RAPID COMMUNICATIONS

PHYSICAL REVIEW D VOLUME 51, NUMBER 2 15 JANUARY 1995

New look at the critical behavior near the threshold of black hole formation in the Russo-Susskind-Thorlacius model

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(Received 27 September 1994)

The dependence of the critical exponent β on the shape of the incoming flux near the threshold of black hole formation is first studied in the context of the Russo-Susskind-Thorlacius (RST) model. In order to describe a generic incoming flux, two parameters (α,n) are first introduced. The critical exponent β is found to be $1/n$, which is parameter dependent. And $\beta=0.5$ is not universal; it is just a special case for $n=2$. The apparent horizon and singularity curves for the generic parameter n are also evaluated in the scaling limit, which do not take the universal form. The singularity curve for $n=1$ even includes the parameter α. All of these indicate that critical phenomena perhaps do not exist in the RST model due to the linear nature of the RST equations which also results in no self-similar oscillations in the RST model.

PACS number(s): 04.70.Dy, 04.60.Kz, 05.70.Jk, 97.60.Lf

Recently, critical phenomena have been shown to occur near the threshold of black hole formation. Choptuik [1] first observed this critical behavior in the spherical collapse of a massless scalar field. Abrahams and Evans [2] found a similar behavior in the axisymmetric collapse of gravitational waves. Critical phenomena have now also been observed in radiation fluid collapse [3]. Some of Choptuik's results have recently been confirmed by Gundlach, Price, and Pullin [4]. All of these seem to indicate that critical behavior is a generic feature of gravity.

Nevertheless, Choptuik [1] only numerically discussed the mass $M_{\rm BH}$ of the resulting black hole as a function of the distance δ in the initial data space from the threshold. For small δ he finds a power law $M_{\rm BH}=K\delta^{\beta}$, where the critical exponent β is near 0.37, and his result appears quite universal. But there is little analytic or conceptual understanding of these interesting phenomena. In order to give an analytical explanation of these phenomena, Strominger and Thorlacius [5] first studied the corresponding power law $M_{\rm BH}=K\delta^{\beta}$, in the context of the two-dimensional Russo-Susskind-Thorlacius (RST) model [6,7]. They discovered the critical exponent β equals 0.5, and though their result was quite universal and insensitive to the precise definition of δ or scalar field couplings. In addition, Strominger and Thorlacius also found there are some apparent differences between their results and those of Ref. [1]. First, their critical exponent is a rational number whereas Choptuik's appears to be irrational. Second, there is no analogue of the self-similar oscillations in the RST model. They guess this may be a special feature arising from the linear nature of the RST equations, and expect the more general two-dimensional models (which are not analytically soluble) might exhibit such oscillations.

As is well known, the self-similar solution which exhibits local self-similarity should be closely related to existence of critical phenomena [3,8]. Because of the absence of the self-similar oscillations for the RST model, one may ask whether the critical phenomena occur near the threshold of black hole formation, that is, whether or not the critical exponent $\beta=0.5$ is universal in the RST model. In order to answer this question, the dependence of the critical exponent β on the shape of the incoming flux near the threshold of black hole formation is first studied in the context of the RST model. And for describing a generic incoming pulse two parameters (α,n) are first introduced. In the scaling limit, we show that

$$M_{\rm BH}=K\delta^{1/n}. \tag{1}$$

So the critical exponent β is $1/n$, which depends on the parameter n of the incoming flux. And $\beta=0.5$ as discussed in Ref. [5] is not universal, it is just a special case for $n=2$. The apparent horizon and singularity curves for the generic parameter n are also evaluated in the scaling limit, which do not take the universal form. The singularity curve for $n=1$ even contains the parameter α. All of these indicate that critical phenomena perhaps do not exist in the RST model. The reason for this is that critical behavior is a nonlinear dynamical behavior, so critical phenomena should not appear from the linear RST equations. And no self-similar oscillations in the RST model also reflect the linear nature of the RST equations.

Now let us consider the RST model; the semiclassical effective action is, in the conformal gauge [6],

$$S=\frac{1}{\pi}\int d^2x\left[\left(2e^{-2\phi}-\frac{N}{12}\phi\right)\partial_+\partial_-\rho + e^{-2\phi}(\lambda^2e^{2\rho}-4\partial_+\phi\partial_-\phi) + \frac{1}{2}\sum_{i=1}^{N}\partial_+f_i\partial_-f_i-\frac{N}{12}\partial_+\rho\partial_-\rho\right], \tag{2}$$

where ρ is the conformal factor, ϕ is the dilation and f_i are N scalar matter fields. This model differs from the original Callan-Giddings-Harvey-Strominger model [9] by a finite local counterterm, which restores a global symmetry of the classical theory.

In the Kruskal gauge with $\rho=\phi+\frac{1}{2}\ln(N/12)$, the semiclassical equations are simply

$$\partial_+\partial_- f_i=0,\quad \partial_+\partial_-\Omega=-\lambda^2,\quad -\partial_\pm^2\Omega=T^f_{\pm\pm}+t_\pm, \tag{3}$$

where

$$\Omega=\frac{12}{N}e^{-2\phi}+\frac{1}{2}\phi+\frac{1}{4}\frac{N}{48}, \tag{4}$$

$$T^f_{\pm\pm}=\frac{6}{N}\sum_{i=1}^{N}(\partial_\pm f_i)^2. \tag{5}$$

The function $t_+(x^+)$ takes the value $t_+=-1/4(x^+)^2$ in Kruskal coordinates. The field redefinition (4) is degenerate at $\Omega=\frac{1}{4}$ and $\Omega<\frac{1}{4}$ does not correspond to a real value of ϕ. RST impose the following boundary conditions at the curve $\Omega=\frac{1}{4}$, wherever it is timelike [6]:

$$\partial_\pm\Omega|_{\Omega=1/4}=0. \tag{6}$$

The boundary conditions (6) ensure semiclassical energy conservation and also that the physical curvature remains finite at the boundary curve as long as it is timelike.

The solution corresponding to incoming matter energy flux is given by [5,6]

$$\Omega(x^+,x^-)=-\lambda x^+\left(\lambda x^-+\frac{1}{\lambda}P_+(x^+)\right)+\frac{1}{\lambda}M(x^+)-\frac{1}{\lambda}M[x_B^+(x^-)]-\frac{1}{4}\ln[x^+/x_B^+(x^-)], \tag{7}$$

where

$$M(x^+)=\lambda\int_0^{x^+}dy^+y^+T^f_{++}(y^+), \tag{8a}$$

$$P_+(x^+)=\int_0^{x^+}dy^+T^f_{++}(y^+), \tag{8b}$$

and $x_B^+(x^-)$ is the x^+ value of the point on the boundary curve from which the reflected signal propagates to (x^+,x^-). Wherever the boundary curve is timelike, it is described by

$$\lambda^2 x_B^-=-P_x(x_B^+)-\frac{1}{4x_B^+}. \tag{9}$$

However, if the incoming energy flux becomes larger than the outgoing Hawking flux of a two-dimensional black hole, $T^f_{++}(x^+)>1/4(x^+)^2$, for some value of x^+, the boundary curve becomes spacelike. The spacelike segments of the boundary curve are curvature singularities. They form inside regions of future trapped points which are bounded by an

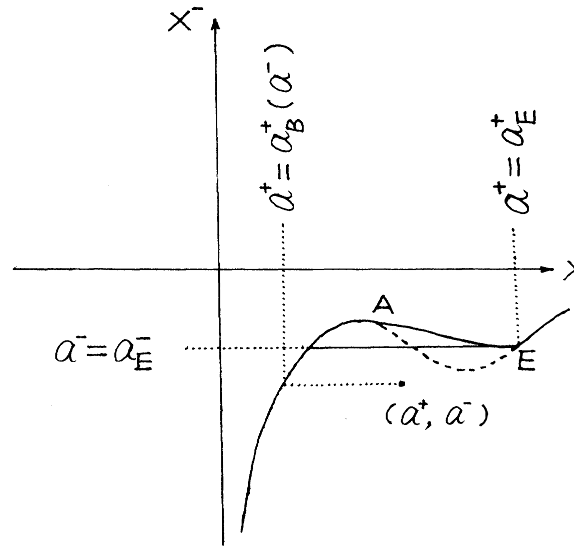

FIG. 1. Kruskal diagram for black hole formation and evaporation in the scaling region. A near-critical flux of matter energy is incident from $x^-=-\infty$. The solid curve is the $\Omega=\frac{1}{4}$ boundary and the dashed curve is the apparent horizon. The spacelike portion of the solid curve is the black hole singularity.

apparent horizon, located at $\partial_+\Omega=0$ [10]. As pointed out in Refs. [11,6], the apparent horizon coincides with the boundary curve where the latter is timelike, but where the boundary becomes spacelike the two curves separate and the apparent horizon cloaks the singularity, as shown in Fig. 1. Once the incoming energy flux falls below the threshold value, the black hole evaporates and the apparent horizon approaches the singularity. The curves join again at the end point of the evaporation, which is denoted by E in Fig. 1.

The null line segment $x^-=x_E^-$, $x_B^+(x_E^-)<x^+<x_E^+$, is the global event horizon of the geometry. We define the black hole mass as the incoming energy of the null matter swallowed by the black hole during its lifetime:

$$M_{\rm BH}=M(x_E^+)-M[x_B^+(x_E^-)]. \tag{10}$$

Since both ends of the global horizon are on the boundary curve and also on the apparent horizon, it follows from (7) and (9) that

$$M_{\rm BH}=\frac{\lambda}{4}\ln[x_E^+/x_B^+(x_E^-)]. \tag{11}$$

In order to discuss arbitrarily small black hole formation just above threshold, we first introduce two parameters (α,n) to describe a generic incoming flux, which is given by

$$T^f_{++}(x^+)=(\tfrac{1}{4}+\delta)[1-\alpha(|\ln\lambda x^+|)^n+\cdots](x^+)^{-2}, \tag{12}$$

where higher orders in the Taylor expansion of T^f_{++} only contribute terms with higher positive powers of δ, which can be ignored in the scaling limit. The incoming flux depicted by (12) has a maximum at $x^+=1/\lambda$, where its value is $\lambda^2(\frac{1}{4}+\delta)$, and it is below threshold everywhere except near $x^+=1/\lambda$ so that only a single small black hole is formed. From the physical viewpoint, the incoming flux with different parameters n give the same physical picture; i.e., they all describe a single small black hole formation near the threshold.

Now let us evaluate the Kruskal coordinate of point A (x_A^+,x_A^-), where the boundary curve turns spacelike. As shown in Fig. 1, x_A^- takes a maximum value on the apparent horizon curve; i.e.,

$$\left.\frac{dx_H^-}{dx_H^+}\right|_{x_H^+=x_A^+,x_H^-=x_A^-}=0. \tag{13}$$

Applying d/dx_H^+ to both sides of (9) yields

$$\lambda^2\frac{dx_H^-}{dx_H^+}=-T_{xx}^f(x_H^+)+\frac{1}{4(x_H^+)^2}. \tag{14}$$

Then we have, from (13),

$$-T_{xx}^f(x_A^+)+\frac{1}{4(x_A^+)^2}=0. \tag{15}$$

To leading order in δ, we obtain

$$\lambda x_A^+=1-4^{1/n}\left(\frac{\delta}{\alpha}\right)^{1/n}. \tag{16}$$

From (9), we get

$$\lambda x_A^-=-\frac{1}{\lambda}P_+(1/\lambda)-\tfrac{1}{4}+\delta\left(\frac{\delta}{\alpha}\right)^{1/n}\frac{n}{n+1}4^{1/n}. \tag{17}$$

Since we are interested in the $\delta\ll 1$ limit, it is convenient to shift and rescale the Kruskal coordinates around point A as follows:

$$\lambda x^+=1+\left(\frac{\delta}{\alpha}\right)^{1/n}[a^+-4^{1/n}], \tag{18a}$$

$$\lambda x^-=-\frac{1}{\lambda}P_+(1/\lambda)-\tfrac{1}{4}+\delta\left(\frac{\delta}{\alpha}\right)^{1/n}\left[a^-+\frac{n}{n+1}4^{1/n}\right]. \tag{18b}$$

The origin of the (a^+,a^-) coordinates system has been chosen where, to leading order in δ, the boundary curve turns spacelike, as shown in Fig. 1.

From (8b) and (12), we evaluate $P_+(x^+)$, then insert $P_+(x^+)$ and (18) into (9). In the scaling region in which higher order terms in δ can be dropped, the apparent horizon curve is

$$a_H^-(a_H^+)=\begin{cases}-(a_H^+-4^{1/n})+\dfrac{1}{4}\dfrac{(-1)^n}{n+1}(a_H^+-4^{1/n})^{n+1}-\dfrac{n}{n+1}4^{1/n} & \text{for } \lambda x^+\leq 1,\\[2ex] -(a_H^+-4^{1/n})+\dfrac{1}{4}\dfrac{1}{n+1}(a_H^+-4^{1/n})^{n+1}-\dfrac{n}{n+1}4^{1/n} & \text{for } \lambda x^+>1.\end{cases} \tag{19}$$

Equation (19) is independent of the parameter α except in the definition of the scaling variables (18), but it includes the parameter n indeed. As an example, we choose $n=2$, then (19) is reduced to

$$a_H^-(a_H^+)=-\tfrac{1}{2}a_H^{+2}+\tfrac{1}{12}a_H^{+3}, \tag{20}$$

which is the same as Eq. (14) of Ref. [5].

Even though (20) does not explicitly contain the parameter α, we still cannot say the apparent horizon is universal, since (20) is just a special case for $n=2$.

Expressing (7) in terms of (a^+,a^-) coordinates, the singularity curve for $n\geq 2$ can be obtained:

$$-\left(a_s^-+\frac{n}{n+1}4^{1/n}\right)(a_s^+-4^{1/n})-\tfrac{1}{2}(a_s^+-4^{1/n})^2+\frac{(-1)^n}{4}\left(-\frac{n+4}{2(n+2)}+\frac{1}{n+1}+\frac{1}{2}\right)(a_s^+-4^{1/n})^{n+2}+\tfrac{1}{2}[a_B^+(a_s^-)-4^{1/n}]^2$$
$$+\frac{(-1)^{n+1}}{8}[a_B^+(a_s^-)-4^{1/n}]^{n+2}=0 \quad \text{for } \lambda x^+\leq 1, \tag{21a}$$

$$-\left(a_s^-+\frac{n}{n+1}4^{1/n}\right)(a_s^+-4^{1/n})-\tfrac{1}{2}(a_s^+-4^{1/n})^2+\frac{1}{4}\left(-\frac{n+4}{2(n+2)}+\frac{1}{n+1}+\frac{1}{2}\right)(a_s^+-4^{1/n})^{n+2}+\tfrac{1}{2}[a_B^+(a_s^-)-4^{1/n}]^2$$
$$+\frac{(-1)^{n+1}}{8}[a_B^+(a_s^-)-4^{1/n}]^{n+2}=0 \quad \text{for } \lambda x^+>1. \tag{21b}$$

Equation (21) does not include the parameter α, but it is directly related to the parameter n. Especially for $n=1$, the singularity curve is also dependent on the parameter α, and this can be seen from the equations

$$-(a_s^-+2)(a_s^+-4)-\tfrac{1}{2}(a_s^+-4)^2-\tfrac{1}{24}(a_s^+-4)^3+\frac{1+\alpha}{2}[a_B^+(a_s^-)-4]^2+\tfrac{1}{8}[a_B^+(a_s^-)-4]^3=0 \quad \text{for } \lambda x^+\leqslant 1, \tag{22a}$$

$$-(a_s^-+2)(a_s^+-4)-\tfrac{1}{2}(a_s^+-4)^2+\tfrac{1}{24}(a_s^+-4)^3+\frac{1-\alpha}{2}[a_B^+(a_s^-)-4]^2+\tfrac{1}{8}[a_B^+(a_s^-)-4]^3=0 \quad \text{for } \lambda x^+>1. \tag{22b}$$

From the above equations, we find the apparent horizon and singularity curves contain the parameter n. Particularly, for $n=1$, the singularity curve is explicitly dependent on the parameter α. These results show that the apparent horizon and singularity curves do not take the universal form in the scaling limit.

Since both end points of the global event horizon are points on the apparent horizon curve (19), and the end point of evaporation (E) is also on the singularity curve, we can determine the coordinates of these two points from (19) and (21); i.e.,

$$a_E^-=-(a_B^+-4^{1/n})+\frac{1}{4}\frac{(-1)^n}{n+1}(a_B^+-4^{1/n})^{n+1}-\frac{n}{n+1}4^{1/n}, \tag{23a}$$

$$a_E^-=-(a_B^+-4^{1/n})+\frac{1}{4}\frac{1}{n+1}(a_B^+-4^{1/n})^{n+1}-\frac{n}{n+1}4^{1/n}, \tag{23b}$$

$$-\left(a_E^-+\frac{n}{n+1}4^{1/n}\right)(a_E^+-4^{1/n})-\tfrac{1}{2}(a_E^+-4^{1/n})^2 +\frac{1}{4}\left(-\frac{n+4}{2(n+2)}+\frac{1}{n+1}+\frac{1}{2}\right)(a_E^+-4^{1/n})^{n+2} +\tfrac{1}{2}(a_B^+-4^{1/n})^2+\frac{(-1)^{n+1}}{8}(a_B^+-4^{1/n})^{n+2}=0, \tag{23c}$$

where we only consider the $n\leqslant 2$ case. For $n=1$, (23c) should be replaced by (22b). Then we have the following solution for (23):

$$a_E^+=f_E^{(n)}, \quad a_B^+=f_B^{(n)} \quad \text{for } n\leqslant 2, \tag{24a}$$

$$a_E^+=f_E^{(1)}=f_E^{(1)}(\alpha), \quad a_B^+=f_B^{(1)}=f_B^{(1)}(\alpha) \quad \text{for } n=1. \tag{24b}$$

It is obvious from (23) and (24) that even though a_E^+, a_B^+ contain the parameter n, they do not include δ. For $n\geqslant 2$, a_E^+, a_B^+ are especially independent of the parameter α, but for $n=1$, a_E^+, a_B^+ are related to the parameter α indeed. Then the black hole mass (11) is to leading order seen to be

$$M_{\rm BH}=K(\alpha,n,\lambda)\,\delta^{1/n} \tag{25}$$

with

$$K(\alpha,n,\lambda)=\frac{\lambda}{4}\,\alpha^{-1/n}[f_E^{(n)}-f_B^{(n)}] \tag{26}$$

where we have used (18a) and (24) to derive (25). Equation (25) shows that the critical exponent β is $1/n$, which is parameter dependent.

So far we have successfully evaluated the apparent horizon and singularity curves in the scaling limit for the generic parameter n, and find the critical exponent β is $1/n$, which is parameter dependent. When $n=2$ is chosen, our result is reduced to the case of Ref. [5], i.e., $\beta=0.5$. From the above calculation, we know that our result is quite sensitive to the precise definition of δ, and the critical exponent β depends on the shape of the incoming flux. Now even though we have studied the dependence of the critical exponent β on the parameter n of the incoming flux, we have not determined whether β is still $1/n$ when the coupling constants of the theory change. A similar critical phenomenon near the threshold of weak coupling singularity formation has been recently discussed in Refs. [12–14]. As pointed out in Ref. [14], the presence of $\ln\delta$ in the expression for $\ln M_{\rm BH}$ may be particular of the exponential couplings. This suggests that $M_{\rm BH}=k\delta^{\beta}$ may be violated for other families of couplings. All of these indicate that critical phenomena which include power-law behavior, discrete scaling relations, and a form of universality might not occur in the RST model due to the linear nature of the RST equations which also results in no self-similar oscillations in the RST model.

One of us (J.-G.Z.) thanks the Alexander von Humboldt Foundation for financial support.

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