RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 51, NUMBER 8 15 APRIL 1995

Phase transition for gravitationally collapsing dust shells in 2+1 dimensions

Yoav Peleg*
Department of Physics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

Alan R. Steif†
Department of Physics, University of California, Davis, California 95616

(Received 15 December 1994)

The collapse of thin dust shells in (2+1)-dimensional gravity with and without a cosmological constant is analyzed. A critical value of the shell's mass as a function of its radius and position is derived. For $\Lambda<0$, a naked singularity or black hole forms depending on whether the shell's mass is below or just above this value. The solution space is divided into four different regions by three critical surfaces. For $\Lambda<0$, two surfaces separate regions of black hole solutions and solutions with naked singularities, while the other surface separates regions of open and closed spaces. Near the transition between a black hole and naked singularity, we find $\mathscr{M}\sim c_p(p-p^*)^\beta$, where $\beta=1/2$ and $\mathscr{M}$ is a naturally defined order parameter. We find no phase transition in crossing from an open to closed space. The critical exponent appears to be universal for spherically symmetric dust. The critical solutions are analogous to higher dimensional extremal black holes. All four phases coexist at one point in solution space corresponding to the static extremal solution.

PACS number(s): 97.60.Lf, 04.20.−q, 04.60.Kz

Following the work of Choptuik [1], critical behavior has been found in several models of black hole formation [2–8]. In these models, the space of solutions is separated by a critical surface into a region of black hole solutions and a region of solutions which are not black holes. There exist continuous parameters with critical values on the boundaries separating these regions. The critical solutions are universal, and scaling laws with critical exponents have been found. For the case of collapsing spherically symmetric inhomogeneous dust, the existence of different phases has been known for some time [9,10], and recently the order parameters and their critical values have been found [11]. So far, the behavior near criticality has not been studied.

Since there is presently no deep understanding of this general phenomenon, the study of simple models of gravitational collapse may be useful. In this paper, we study collapsing spherical thin dust shells in (2+1)-dimensional gravity with both a vanishing and negative cosmological constant. Imposing junction conditions across the dust shell, we derive a relation for the total mass of the system in terms of the rest mass of the dust shell μ, its initial radius r_0, and its initial velocity $\dot{r}_0$. We show that the solution space is divided into four regions. For $\Lambda=0$, the solutions in the four regions are (1) open conical spaces [12], (2) open three-dimensional Misner-Taub-NUT (Newman-Unti-Tamburino) (MTN) spacetimes, (3) closed conical spaces [12], and (4) closed MTN spacetimes. For $\Lambda<0$, the solutions in the four regions are (1) open anti–de Sitter (AdS) conical spaces [13], (2) exteriors of three-dimensional black holes [14], (3) closed conical AdS spaces [13], and (4) interiors of three-dimensional black holes. We find the critical surfaces separating the four regions and study the critical behavior in their vicinity. Following [1–7], we define the order parameter $\mathscr{M}$ as the total mass of the system. For $\Lambda<0$, we show that near the two critical surfaces separating black holes and solutions with naked singularities, the order parameter behaves as

$$\mathscr{M}\sim c_p(p-p^*)^\beta, \tag{1}$$

where $\beta=1/2$ for both surfaces. p is an affine parameter along *any* curve in the space of solutions that crosses the critical surface at $p=p^*$, and c_p is a constant. For $\Lambda=0$, there exists an analogous phase transition with the same exponent. In going from an open to closed space, we find $\beta=1$ suggesting that in that case there is no phase transition.

We now consider a collapsing spherical shell of dust in 2+1 dimensions and derive the junction conditions which allow us to relate the exterior and interior geometries. The trajectory of the shell forms a two-dimensional hypersurface Σ in the (2+1)-dimensional spacetime. Let the exterior (+) and interior (−) spherically symmetric metrics be given by

$$ds_\pm^2=-A_\pm^2(r_\pm)dt_\pm^2+B_\pm^2(r_\pm)dr_\pm^2+r_\pm^2d\phi^2. \tag{2}$$

The surface stress tensor for dust is $S_{\mu\nu}=\sigma u_\mu u_\nu$ where $u^\mu\equiv(\partial/\partial\tau)^\mu$ and σ are the three-velocity and the mass density of the shell with τ the proper time. The junction conditions across the shell are (1) continuity of the induced two-dimensional metric h_{ij} on Σ and (2) a discontinuity of the extrinsic curvature determined by the shell's stress tensor. The components of u^μ are $\dot{t}_\pm,\dot{r}_\pm$ inside and outside the shell where the overdot is equivalent to $d/d\tau$. The induced two-metric is given by

$$dl^2=-d\tau^2+r_\pm^2(\tau)d\phi^2. \tag{3}$$

Continuity of h_{ij} across the shell yields

$$r_+(\tau)=r_-(\tau)\equiv R(\tau). \tag{4}$$

*Electronic address: yoav@alpha2.csd.uwm.edu
†Electronic address: steif@dirac.ucdavis.edu

The second junction condition can be obtained by decomposing Einstein's equation into components normal and tangential to Σ. The components of the normal n^μ to the hypersurface are $n^t = \pm(B/A)u^r$ and $n^r = \mp(A/B)u^t$. Integrating the tangential components of Einstein's equations across Σ, one obtains the junction condition [15]:

$$[K_{ij} - h_{ij}K^l_l] = 8\pi G S_{ij} \tag{5}$$

where $K_{ij} \equiv h^k_i h^l_j \nabla_k n_l$ is the extrinsic curvature and $i,j,\ldots$ are tangential components. The brackets denote the discontinuity of the enclosed expression across the shell. From $[G_{ni}]=0$ and the junction condition (5), one obtains the conservation equation

$$S^j_{i|j} = 0 \tag{6}$$

where $|j$ is the covariant derivative in Σ.

The $\tau\tau$ component of (5) for the metric (2) becomes

$$\frac{1}{r}\left[\left(\frac{1}{B^2} + (u^r)^2\right)^{1/2}\right] = 8\pi G\sigma. \tag{7}$$

Projecting the conservation equation (6) onto u^μ produces an equation for σ:

$$d\sigma/d\tau + u^i_{|i}\sigma = 0. \tag{8}$$

From the two-metric (3), one finds $u^i_{|i} = \dot{r}/r$. Substituting into (8) and solving for σ leads to

$$\sigma = \mu/2\pi r \tag{9}$$

where the constant μ is the rest mass of the shell. The other component of (6) implies that the dust trajectory is a geodesic of the two-metric which is already clear from the form of (3). Substituting (9) into (7) yields

$$\mathrm{sgn}(n_-)\left(\frac{1}{B_-^2} + \dot{r}^2\right)^{1/2} + \mathrm{sgn}(n_+)\left(\frac{1}{B_+^2} + \dot{r}^2\right)^{1/2} = 4G\mu, \tag{10}$$

where $\mathrm{sgn}(n_{-/+})$ is the orientation of the normal to Σ in the inside and outside spaces, respectively [16]. One can easily see that the accelerations on both sides of the shell vanish. This contrasts with 3+1 dimensions where the accelerations are nonzero, but equal and opposite on the two sides of the shell.

All spacetime solutions to vacuum 2+1 Einstein gravity with vanishing cosmological constant are locally flat, but can have nontrivial global identifications. Consider the general static spherically symmetric spacetime [12]

$$ds^2 = -\gamma dt^2 + dr^2/\gamma + r^2 d\phi^2, \quad 0 \leq \phi \leq 2\pi \tag{11}$$

where γ is a constant. $\gamma = 1$ corresponds to Minkowski space. For $0 < \gamma \equiv \alpha^2 < 1$, the spatial geometry of (11) describes a cone with a deficit angle $\Delta\phi = (1-\alpha)2\pi$ and mass $m = (1-\alpha)/4G$ [12]. For $\gamma < 0$, (11) describes Taub-NUT (Misner) space [17] with t spacelike and r timelike. Now, consider a dust shell with a flat interior geometry $[B_- = 1,\ \mathrm{sgn}(n_-) = 1]$ and with exterior geometry given by (11). (10) then yields

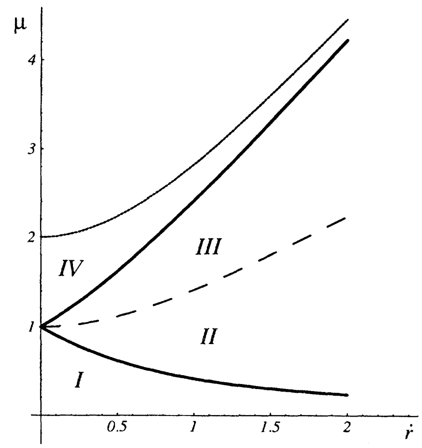

FIG. 1. Solution space for $\Lambda = 0$ with $G = 1/4$. The lower and upper solid lines correspond to $\mu = \mu_{c1}$ ($\gamma = 0$) and $\mu = \mu_{c3}$ ($\gamma = 0$), respectively. The dashed line corresponds to $\mu = \mu_{c2}$ ($\gamma = -\dot{r}^2$). The upper dotted line corresponds to $\mu = \mu_{c4}$. The solutions in the four regions are (I) open cones, (II) open Taub-NUT spaces, (III) closed Taub-NUT spaces, and (IV) closed cones.

$$(1 + \dot{r}^2)^{1/2} + \mathrm{sgn}(n_+)(\gamma + \dot{r}^2)^{1/2} = 4G\mu. \tag{12}$$

Since μ and γ are constants, we recover the radial geodesic equation, $\ddot{r} = 0$.

For a given μ and $\dot{r}$, (12) determines the exterior geometry. There is no dependence on the radius of the shell because there is no length scale in 2+1 gravity with $\Lambda = 0$. Consider fixed $\dot{r}$ and increase μ from zero. There are four ranges of μ describing qualitatively different exterior geometries shown in Fig. 1.

The four regions are as follows.

$$\text{(I) } 0 < \mu < \mu_{c1} = (4G)^{-1}[(1 + \dot{r}^2)^{1/2} - |\dot{r}|] \quad [\mathrm{sgn}(n_+) = -1].$$

The exterior geometry is an open cone with mass m given by

$$m = (4G)^{-1}\{1 - [1 - 8G\mu(1 + \dot{r}^2)^{1/2} + 16G^2\mu^2]^{1/2}\}. \tag{13}$$

At each time t, space is described geometrically by a truncated cone of deficit angle $\Delta\phi = 8\pi G m$. The top of the cone corresponding to the inside of the shell is replaced by a flat disk [Fig. 2(a)].

For a shell at rest ($\dot{r} = 0$), $m = \mu$ is recovered. As $\mu \to \mu_c = 1/4G$, the cone limits to an infinite cylinder [Fig. 2(b)]. The coordinates (t, r, ϕ) are singular in that case. One can define the coordinates $T = \alpha t$ and $\rho = (r_s - r)/\alpha$ (where r_s is the constant radius of the shell). In the limit $\alpha \to 0$ we get the infinite cylinder metric $ds^2 = -dT^2 + d\rho^2 + r_s^2 d\phi^2$. For $\dot{r} \neq 0$, the mass (13) depends on the kinetic energy of the shell as well. In this case, we can obtain the limiting geometry for $\mu \to \mu_{c1}$ by defining the coordinates $u = \alpha^2 t$ and[1]

[1]We would like to thank Jorma Louko for this important observation.

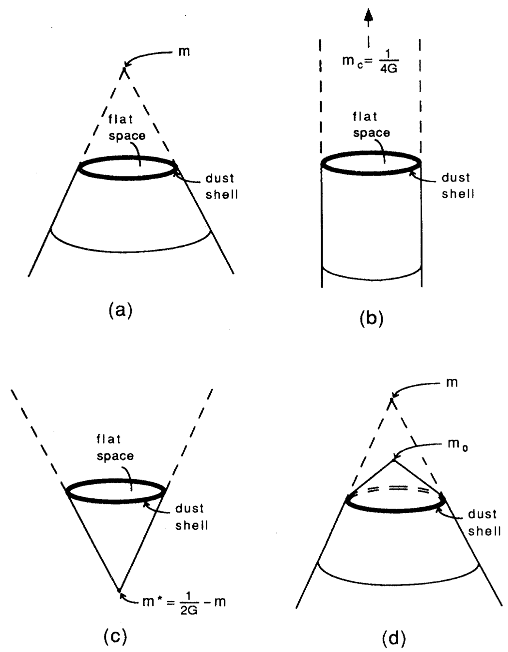

FIG. 2. Spatial 2 geometries for $\Lambda=0$ and $\dot{r}=0$: (a) open conical space ($m<m_c=1/4G$), (b) the infinite static cylinder ($m=m_c$), (c) closed conical space ($m>m_c$), and (d) collapse onto a particle of mass m_0.

$$v=r/\alpha^2-|\dot{r}_s|t/\sqrt{\alpha^2+\dot{r}_s^2}.$$

The metric has a smooth $\alpha\to 0$ limit approaching $ds^2=-(\dot{r}_s)^{-2}du^2+2du\,dv+u^2d\phi^2$. The vector $\partial/\partial\phi$ is a null-rotation generator, and in Minkowskian coordinates (x^0,x^1,x^2), it can indeed be expressed as a linear combination of rotation and boost generators

$$\frac{\partial}{\partial\phi}=\left(x^2\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^2}\right)+\left(x^2\frac{\partial}{\partial x^0}+x^0\frac{\partial}{\partial x^2}\right). \quad (14)$$

The spacetime outside the shell is therefore three-dimensional Minkowski space with the null-rotation identification $\phi\sim\phi+2\pi$ [18].

$$\text{(II)}\quad \mu_{c1}<\mu<\mu_{c2}=\frac{1}{4G}(1+\dot{r}^2)^{1/2}\quad [\mathrm{sgn}(n_+)=-1].$$

The exterior geometry is Taub-NUT, described by (11) with $\gamma<0$ and $r>r_s$.

$$\text{(III)}\quad \mu_{c2}<\mu<\mu_{c3}=\frac{1}{4G}[(1+\dot{r}^2)^{1/2}+|\dot{r}|]\qquad [\mathrm{sgn}(n_+)=+1].$$

The exterior geometry is closed Taub-NUT, described by (11) with $\gamma<0$ and $r<r_s$.

$$\text{(IV)}\quad \mu_{c3}<\mu<\mu_{c4}=\frac{1}{2G}(1+\dot{r}^2)^{1/2}\quad [\mathrm{sgn}(n_+)=+1].$$

The exterior geometry is a closed cone [see Fig. 2(c)] with a mass

$$m=\frac{1}{4G}\{1+[1-8G\mu(1+\dot{r}^2)^{1/2}+16G^2\mu^2]^{1/2}\}. \quad (15)$$

Now there is an additional conical singularity outside the shell with mass $m^*=1/2G-m$ fixed by the Euler number of the space [see Fig. 2(c)]. As $\mu\to\mu_{c4}$, the geometry degenerates to a disk. For $\mu>\mu_{c4}$, m^* becomes negative so the physical solution space is the region $\mu<\mu_{c4}$ in Fig. 1.

From Fig. 1, one observes that ($\mu=1/4G$, $\dot{r}=0$) corresponding to the static infinite cylinder solution [Fig. 2(b)] is a special point in the solution space, in which *all* four phases coexist. For a shell at rest ($\dot{r}=0$) regions (II) and (III) are absent. As μ exceeds $\mu_c=1/4G$, the geometry goes directly from an open cone to a closed cone with the infinite cylindrical geometry at $\mu=\mu_c$. The mass m_c is exactly the bound that was found in the general case of (2+1)-dimensional gravity [19]. As was observed in [12], one can go beyond $m=m_c$.

All classical solutions to 2+1 gravity with $\Lambda<0$ correspond to anti–de Sitter (AdS) space locally. However, nontrivial global identifications can lead to very different properties including different values of the mass as well as the existence of event horizons or naked singularities. For the analysis of collapsing dust shells, it is most convenient to describe the solutions analytically.

The general spherically symmetric static solution in (2+1)-dimensional cosmological gravity can be written as

$$ds^2=-\left(\frac{r^2}{l^2}+\gamma\right)dt^2+\left(\frac{r^2}{l^2}+\gamma\right)^{-1}dr^2+r^2d\phi^2,$$

$$l=(-\Lambda)^{-1/2},\quad 0<\phi<2\pi \quad (16)$$

where γ is a constant. $\gamma=1$ corresponds to anti–de Sitter (AdS) space. For $0<\gamma\equiv\alpha^2<1$, we have anti–de Sitter space with a deficit angle $\Delta\phi=2\pi(1-\alpha)$ describing a point particle with a mass $m=(1-\alpha)/4G$ [13]. In the limit $\Lambda\to 0$, one recovers the point particle solutions in [12]. For $\gamma<0$ the solution (16) is a black hole with an Arnowitt-Deser-Misner (ADM) mass $M=-\gamma/8G$ defined relative to the $M=0$ vacuum. The singularity $r=0$ is located behind an event horizon at $r_H=\sqrt{8GMl}$. The black hole solutions approach anti–de Sitter space asymptotically. They can also be obtained geometrically from anti–de Sitter space by identifying under the action of a boost. The $M=0$ solution has an infinite throat of vanishing radius.

One should distinguish the Banados-Teitelboim-Zanelli (BTZ) mass M (which is in fact the ADM [20] mass) from the Deser–Jackiw–'t Hooft (DJ'tH) mass m defined in [12,13]. The difference is due not only to different choices of vacua, but to different choices of time slices as well. The DJ'tH mass is defined as [12] $m\equiv\int\sqrt{g^{(2)}}T^0_0d^2x$, where $g^{(2)}$ is the *two*-dimensional metric on a spacelike surface. On the other hand, the BTZ mass is defined as $M\equiv\int\sqrt{-\bar{g}^{(3)}}T^0_0d^2x$ where $\bar{g}^{(3)}$ is the *three*-dimensional background metric. In the definition of the BTZ mass, t in (16) is the time parameter, while in the definition of the DJ'tH mass, it is αt.

Let us now consider the collapse of a dust shell in the geometry (16). Let γ_+ and γ_- denote the values of γ for the exterior and interior geometries. Equation (10) yields

$$\mathrm{sgn}(n_-)(\gamma_- + r^2/l^2 + \dot{r}^2)^{1/2} + \mathrm{sgn}(n_+)(\gamma_+ + r^2/l^2 + \dot{r}^2)^{1/2} = 4G\mu. \quad (17)$$

Equation (17) implies $r^2/l^2 + \dot{r}^2$ is a constant and is consistent with the radial geodesic equation $\ddot{r} = -(1/l^2)r$. $\gamma_+ > 0$ and $\gamma_+ < 0$ correspond to the formation of a conical naked singularity and a black hole, respectively.

For the case of collapse into the anti–de Sitter vacuum $[\gamma_- = 1,\ \mathrm{sgn}(n_-) = 1]$ with an exterior open space $[\gamma_+ = -8GM,\ \mathrm{sgn}(n_+) = -1]$, (17) yields

$$M = -\frac{1}{8G} + (1 + r^2/l^2 + \dot{r}^2)^{1/2}\mu - 2G\mu^2 \quad \text{(collapse in AdS space)}. \quad (18)$$

M increases with $\dot{r}$, the kinetic energy of the shell as well as with r due to the increasing anti–de Sitter potential energy. We see that M can be negative (naked conical singularity) or positive (black hole), depending on μ, r, and $\dot{r}$. For $M < 0$, the corresponding DJ'tH mass is well defined and is given by

$$m = \frac{1}{4G}\{1 - [1 - 8G\mu(1 + r^2/l^2 + \dot{r}^2)^{1/2} + 16G^2\mu^2]^{1/2}\} \quad \text{(collapse in AdS space)}. \quad (19)$$

For an exterior closed space $[\mathrm{sgn}(n_+) = 1]$ (10) yields

$$(\gamma_- + r^2/l^2 + \dot{r}^2)^{1/2} + (\gamma_+ + r^2/l^2 + \dot{r}^2)^{1/2} = 4G\mu. \quad (20)$$

As a function of μ, r, and $\dot{r}$, the BTZ mass is again given by (18) while the DJ'tH mass is now

$$m = (4G)^{-1}\{1 + [1 - 8G\mu(1 + r^2/l^2 + \dot{r}^2)^{1/2} + 16G^2\mu^2]^{1/2}\}. \quad (21)$$

The total mass (18) can be viewed as a function of three parameters: the dust shell rest mass μ, its initial radius $r_0 = r(0)$, and its initial velocity $\dot{r}_0 = \dot{r}(0)$. The space of solutions is the three-dimensional space $\mathscr{S} \equiv \{\mu, r_0, \dot{r}_0\}$, shown in Fig. 3. The solution space for $\Lambda = 0$ (Fig. 1) is the restriction of $\mathscr{S}$ to the $r_0 = 0$ plane.

As in the $\Lambda = 0$ case, there are four qualitatively different regions in $\mathscr{S}$ corresponding to four ranges of μ as a function of r and $\dot{r}$. The solutions in the four regions are (I) $0 < \mu < \mu_{c1}$, open conical AdS spaces, (II) $\mu_{c1} < \mu < \mu_{c2}$, exteriors of BTZ black holes $(r > r_s)$, (III) $\mu_{c2} < \mu < \mu_{c3}$, interiors of BTZ black holes $(r < r_s)$, and (IV) $\mu_{c3} < \mu < \mu_{c4}$, closed conical AdS spaces. The critical values of μ separating the regions can be obtained from those for $\Lambda = 0$ by the replacement $\dot{r}^2 \to \dot{r}^2 + r^2/l^2$. On the surface separating closed and open spaces II and III, $\gamma = -(r_0^2/l^2 + \dot{r}_0^2)$. The exterior geometries associated with the solutions on the critical surface separating regions I and II (III and IV) are the *exterior* $(r > r_s)$ [*interior* $(r < r_s)$] of the infinite throat $M = 0$ black hole solution. The point

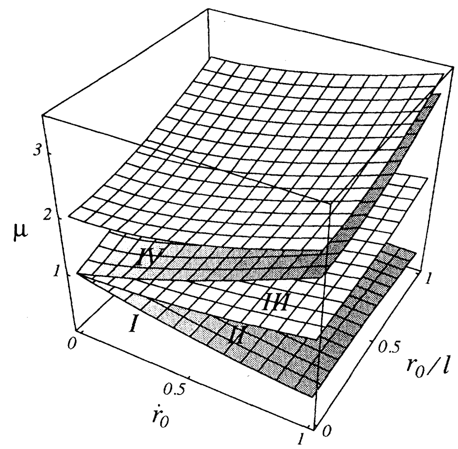

FIG. 3. Solution space $\mathscr{S}$ for shell collapse in AdS vacuum $(G = 1/4)$.

$(\mu = 1/4G,\ r_0 = 0,\ \dot{r}_0 = 0)$ is the *complete* infinite throat $M = 0$ vacuum solution. This is a special point in solution space in which *all* the four phases coexist.[2]

Collapse in the $M = 0$ vacuum $(\gamma_- = 0)$ results in one of the following three possibilities: (1) $0 < \mu < (1/4G) \times (r^2/l^2 + \dot{r}^2)^{1/2}$, an open BTZ black hole; (2) $(1/4G) \times (r^2/l^2 + \dot{r}^2)^{1/2} < \mu < (1/2G)(r^2/l^2 + \dot{r}^2)^{1/2}$, a closed BTZ black hole; (3) $(1/2G)(r^2/l^2 + \dot{r}^2)^{1/2} < \mu$, a closed conical AdS space.

We now study the structure of the phase transition as one crosses the two $\gamma = 0$ critical surfaces separating regions (I) and (II) and regions (III) and (IV). Following [1–7], we look for an order parameter related to the total mass. In approaching the critical surfaces from regions I and IV, we use the following order parameter which is related to the DJ'tH mass and vanishes on both critical surfaces $(m_c = 1/4G)$:

$$\mathscr{M} \equiv \alpha = 1 - 4Gm. \quad (22)$$

The Minkowski space (for $\Lambda = 0$) and AdS space (for $\Lambda < 0$) solutions with $\mathscr{M} = 1$ $(m = 0)$ might be thought of as the ordered vacua.

From (19) and (21) one can obtain the behavior of $\mathscr{M}$ near the critical surfaces. In the direction of the principle axes, $\mathscr{M}$ takes the form

$$\mathscr{M} \sim \begin{cases} \mp(8G)^{1/2}[(r_{0c}^{\pm})^2/l^2 + (\dot{r}_{0c}^{\pm})^2]^{1/4}|\mu - \mu_c^{\pm}|^{1/2}, \\ \mp\left(\dfrac{8G\mu_c^{\pm}\dot{r}_{0c}^{\pm}}{1 + (r_{0c}^{\pm})^2/l^2 + (\dot{r}_{0c}^{\pm})^2}\right)^{1/2}|\dot{r}_0 - \dot{r}_{0c}^{\pm}|^{1/2}, \\ \mp\left(\dfrac{8G\mu_c^{\pm}r_{0c}^{\pm}l^{-2}}{1 + (r_{0c}^{\pm})^2/l^2 + (\dot{r}_{0c}^{\pm})^2}\right)^{1/2}|r_0 - r_{0c}^{\pm}|^{1/2}, \end{cases} \quad (23)$$

where $\mu_c^{\pm}(r_{0c}^{\pm}, \dot{r}_{0c}^{\pm})$ can be obtained from (18) by setting $M = 0$. The lower (upper) signs in (23) refer to the lower

[2]The $M = 0$ solution is very similar to higher-dimensional extremal black holes and its special role in the solution space raises the question whether this is a general feature of extremal black holes.

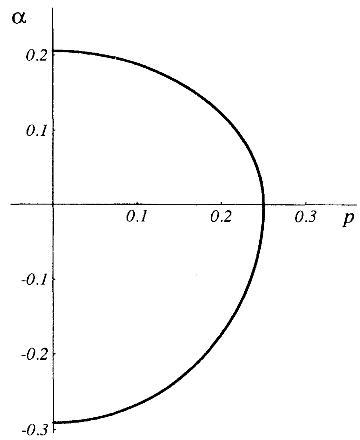

FIG. 4. The order parameter α as a function of $p=\dot{r}_0$ with $r_0^+/l=r_0^-/l=1/4$ and $4G\mu^+=(4G\mu^-)^{-1}=2^{1/2}$. The upper part, $\alpha>0$, corresponds to region I in Fig. 3, and the lower part, $\alpha<0$, corresponds to region IV.

(upper) shaded surfaces in Fig. 3 (see also Fig. 4). From (23), we see that the critical exponent is $\beta=1/2$ as one approaches the critical surface from *any* direction.

Consider approaching the critical surfaces from regions II and III. The DJ'tH mass in those regions is ill defined. A geometrical quantity which is sensitive to the phase transition and is a continuation of α to regions II and III is the black hole radius[3] r_H. Using (18), we find $r_H \sim c_p(p-p^*)^{1/2}$ and thus again obtaining the same exponent $\beta=1/2$ as in (23).

A value of 1/2 for the critical exponent β was found in certain models in 3+1 dimensions [5–7] and also in 1+1 dimensions [4]. Our critical solution does not appear to be self-similar. So it is not clear if there is a deeper relation between these models and the one considered here. However, since our critical solution is very similar to a higher dimensional extremal black hole, a natural question is whether a phase transition also occurs in other models of the formation of extremal black holes. Some indications for that can be found in [5].

Near the $\gamma=-(r_0^2/l^2+\dot{r}_0^2)$ surface, one finds $\beta=1$ using either α or r_H or M as order parameter. Thus, there is no phase transition in crossing from an open to closed space. A way to see the distinct behavior of the transition between open and closed spaces is to consider the static case with $\Lambda=0$. Now, the only parameter is μ, and the transition connects regions I and IV directly. We find $\mathscr{M}=c_\mu(\mu-\mu^*)$, and thus indeed $\beta=1$.

In this paper, we described the phase transition which occurs in the gravitational collapse of dust shells in (2+1)-dimensional gravity. We found that the solution space is divided into four qualitatively distinct regions. There is critical behavior near the transition between black holes and solutions with naked singularities. One can easily generalize our results to higher (or lower) dimensional solution spaces. Consider for example collapse into a conical space, with interior mass m_0 [Fig. 2(d)]. We then have a four-dimensional solution space, and furthermore, one finds that $\mathscr{M}\sim c_{m_0}(m_0-m_{0c})^{1/2}$ as well. On the other hand, in the case of the collapse of a homogeneous ball of dust in 2+1 dimensions [21], the total mass is given by $M=\mu-1/8G$ and is therefore only a function of μ. So we have effectively a one-dimensional solution space, and if we use our definition for the order parameter, we find $\mathscr{M}=(\mu-\tilde{\mu}_c)^{1/2}$, where $\tilde{\mu}_c=1/8G$. So, it seems that our results and in particular the value of the critical exponent are quite general in the case of spherical dust collapse in 2+1 dimensions.

We would like to thank S. Carlip, S. Deser, J. Louko, R. Mann, and L. Parker for very helpful discussions. Y.P. would like to acknowledge the financial support of NSF Grants No. NSF-91-05935 and No. NSF-93-15811. A.R.S. would like to acknowledge the financial support of the SERC at Cambridge and NSF Grant No. PHY-93-57203 at Davis.

[3]In [1–7], the black hole mass was used as order parameter, but in four dimensions, this is proportional to the black hole radius.

[1] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[2] A. M. Abrahams and C. R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[3] J. Coleman and C. R. Evans, Phys. Rev. Lett. **72**, 1842 (1994).
[4] A. Strominger and L. Thorlacius, Phys. Rev. Lett. **72**, 1584 (1994).
[5] J. Traschen, Phys. Rev. D **50**, 7144 (1994).
[6] V. Husain *et al.*, Phys. Rev. D **50**, 3783 (1994).
[7] P. R. Brady, Phys. Rev. D (to be published).
[8] T. Koike and T. Mishima, Phys. Rev. D (to be published).
[9] D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979).
[10] D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984).
[11] P. S. Joshi and T. P. Singh, Phys. Rev. D (to be published); I. H. Dwivedi and P. S. Joshi, Report No. gr-qc/9405049 (unpublished); T. P. Singh and P. S. Joshi, Report No. gr-qc/9409062 (unpublished).
[12] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[13] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).
[14] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[15] W. Israel, Nuovo Cimento B **44**, 1 (1966).
[16] D. S. Goldwirth and J. Katz, Tel-Aviv University and Hebrew University Report No. gr-qc/9408034 (unpublished).
[17] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973).
[18] G. Horowitz and A. Steif, Phys. Lett. B **258**, 91 (1991).
[19] A. Ashtekar and M. Varadarajan, Phys. Rev. D **50**, 4944 (1994).
[20] L. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).
[21] R. B. Mann and S. F. Ross, Phys. Rev. D **47**, 3319 (1993).