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Gravitational Lorentz anomaly from the overlap formula in two dimensions

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In this Rapid Communication we show that the overlap formulation of chiral gauge theories correctly reproduces the gravitational Lorentz anomaly in two dimensions. This formulation has been recently suggested as a solution to the fermion doubling problem on the lattice. The well-known response to general coordinate transformations of the effective action of Weyl fermions coupled to gravity in two dimensions can also be recovered.

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The formulation of lattice chiral gauge theories has been an outstanding problem for many years [1]. In a recent paper a new formulation of this problem has been suggested [2]. One of the necessary steps in checking the viability of this suggestion is to show that it correctly reproduces the chiral anomalies in the continuum limit. For the case of a U(1) gauge theory in two dimensions this was done in [3], and its generalization to non-Abelian chiral anomalies in four dimensions is contained in [4].

In this Rapid Communication we would like to use the notation and formalism developed in [4] to examine the anomalous coupling of Weyl fermions to a background gravitational field in two dimensions. It will be shown that the overlap formalism proposed in [2] correctly reproduces the chiral anomaly in this system.

Our starting point will be the Hamiltonian of a two-component "massive" fermion coupled to a gravitational field e_a^μ in $2+1$ dimensions:

$$H=\int d^2x\,\psi^\dagger(x)\sigma_3(\sigma_a e_a^\mu \nabla_\mu+\Lambda)\psi(x), \qquad (1)$$

where σ_a, $a=1,2$ and σ_3 are the Pauli spin matrices and $\nabla_\mu=\partial_\mu-(i/2)\omega_\mu-\frac{1}{2}\partial_\mu \ln e$, with ω_μ being the spin connection derived from the zweibein e_a^μ and $e^{-1}=\det e_a^\mu$. All the fields in (1) depend only on the two-dimensional coordinates.

The Hamiltonian (1) is invariant under general coordinate transformations $x^\mu \to f^\mu(x^1,x^2)$ provided the two-component spinor field ψ transforms as a scalar density of weight $\frac{1}{2}$. It is also invariant under the local O(2) frame rotations, $\psi(x)\to e^{(i/2)\theta(x)\sigma_3}\psi(x)$.

It was shown in [4] that in the limit of $|\Lambda|\to\infty$ the Hamiltonians of the type (1) can be used to recover the Green's functions of a massless chiral gauge theory. It is well known that the gravitational coupling of such fermions is anomalous [5]. Here we would like to rederive this anomaly in the local frame rotations as $|\Lambda|\to\infty$.

The overlap formulation of [2] defines the effective action $\Gamma[e]$ by

$$\Gamma[e]=-\ln\langle e;+|e;-\rangle,$$

where $|e;+\rangle$ and $|e;-\rangle$ are the Dirac ground states of the two Hamiltonians $H(+\Lambda)$ and $H(-\Lambda)$, respectively. To study the behavior of $\Gamma[e]$ under local frame rotations we must test the response of the two ground states with respect to such transformations. Acting on the Schrödinger picture fields these transformations are realized by unitary operators $U(\theta)$:

$$U(\theta)^{-1}\psi(x)U(\theta)=e^{(i/2)\,\theta(x)\sigma_3}\psi(x).$$

Since the free Hamiltonian $H_0=\int d^2x\,\psi^\dagger(x)\sigma_3(\sigma_\mu\partial_\mu+\Lambda)\psi(x)$ is invariant under the constant gauge transformations it follows that

$$U(\theta)^{-1}H(e)U(\theta)=H(e^\theta),$$

where e^θ denotes the rotated frame. We shall regard $|e;\pm\rangle$ as the perturbed vacua of the Dirac ground states $|\pm\rangle$ of $H_0(\pm\Lambda)$. For weak fields it then follows

$$U(\theta)|e;\pm\rangle=|e^\theta;\pm\rangle e^{i\Phi_\pm(\theta;e)},$$

where the angles $\Phi_\pm(\theta;e)$ are real. It is possible to compute these angles by applying "time"-independent perturbation theory. To first order in θ this gives [4]

$$\Phi_\pm(\theta;e)=\int d^2x\left\langle \pm\left|\psi^\dagger(x)\frac{1}{2}\sigma_3\theta(x)\psi(x)\left[1-\frac{1}{E_{0\pm}-H_{0\pm}}\Pi_\pm(V-\Delta E_\pm)\right]^{-1}\right|\pm\right\rangle,$$

where $\Pi_\pm=1-|\pm\rangle\langle\pm|$ and

$$V=\int d^2x\,\psi^\dagger(x)\sigma_3\left[\sigma_\mu\left(-\frac{i}{2}\omega_\mu+\frac{1}{2}\partial_\mu h\right)+\sigma_a h_a^\mu\partial_\mu\right]\psi(x).$$

Here the weak gravitational perturbations around the flat space are given by $h^{\mu}_{a}=e^{\mu}_{a}-\delta^{\mu}_{a}$ and $h=\delta^{a}_{\mu}h^{\mu}_{a}$. The actual values of the vacuum shifts $\Delta E_{\pm}$ are irrelevant for the present discussion. For further detail of the notation Ref. [4] can be consulted. Since $U(\theta)$ is unitary it follows that the overlap must satisfy

$$\langle e^{\theta};+|e^{\theta};-\rangle=\langle e;+|e;-\rangle e^{i(\Phi_{+}-\Phi_{-})},$$

which implies an anomaly if $\Phi_{+}-\Phi_{-}\neq 0$. We compute this difference perturbatively and show that in the limit of $|\Lambda|\to\infty$ it contains the usual Lorentz anomaly.

The first-order part of Φ_{+} is given by

$$\Phi^{(1)}_{+}=\sum_{n}\int_{\Omega}d^{2}x\langle +|\psi^{\dagger}(x)\tfrac{1}{2}\sigma_{3}\theta(x)\psi(x)|n\rangle \times\frac{1}{E_{0+}-E_{n}}\langle n|V|+\rangle \qquad (2)$$

where the sums are restricted to two-particle intermediate states,

$$\sum_{n}|n\rangle\langle n|=\frac{1}{\Omega^{2}}\sum_{k_{1},k_{2}}|k_{1},k_{2}\rangle\langle k_{1},k_{2}|,$$

and where

$$|k_{1},k_{2}\rangle=b^{\dagger}_{+}(k_{1})d^{\dagger}_{+}(k_{2})|+\rangle.$$

The operators $b_{\pm}$ and $d_{\pm}$ are defined by [4]

$$\psi(x)=\frac{1}{\Omega}\sum_{k}[b_{\pm}u_{\pm}(k)+d^{\dagger}_{\pm}v_{\pm}(k)]e^{ikx},$$

where Ω is the volume of a two-box and u's and v's are the positive and the negative energy eigenvectors of $H_{0\pm}(k)=\sigma_{3}(i\sigma_{\mu}k_{\mu}\pm\Lambda)$:

$$H_{0\pm}(k)u_{\pm}(k)=\omega(k)u_{\pm}(k),$$

$$H_{0\pm}(k)v_{\pm}(k)=-\omega(k)v_{\pm}(k),$$

where $\omega(k)=(k^{2}+\Lambda^{2})^{1/2}$.

By making use of these results in (2) we obtain

$$\frac{\delta\Phi^{(1)}_{+}}{\delta\theta(x)}=\frac{i}{4\Omega^{2}}\sum_{k_{1},k_{2}}\frac{e^{i(k_{1}+k_{2})x}}{\omega(k_{1})+\omega(k_{2})}\mathrm{Tr}[\sigma_{3}U(k_{1})\sigma_{3}\sigma^{\mu}\{\tilde{\omega}(k_{1}-k_{2})\sigma_{3}+(k_{1}-k_{2})_{\mu}\tilde{h}(k_{1}-k_{2})-2k_{2\nu}\tilde{h}^{\nu}_{\mu}(k_{1}-k_{2})\}V(k_{2})],$$

where $U(k)=[\omega(k)+\sigma_{3}(i\sigma_{\mu}k_{\mu}+\Lambda)]/2\omega(k)=1-V(k)$ and the Fourier transforms are defined in the usual way, e.g., $\tilde{h}(k)=\int_{\Omega}d^{2}xe^{-ikx}h(x)$. To obtain $\Phi^{(1)}_{-}$ we need to change the sign of Λ. Evaluating the Dirac traces and letting $\Omega\to\infty$ we obtain

$$\frac{\delta(\Phi^{(1)}_{+}-\Phi^{(1)}_{-})}{\delta\theta(x)}=\Lambda\int\frac{d^{2}p}{(2\pi)^{2}}e^{ikx}F_{\mu\nu}(p)\tilde{h}_{\nu\mu}(p), \qquad (3)$$

where

$$F_{\mu\nu}(p)=\int\frac{d^{2}k}{(2\pi)^{2}}\frac{k_{\mu}k_{\nu}}{\omega\left(\frac{k+p}{2}\right)\omega\left(\frac{k-p}{2}\right)\left[\omega\left(\frac{k+p}{2}\right)+\omega\left(\frac{k-p}{2}\right)\right]}. \qquad (4)$$

Equation (3) should yield the gravitational Lorentz anomaly. To evaluate it we expand $F_{\mu\nu}(p)$ in powers of $1/|\Lambda|$ and obtain

$$F_{\mu\nu}(p)=\frac{|\Lambda|}{2}\int\frac{d^{2}q}{(2\pi)^{2}}\frac{q_{\mu}q_{\nu}}{(q^{2}+1)^{3/2}}\left[1-\frac{3}{8}\frac{p^{2}}{\Lambda^{2}(q^{2}+1)}+\frac{5}{8}\frac{(q\cdot p)^{2}}{\Lambda^{2}(q^{2}+1)^{2}}+\cdots\right], \qquad (5)$$

where all the terms not indicated explicitly are given by convergent integrals and vanish as $|\Lambda|\to\infty$. The leading term inside the bracket on the right-hand side makes a divergent contribution which must be subtracted in the usual way by a suitable counterterm. It should be noted that this contribution is independent of p and therefore will not contribute to the terms involving the derivatives of $h_{\mu\nu}$. Those of the convergent integrals which contribute in the limit of $\Lambda\to\infty$ produce

$$F_{\mu\nu}(p)=\frac{1}{48\pi|\Lambda|}(p_{\mu}p_{\nu}-p^{2}\delta_{\mu\nu}). \qquad (6)$$

Upon substitution of this result in (3) we obtain

$$\frac{\delta(\Phi^{(1)}_{+}-\Phi^{(1)}_{-})}{\delta\theta(x)}=\frac{\Lambda}{|\Lambda|}\frac{1}{48\pi}(\partial^{2}\delta_{\mu\nu}-\partial_{\mu}\partial_{\nu})h_{\nu\mu}(x). \qquad (7)$$

To see that this is the standard result we only need to make use of the geometric relations $\omega_{\mu}=-\frac{1}{2}e^{a}_{\mu}(\varepsilon^{\alpha\beta}/e)\partial_{\beta}e^{a}_{\alpha}$ and $\partial_{\mu}\omega_{\nu}-\partial_{\nu}\omega_{\mu}=\varepsilon_{\mu\nu}R/4e$ to obtain $R=2(\partial^{2}\delta_{\mu\nu}-\partial_{\mu}\partial_{\nu})\times h_{\nu\mu}(x)$, up to the first order terms in h. Thus (7) becomes

$$\frac{\delta(\Phi^{(1)}_{+}-\Phi^{(1)}_{-})}{\delta\theta(x)}=\frac{\Lambda}{|\Lambda|}\frac{1}{96\pi}R. \qquad (8)$$

This agrees with the well-known result for the Lorentz anomaly [6].

In a similar way we can study the response of the overlap to general coordinate transformations of two-dimensional manifold. If we denote the parameter of this transformation by $\xi^{\mu}(x)$, it can be shown that

$$\frac{\delta(\Phi_+ - \Phi_-)}{\delta\xi^{\mu}(x)} = \frac{\Lambda}{|\Lambda|}\frac{1}{48\pi}\varepsilon_{\sigma\nu}(\partial^2\delta_{\mu\lambda} - \partial_\mu\partial_{\nu\lambda})\partial_\nu h_{\sigma\lambda}(x).$$

It is not hard to verify that this agrees with the result of [6] which is expressed as

$$\nabla_\lambda T^{\lambda\mu} = \frac{1}{96\pi} e^{-1}\varepsilon^{\mu\nu}\partial_\nu R,$$

where $T^{\lambda\mu}$ is defined by $\delta \ln \mathrm{Det} D = \int d^2x \frac{1}{2}\delta g_{\mu\nu}T^{\mu\nu}$, with D being the Weyl operator.

Further support for the overlap approach is contained in a recent report concerning the chiral determinant on a two-dimensional torus in the presence of nontrivial background Polyakov loop variable [7]. A lattice derivation of gravitational chiral anomalies, as far as we know, does not exist in the literature.

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[1] L. H. Karsten, Phys. Lett. **104B**, 315 (1981); L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981); H. B. Nielsen and M. Ninomya, *ibid.* **B185**, 20 (1981); **B193**, 173 (1981); Phys. Lett. **105B**, 219 (1981).

[2] R. Narayanan and H. Neuberger, "A construction of lattice chiral gauge theories," Report No. ISSNS-HEP-94/99, RU-94-93 (unpublished).

[3] R. Narayanan and H. Neuberger, Nucl. Phys. **B412**, 574 (1994).

[4] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348**, 543 (1995); S. Randjbar-Daemi and J. Strathdee, Nucl. Phys. B (to be published).

[5] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1983); W. Bardeen and B. Zumino, *ibid.* **B244**, 421 (1984).

[6] S. H. Leutwyler, Phys. Lett. **153B**, 65 (1985).

[7] R. Narayanan and H. Neuberger, Phys. Lett. B **348**, 549 (1995).