 1, 2, ..., $(D-1)$: i.e.,

$$m^2 \alpha'^2 = 2 \sum_n \left(2n^2 + \frac{m^2 \alpha'^2}{l^2} \right) \sum_{R=1}^{D-1} A_n^R \tilde{A}_{-n}^R. \quad (2.18)$$

III. QUANTIZATION IN ANTI-DE SITTER SPACE

In this section we perform the canonical quantization of the closed bosonic string in D -dimensional anti-de Sitter space using the results of Sec. II. The first-order perturbations (2.9) correspond to the action

$$S^{(2)} = \frac{-1}{2\pi\alpha'} \int d\tau d\sigma \sum_{R=1}^{D-1} \left[\eta^{ab} (\delta x^R)_{,a} (\delta x^R)_{,b} + \frac{m^2 \alpha'^2}{l^2} \delta x^R \delta x^R \right]. \quad (3.1)$$

The momentum conjugate to δx^R is

$$\Pi_R \equiv \frac{\partial S^{(2)}}{\partial (\delta x^R)_{,\tau}} = \frac{1}{\pi\alpha'} (\delta x^R)_{,\tau}, \quad (3.2)$$

and then the canonical commutation relations become

$$[\delta x^R, \delta x^S] = [\Pi_R, \Pi_S] = 0,$$

$$[\Pi_R, \delta x^S] = -i \delta_R^S \delta(\sigma - \sigma'). \quad (3.3)$$

The constants A_n^R and \tilde{A}_{-n}^R introduced in Eqs. (2.9)–(2.11), which are now considered as quantum operators, do not represent independent left and right movers since the frequency ω_n is always positive. It is therefore convenient to make the redefinitions

$$A_n^R \equiv \begin{cases} \tilde{a}_n^R, & n > 0, \\ a_{-n}^R, & n < 0, \end{cases}$$

$$\tilde{A}_{-n}^R \equiv \begin{cases} a_{-n}^R, & n > 0, \\ \tilde{a}_n^R, & n < 0, \end{cases} \quad (3.4)$$

so that the first-order perturbations take the form

$$\delta x^R = \sum_{n \neq 0} [a_n^R e^{-in(\Omega_n \tau - \sigma)} + \tilde{a}_n^R e^{-in(\Omega_n \tau + \sigma)}] + A_0^R e^{-i \frac{m \alpha'}{l} \tau} + \tilde{A}_0^R e^{i \frac{m \alpha'}{l} \tau}, \quad (3.5)$$

where

$$\Omega_n = \sqrt{1 + \frac{m^2 \alpha'^2}{n^2 l^2}}, \quad a_n^R = (a_{-n}^R)^\dagger, \quad \tilde{a}_n^R = (\tilde{a}_{-n}^R)^\dagger, \quad \tilde{A}_0^R = (A_0^R)^\dagger. \quad (3.6)$$

The nonvanishing commutators now take the form

$$[a_n^R, (a_n^R)^\dagger] = [\tilde{a}_n^R, (\tilde{a}_n^R)^\dagger] = \frac{\alpha'}{2n\Omega_n}, \quad [A_0^R, (A_0^R)^\dagger] = \frac{l}{2m}, \quad (3.7)$$

and the mass formula (2.18), with symmetric order of the operators, becomes

$$m^2 \alpha'^2 = \sum_{n > 0} \left(2n^2 + \frac{m^2 \alpha'^2}{l^2} \right) \sum_{R=1}^{D-1} [(a_n^R)^\dagger a_n^R + a_n^R (a_n^R)^\dagger + (\tilde{a}_n^R)^\dagger \tilde{a}_n^R + \tilde{a}_n^R (\tilde{a}_n^R)^\dagger], \quad (3.8)$$

where we have eliminated the zero modes. We finally introduce the more conventionally normalized oscillators:

$$a_n^R = \sqrt{\frac{\alpha'}{2n\Omega_n}} \alpha_n^R, \quad \tilde{a}_n^R = \sqrt{\frac{\alpha'}{2n\Omega_n}} \tilde{\alpha}_n^R, \quad \text{for all } n > 0. \quad (3.9)$$

The nonvanishing commutators are then represented by

$$[\alpha_n^R, (\alpha_n^R)^\dagger] = [\tilde{\alpha}_n^R, (\tilde{\alpha}_n^R)^\dagger] = 1, \quad \text{for all } n > 0. \quad (3.10)$$

The classical constraints $L_0 = \tilde{L}_0 = 0$ in the quantum theory take the form

$$(L_0 - 2\pi\alpha'a) | \psi \rangle = (\tilde{L}_0 - 2\pi\alpha'a) | \psi \rangle = 0, \quad (3.11)$$

where a is the normal-ordering constant and the factor $2\pi\alpha'$ is introduced for later convenience. The normal-ordering constant is most easily obtained by symmetrization of the oscillator products as in Eq. (3.8). Using Eqs. (2.14) and (2.15), the physical state conditions (3.11), in terms of the conventionally normalized oscillators, become

$$m^2\alpha' = (D-1) \sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} + \sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} \sum_{R=1}^{D-1} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R] \quad (3.12)$$

and

$$\sum_{n>0} n \sum_{R=1}^{D-1} [(\alpha_n^R)^\dagger \alpha_n^R - (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R] = 0, \quad (3.13)$$

where we introduced the dimensionless positive parameter $\lambda = \alpha'/l^2$. The first term in the mass formula (3.12) obviously needs ζ -function regularization. Assuming $\lambda \ll 1$, which is clearly satisfied in most interesting cases, we find, for the lower mass states $m^2\alpha'\lambda \ll 1$,

$$\sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} = -\frac{1}{6} + \frac{(m^2\alpha')^2}{4} \zeta(3)\lambda^2 - \frac{(m^2\alpha')^3}{4} \zeta(5)\lambda^3 + O((m^2\alpha'\lambda)^4). \quad (3.14)$$

For the very high mass states $m^2\alpha'\lambda \gg 1$ (but still assuming $\lambda \ll 1$) we find instead

$$\sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} = \frac{m^2\alpha'\lambda}{2} - \frac{1}{2} \sqrt{m^2\alpha'\lambda} + O(1). \quad (3.15)$$

In de Sitter space the mass formula takes the form (3.12) but with λ being negative [1]. This means that there is only a *finite* number of states in de Sitter space. Beyond some maximal mass $m^2\alpha' \sim 1/\lambda$, the strings become unstable and no real solutions to the analogue of Eq. (3.12) can be found. Here in anti-de Sitter space λ is positive so that arbitrarily high mass solutions of Eq. (3.12) can be found. We therefore find *infinitely* many states in anti-de Sitter space.

Let us first consider the lower mass spectrum in a little more detail. As in Minkowski space, it is convenient to characterize the physical states by the eigenvalue of the number operator

$$N = \frac{1}{2} \sum_{n>0} n \sum_{R=1}^{D-1} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R]. \quad (3.16)$$

For $N = 0$ we have the vacuum state $|0\rangle$ defined by

$$\alpha_n^R |0\rangle = \tilde{\alpha}_n^R |0\rangle = 0 \quad \text{for all } n > 0. \quad (3.17)$$

Using Eqs. (3.12) and (3.14) we find that there is a tachyon with mass

$$\langle 0 | m^2\alpha' | 0 \rangle = (D-1) \left[-\frac{1}{6} + \frac{(D-1)^2}{144} \zeta(3)\lambda^2 - \frac{(D-1)^3}{864} \zeta(5)\lambda^3 + O(\lambda^4) \right]. \quad (3.18)$$

At the first excited level ($N = 1$) we have states of the form

$$(\tilde{\alpha}_1^R)^\dagger (\alpha_1^S)^\dagger |0\rangle \equiv |\Omega_{11}^{\tilde{R}S}\rangle. \quad (3.19)$$

For $D = 25$ it yields the graviton

$$\langle \Omega_{11}^{\tilde{R}S} | m^2 \alpha' | \Omega_{11}^{\tilde{R}S} \rangle = 0 . \tag{3.20}$$

Notice that $m^2 \alpha' = 0$ is an *exact* solution of the mass formula in $D = 25$ dimensions. As in de Sitter space [1], this indicates that the critical dimension in anti-de Sitter space is 25. It should be stressed, however, that it is not known how to obtain de Sitter space from the β -function equations. In the further analysis we take $D = 25$.

At the next excited level ($N = 2$) we have states of the form

$$\begin{aligned} (\tilde{\alpha}_1^R)^\dagger (\alpha_1^S)^\dagger (\tilde{\alpha}_1^T)^\dagger (\alpha_1^U)^\dagger | 0 \rangle &\equiv | \Omega_{1111}^{\tilde{R}S\tilde{T}U} \rangle, \\ (\tilde{\alpha}_1^R)^\dagger (\tilde{\alpha}_1^S)^\dagger (\alpha_2^T)^\dagger | 0 \rangle &\equiv | \Omega_{112}^{\tilde{R}\tilde{S}T} \rangle, \quad (\alpha_1^R)^\dagger (\alpha_1^S)^\dagger (\tilde{\alpha}_2^T)^\dagger | 0 \rangle \equiv | \Omega_{112}^{RS\tilde{T}} \rangle, \\ (\tilde{\alpha}_2^R)^\dagger (\alpha_2^S)^\dagger | 0 \rangle &\equiv | \Omega_{22}^{\tilde{R}S} \rangle. \end{aligned} \tag{3.21}$$

In Minkowski spacetime the corresponding states all have $m^2 \alpha' = 4$. In fact, the mass operator and the number operator are related by $m^2 \alpha' = -4 + 4N$ in Minkowski space. Here in anti-de Sitter space there is no such simple relation. Using Eqs. (3.12) and (3.14) we find the following masses of the states (3.21) when $D = 25$:

$$\begin{aligned} \langle \Omega_{1111}^{\tilde{R}S\tilde{T}U} | m^2 \alpha' | \Omega_{1111}^{\tilde{R}S\tilde{T}U} \rangle &= 4 + [16 + 96\zeta(3)]\lambda^2 - [64 + 384\zeta(5)]\lambda^3 + O(\lambda^4) , \\ \langle \Omega_{112}^{\tilde{R}\tilde{S}T} | m^2 \alpha' | \Omega_{112}^{\tilde{R}\tilde{S}T} \rangle &= \langle \Omega_{112}^{RS\tilde{T}} | m^2 \alpha' | \Omega_{112}^{RS\tilde{T}} \rangle = 4 + [17/2 + 96\zeta(3)]\lambda^2 - [65/2 + 384\zeta(5)]\lambda^3 + O(\lambda^4) , \\ \langle \Omega_{22}^{\tilde{R}S} | m^2 \alpha' | \Omega_{22}^{\tilde{R}S} \rangle &= 4 + [1 + 96\zeta(3)]\lambda^2 - [1 + 384\zeta(5)]\lambda^3 + O(\lambda^4) . \end{aligned} \tag{3.22}$$

We therefore reach the interesting conclusion that the coupling to the gravitational background (here anti-de Sitter spacetime) gives rise to a *fine structure* in the string mass spectrum. This turns out to be a general feature at all excited levels beyond the graviton, i.e., for $N > 1$. To zeroth order in the expansion parameter λ we recover, of course, the flat Minkowski space spectrum. The first corrections appear to order λ^2 . The leading Regge trajectory for the lower mass states therefore takes the form

$$J = 2 + \frac{1}{2} m^2 \alpha' + O(\lambda^2). \tag{3.23}$$

To second order in λ the masses are the same as in de Sitter space [1]. The difference in the lower mass spectrum between de Sitter space and anti-de Sitter space is of order λ^3 .

For the very high mass spectrum the situation changes drastically. Consider first excited states of the form

$$(\tilde{\alpha}_1^{R_1})^\dagger (\alpha_1^{S_1})^\dagger, \dots, (\tilde{\alpha}_1^{R_N})^\dagger (\alpha_1^{S_N})^\dagger | 0 \rangle \equiv | \Omega_{11\dots 11}^{\tilde{R}_1 S_1 \dots \tilde{R}_N S_N} \rangle . \tag{3.24}$$

This is a state with eigenvalue N of the number operator, and we will consider very large N , say, $N \gg \lambda^{-1}$. Using Eqs. (3.12) and (3.15) we find the approximate value of the mass:

$$\langle \Omega_{11\dots 11}^{\tilde{R}_1 S_1 \dots \tilde{R}_N S_N} | m^2 \alpha' | \Omega_{11\dots 11}^{\tilde{R}_1 S_1 \dots \tilde{R}_N S_N} \rangle \approx 4\lambda N^2 . \tag{3.25}$$

The Regge trajectory now takes the form

$$J^2 \approx \frac{1}{\lambda} m^2 \alpha' \text{ for } N \gg \lambda^{-1} . \tag{3.26}$$

This is significantly different from the lower mass relation, as compared with Eq. (3.23). Considering instead the state (for even N)

$$\begin{aligned} (\tilde{\alpha}_2^{R_1})^\dagger (\alpha_2^{S_1})^\dagger, \dots, (\tilde{\alpha}_2^{R_{N/2}})^\dagger (\alpha_2^{S_{N/2}})^\dagger | 0 \rangle \\ \equiv | \Omega_{22\dots 22}^{\tilde{R}_1 S_1 \dots \tilde{R}_{N/2} S_{N/2}} \rangle , \end{aligned} \tag{3.27}$$

and taking again $N \gg \lambda^{-1}$, we find

$$\langle \Omega_{22\dots 22}^{\tilde{R}_1 S_1 \dots \tilde{R}_{N/2} S_{N/2}} | m^2 \alpha' | \Omega_{22\dots 22}^{\tilde{R}_1 S_1 \dots \tilde{R}_{N/2} S_{N/2}} \rangle \approx \lambda N^2 . \tag{3.28}$$

In flat Minkowski space the states Eqs. (3.24) and (3.27) have the same mass ($m^2 \alpha' = -4 + 4N \approx 4N$) but here in anti-de Sitter space the masses are different by a factor of 4. The fine structure we found in the lower mass spectrum completely changes the very high mass spectrum. States with the same eigenvalue of the number operator can have considerably different masses. Furthermore, states with different eigenvalues of the number operator can get mixed up in the mass spectrum. For very high mass states of the form Eq. (3.24) or Eq. (3.27), the level spacing $[\Delta(m^2 \alpha') \text{ as a function of } N]$ grows proportionally to N . This should be contrasted with the situation in Minkowski space where the level spacing is constant. This suggests that in anti-de Sitter space the partition function can be defined for any temperature, as opposed to Minkowski space where there is a critical temperature (the Hagedorn temperature) of the order $(\alpha')^{-1/2}$. In Minkowski space, for very large N , the number of states

d_N with eigenvalue N of the number operator is roughly growing like [28]

$$d_N \sim \frac{e^{\sqrt{N}}}{N^p}, \quad (3.29)$$

where p is some number larger than 1. This holds for anti-de Sitter space as well. In anti-de Sitter space, for the states in the form (3.24) or (3.27), this leads to the following density of levels as a function of mass:

$$\rho(m) \sim \frac{e^{\beta_l \sqrt{m}}}{m^{p-1}}, \quad (3.30)$$

where $\beta_l \sim \sqrt{l}$ is *independent* of the string tension. Therefore, for a gas of strings in anti-de Sitter space at temperature β^{-1} , the partition function behaves like

$$\begin{aligned} Z(\beta) &= \int_0^\infty dm \rho(m) e^{-\beta m} \\ &\sim \int_0^\infty dm \frac{e^{\beta_l \sqrt{m}(1 - \frac{\beta}{\beta_l} \sqrt{m})}}{m^{p-1}}. \end{aligned} \quad (3.31)$$

This integral is finite for any value of β ; thus, we find no Hagedorn temperature in anti-de Sitter space.

We close this section with some interesting remarks on the string masses in the two regimes considered. For the low mass states ($m^2 \alpha' \lambda \ll 1$) our results can be written as

$$\langle j | m^2(\alpha', l) | j \rangle = \frac{4(N-1)}{\alpha'} + \frac{1}{l^2} \sum_{n=0}^{\infty} a_{jn} (\alpha'/l^2)^n, \quad (3.32)$$

where j is a collective index labeling the state $|j\rangle$. It is now important to notice that $a_{j0} = 0$ for *all* the low mass states; i.e., there is no ‘‘constant’’ term on the right-hand side of Eq. (3.32). A nonzero a_{j0} term would give rise to an α' -independent contribution to the string mass. Its absence, on the other hand, means that the first term on the right-hand side of Eq. (3.32) is superdominant (since, in all cases, $\alpha'/l^2 = \lambda \ll 1$) and that the string scale is therefore set by $1/\alpha'$. For the high mass states ($m^2 \alpha' \lambda \gg 1$) we found instead

$$\langle j | m^2(\alpha', l) | j \rangle \approx \frac{d_j}{l^2} N^2 \quad \text{for } N \gg l^2/\alpha', \quad (3.33)$$

where the number d_j depends on the state. The masses of the high mass states are therefore *independent* of α' . Moreover, the right-hand side of Eq. (3.33) is exactly like a nonzero dominant a_{j0} term in Eq. (3.32). For the high mass states the scale is therefore set by $1/l^2$ which is equal to the absolute value of the cosmological constant Λ (up to a geometrical factor). This suggests that for $\lambda \ll 1$, the masses of *all* string states can be represented by a formula of the form (3.32). For the low mass states $a_{j0} = 0$, while for the high mass states a_{j0} becomes a large positive number.

IV. BLACK STRING BACKGROUND

By a duality transformation [26] the $(2+1)$ -dimensional black hole anti-de Sitter spacetime becomes the black string of Horne and Horowitz [27]. It is therefore interesting to compare the string propagation in these two spacetimes. In Sec. II we presented the main results of the string perturbation approach in the case of the $(2+1)$ -dimensional black hole anti-de Sitter spacetime; the details can be found in Ref. [5]. In Ref. [5] we considered also the first-order string perturbations in the black string background, with special interest in the behavior near the physical singularity. In this section we perform a more complete analysis of the string propagation in the black string background. We consider all possible string center-of-mass geodesics (bounded and unbounded) and we calculate explicitly the first- and second-order string perturbations around these solutions. In the asymptotic region we then calculate the worldsheet energy-momentum tensor and we derive the mass formula of the string. To allow for comparison with the results of Sec. I we consider the uncharged black string with a line element [27]:

$$ds^2 = - \left(1 - \frac{Ml}{r}\right) dt^2 + \left(1 - \frac{Ml}{r}\right)^{-1} \frac{l^2 dr^2}{4r^2} + dx^2. \quad (4.1)$$

In this form the spacetime is just the product of Witten's two-dimensional black hole [22] and the real line space. It has a horizon at $r = Ml$ and, contrary to its dual the $(2+1)$ -dimensional black hole anti-de Sitter spacetime, it has a strong curvature singularity at $r = 0$. For a radially infalling string ($x = \text{const}$) the geodesic equations (2.6), determining the string center-of-mass motion, are integrated to

$$\dot{r}^2 = \frac{4r^2 \alpha'^2}{l^2} \left[E^2 - m^2 + \frac{m^2 Ml}{r} \right], \quad (4.2)$$

$$\dot{t} = \frac{E \alpha'}{1 - Ml/r}, \quad (4.3)$$

where E is the integration constant. These equations can be solved in terms of the elementary functions

$$r(\tau) = \frac{m^2 Ml}{E^2 - m^2} \sinh^2 \left(\frac{\alpha' \sqrt{E^2 - m^2}}{l} \tau \right), \quad (4.4)$$

$$\begin{aligned} t(\tau) &= E \alpha' \tau + l \operatorname{arctanh} \left[\frac{E}{\sqrt{E^2 - m^2}} \right. \\ &\quad \left. \times \tanh \left(\frac{\alpha' \sqrt{E^2 - m^2}}{l} \tau \right) \right]. \end{aligned} \quad (4.5)$$

Notice that these relations are well defined for any relation between E^2 and m^2 . For $E^2 - m^2 < 0$, $r(\tau)$ becomes a trigonometric function describing the bounded solutions.

For $E^2 = m^2$ we find that $r(\tau)$ is simply proportional to τ^2 , $r(\tau) = M(m\alpha'\tau)^2/l$.

Two covariantly constant normal vectors satisfying Eqs. (2.7) are given by

$$n_{\perp}^{\mu} = (0, 0, 1), \quad n_{\parallel}^{\mu} = \left(\frac{l\dot{r}}{2m\alpha'(r - Ml)}, \frac{2Er}{ml}, 0 \right). \tag{4.6}$$

It can now be shown [5] that the first-order perturbations (2.8) take the form

$$\begin{aligned} \delta x^{\perp}(\tau, \sigma) &= \sum_n C_n^{\perp}(\tau) e^{-in\sigma}, \\ \delta x^{\parallel}(\tau, \sigma) &= \sum_n C_n^{\parallel}(\tau) e^{-in\sigma}, \end{aligned} \tag{4.7}$$

where C_n^{\perp} and C_n^{\parallel} are solutions of the ‘‘Schrödinger equations’’ in τ :

$$\begin{aligned} \ddot{C}_n^{\perp} + n^2 C_n^{\perp} &= 0, \\ \ddot{C}_n^{\parallel} + \left(n^2 - \frac{2Mm^2\alpha'^2}{lr} \right) C_n^{\parallel} &= 0. \end{aligned} \tag{4.8}$$

Not surprisingly the perturbations in the ‘‘transverse’’ direction are the completely finite and regular trigonometric functions

$$C_n^{\perp}(\tau) = A_n^{\perp} e^{-in\tau} + B_n^{\perp} e^{in\tau}, \tag{4.10}$$

where $A_n^{\perp} = (A_{-n}^{\perp})^{\dagger}$, $B_n^{\perp} = (B_{-n}^{\perp})^{\dagger}$. In Ref. [5] we considered the solution of Eq. (4.9) for $r \rightarrow 0$ and we found that the perturbations blow up. We now give the complete solution in explicit form for all r .

For $E^2 = m^2$, using Eq. (4.4) we find

$$\ddot{C}_n^{\parallel} + \left(n^2 - \frac{2}{\tau^2} \right) C_n^{\parallel} = 0. \tag{4.11}$$

This equation is solved by

$$C_n^{\parallel}(\tau) = A_n^{\parallel} \left(1 - \frac{i}{n\tau} \right) e^{-in\tau} + B_n^{\parallel} \left(1 + \frac{i}{n\tau} \right) e^{in\tau}, \tag{4.12}$$

where $A_n^{\parallel} = (A_{-n}^{\parallel})^{\dagger}$, $B_n^{\parallel} = (B_{-n}^{\parallel})^{\dagger}$. Notice in particular the behavior

$$C_n^{\parallel}(\tau) \rightarrow \begin{cases} A_n^{\parallel} e^{-in\tau} + B_n^{\parallel} e^{in\tau} & \text{for } \tau \rightarrow -\infty \ (r \rightarrow \infty), \\ \frac{i}{n\tau} (B_n^{\parallel} - A_n^{\parallel}) & \text{for } \tau \rightarrow 0_+ \ (r \rightarrow 0). \end{cases} \tag{4.13}$$

Asymptotically this is a plane wave while near the singularity the perturbations blow up. This in agreement with the results obtained in Ref. [5].

For $E^2 \neq m^2$ we introduce a real parameter z and a function $g(z)$:

$$z \equiv -\sinh^2 \left(\frac{\alpha' \sqrt{E^2 - m^2}}{l} \tau \right), \quad g(z) \equiv \frac{C_n^{\parallel}(z)}{z}. \tag{4.14}$$

Equation (4.9) then reduces to the hypergeometric equation [29]

$$z(z-1) \frac{d^2 g(z)}{dz^2} + \left[c - (a+b+1)z \right] \frac{dg(z)}{dz} - ab g(z) = 0, \tag{4.15}$$

with parameters

$$\begin{aligned} a &= 1 + \frac{inl}{2\alpha' \sqrt{E^2 - m^2}}, \\ b &= 1 - \frac{inl}{2\alpha' \sqrt{E^2 - m^2}}, \\ c &= \frac{5}{2}. \end{aligned} \tag{4.16}$$

For $|z| \leq 1$ the solution $C_n^{\parallel}(z)$ is a linear combination of the functions

$$zF(a, b, c; z) \quad \text{and} \quad z^{2-c} F(a-c+1, b-c+1, 2-c; z). \tag{4.17}$$

In the case where $E^2 < m^2$ [where $r(\tau)$ reduces to a trigonometric function] we find that $z \in [0, 1]$ and therefore Eq. (4.17) gives the full solution. For $E^2 > m^2$, on the other hand, we have $z \in]-\infty, 0]$ and the solutions (4.17) have to be matched with solutions for $|z| > 1$ using analytical continuation [29]. This leads to the expression for $C_n^{\parallel}(z)$ when $|z| > 1$:

$$\begin{aligned} C_n^{\parallel}(z) &= 2^{\frac{inl}{\alpha' \sqrt{E^2 - m^2}}} A_n^{\parallel} (-z)^{1-b} F(b, 1-c+b, 1-a+b; 1/z) \\ &\quad + 2^{\frac{-inl}{\alpha' \sqrt{E^2 - m^2}}} B_n^{\parallel} (-z)^{1-a} F(a, 1-c+a, 1-b+a; 1/z), \end{aligned} \tag{4.18}$$

where again $A_n^{\parallel} = (A_{-n}^{\parallel})^{\dagger}$, $B_n^{\parallel} = (B_{-n}^{\parallel})^{\dagger}$. The constant factors in front of A_n^{\parallel} and B_n^{\parallel} were included to ensure the asymptotic behavior

$$C_n^{\parallel}(\tau) \rightarrow A_n^{\parallel} e^{-in\tau} + B_n^{\parallel} e^{in\tau} \quad \text{for } \tau \rightarrow -\infty \ (r \rightarrow \infty). \tag{4.19}$$

In terms of the constants A_n^{\parallel} and B_n^{\parallel} the solution for $|z| \leq 1$ reads

$$C_n^{\parallel}(z) = -\frac{2nl}{3\alpha'\sqrt{E^2-m^2}} \frac{inl}{\sqrt{E^2-m^2}} A_n^{\parallel} \left[i \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(b-c+1)\Gamma(1-a)} z F(a, b, c; z) + \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{2-c} F(a-c+1, b-c+1, 2-c; z) \right] \\ + \frac{2nl}{3\alpha'\sqrt{E^2-m^2}} \frac{-inl}{\sqrt{E^2-m^2}} B_n^{\parallel} \left[i \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(a-c+1)\Gamma(1-b)} z F(a, b, c; z) \right. \\ \left. + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{2-c} F(a-c+1, b-c+1, 2-c; z) \right]. \quad (4.20)$$

For $r \rightarrow 0$ ($\tau \rightarrow 0_-$) we find

$$C_n^{\parallel}(\tau \rightarrow 0_-) \rightarrow -\frac{(\alpha'\sqrt{E^2-m^2} - inl)B_n^{\parallel} + (\alpha'\sqrt{E^2-m^2} + inl)A_n^{\parallel}}{\alpha'^2(E^2-m^2) + n^2l^2} \frac{l}{\tau}; \quad (4.21)$$

i.e., the perturbations blow up, in agreement with the result found in Ref. [5]. The second-order perturbations $\xi^\mu(\tau, \sigma)$ are determined by [1,3]

$$\dot{q}^\lambda \nabla_\lambda (\dot{q}^\delta \nabla_\delta \xi^\mu) - R_{\epsilon\delta\lambda}^\mu \dot{q}^\epsilon \dot{q}^\delta \xi^\lambda - \xi''^\mu = U^\mu, \quad (4.22)$$

where the source U^μ is bilinear in the first-order perturbations and explicitly given by

$$U^\mu = -\Gamma_{\rho\sigma}^\mu (\dot{\eta}^\rho \dot{\eta}^\sigma - \eta'^\rho \eta'^\sigma) - 2\Gamma_{\rho\sigma,\lambda}^\mu \dot{q}^\rho \eta^\lambda \dot{\eta}^\sigma - \frac{1}{2} \Gamma_{\rho\sigma,\lambda\delta}^\mu \dot{q}^\rho \dot{q}^\sigma \eta^\lambda \eta^\delta. \quad (4.23)$$

The ξ^x perturbations decouple and Eq. (4.22) reduces to the free wave equation. The ξ^x perturbations are then given explicitly by

$$\xi^x(\tau, \sigma) = \sum_n [\tilde{A}_n^x e^{-in(\tau+\sigma)} + \tilde{B}_n^x e^{-in(\sigma-\tau)}]. \quad (4.24)$$

where $\tilde{A}_n^x = (\tilde{A}_{-n}^x)^\dagger$, $\tilde{B}_n^x = (\tilde{B}_{-n}^x)^\dagger$. The perturbations ξ^t and ξ^r are somewhat more complicated to derive. By redefining ξ^r and U^r ,

$$\xi^r \equiv \frac{2r}{l} \left(1 - \frac{Ml}{r} \right) \xi^*, \quad (4.25)$$

$$U^r \equiv \frac{2r}{l} \left(1 - \frac{Ml}{r} \right) U^*, \quad (4.26)$$

we find, from Eq. (4.22),

$$\begin{pmatrix} \ddot{\xi}^t \\ \ddot{\xi}^* \end{pmatrix} - \begin{pmatrix} \xi'''^t \\ \xi'''^* \end{pmatrix} + 2\mathcal{A} \begin{pmatrix} \dot{\xi}^t \\ \dot{\xi}^* \end{pmatrix} + \mathcal{B} \begin{pmatrix} \xi^t \\ \xi^* \end{pmatrix} = \begin{pmatrix} U^t \\ U^* \end{pmatrix}, \quad (4.27)$$

where the matrices \mathcal{A} and \mathcal{B} are given by

$$\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & \gamma \\ 0 & \delta \end{pmatrix}, \quad (4.28)$$

$$\alpha = \frac{Ml\dot{r}}{2r^2} \left(1 - \frac{Ml}{r} \right)^{-1}, \quad \beta = \frac{ME\alpha'}{r} \left(1 - \frac{Ml}{r} \right)^{-1},$$

$$\gamma = \frac{-2EM\alpha'\dot{r}}{r^2(1-Ml/r)}, \quad \delta = \frac{-4ME^2\alpha'^2}{lr(1-Ml/r)} + \frac{2Mm^2\alpha'^2}{lr}.$$

The first-order τ derivatives in Eq. (4.27) are eliminated by the transformation

$$\begin{pmatrix} \xi^t \\ \xi^* \end{pmatrix} \equiv \mathcal{G} \begin{pmatrix} \Sigma^t \\ \Sigma^* \end{pmatrix}, \quad \mathcal{G} = \exp\left(-\int^\tau \mathcal{A}(\tau') d\tau'\right); \quad (4.29)$$

i.e.,

$$\mathcal{G} = (1 - Ml/r)^{-1} \begin{pmatrix} E/m & l\dot{r}/2m\alpha'r \\ l\dot{r}/2m\alpha'r & E/m \end{pmatrix}. \quad (4.30)$$

We now Fourier expand the second-order perturbations and the sources:

$$\Sigma^t(\tau, \sigma) = \sum_n \Sigma_n^t(\tau) e^{-in\sigma}, \quad \Sigma^*(\tau, \sigma) = \sum_n \Sigma_n^*(\tau) e^{-in\sigma}, \quad (4.31)$$

$$U^t(\tau, \sigma) = \sum_n U_n^t(\tau) e^{-in\sigma}, \quad U^*(\tau, \sigma) = \sum_n U_n^*(\tau) e^{-in\sigma}, \quad (4.32)$$

and the matrix equation (4.27) reduces to

$$\begin{pmatrix} \ddot{\Sigma}_n^t \\ \ddot{\Sigma}_n^* \end{pmatrix} + \mathcal{V} \begin{pmatrix} \Sigma_n^t \\ \Sigma_n^* \end{pmatrix} = \mathcal{G}^{-1} \begin{pmatrix} U_n^t \\ U_n^* \end{pmatrix} \equiv \begin{pmatrix} \tilde{U}_n^t \\ \tilde{U}_n^* \end{pmatrix}, \quad (4.33)$$

where

$$\mathcal{V} = \mathcal{G}^{-1}(n^2 I + \mathcal{B} - \mathcal{A}^2 - \dot{\mathcal{A}})\mathcal{G} = \begin{pmatrix} n^2 & 0 \\ 0 & n^2 - \frac{2Mm^2\alpha'^2}{lr} \end{pmatrix}, \quad (4.34)$$

i.e., two decoupled inhomogeneous second-order linear differential equations. These equations can easily be integrated by noticing that the corresponding homogeneous equations take exactly the same form as Eqs. (4.8) and (4.9), which we have already solved explicitly. Σ_n^t and Σ_n^* will then take the same form, plus extra terms involving suitable integrals of the sources. By transforming backwards, using Eqs. (4.25)–(4.32), we finally get the explicit expressions for ξ^r and ξ^t which we shall, however, not write down here. It is instructive to consider the results in the asymptotic region $r \rightarrow \infty$. This region can only be reached if $E^2 \geq m^2$ and for the following computations we therefore assume $E^2 > m^2$, although the results will hold for $E^2 = m^2$, too. Using the results of the analysis of the first-order perturbations we find, for $r \rightarrow \infty$ ($\tau \rightarrow -\infty$),

$$\eta^t(\tau, \sigma) = -\frac{\sqrt{E^2 - m^2}}{m} \sum_n [A_n^\parallel e^{-in(\tau+\sigma)} + B_n^\parallel e^{-in(\sigma-\tau)}] + O(1/r), \quad (4.35)$$

$$\eta^r(\tau, \sigma) = \frac{2Er}{ml} \sum_n [A_n^\parallel e^{-in(\tau+\sigma)} + B_n^\parallel e^{-in(\sigma-\tau)}] + O(1). \quad (4.36)$$

The sources then take the asymptotic form

$$\begin{pmatrix} \tilde{U}^t \\ \tilde{U}^* \end{pmatrix} = \frac{2E^2}{lm^2} \left[\left(\frac{\partial \delta x^\parallel}{\partial \tau} \right)^2 - \left(\frac{\partial \delta x^\parallel}{\partial \sigma} \right)^2 \right] \begin{pmatrix} \sqrt{E^2 - m^2}/m \\ E/m \end{pmatrix} + O(1/r), \quad (4.37)$$

and Eqs. (4.33) are solved by

$$\begin{aligned} \Sigma_n^t(\tau) &= \tilde{A}_n^\parallel e^{-in\tau} + \tilde{B}_n^\parallel e^{in\tau} + \frac{2E^2\sqrt{E^2 - m^2}}{lm^3} \frac{e^{in\tau}}{n} \sum_p (n-p) A_p^\parallel B_{n-p}^\parallel e^{-2ip\tau} \\ &+ \frac{2E^2\sqrt{E^2 - m^2}}{lm^3} \frac{e^{-in\tau}}{n} \sum_p (n-p) A_{n-p}^\parallel B_p^\parallel e^{2ip\tau} + O(1/r), \end{aligned} \quad (4.38)$$

$$\begin{aligned} \Sigma_n^*(\tau) &= \tilde{C}_n^\parallel e^{-in\tau} + \tilde{D}_n^\parallel e^{in\tau} + \frac{2E^3}{lm^3} \frac{e^{in\tau}}{n} \sum_p (n-p) A_p^\parallel B_{n-p}^\parallel e^{-2ip\tau} \\ &\quad + \frac{2E^3}{lm^3} \frac{e^{-in\tau}}{n} \sum_p (n-p) A_{n-p}^\parallel B_p^\parallel e^{2ip\tau} + O(1/r). \end{aligned} \quad (4.39)$$

The second-order perturbations become

$$\xi^t(\tau, \sigma) = \sum_n \left[\left(\frac{E}{m} \tilde{A}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{C}_n^\parallel \right) e^{-in(\tau+\sigma)} + \left(\frac{E}{m} \tilde{B}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{D}_n^\parallel \right) e^{-in(\sigma-\tau)} \right] + O(1/r), \quad (4.40)$$

$$\begin{aligned} \xi^r(\tau, \sigma) &= \frac{2r}{l} \sum_n \left[\left(\frac{E}{m} \tilde{C}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{A}_n^\parallel \right) e^{-in(\tau+\sigma)} + \left(\frac{E}{m} \tilde{D}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{B}_n^\parallel \right) e^{-in(\sigma-\tau)} \right. \\ &\quad \left. + \frac{2E^2}{lm^2} e^{-in(\sigma-\tau)} \sum_p A_p^\parallel B_{n-p}^\parallel e^{-2ip\tau} \right] + O(1). \end{aligned} \quad (4.41)$$

η^r and ξ^r are ordinary plane waves in the coordinate x^R defined by

$$x^R = R + \eta^R + \xi^R + \dots \equiv \frac{l}{2} \ln \frac{x^r}{l} = \frac{l}{2} \ln \frac{r + \eta^r + \xi^r + \dots}{l}. \quad (4.42)$$

We find

$$\eta^R(\tau, \sigma) = \frac{E}{m} \sum_n [A_n^\parallel e^{-in(\tau+\sigma)} + B_n^\parallel e^{-in(\sigma-\tau)}] + O(e^{-2R/l}), \quad (4.43)$$

$$\begin{aligned} \xi^R(\tau, \sigma) &= \sum_n \left[\left(\frac{E}{m} \tilde{C}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{A}_n^\parallel - \frac{E^2}{lm^2} \sum_p A_{n-p}^\parallel A_p^\parallel \right) e^{-in(\tau+\sigma)} \right. \\ &\quad \left. + \left(\frac{E}{m} \tilde{D}_n^\parallel - \frac{\sqrt{E^2 - m^2}}{m} \tilde{B}_n^\parallel - \frac{E^2}{lm^2} \sum_p B_{n-p}^\parallel B_p^\parallel \right) e^{-in(\sigma-\tau)} \right] + O(e^{-2R/l}). \end{aligned} \quad (4.44)$$

From the expressions of (η^x, η^t, η^R) and (ξ^x, ξ^t, ξ^R) , it follows that the center-of-mass solution and the first-order perturbations already give the complete solution in the asymptotic region. This is what it should be since the black string background is asymptotically flat. We therefore choose the initial conditions such that $\xi^x = \xi^t = \xi^R = 0$; i.e., we take

$$\begin{aligned} \tilde{A}_n^\perp = \tilde{B}_n^\perp = 0, \quad E \tilde{A}_n^\parallel = \sqrt{E^2 - m^2} \tilde{C}_n^\parallel, \quad E \tilde{B}_n^\parallel = \sqrt{E^2 - m^2} \tilde{D}_n^\parallel, \\ \tilde{C}_n^\parallel = \frac{E^3}{lm^3} \sum_p A_{n-p}^\parallel A_p^\parallel, \quad \tilde{D}_n^\parallel = \frac{E^3}{lm^3} \sum_p B_{n-p}^\parallel B_p^\parallel. \end{aligned} \quad (4.45)$$

Let us finally consider the world-sheet energy-momentum tensor that was introduced in Eq. (2.7). Up to second order in the expansion around the string center of mass we find

$$\begin{aligned} T_{\pm\pm} &= -\frac{1}{4} m^2 \alpha'^2 + g_{\mu\nu} \dot{q}^\mu \partial_\pm \eta^\nu + \frac{1}{4} g_{\mu\nu, \rho} \dot{q}^\mu \dot{q}^\nu \eta^\rho + g_{\mu\nu} \dot{q}^\mu \partial_\pm \xi^\nu + g_{\mu\nu} \partial_\pm \eta^\mu \partial_\pm \eta^\nu + g_{\mu\nu, \rho} \dot{q}^\mu \eta^\rho \partial_\pm \eta^\nu \\ &\quad + \frac{1}{4} g_{\mu\nu, \rho} \dot{q}^\mu \dot{q}^\nu \xi^\rho + \frac{1}{8} g_{\mu\nu, \rho\sigma} \dot{q}^\mu \dot{q}^\nu \eta^\rho \eta^\sigma. \end{aligned} \quad (4.46)$$

Using the expressions for the first- and second-order perturbations, Eqs. (4.35), (4.36) and Eqs. (4.40), (4.41), as well as the conditions (4.45), it is straightforward now to calculate T_{++} and T_{--} in the asymptotic region ($r \rightarrow \infty$):

$$T_{++} = -\frac{1}{4} m^2 \alpha'^2 - \sum_{R=\pm, \parallel} \sum_{n,p} p(n-p) A_p^R A_{n-p}^R e^{-in(\tau+\sigma)}, \quad (4.47)$$

$$T_{--} = -\frac{1}{4}m^2\alpha'^2 - \sum_{R=\perp,\parallel} \sum_{n,p} p(n-p)B_p^R B_{n-p}^R e^{-in(\sigma-\tau)}, \tag{4.48}$$

and we get the usual flat spacetime constraints. The mass formula in particular takes the form

$$m^2\alpha'^2 = 4 \sum_n n^2 \sum_{R=\perp,\parallel} A_n^R A_{-n}^R = 4 \sum_n n^2 \sum_{R=\perp,\parallel} B_n^R B_{-n}^R. \tag{4.49}$$

This is completely different from the result obtained for the 2 + 1 black hole anti-de Sitter spacetime [compare with Eq. (2.16)] which is dual to the black string. It should be stressed, however, that the formula (4.49) was obtained in the asymptotic region $r \rightarrow \infty$ of the black string background, and therefore does not include any contribution from the bounded solutions found when $E^2 < m^2$. The expression (2.16), on the other hand, is general for the 2 + 1 black hole anti-de Sitter spacetime.

For the black string, in the asymptotic region, we therefore obtain the density of levels

$$\rho(m) \sim e^{m\sqrt{\alpha'}}, \tag{4.50}$$

similar to the expression in flat Minkowski space, up to multiplication by a polynomial in m . The partition function for a gas of strings at temperature β^{-1} , in the asymptotic region of the black string background, therefore goes like

$$Z(\beta) \sim \int_0^\infty dm e^{-m(\beta-\sqrt{\alpha'})}, \tag{4.51}$$

which is only defined for $\beta > \sqrt{\alpha'}$; i.e., there is a Hagedorn temperature

$$T_{\text{Hg}} = (\alpha')^{-1/2}. \tag{4.52}$$

TABLE I. Characteristic features of the quantum string mass spectrum in anti-de Sitter (AdS) and de Sitter (dS) spacetimes. Notice the difference in the high mass spectrum: In AdS spacetime the masses and level density become independent of α' . In dS spacetime there is no such high mass spectrum at all.

Anti-de Sitter spacetime (AdS)	de Sitter spacetime (dS)
Classical motion is stable and oscillatory in time with real frequencies $\omega_n = \sqrt{n^2 + m^2\alpha'^2 H^2}$.	Classical motion is unstable with frequencies $\omega_n = \sqrt{n^2 - m^2\alpha'^2 H^2}$. Unbounded string size and energy for large de Sitter radius, $R \rightarrow \infty$.
The mass formula is well defined for all m . There is an infinite number of states with arbitrary high masses. $m^2\alpha' = 0$ is an exact solution at $D = 25$.	Real mass solutions only for $m < 1/(\alpha'H)$. Finite number of states, $N_{\text{max}} \approx 0.15/(\alpha'H^2)$. $m^2\alpha' = 0$ is an exact solution at $D = 25$.
The coupling to the gravitational background produces a Fine structure effect at all levels in the mass spectrum. The number of levels considerably increases with respect to flat space.	Fine structure effect appears in low mass spectrum. Is similar to AdS; the differences appear to order $(\alpha'H^2)^3$.
For the high mass states, $\langle m^2 \rangle \sim \Lambda N^2$, $J^2 \sim m^2 / \Lambda $. Both are independent of α' . The level spacing grows with N . $\rho(m) \sim \text{Exp}[m/\sqrt{ \Lambda }]^{1/2}$, high m . No Hagedorn temperature exists.	The similar region of high mass states does not exist in the de Sitter spacetime.

In higher-dimensional ($D \geq 4$) black hole spacetimes the next step now would be to set up a scattering formalism, where a string from an asymptotic in-state interacts with the gravitational field of the black hole and reappears in an asymptotic out-state [11]. However, this is not possible in the black string background. In the uncharged black string background under consideration here, *all* null and timelike geodesics incoming from spatial infinity pass through the horizon and fall into the physical singularity [27]. No “angular momentum,” as in the case of scattering off the ordinary Schwarzschild black hole, can prevent a point particle from falling into the singularity. The string solutions considered in the present paper are based on perturbations around the string center of mass which follows, at least approximately, a point particle geodesic. A string incoming from spatial infinity therefore inevitably falls into the singularity in the black string background.

V. CONCLUDING REMARKS

The classical string motion in anti-de Sitter spacetime is stable in the sense that it is oscillatory in time with real frequencies and the string size and energy are bounded. Quantum mechanically, this is reflected in the mass operator, which is well defined for any value of the wave number n , and arbitrary high mass states (and therefore an infinite number of states) can be constructed. This is to be contrasted with de Sitter spacetime, where string instabilities develop, in the sense that the string size and energy become unbounded for large de Sitter radius. For low mass states (the stable regime), the mass operator in de Sitter spacetime is given by Eq. (1.1) but with

$$\Omega_n(\lambda)_{\text{dS}} = \frac{2n^2 - m^2\alpha'\lambda}{\sqrt{n^2 - m^2\alpha'\lambda}}.$$

Real mass solutions can be defined only up to some *maximal mass* of the order $m^2\alpha' \approx 1/\lambda$ [1]. For $\lambda \ll 1$,

real mass solutions can be defined only for $N \leq N_{\text{max}} \sim 0.15/\lambda$ (where N is the eigenvalue of the number operator) and therefore there exists a *finite* number of states only. These features of strings in de Sitter spacetime have been recently confirmed within a different (semiclassical) quantization approach based on *exact* circular string solutions [30].

The presence of a cosmological constant Λ (positive or negative) increases considerably the number of levels of different eigenvalue of the mass operator (there is a splitting of levels) with respect to flat spacetime. That is, a nonzero cosmological constant *decreases* (although does not remove) the degeneracy of the string mass states, introducing a fine structure effect. For the low mass states the level spacing is approximately constant (up to corrections of the order λ^2). For the high mass states, the changes are more drastic and they depend crucially on the sign of Λ . A value $\Lambda < 0$ causes the *growing* of the level spacing linearly with N instead of being constant as in Minkowski space. Consequently, the density of states $\rho(m)$ grows with the exponential of \sqrt{m} (instead of m as in Minkowski space), discarding the existence of a Hagedorn temperature in AdS spacetime and the possibility of a phase transition. In addition, another important feature of the high mass string spectrum in AdS spacetime is that it becomes independent of α' . The string scale for the high mass states is given by $|\Lambda|$, instead of $1/\alpha'$ for the low mass states, as discussed at the end of Sec. III, Eqs. (3.32), (3.33).

The main physical features found in this paper are summarized in Table I.

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