Phase averaging and generalized eikonal representations

H. M. Fried
Physics Department, Brown University, Providence, Rhode Island 02912

Y. M. Gabellini
Institut Non Linéaire de Nice, Université de Nice-Sophia Antipolis, UMR 129, Route des Lucioles, 06560 Valbonne, France
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A new, "phase-averaged" approximation to the coupled Green's functions of potential theory and quantum field theory is shown to lead to generalized eikonal approximations in which the nonperturbative effects of low-frequency virtual quanta can be extracted and represented for all processes, including determinantal factors not necessarily associated with particle scattering, in terms of a finite number of quadratures, depending on the interaction, the number of space-time dimensions, and the process considered. Derivations and discussions are given for potential theory and scalar and gauge field theory.

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I. INTRODUCTION

This is the first of what is intended to be a sequence of new and useful approximations to the interacting Green's functions of potential theory and quantum field theory. In this paper we describe the simplest, "zeroth" approximation to a new version of the exact Fradkin representation for the causal propagator $G_c(x,y|A)$, of a particle moving in a fictitious (or "background") field $A(z)$, which generates a (properly) symmetrized version of older no-recoil "Bloch-Nordsieck" (BN) models. In the second paper of this series, we define a new and exact representation, equivalent to the conventional Fradkin representation, and explain why the first few corrections to the present "zeroth" order, or "phase-averaged" approximation of this paper may be expected to yield quite respectable results for $G_c[A]$ well away from the usual limits of the eikonal approximation; these corrections, as well as the present phase-averaged approximation, may be expressed in terms of a finite number of quadratures. In subsequent papers we intend to apply these new forms to the calculation of various quantum field theory processes whose nonperturbative description had previously been infrared limited.

One of the long-standing goals of (a subset of) those who pursue nonperturbative, analytic approximations in quantum field theory has been the construction of eikonal approximations for both scattering and nonscattering processes. For the latter, the aim has been to extract, in a systematic way, the effects of low-frequency (or large-scale) virtual quanta exchanged when specified, external particle momenta of a specific process are absent. For scattering situations, there exists a standard set of techniques [1–3] for the representation and partial extraction of unitary scattering and production amplitudes in close-to-forward directions; but to the best of the authors' knowledge, only one such method, the "infrared (IR) method," has been devised and applied to the nonperturbative estimates of nonscattering processes [4].

The IR method was based upon reasonable and intuitive approximations to the exact Fradkin representation for coupled Green's functions in a variety of theories and applications [5], as will be the phase-averaged construction devised here. In the IR method applications of Ref. [4], a two-step approximation scheme was necessary, involving firstly a restriction to suitably low-frequency virtual quanta, and, secondly, the adoption of a multipole expansion whose output is expressed in terms of a numerical parameter of order unity, that can only be determined by arguments external to any such calculation. In effect, the results of Ref. [4] are estimates, rather than precise calculations. The present, phase-averaged method, which we henceforth denote by the symbol $\langle \mathrm{ph} \rangle$, will also require a restriction to suitably low frequencies, upon calculating any given effect, but there is no ambiguity whatsoever in any particular calculation. The $\langle \mathrm{ph} \rangle$ method rewrites the functions of any given Feynman graph in such a way that a suitably generalized (symmetrized) nonperturbative eikonal form appears for all relevant, internal propagators. One will find the apparently paradoxical statement that overall four-momentum conservation is maintained but four-momentum conservation is not explicitly satisfied at each internal vertex. Nevertheless, in a very real sense this $\langle \mathrm{ph} \rangle$ method (of the many variants possible) is unique, for it does produce overall four-momentum conservation, translational invariance as appropriate, and, for time-independent Hamiltonians, generates scattering amplitudes which satisfy time-reversal invariance. It should also be noted that corrections to this $\langle \mathrm{ph} \rangle$ approximation can be defined in a systematic way, although that subject will not be the burden of this paper. No undetermined proportionality constants appear; one calculates rather than estimates.

All such individual eikonalized Feynman graphs can then be summed, and the results written in terms of a few quadratures, at first for the coupled Green's functions of potential theory entering into the description of the field theory process, and then for the complete field theory

process itself. More precisely, a finite number of quadratures are necessary for such an eikonalized treatment involving a finite number of cluster-expansion coefficients and the insertions into other (scattering) processes generated by such closed-loop determinantal factors; but the sum over an infinite number of such connected, cluster-expansion coefficients would still require a correspondingly infinite number of such quadratures. However, all processes involving a finite set of such cluster coefficients can be expressed in this generalized eikonal method in terms of a finite number of quadratures.

For clarity and simplicity, consider a pair of interacting scalar quantum fields Φ and A, described by the simplest Lagrangian density in $D = d + 1$ (spatial+time) dimensions:

$$\mathcal{L} = -\frac{1}{2}\left[(\partial_\mu \Phi)^2 + m^2\Phi^2\right] - \frac{1}{2}\left[(\partial_\mu A)^2 + \mu^2 A^2\right] - g\Phi^2 A \ , \tag{1}$$

where g is a bare coupling constant (of dimension $[\text{mass}]^{(5-d)/2}$). The generating functional $\mathcal{Z}\{j,n\} \equiv \langle 0|(\exp[i\int[jA + \eta\Phi]])_+|0\rangle$ for this process can be expressed in terms of the c-number sources $j(x), \eta(y)$ and operator fields A, Φ in many ways, [6] and we shall here employ that variant previously used in the papers of Ref. [4] and by Fried [5]:

$$\langle S\rangle \mathcal{Z}\{j,n\} = \exp\left[\frac{i}{2}\int j\Delta_c^{(\mu)} j\right] \exp\left[-\frac{i}{2}\int \frac{\delta}{\delta A}\Delta_c^{(\mu)}\frac{\delta}{\delta A}\right] \times \exp\left[\frac{i}{2}\int \eta G_c[A]\eta\right] e^{L[A]}|_{A=\int \Delta_c^{(\mu)} j} \ , \tag{2}$$

where $\Delta_c^{(\mu)}$ and $G_c^{(m)}$ are bare, causal, fully relativistic propagators associated with the quanta of the fields A and Φ, respectively. Here, $G_c[A]$ denotes the Φ propagator defined in the presence of an effective c-number "background" field $A(z), G_c[A] = G_c^{(m)}[1 + gAG_c^{(m)}]^{-1}$, while the determinantal, or closed- (boson-) loop factor is given by $L[A] = -\frac{1}{2}\text{Trln}[1 + gAG_c^{(m)}]$. Equation (2) is a convenient way of representing the Gaussian-weighted functional integral of the exact field theory in terms of a conceptually simple "linkage" operator, $\exp(\mathcal{D}) = \exp[-\frac{i}{2}\int(\delta/\delta A)\Delta_c^{(\mu)}(\delta/\delta A)]$, acting upon the potential theory (or first quantized) forms involving $G[A]$ and $L[A]$ which stand to its right. The quantity $\langle S\rangle$ is the complete vacuum-to-vacuum probability amplitude which here appears as a normalization factor:

$$\langle S\rangle = e^{\mathcal{D}} e^{L[A]}|_{A=0} \ , \tag{3}$$

while the functional derivative involved in the computation of any process are to be followed by setting all sources equal to zero.

For example, the scattering of a pair of Φ particles, interacting by the exchange of virtual A quanta, may at first be represented by the configuration space amplitude:

$$M(x_1,y_1,x_2,y_2) = e^{\mathcal{D}} G_c(x_1,y_1|A)G_c(x_2,y_2|A) \times \frac{e^{L[A]}}{\langle S\rangle}\bigg|_{A=0} \tag{4}$$

in which, for simplicity, we suppress a symmetry involving permutation of the configuration indices. A mass-shell amputation performed on each configuration variable of (4) then generates the S-matrix element for this scattering process.

Neglect of the determinantal factor $L[A]$, which represents the sum of all closed Φ loops containing all powers of the background field $A(z)$, generates the amplitudes corresponding to the sum of all ladders and crossed graphs between this pair of Φ particles, as well as "self-linkages" on and along each Φ line; neglect of the latter, together with the replacement of bare mass and coupling by their renormalized values, is permissible [7] in the forward scattering direction, where $(-t) \ll s$, and one builds the so-called "relativistic eikonal approximation" of several decades ago [8]. Including the subset of "tower graphs" constructed in a well-defined way [9] from insertion of $L[A]$, one reaches the "multiperipheral" eikonal models still in use because of their ability to produce increasing total scattering cross-sections in a unitarily acceptable way, in conjunction with observed multiperipheral inelastic emissions [10].

The $\langle$ph$\rangle$ approximation defined in this paper will provide generalized eikonal representations for both $G[A]$ and $L[A]$, in terms of a few, relevant, nonperturbative quadratures for each functional. It may be convenient to rewrite the complete M of (4) as

$$M = \exp[\mathcal{D}_{12} + \mathcal{D}_{13} + \mathcal{D}_{23}]\bar{G}(x_1,y_1|A_1)\bar{G}(x_2,y_2|A_2) \times \left[\sum_1^\infty Q_n[A_3]/n!\right]\bigg|_{A_1=A_2=A_3=0} \ , \tag{5}$$

where $\bar{G}[A] \equiv e^{\mathcal{D}}G[A], \mathcal{D}_{ij} = -i\int(\delta/\delta A_i)\Delta_c^{(\mu)}(\delta/\delta A_j)$, and the cluster expansion coefficients are defined as $Q_n[A] = e^{\mathcal{D}}(L[A])^n|_{\text{conn}}$, so that $e^{\mathcal{D}}e^L = \exp[\sum Q_n[A]/n!]$, where the subscript conn indicates that only the connected parts of each such expression are to be retained. The difference between $G[A]$ and $\bar{G}[A]$ is that the latter contains all possible self-linkages, while the $\exp[\mathcal{D}_{ij}]$ operation express all possible cross-linkage exchange of quanta between the ith and jth operands. In conventional eikonal scattering models, $\bar{G}[A]$ is approximated by $G[A]$, but the "tower" of presumed importance in current high-energy scattering models comes from every Q_n. In other, related theories (such as the old "hybrid," or "checkerboard" model [11] mentioned below), the eikonal function $\chi_n(b,s)$ corresponds to the exchange of n-virtual, t-channel NVM quanta which then exchange all possible number of "scalar pions," and will be able to be calculated in a generalized eikonal approximation in terms of a finite number of quadratures. The number of needed quadratures, however, grows with n,

so that the complete sum over all χ_n, which might be inferred from its leading, s and b dependence, or as the solution of a Bethe-Salpeter equation, cannot, by the present method, be reduced to a finite number of quadratures.

From the generating functional of (2) and the example of (5), it is clear that knowledge of the $G_c[A]$ relevant to the particular interaction is an essential part of the realization of these formal field solutions for the n-point functions of quantum field theory. Thus, the $G_c[A]$ and its construct $L[A]$ which, for specific fields $A(z)$ define solutions in potential theory, are in field theory to be written for arbitrary fields $A(z)$ upon which the linkage operators are to act, and which express the relativistic quantum fluctuation of the field operator [to reach (2) the Φ fluctuations have been integrated out]; this is just the difference between first and second quantization. It is, therefore, with $G_c[A]$ that the $\langle \text{ph} \rangle$ construction begins in the following section, with the aid of the elegant and most useful Fradkin representation [5]. For clarity, a brief demonstration of that representation is provided in Appendix A, and the $\langle \text{ph} \rangle$ approximation defined in terms of an approximation to the exact Fradkin forms. In Sec. III, the properties of $G_c^{\langle \text{ph} \rangle}[A]$ are sketched, including mass-shell (or energy-shell) amputation, and a demonstration of time-reversal invariance. A derivation of the standard eikonal scattering representation in potential theory has been relegated to Appendix B; while a configuration-space variant of the more usual, momentum-space, no recoil approximations to $G_c[A]$ is constructed in Appendix C. Section IV contains applications to QED and, briefly, to QCD, which maintain gauge covariance and/or invariance, as appropriate. A final section discusses corrections to this $\langle \text{ph} \rangle$ approximation, and other possible applications and generalizations of the $\langle \text{ph} \rangle$ method.

II. CONSTRUCTION OF THE PHASE-AVERAGED APPROXIMATION

We begin by writing the exact Fradkin representation [5] for the causal Green's function $G_c(x, y|A)$, which satisfies the differential equation

$$[-\partial_x^2 + m^2 + gA(x)]G_c(x, y|A) = \delta^4(x - y) \tag{6}$$

or the integral equation

$$G_c(x, y|A) = G_c^{(0)}(x - y) - g \int d^4z G_c^{(0)}(x - z) \times A(z) G_c(z, y|A) \tag{7}$$

given in terms of the "free" Green's function

$$G_c^{(0)}(z) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikz}}{k^2 + m^2 - i\epsilon}\Big|_{\epsilon \to 0^+} = \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-ism^2} e^{i(z^2 + i\epsilon)/4s}\Big|_{\epsilon \to 0^+} , \tag{8}$$

which may be written, in terms of the timelike, spacelike, and light-cone behavior of z^2 as

$$G_c^{(0)}(z) = im\theta(z^2) K_1(m\sqrt{z^2})/4\pi^2\sqrt{z^2} - m\theta(-z^2) H_1^{(2)}(m\sqrt{-z^2})/8\pi\sqrt{-z^2} + \delta(z^2)/4\pi , \tag{9}$$

where $\theta(x)$ is the step function.

In this paper, we shall use the simplest relativistic metric, $x_\mu y_\mu = x \cdot y = \mathbf{x} \cdot \mathbf{y} - x_0 y_0$, with $x_\mu = (\mathbf{x}, ix_0)$. In writing (1)–(4), we have chosen 3+1=4 dimensions, but with obvious modifications one can repeat each piece of analysis for arbitrary dimensions. Note also that in a nonrelativistic context, causal Green's functions become retarded Green's functions. In time-independent potential theory, with a trivial change of parameters, these techniques will define the $\langle \text{ph} \rangle$ approximation to the $G_{\text{out}}(\mathbf{r}, \mathbf{r}'|A)$ and $G_{\text{in}}(\mathbf{r}, \mathbf{r}'|A)$ used in three-dimensional (spatial) scattering.

As sketched in Appendix A, the Fradkin representation provides an explicit functional integral, or "linkage-operator" representation of the solutions to (1), and in particular, to (2). Starting from the exact form

$$G_c(x, y|A) = i \int_0^\infty ds\, e^{-ism^2} \exp\left[i \int_0^s ds' \delta^2/\delta v^2(s') \right] \exp\left[-ig \int_0^s ds' A\left(y - \int_0^{s'} ds'' v(s'') \right) \right] \times \delta^{(4)}\left(x - y + \int_0^s ds' v(s') \right)\Big|_{v=0} , \tag{10}$$

one inserts an exponential representation for $\delta^{(4)}(x - y + \int_0^s v)$, and expands the A dependence of (10) so that the nth term of that expansion takes the form

$$G_c(x, y|A) = i \int_0^\infty ds\, e^{-ism^2} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \sum_{n=0}^\infty \frac{(-ig)^n}{n!} \int_0^s ds_1 \times \cdots \times \int_0^s ds_n \times \int \frac{d^4k_1}{(2\pi)^4} \tilde{A}(k_1) \times \cdots \times \int \frac{d^4k_n}{(2\pi)^4} \tilde{A}(k_n) \exp\left[i \sum_{l=1}^n k_1 \cdot y \right] \times \exp\left[i \int_0^s ds' \delta^2/\delta v^2(s') \right] \exp\left[i \int_0^s ds' v(s') \cdot \left(p - \sum_{l=1}^n k_l \theta(s_l - s') \right) \right]\Big|_{v=0} . \tag{11}$$

The functional operation of the last line of (11) is now trivial and yields

$$\exp\left[-i\int_0^s ds'\left(p-\sum_{l=1}^n k_l\theta(s_l-s')\right)^2\right]=\exp\left[-isp^2+2ip\cdot\sum_{l=1}^n k_l s_l - i\sum_{l,m=1}^n k_l\cdot k_m\int_0^s ds'\theta(s_l-s')\theta(s_m-s')\right]. \tag{12}$$

The integral of the last term of (12) may be written as

$$h(s_l,s_m)=s_l\theta(s_m-s_l)+s_m\theta(s_l-s_m)=\frac{1}{2}[s_l+s_m-|s_l-s_m|]\ , \tag{13}$$

and its peculiar form carries the condition that (exact) four-momentum conservation is to be enforced at every $\tilde{A}$ vertex in the expansion of $G_c[A]$.

Before motivating and defining the $\langle$ph$\rangle$ approximation, it may be worthwhile to see how the exact, perturbative forms follow from (13); and for this, it is simplest to begin with the first nontrivial term, $n=2$, for which (10) contains one factor

$$i\int_0^\infty ds\, e^{-is[p^2+m^2]}\int_0^s ds_1\int_0^s ds_2 e^{2ipk_1s_1+2ipk_2s_2-ik_1^2s_1-ik_2^2s_2}\exp\{-2ik_1\cdot k_2[s_1\theta(s_2-s_1)+s_2\theta(s_1-s_2)]\}\ . \tag{14}$$

To obtain the usual perturbative result, one may most simply divide and multiply by the factor $[p^2+m^2]$, so that (14) may be replaced by

$$-\frac{1}{p^2+m^2}\int_0^\infty ds\left(\frac{\partial}{\partial s}e^{-is[p^2+m^2]}\right)\times\int_0^s ds_1\int_0^s ds_2\Phi(s_1,s_2)\ , \tag{15}$$

where $\Phi(s_1,s_2)$ denotes the integrand of (14). An integration by parts now converts (15) to

$$\frac{1}{p^2+m^2}\int_0^\infty ds\, e^{-is[p^2+m^2]}\left\{\int_0^s ds_2\Phi(s,s_2)+\int_0^s ds_1\Phi(s_1,s)\right\} \tag{16}$$

and we discuss these two permutations separately. The first generates

$$\frac{1}{p^2+m^2}\int_0^\infty ds\, e^{-is[(p-k_1)^2+m^2]}\times\int_0^s ds_2 e^{-is_2[k_2^2+2k_1\cdot k_2-2p\cdot k_2]} \tag{17}$$

and repetition of the same integration-by-parts techniques then yields

$$\frac{1}{[p^2+m^2]}\frac{1}{[(p-k_1)^2+m^2]}i\int_0^\infty ds\, e^{-is[(p-k_1-k_2)^2+m^2]}$$
$$=[p^2+m^2]^{-1}[(p-k_1)^2+m^2]^{-1}\times[(p-k_1-k_2)^2+m^2]^{-1}\ . \tag{18}$$

Obviously, the second permutation of (16) must provide the same result as that of (18) with k_1 replaced by k_2; since the remaining integrand of (11) is symmetric under the interchange of k_1 and k_2, both permutations yield the same result, thereby removing the multiplicative factor of $1/2!$. Clearly, the same effect will happen in the nth perturbative term, with the $n!$ identical permutations removing the factor $1/n!$, and leading to the expected result

$$[p^2+m^2]^{-1}[(p-k_1)^2+m^2]^{-1}\times\cdots\times\left[\left(p-\sum_{l=1}^{n-1}k_l\right)^2+m^2\right]^{-1}\left[\left(p-\sum_{l=1}^{n}k_l\right)^2+m^2\right]^{-1}\ . \tag{19}$$

One notes that exact four-momentum conservation exists at every $\tilde{A}(k)$ vertex, resulting in overall four-momentum conservation. The last factor takes on the value associated with the "final" four-momentum, $p'=p-\sum_{l=1}^n k_l$. This result is, of course, very well known, and serves here to demonstrate how the exact Fradkin representation contains the conventional perturbative expansion.

It also serves to clarify the reason for the basic complexity of potential theory, and subsequently of field theory, for the (s_l,s_m) dependence of (12) is sufficiently nonlinear to preclude the existence of any simple factorization, an attribute which would be necessary to convert these perturbative structures into a summable form of simple, nonperturbative expression. What we shall do is to define an approximation procedure which affects only the offending terms, as expressed in (13), of any such perturbative expansion; and then calculate the result which would replace (19) if such a simplifying average over the s_l, s_m variables of (13) were to be used.

Of course, there are many types of averages that can be

contemplated, and we here discuss the one which seems to be the simplest, seems to be unique and has immediate applications to eikonal physics. The function $h(s_l, s_m)$ takes on the value of its smallest argument, and is either given by s_l or by s_m. The "phase-averaged" approximation we now define replaces $h(s_l, s_m)$ by the simplest average,

$$h \to \langle h(s_l, s_m)\rangle = \tfrac{1}{2}[s_l + s_m] \ , \tag{20}$$

which in essence, neglects the remaining, and complicated, term of $h(s_l, s_m) : -|s_l - s_m|/2$. The reason that this latter term is complicated lies in the fact that its value cannot be specified unless one knows both s_l and s_m, simultaneously. Its neglect here corresponds to the "zeroth" approximation of retaining only the contribution $(s_l + s_m)/2$, of a related and exact representation described in the second paper of this series.

Aside from simplicity, the beauty of (20) is that all terms of the perturbative expansion now factorize, so that the result can be given a simple nonperturbative representation; and that result turns out to be exactly what is needed to define generalized eikonal representations everywhere it is employed, in both $G_c[A]$ and $L[A]$.

The reason this is true is that (12) may now be replaced by

$$\exp\left[-isp^2 + 2ip\cdot\sum_{l=1}^{n} k_l s_l - i\left(\sum_{m=1}^{n} k_m\right)\cdot\left(\sum_{l=1}^{n} k_l s_l\right)\right], \tag{21}$$

which can be rewritten in terms of the variable $z = \sum_{l=1}^{n} k_l s_l$, as

$$\int d^4z\,\delta^{(4)}\left(z - \sum_{l=1}^{n} k_l s_l\right)\exp\left[-isp^2 + 2ip\cdot z - iz\cdot\left(\sum_{l=1}^{n} k_l\right)\right] . \tag{22}$$

Introducing the representation

$$\delta^{(4)}\left(z - \sum_{l=1}^{n} k_l s_l\right) = \int \frac{d^4P}{(2\pi)^4}\exp\left[i\left(z - \sum_{l} k_l s_l\right)\cdot P\right], \tag{23}$$

and inserting the two previous formulas into (21), one obtains

$$G_c^{\langle \mathrm{ph}\rangle}(x,y|A) = i\int_0^\infty ds\, e^{-ism^2}\int\frac{d^4p}{(2\pi)^4}e^{-isp^2+ip(x-y)}\int\frac{d^4z\,d^4P}{(2\pi)^4}e^{iz\cdot(P+2p)} \times\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\prod_{l=1}^{n}\int_0^s ds_l\int\frac{d^4k_l}{(2\pi)^4}\tilde{A}(k_l)e^{ik_l\cdot(y-z-s_lP)} \ . \tag{24}$$

Notice that one does not need to append a subscript n to the z, P variables, for the representation (23) will produce exactly the form (21) for any and every n. In this sense, the A dependence of $G_c^{\langle \mathrm{ph}\rangle}$ now factorizes, and one finds that the sum over n can be reexpressed as

$$G_c^{\langle \mathrm{ph}\rangle}(x,y|A) = \int_0^\infty ds\, e^{-ism^2}\int\frac{d^4p}{(2\pi)^4}e^{-isp^2+ip\cdot(x-y)}\int\frac{d^4z\,d^4P}{(2\pi)^4}e^{iz\cdot(P+2p)}\exp\left[-ig\int_0^s ds' A(y-z-s'P)\right] . \tag{25}$$

A convenient form is obtained under the variable change: $y - z \to z, P \to -P$, which, after integration over p leads to

$$G_c^{\langle \mathrm{ph}\rangle}(x,y|A) = \frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-ism^2}e^{i(x-y)^2/4s}\int\frac{d^4z d^4P}{(2\pi)^4}e^{i(z-y)\cdot(z+sP-x)}\exp\left[-ig\int_0^s ds' A(z+s'P)\right] . \tag{26}$$

In momentum space, we obtain

$$\langle p|G_c^{\langle \mathrm{ph}\rangle}[A]|p'\rangle = \int\frac{d^4x\,d^4y}{(2\pi)^4}e^{-i(p\cdot x - p'\cdot y)}\langle x|G_c^{\langle \mathrm{ph}\rangle}[A]|y\rangle = i\int_0^\infty ds\, e^{-is[p^2+m^2]}\int\frac{d^4z}{(2\pi)^4}e^{iz\cdot(p'-p)}\exp\left[-ig\int_0^s ds' A[z+s'(p+p')]\right] , \tag{27}$$

which will be of use later on.

This $\langle \mathrm{ph}\rangle$ approximation should not be confused with the random phase approximation (RPA) of field-theoretic models in solid-state physics [12], which is an expression of a Hartree-Fock, or self-consistent mean-field approximation. Other forms of the $\langle \mathrm{ph}\rangle$ approximation are possible, as discussed in Sec. V. Equation (27) is, of course, quite similar to the G_{BN} derived in Chap. 8 of Ref. [3]; the difference is the method of derivation, a clear representation of the neglected terms, $h(s_l, s_m) - \langle h(s_l, s_m)\rangle$, and most importantly, the exact time-reversal (TR) symmetry discussed below, $\langle p|G_c^{\langle \mathrm{ph}\rangle}[A]|p'\rangle = \langle -p'|G_c^{\langle \mathrm{ph}\rangle}[A]|-p\rangle$.

III. PROPERTIES OF THE ⟨ph⟩ APPROXIMATION

We first consider the perturbative form contained in the ⟨ph⟩ solutions above, and in particular, that of (27). The nth term in that expansion generates

$$\frac{(-ig)^n}{n!} i \int_0^\infty ds\, e^{-is[p^2+m^2]} \int \frac{d^4z}{(2\pi)^4} e^{iq\cdot z}$$

$$\times \prod_{l=1}^{n} \int \frac{d^4k_l}{(2\pi)^4} \tilde{A}(k_l) e^{ik_l\cdot z} \int_0^s ds_l e^{is_l k_l\cdot(p+p')} \quad (28)$$

with $q = p' - p$.

Again, the integration-by-parts techniques used following (14) will produce the appropriate denominator factors; and again, it is simplest to begin with $n = 2$, where one easily obtains the factors

$$\frac{1}{2!}\frac{1}{p^2+m^2}\left[\frac{1}{p^2 - k_1\cdot(p+p') + m^2} + \frac{1}{p^2 - k_2\cdot(p+p') + m^2}\right]\frac{1}{p'^2+m^2}\ , \quad (29)$$

where the last denominator $p^2 - (k_1 + k_2)(p + p') + m^2$ has been evaluated using the overall four-momentum conservation following from integration over z:$p' = p - (k_1 + k_2)$. The two permutations inside the brackets of (28) are precisely those approximated terms which would result from an "averaged" BN, or no-recoil approximation [13]; and yet, the result contains the proper $(p^2 + m^2)^{-1}(p'^2 + m^2)^{-1}$ factors, together with proper four-momentum conservation. In effect, four-momentum is not conserved at intermediate vertices, but it *is* conserved, overall, between initial and final scattering states.

As an aside relevant to the construction of subsequent eikonal scattering amplitudes, where mass and charge renormalization is suppressed, consider the case where mass-shell amputation is to be performed directly on (again for $n = 2$) one of the two factors of $(p^2 + m^2)^{-1}$ of expression (28). Calling $\Delta = p^2 + m^2$ and neglecting q with respect to p, one will need to calculate

$$\Delta\frac{1}{2!}\frac{1}{\Delta}\left[\frac{1}{\Delta - 2p\cdot k_1} + \frac{1}{\Delta - 2p\cdot k_2}\right]\frac{1}{\Delta - 2p\cdot(k_1+k_2)}\bigg|_{\Delta\to 0} = \frac{1}{2!}(-2p\cdot k_1)^{-1}(-2p\cdot k_2)^{-1}\ , \quad (30)$$

which represents the quadratic term of the standard eikonal exponentiation. In a similar way, for $n = 3$ one finds the appropriate version of (29):

$$\frac{1}{3!}(p^2+m^2)^{-1}\left\{\sum_{\text{perm}}[p^2 - (p+p')\cdot k_i + m^2]^{-1}\right.$$

$$\left.\times[p^2 - (p+p')\cdot(k_i + kj) + m^2]^{-1}\right\}(p'^2+m^2)^{-1}\ , \quad (31)$$

where $\sum_{\text{perm}}$ denotes a sum of 3! distinct permutations of the three-momenta. Again, overall four-momentum conservation is guaranteed; and again, mass-shell amputation on the factor (p^2+m^2), for $q \ll p$ will yield exactly: $1/3! \times [1/(-2p\cdot k_1)] \times [1/(-2p\cdot k_2)] \times [1/(-2p\cdot k_3)]$.

Clearly, these constructions can be carried through for arbitrary n, and represent just the correct sum of $n!$ distinct permutations in that special "generalized eikonal" case where the exact, internal denominator $[(p - \sum_{l=1}^{j} k_l)^2 + m^2]^{-1}$ is replaced by

$$\left[p^2 + m^2 - (p+p')\cdot\left(\sum_{l=1}^{j} k_l\right)\right]^{-1}\ .$$

Replacing the p' of this expression by $p - \sum_{l=1}^{n} k_l$ shows that these terms have a certain relevance to the large-k limit as well, for this factor can be rewritten as

$$\left[m^2 + \left(p - \sum_{l=1}^{j} k_l\right)^2 + \left(\sum_{l=1}^{j} k\right)\left(\sum_{l=j+1}^{n} k\right)\right]^{-1}\ ;$$

that is, as an unwanted "correction" added to the exact value. From this, one sees that if but one k_m is large, the string of denominators is esentially correct, for quadratic dependence on that k_m appears only in its proper place; if more than one k_m is large, the representation is wrong. Since one expects that scattering at large angles is most probable when a single, large-momentum-transfer is involved, together with many, small, eikonal-style, momentum transfers, the ⟨ph⟩ approximation is then one in which scattering at both small and large momentum transfers may be expected to be given correctly, although this is not true at intermediate momentum transfers. Such a property is still inadequate for most virtual processes of importance in renormalization studies, where more than one momenta can become ultraviolet.

One may apply these constructions to the case of nonrelativistic potential-theory scattering by replacing $A(x)$ by the scattering potential $V(\mathbf{r})$, by replacing $p^2 + m^2$ by $\mathbf{p}^2 - k_0^2$, where $k_0^2/2m$ denotes the conserved kinetic energy of the initial particle, and by replacing all four-dimensional (4D) integrations by corresponding 3D integrations. Since the relation of the exact Green's functions $\langle\mathbf{p}|G_c[V]|\mathbf{p}'\rangle$ and scattering amplitudes $f(\mathbf{p},\mathbf{p}')$ is given by

$$\langle \mathbf{p}|G_c[V]|\mathbf{p}'\rangle = \frac{\delta(\mathbf{p}-\mathbf{p}')}{\mathbf{p}^2-k_0^2} + \xi \frac{1}{\mathbf{p}^2-k_0^2} f(\mathbf{p}\cdot\mathbf{p}') \frac{1}{\mathbf{p}'^2-k_0^2} \tag{32}$$

with ξ an appropriate normalization constant, it is clear that the requirement of invariance under time reversal, $f(\mathbf{p},\mathbf{p}') = f(-\mathbf{p}',-\mathbf{p})$ (in this spinless case) will be automatically satisfied if the same behavior is true for $\langle \mathbf{p}|G_c[V]|\mathbf{p}'\rangle$. From the perturbative expansion of $G_c^{\langle \mathrm{ph}\rangle}(x,y|A)$ above, transcribed to this 3D case, it is clear that our $\langle$ph$\rangle$ Green's functions satisfy this property. In fact, this invariance can be demonstrated directly from the nonperturbative representation (26) by performing the simple variable changes $z \to z - s(p+p')$, followed by $s' \to s - s'$.

One further comment on potential theory may be useful. Equation (27) expresses the nonperturbative $\langle$ph$\rangle$ solution for the complete propagator $\langle \mathbf{p}|G[V]|\mathbf{p}'\rangle$. For conversion to the scattering amplitude $f(\mathbf{p},\mathbf{p}')$, one requires energy-shell amputation on both $\mathbf{p}$ and $\mathbf{p}'$ coordinates, which may be done in a time-reversal-invariant way by the method sketched in Appendix B. But for many problems, one might like to have a nonperturbative approximation to a relevant Green's function, and in terms of a few quadratures, just this is provided by the $\langle$ph$\rangle$ method. It is a sensible approximation for those situations where Fourier components $\mathbf{k}$ of the potential are small compared to the "natural" scale of the problem, $\mathbf{k} < k_0 \equiv [2m|E|]^{1/2}$, where E is either a scattering or bound-state energy.

For the time-independent scattering situation, one typically expresses the exact Green's functions of the problem,

$$(E-H)G_{\pm}(\mathbf{r},\mathbf{r}'|V) = \delta(\mathbf{r}-\mathbf{r}'), \quad H = H_0 + V \ ,$$

in terms of the sum over a complete, orthonormal set of wave functions $u_n(\mathbf{r})$,

$$G_{\pm}(\mathbf{r},\mathbf{r}'|V) = \sum_n \frac{u_n(\mathbf{r})u_n^*(\mathbf{r}')}{E-E_n \pm i\epsilon} \ ,$$

where the $\pm$ signs distinguish outgoing and incoming scattered waves. If one has in hand a nonperturbative expression for $G_{\pm}$, as is given by (27), one may contemplate the calculation of isolated, bound-state energy levels by continuing the energy E below threshold, and identifying the approximate wave functions of the problem as the residues of such isolated poles. There is, of course, no guarantee that the $\langle$ph$\rangle$ approximation to $G_{\pm}$ will preserve the proper analytic structure; but it is an approach which might be useful to consider. This continuation method is certainly well known for eikonal scattering amplitudes where such pole structure is seen for the Coulomb problem [3], which suggests that it might be possible to extract both wave functions and energy eigenvalues directly from $G^{\langle \mathrm{ph}\rangle}$. However, the scattered phase calculated directly from $G_{\mathrm{out}}^{\langle \mathrm{ph}\rangle}$ will generate only one-half the correct eikonal phase obtained in Appendix B.

Returning to the 4D scalar problem, we now discuss the closed-loop determinantal factor $L[A]$, which may be similarly exhibited in this $\langle$ph$\rangle$ approximation. The exact relation between $L[A]$ and $G_c[A]$ can (in this spinless case) be written as

$$\begin{aligned} L[A] &= -\frac{1}{2}\int_0^1 d\lambda \,\mathrm{Tr}(AG_c[\lambda A]) \\ &= -\frac{1}{2}\int_0^1 d\lambda \int d^4x\, A(x) G_c(x,x|\lambda A) \end{aligned} \tag{33}$$

or

$$L[A] = -\frac{1}{2}\int_0^1 d\lambda \int\int \frac{d^4p\, d^4p'}{(2\pi)^2} \tilde{A}(p'-p)\langle p|G[\lambda A]|p'\rangle \ , \tag{34}$$

and, therefore,

$$L^{\langle \mathrm{ph}\rangle}[A] = -\frac{1}{2}\int_0^1 d\lambda \int\int \frac{d^4p\, d^4p'}{(2\pi)^4} \tilde{A}(p'-p) \times \langle p|G^{\langle \mathrm{ph}\rangle}[\lambda A]|p'\rangle \ . \tag{35}$$

The difference between (34) and (35) is that the former is exact, but the latter can be expressed, to all orders in gA, by means of a few quadratures, which provide a physically correct description of those situations when the Fourier components k of $\tilde{A}(k)$ are essentially less than those of the p,p' variables. Upon integration, the latter take on the qualitative size of relevant mass factors—either m, for finite contributions, or an UV cutoff Λ for divergent ones—and so can be expected to yield qualitatively correct results in those cases where "infrared," or low-frequency k are important; that is, where the relevant $k \ll m$.

The interested reader is urged to compare the first few perturbative orders of (34) and (35), to become convinced of the truth of this statement. In every case, one will find "internal" factors of

$$\left[p^2 - (p+p')\cdot\left(\sum_{l=1}^{j} k_l\right) + m^2\right]^{-1}$$

replacing the exact

$$\left[\left(p - \sum_{l=1}^{j} k_l\right)^2 + m^2\right]^{-1} \ ;$$

all other parts of both expressions are identical. For example, to order A^3, that exact term in the perturbative expansion of $L[A]$ is, with $\int dp \equiv \int dp/(2\pi)^4$, given by

$$\begin{aligned} -\frac{1}{6}\int dp \int dp' \int dk_1 \int dk_2 (2\pi)^4 \delta^{(4)}(p'-p+k_1+k_2) \\ \times \tilde{A}(k_1)\tilde{A}(k_2)\tilde{A}(p-p')[p^2+m^2]^{-1}[(p-k_1)^2+m^2]^{-1} \\ \times [p'^2+m^2]^{-1} \end{aligned} \tag{36}$$

while the corresponding contribution of $L^{\langle \mathrm{ph}\rangle}[A]$ is the

same as that of (36), except for the replacement of the denominator factors by

$$[p^2+m^2]^{-1}[p^2-(p+p')\cdot k_1+m^2]^{-1}[p'^2+m^2]^{-1}\ .$$

IV. THE ⟨ph⟩ APPROXIMATION IN GAUGE THEORY

It is almost a straightforward generalization to write down ⟨ph⟩ approximations in QED and QCD; for simplicity and clarity, we shall write all expressions first in QED, and then describe the changes needed for QCD. In fact, because of the different sorts of local gauge transformations involved, gauge invariance of any IR approximation in QCD is a more delicate and conditional affair, as described below. And even the relatively simple U(1) situation of QED (where $F_{\mu\nu}$ itself is gauge invariant) is not completely trivial.

A. QED

In order to preserve the QED gauge covariance of $G_c(x,y|A)$,

$$G_c(x,y|A+\partial\Lambda)=e^{ig[\Lambda(x)-\Lambda(y)]}G_c(x,y|A)\ , \tag{37}$$

and the gauge invariance of $L[A]$, which will subsequently be denoted by $L[F]$, it is wisest to begin with the exact forms generated by the Fradkin representation [5]. For the causal soluton of the differential equation

$$[m+\gamma(\partial_x-igA(x))]G_c(x,y|A)=\delta^{(4)}(x-y)\ ,$$

one writes

$$G_c(x,y|A)=i\int_0^\infty ds\,e^{-ism^2}\exp\left[i\int_0^s ds'\left(\frac{\delta}{\delta v}\right)^2\right]\times\left(m-\gamma\frac{\delta}{\delta v(s)}\right)\langle x|W(s)|y\rangle|_{v\to 0} \tag{38}$$

and, simultaneously for the log of the fermion determinant,

$$L[A]=\mathrm{Trln}[1-ig\gamma\cdot AS_c],S_c=G_c[A=0]\ ,$$

$$L[A]=-\frac{1}{2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\mathrm{tr}\int d^4x\,\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)\left\{\langle x|W(s)|x\rangle-\langle x|W(s)|x\rangle|_{g\to 0}\right\}\Big|_{v\to 0}\ , \tag{39}$$

with

$$\langle x|W(s)|y\rangle=\exp\left[-ig\int_0^s ds'v_\mu(s')\cdot A_\mu\left(y-\int_0^{s'}v\right)\right]\left(\exp\left[g\int_0^s ds'\sigma\cdot F\left(y-\int_0^{s'}v\right)\right]\right)_+\delta\left(x-y+\int_0^s v\right)\ . \tag{40}$$

Equations (38)–(40) can be reexpressed in terms of functional integrals over the "four-velocities" $v_\mu(s')$, whose fluctuations provide, as originally noted by Fradkin, an exact representation for the proper-time formalism originally introduced by Schwinger [14]. Here, $\sigma_{\mu\nu}=(1/4)[\gamma_\mu,\gamma_\nu]$; and the ordered exponential (OE) of (4.4) is necessary for fields $F_{\mu\nu}(z)$ such that $[\sigma\cdot F(z),\sigma\cdot F(z')]\neq 0$. In QCD, however, where $A_\mu\to A_\mu^a\lambda_a$, with λ_a a Gell-Mann SU(N) matrix, the entire exponent of (40) must be included in the OE.

Before performing any approximation, it is appropriate to rewrite the first exponential factor of (40), $\exp[-ig\int_0^s ds'v\cdot A(y-\int_0^{s'}v)]$ as

$$\exp\left[ig\int_y^x d\xi_\mu A_\mu(\xi)-ig\int_0^s ds'v_\mu(s')\int_0^{s'}ds''v_\nu(s'')\times\int_0^1\lambda\,d\lambda\,F_{\mu\nu}\left(y-\lambda\int_0^{s'}v\right)\right]\ , \tag{41}$$

where $\xi_\mu=\lambda x_\mu+(1-\lambda)y_\mu$ denotes the straight-line path, parametrized by $0\le\lambda\le 1$, connecting y_μ and x_μ. This replacement, obtained from a simple integration by parts, requires the restriction $\int_0^s ds'v(s')=y-x$, as expressed by the δ function of (40). It generates the gauge dependence of (37), and explicitly shows that $L[A]$ is independent of gauge. In QCD, the operations leading to (41) are not possible without further approximations [15], even though the gauge invariance of the exact $L[A]$ written in the Fradkin form can be made explicit [16].

The gauge-dependent line integral of (41) is absent in (39), while its removal in (38) leads to a gauge-invariant object of the old Valatin form [17], here called $G_c[F]$:

$$G_c(x,y|F\rangle = i\int_0^\infty ds\, e^{-ism^2}\left(m-\gamma\cdot\frac{\delta}{\delta v(s)}\right)\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)$$
$$\times\exp\left[-ig\int_0^s ds' v_\mu(s')\int_0^{s'} ds'' v_\nu(s'')\int_0^1 \lambda\, d\lambda F_{\mu\nu}\left(y-\lambda\int_0^{s'} v\right)\right]$$
$$\times\left(\exp\left[g\int_0^s ds'\sigma\cdot F\left(y-\int_0^{s'} v\right)\right]\right)_+\delta\left(x-y+\int_0^s v\right)\Big|_{v\to 0}\,. \tag{42}$$

When $F_{\mu\nu}(z)$ is independent of z, the functional operations of (42) are equivalent to a Gaussian functional integration, which can be done exactly and which produces Schwinger's well-known solutions (in any number of space-time dimensions) [14]. When the $F_{\mu\nu}$ are not constants but are slowly varying, when the momenta k of their Fourier transform are small with respect to an appropriate scale (e.g., m), the IR method generates an estimate of this functional integral (FI), in effect by replacing the v dependence inside the argument of each F of (42) by a related, s-dependent factor; the FI is still Gaussian, but the proper-time dependence of the integrand becomes more complex.

In the present case, we seek to produce a $\langle$ph$\rangle$ approximation for $G_c[F]$ and for $L[F]$, but immediately run into a complication not found in the scalar analysis of Sec. II. In the language of "linkages," generated by the operator $\exp[i\int_0^s ds'\frac{\delta^2}{\delta v^2}]$, we are to calculate "cross linkages" between the $v_\mu(s')v_\nu(s'')$ terms of (41) and the remaining v dependence in the arguments of F. Upon expanding in powers of g, and introducing the $\langle$ph$\rangle$ approximation, one finds that cross linkages between $v_\mu(s')v_\nu(s'')$ and the v dependence of the F's do not appear to factorize (as do the linkages between the arguments of the F factors, exactly as in the scalar case). However, these awkward terms take the form of effective k_μ dependence added to a factor of p_μ; and if the entire $\langle$ph$\rangle$ approximation is sensible only for those situations where $p \gg k$, such k dependence may be comfortably neglected. The result is that the computation effectively splits into two parts, the first involving fluctuations of the $v_\mu(s')v_\nu(s'')$ dependence (which for constant F would generate the familiar Schwinger solution), while the second part is concerned with linkages of the v dependence arising from the arguments of the F's. Proper linkage to the factor $\exp[ip\int_0^s v]$ is of course maintained. The net result is to blend the two parts of the computation into one relation which automatically reproduces Schwinger's form in the limit of constant F, but which contains relevant, eikonal dependence for smoothly varying F.

To show this in a symbolic but clear way, consider the set of terms of order g^{n+N} in the expansion of (42), after a Fourier exponential representation has been written for $\delta(x-y+\int_0^s v)$. With $\mathcal{D}$ now representing $i\int_0^s ds'\frac{\delta^2}{\delta v^2}$, the v dependence of (42) can be written in the form

$$e^{\mathcal{D}}(A_n[v]B_{n,N}[v]e^{i\int_0^s v(s')ds'\cdot p})|_{v\to 0'}\,, \tag{43}$$

where $A_n[v]$ denotes the n factors each of form $v_\mu(s')v_\nu(s'')$, and $B_{n,N}[v]$ denotes all the remaining v dependence inside the associated F terms, of arguments $(y-\lambda\int_0^{s'} v)$ or $(y-\int_0^{s'} v)$. With $\mathcal{D}_{ij}=2i\int_0^s \frac{\delta}{dv_i}\frac{\delta}{dv_j}$, (43) may be rewritten as

$$e^{\mathcal{D}_{12}+\mathcal{D}_{23}+\mathcal{D}_{13}}(e^{\mathcal{D}}A_n[v_1])(e^{\mathcal{D}}B_{n,N}[v_2])$$
$$\times(e^{\mathcal{D}}e^{i\int_0^s v_3\cdot p})|_{v_i\to 0}\,. \tag{44}$$

The $\mathcal{D}_{12}$ operations will always generate k dependence that can be neglected compared to the p dependence generated by the D_{23} operator. We therefore replace (44) by

$$(e^{\mathcal{D}}A_n[v]e^{i\int_0^s v\cdot p})|_{v\to 0}(e^{\mathcal{D}}[B_{n,N}[v]e^{i\int_0^s v\cdot p}])|_{v\to 0}e^{+isp^2}\,, \tag{45}$$

and in this way the v fluctuations of $A_n[v]$ are calculated independently of those of $B_{n,N}[v]$. In every order, the latter produce coefficients which are analogous to those found in the factorized, $\langle$ph$\rangle$ analysis of Sec. II, although care must be taken to distinguish the two different F arguments of $B_{n,N}$. The remaining $e^{\mathcal{D}}A_n[v]$ dependence builds precisely the Schwinger forms found when the F's are constant, although they are not constants here, so that the entire grouping may be written as

$$G_c^{\langle\mathrm{ph}\rangle}(x,y|F) = i\int_0^\infty ds\, e^{-ism^2}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}(m-i\gamma\cdot p)$$
$$\times\int\frac{d^4z\,d^4P}{(2\pi)^4}e^{iz\cdot(P-2p)}e^{-i\int_0^s\int_0^s p_\mu\langle s_1|(1+2K)^{-1}_{\mu\nu}|s_2\rangle p_\nu}(e^{q\int_0^s ds'\sigma\cdot F(y+z+s'P)})_+\,, \tag{46}$$

where

$$K_{\mu\nu}(s_1,s_2) = -2g\theta(s_1-s_2)\int_0^1\lambda\, d\lambda\, F_{\mu\nu}(y+\lambda z+\lambda s_1 P)\,,$$

and the log of the determinantal factor of this computation vanishes because of the "retardedness" of $K_{\mu\nu}$. Again, when the F's are constant, this reduces exactly to the Schwinger result; but for variable F's of Fourier momenta k, (46) represents the eikonal generalization appropriate to the case $k < m$.

The analogous result may be written for $L[F]$:

$$L^{\langle \mathrm{ph} \rangle}[F] = -\frac{1}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2} \int \frac{d^4p}{(2\pi)^4} d^4y \int \frac{d^4z\, d^4P}{(2\pi)^4} e^{iz\cdot(P-2p)} \times \{[e^{-i\int_0^s ds_1 \int_0^s ds_2 p_\mu \langle s_1|(1+2K)^{-1}_{\mu\nu}[s_2]p_\nu} \mathrm{tr}(e^{g\int_0^s ds'\sigma\cdot F(y+z+s'P)})_+] - [\ ]|_{g\to 0}\} , \tag{47}$$

where tr denotes the trace over Dirac indices.

B. QCD

Here, there is no gauge-invariant $G_c[F]$, and we proceed directly to the $\langle \mathrm{ph} \rangle$ approximation of the gauge-invariant $L[A]$. With a summation now over color and Dirac coordinates, we replace the functional operation

$$e^{i\int_0^s ds' \frac{\delta^2}{\delta v^2}} \mathrm{tr}\left(\exp\left\{ -ig\int_0^s ds' \left[v_\mu(s') A_\mu\left(x - \int_0^{s'} v \right) + i\sigma\cdot F\left(x - \int_0^{s'} v \right) \right] \right\} \right)_+ (e^{i\int_0^s v\cdot p})|_{v\to 0} \tag{48}$$

with the $\langle \mathrm{ph} \rangle$ approximated form

$$\frac{\int d^4z\, d^4P}{(2\pi)^4} e^{iz\cdot(P-2p)} e^{-isp^2} \cdot \mathrm{tr}\left[\exp\left(-ig\int_0^s ds'[(-P_\mu)A_\mu(x+z+s'P) + i\sigma\cdot F(x+z+s'P)] \right) \right]_+ , \tag{49}$$

which, after a variable change and integration, is equivalent to

$$\mathrm{tr}\left[\exp\left(-ig\int_0^s ds'[(-2p_\mu)A_\mu(x+2s'p) + i\sigma\cdot F(x+2s'p)] \right) \right]_+ . \tag{50}$$

The steps leading to (50) mirror those of the previous section, including the neglect of k dependence compared to p dependence. Here, however, there has been no principle of invariance to guide the analysis. What is required of any approximation to $L[A]$, as is easily demonstrated [5] for the representations which define the exact quantity, is invariance under local, configuration-space transformations of the SU(N) variables. But the IR approximation corresponds to a local restriction in momentum space, which is nonlinear in configuration space. How can the equivalent $\langle \mathrm{ph} \rangle$ approximation be compatible with invariance under local, configuration-space transformations?

If we denote by $U(s)$ the OE factor of (50), consider the effect of the local, SU(N) transformation induced by a unitary matrix $V(x)$. Here, $U(s) \to U'(s)$, where

$$U'(s) = \left(\exp\left\{ -ig\int_0^s ds' \left((-2p_\mu)V^\dagger(x+2s'p)\left[A_\mu V + \frac{i}{g}\partial_\mu V \right] + i\sigma\cdot V^\dagger(x+2s'p) F V \right) \right\} \right)_+ \tag{51}$$

and where the value of the configuration-space argument in a product of factors is exhibited only in the first factor. To determine the relation of U' to U, set $U'(s) = V^\dagger(s)Z(s)$, and calculate the differential equation satisfied by $Z(s)$,

$$\frac{\partial Z}{\partial s} = -ig[(-2p_\mu)A_\mu(x+2sp) + i\sigma\cdot F(x+2sp)]\cdot Z ,$$

which turns out to be exactly that satisfied by $U(s)$; in the process of demonstrating this, one uses the relation

$$2p_\mu\partial_\mu V(x+2sp) = \frac{\partial}{\partial s} V(x+2sp)$$

and the unitarity of $V(s)$. Since functions which are the solution of the same, first-order, differential equation are the same to within initial conditions, one has $Z(s) = U(s)Z(0)$. By its original definition, $Z(0) = V(0)$, and one therefore finds that

$$\mathrm{tr}[U'(s)] = \mathrm{tr}[V(0)V^\dagger(s)U(s)] , \tag{52}$$

which describes the behavior of the $\langle \mathrm{ph} \rangle$ approximation under SU(N) gauge transformation. Equation (52) is not the full statement of invariance, $\mathrm{tr}[U'(s)] = \mathrm{tr}[U(s)]$, which is satisfied by the exact function, but it can represent a "conditional" form of invariance as long as one employs these quantities in a sensible and consistent way. Namely, the argument of these functions is $(x + 2s'p)$, with integrations over all variables. In every perturbation order, however, the size of s (or of its analytically continued variable) is $O(s) \sim m^{-2}$, while $O(p) \sim s^{-1/2} \sim m$, so

that $O(2s'p) \sim m^{-1}$. If the x variable of that argument is maintained at values $\gg m^{-1}$, the product $V(s)V^+(0)$ may be replaced by unity, and one then has effective, or conditional, invariance. What this means in practice is that only soft momenta are permitted to flow through these closed-quark loops, which are imagined to have an extension $> m^{-1}$, and are never permitted to approach each other at distances smaller than m^{-1}. These restrictions can be built in "by hand" if necessary [15], so that the IR sectors do contain the necessary, if conditional, gauge invariance.

V. APPLICATIONS, GENERALIZATIONS, AND CORRECTIONS

These three topics underlie the possible importance, generalization, and accuracy of the ⟨ph⟩ approximation method described in this paper, and we discuss them briefly in this sequence.

A. Applications

Eikonal approximations found their first quantum employment in scattering problems [2,3,5], and it is most interesting to see that appropriate generalizations are still of paramount interest in high-energy particle scattering reactions [18]. For example, a topic of current interest is the construction of "hard" and "soft" Pomerons [19], and their possible relation to phenomenological (and very successful) eikonal forms [20]. Eikonal problems of two decades ago, never completely resolved, have resurfaced in a somewhat different guise. For example, construction of the "Lipatov Pomeron" appears to resemble attempts made for the old "checkerboard" or "hybrid" eikonal models, with the exchange of "elementary particles" now replaced by multiple gluon exchange. One wonders if the symmetries and factorization properties found in these more modern attempts could be related to the underlying factorization of an appropriate ⟨ph⟩ model, where the desired eikonal function $\chi_n(b,s)$ corresponds to the exchange between scattering objects of n particles, or Reggeons, or gluons, each of which communicates with the others by the exchange of an arbitrary number of similar objects. With the ⟨ph⟩ approximation of this paper, one now has a starting point for the functional computation of χ_n, with the assurance that the propagator of each of the n,t-channel gluons exchanged between scatterers contains the proper (and symmetrized!) functional dependence on the sources used to describe the s-channel gluons multiply-exchanged between those t-channel gluons; and this dependence is explicitly given, in terms of a few, relevant quadratures for each of the "Pomerons." The calculation of such a χ_n will not be simple; but at least the ⟨ph⟩ method provides a reasonable starting point.

Another application is a "loop bremsstrahlung" calculation better than that previously given [21], in order to test whether the sharp e^+e^- "resonances" produced by that estimate do really occur during the scattering of heavy, nonrelativistic ions. In eikonal language, that calculation corresponded to the insertion of a closed-electron loop between a pair of scattering ions, and the use of low-frequency, or IR virtual photons across the loop as well as connecting the loop to the scattering ions. Because of one technical complication, that calculation was forced to rely on an added "factorization" approximation; and it would be very interesting to attempt the analysis starting from the more systematic (and symmetrized) ⟨ph⟩ approximation.

It should be clear that many other, more complicated eikonal calculations can now be attempted, for a variety of processes and fields.

B. Generalizations and corrections

The ⟨ph⟩ method presented in this paper is certainly not the only one, for there are many ways of introducing a "factorization" approximation into the Green's functions of potential theory and field theory. Indeed, the first method tried by the present authors in this direction was the replacement of the function $h(s_1,s_2)$ of (13) by its average over both variables, each from 0 to s. This produces a factorized form for $G_c[A]$, but one with a crude form of overall four-momentum conservation. Nevertheless, it can be a useful beginning for certain problems, and has the virtue that it gives exactly the correct result for $G_c(x,y|A)$ when $A(z)$ is a linear function of its argument.

Other forms of phase averaging will have other outputs, and no effort has as yet been made to classify these possibilities. It would be extremely interesting if one could manufacture an equivalent, factorized approximation which would be approximately correct for high frequencies. This can be easily done in a crude "averaged" approximation, in which $h(s_l,s_m)$ is replaced by its "diagonal" value $\delta_{lm}\cdot s_l$, thereby inserting a corresponding k_l^2 into the lth denominator of (18). The only trouble with doing this is that (19) will be replaced by

$$[m^2+p^2]^{-1}[m^2+(p-h_1)^2]^{-1}\cdots\left[m^2+p^2+\sum_l k_l^2 - 2p\cdot\sum_l k_l\right]^{-1}, \quad (19a)$$

and that the last factor of (19a) is not the same thing as $[m^2+p'^2]^{-1}$, although one still has $p'=p-\sum_l k_l$.

A large value of one k_l^2 can in this way be correctly described, even though this form can be wrong if more than one k_l is large. But only one denominator factor, the left-hand side (LHS) $[m^2+p^2]^{-1}$, is now correct; and care must be taken when subsequent mass-shell amputations are performed. The forms which will result from this approximation can be written as a "factorized" form similar to those of (26) or (27), except that each $\tilde{A}(k)$ should be replaced by $\exp[-is'k^2]\tilde{A}(k)$.

The difficulty with the improved version of (19a) is

that all $(k_l \cdot k_m)_{l \neq m}$ dependence has been dropped; and the obvious next step is to try to retain some measure of correlation between the different k_l. The simplest device, which still retains factorization, would be to replace the exact quantity

$$\exp\left[-i \sum_{l,m} k_l \cdot k_m h(s_l, s_m)\right]$$

of (12) with

$$\exp\left[-i \sum_{l,m} k_l \cdot k_m \left\{ \frac{1}{2}(s_l + s_m) - \left\langle \frac{1}{2} |s_l - s_m| \right\rangle \right\}\right] = \exp\left[-i \sum_l s_l k_l^2 - \frac{i}{2} \sum_{l \neq m} k_l \cdot k_m (s_l + s_m - s/2)\right] , \quad (21a)$$

where the phase averaging of (21a) is defined by the replacement $\langle |s_l - s_m| \rangle \to \frac{1}{s} \int_0^s ds' s' = s/2$. This corresponds to the same coefficient multiplying every $k_l \cdot k_m$ correlation, and is only provisionally acceptable because a factor proportional to $\sum_{l \neq m} k_l \cdot k_m$ will now appear in every denominator term of (19).

Clearly, one proper way of introducing systematic corrections to the $\langle \text{ph} \rangle$ definition lies in finding a "factorized" way of sequentially approximating the neglected part of $h(s_1, s_m)$; this approach will be treated in the second paper of this series. As work in the correction of IR approximations to solutions of nonlinear differential equations has shown [22], it is not the only way, for there a rescaling of an initial IR approximation leads to results dramatically close to the exact solutions. In its own right, the $\langle \text{ph} \rangle$ approximation of (20) adopted in this paper is immediately applicable to all eikonal, or IR approximations, in potential theory and in field theory.

One final comment may be in order, concerning another, possible method of computation [23] of $G_c[A]$. If one calculates the solution of the classical field equation

$$[m^2 - \partial^2]\varphi + \frac{g}{2}\varphi^2 = j(x) , \quad (53)$$

then simple functional differentiation of this solution (with proper attention to causality) implies that

$$G_c(x, y|\varphi) = \frac{\delta \varphi(x)}{\delta j(y)} . \quad (54)$$

Solving (53) for $\varphi(x|j)$ is then one way of obtaining the Green's function $G_c[\varphi]$, but only as a functional of that field φ which is a solution of (53). The $G_c[A]$ constructions described here are designed to yield the appropriate functionals of arbitrary fields $A(z)$, rather than those of fields which satisfy a (related) differential equation.

In summary, one might express the guarded hope that the $\langle \text{ph} \rangle$ approximation described here may eventually be looked upon as generating the simplest of nontrivial approximations to the n-point functions of quantum field theory, an approximation which, for certain, well-defined computations, can act as the field theory equivalent of the simple harmonic oscillator of ordinary quantum mechanics. Determinantal factors in a gauge theory, as well as the relevant Green's functions themselves, may be written in terms of a few, conceptually simple quadratures, providing a new and analytic framework within which one may try to address increasingly complex problems.

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APPENDIX A

We give a brief demonstration of Fradkin's representation for the Green's functions satisfying (6).

Consider the function $G_c(x, y|A)$ as a matrix element of the operator $\mathcal{G}[\mathcal{A}] : G_c(x, y|A) = \langle x|\mathcal{G}[\mathcal{A}]|y\rangle$, $\mathcal{A}$ being the operator associated with the local A field: $\langle x|\mathcal{A} = A(x)\langle x|$. In that case, (1) is equivalent to

$$[\mathcal{P}^2 + m^2 \mathbf{I} + g\mathcal{A}]\mathcal{G}[\mathcal{A}] = \mathbf{I} \quad (A1)$$

with $\langle x|\mathcal{P} = i\partial_x \langle x|$ and $\mathbf{I}$ is the identity operator.

Using Schwinger's representation, one writes

$$\mathcal{G}[\mathcal{A}] = i \int_0^\infty ds\, e^{-ism^2} e^{-is[\mathcal{P}^2 + g\mathcal{A}]} . \quad (A2)$$

Using now the functional formula

$$F\left[\frac{\delta}{\delta j}\right] e^{i\int jf} = F[if] e^{i\int jf} , \quad (A3)$$

one has

$$e^{i\int_0^s ds'[\delta^2/\delta v^2(s')]} e^{-i\int_0^s ds' \mathcal{P} \cdot v(s')} = e^{-i\int_0^s ds' \mathcal{P}^2} e^{-i\int_0^s ds' \mathcal{P} \cdot v(s')} , \quad (A4)$$

and then (A2) takes the form

$$\mathcal{G}[\mathcal{A}] = i\int_0^\infty ds\, e^{-ism^2} e^{i\int_0^s ds'[\delta^2/\delta v^2(s')]} e^{-i\mathcal{P}\cdot\int_0^s ds' v(s') - ig\mathcal{A}}|_{v=0}$$
$$= i\int_0^\infty ds\, e^{-ism^2} e^{i\int_0^s ds'[\delta^2/\delta v^2(s')]} e^{-i\mathcal{P}\cdot\int_0^s ds' v(s')} \mathcal{F}(s)|_{v=0} \ , \tag{A5}$$

where it can be easily shown that $\mathcal{F}(s) = e^{i\mathcal{P}\cdot\int v} e^{-i\mathcal{P}\cdot\int v - ig\mathcal{A}}$ satisfies the equation

$$\frac{\partial}{\partial s}\mathcal{F}(s) = e^{i\mathcal{P}\cdot\int_0^s ds' v(s')(-ig\mathcal{A})} e^{-i\mathcal{P}\cdot\int_0^s ds' v(s')}\mathcal{F}(s) \ . \tag{A6}$$

Applying $\langle x|$ to $\mathcal{G}[\mathcal{A}]$ yields

$$\langle x|\mathcal{G}[\mathcal{A}] = i\int_0^\infty ds\, e^{-ism^2} \exp\left(i\int_0^s ds'[\delta^2/\delta v^2(s')]\right)$$
$$\times \exp\left(\partial_x \int_0^s ds' v(s')\right)\langle x|\mathcal{F}(s)|_{v=0} \tag{A7}$$

where, following (A6), $\langle x|\mathcal{F}(s)$ obeys the differential equation

$$\frac{\partial}{\partial s}\langle x|\mathcal{F}(s) = e^{-\partial_x\int_0^s ds' v(s')}[-igA(x)]e^{\partial_x\int_0^s ds' v(s')}\langle x|\mathcal{F}(s)$$
$$= -igA\left(x - \int_0^s ds' v(s')\right)\langle x|\mathcal{F}(s) \ , \tag{A8}$$

which, with the condition $\mathcal{F}(0) = \mathbf{I}$, leads to

$$\langle x|\mathcal{F}(s) = \exp\left[-ig\int_0^s ds' A\left(x - \int_0^{s'} ds'' v(s'')\right)\right]\langle x| \ . \tag{A9}$$

Applying $|y\rangle$ to (A7) in which (A9) has been inserted, one recovers formula (10).

APPENDIX B

We here give a brief derivation of the potential theory eikonal representation, in terms of the $\langle \mathrm{ph}\rangle$ forms of Sec. II. This treatment, which preserves time-reversal invariance at each stage, is quite similar in spirit to previous field-theory derivations [3]. One begins by calculating the variation of the exact $G_c[V]$ with respect to the coupling parameter,

$$\frac{\partial}{\partial g}\langle \mathbf{p}|G_c[V]|\mathbf{p}'\rangle = -\int d^3y \langle \mathbf{p}|G_c[V]|y\rangle V(y)$$
$$\times\langle y|G_c[V]|\mathbf{p}'\rangle \ , \tag{B1}$$

and into this expression one inserts the $\langle \mathrm{ph}\rangle$ forms for each Green's function. The essential point is to perform energy-shell amputation but once on each Green's function, using

$$(\mathbf{p}^2 - k_0^2)\langle \mathbf{p}|G_c^{\langle \mathrm{ph}\rangle}[V]|\mathbf{p}'\rangle|_{p=k_0} = \int d^3z \exp\left(i(\mathbf{p}'-\mathbf{p})\cdot z - ig\int_0^\infty ds\, V(z + s(\mathbf{p}+\mathbf{p}'))\right) \tag{B2}$$

and

$$\langle \mathbf{p}|G_c^{\langle \mathrm{ph}\rangle}[V]|\mathbf{p}'\rangle(\mathbf{p}'^2 - k_0^2)|_{p'=k_0} = \int d^3w \exp\left(i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{w} - ig\int_0^{\inf} ds\, V(\mathbf{w} - s(\mathbf{p}+\mathbf{p}'))\right) \ . \tag{B3}$$

From (B1) and (32), the energy-shell scattering amplitude satisfies the relation

$$\xi\frac{\partial}{\partial g}f(\mathbf{p},\mathbf{p}') = -\int d^3y\, V(\mathbf{y})\int\frac{d^3p''}{(2\pi)^3}e^{-i\mathbf{p}''\cdot\mathbf{y}}\int d^3z \exp\left(i(\mathbf{p}''-\mathbf{p})\cdot z - ig\int_0^\infty ds\, V(z+s(\mathbf{p}+\mathbf{p}''))\right)$$
$$\times\int\frac{d^3\bar{p}}{(2\pi)^3}e^{i\bar{\mathbf{p}}\cdot\mathbf{y}}\int d^3w \exp\left(i(p'-\bar{p})\cdot w - ig\int_0^\infty ds\, V(w - s(\bar{p}+p'))\right) \ , \tag{B4}$$

and in order to proceed we must neglect q with respect to p' and/or p'', which entails the replacements

$$V(z + s(p+p'')) \to V(z+2sp), V(w - s(\bar{p}+p')) \to V(w - 2sp') \ .$$

Here, the integrated momentum variable of each Green's function can have contributions only from the close-to-forward directions specified by the other, fixed momentum variable; this is true because of the oscillatory factors $\exp[i(p'-\bar{p})\cdot w]$ and $\exp[i(p''-p)\cdot z]$ and the assumed, smooth spatial behavior of the potential. The result is to

generate the simplifying replacements

$$\int \frac{d^3p''}{(2\pi)^3} e^{i\mathbf{p}''\cdot(\mathbf{z}-\mathbf{y})} = \delta(\mathbf{z}-\mathbf{y}), \int \frac{d^3\bar{p}}{(2\pi)^3} e^{i\bar{\mathbf{p}}\cdot(\mathbf{y}-\mathbf{w})} = \delta(\mathbf{y}-\mathbf{w}) \ ,$$

so that

$$\xi \frac{\partial}{\partial g} f(\mathbf{p},\mathbf{p}') = -\int d^3y\, V(\mathbf{y}) e^{-i(\mathbf{p}-\mathbf{p}')\cdot y} \exp\left(-ig\int_0^\infty ds\, V(y+2sp)\right) \exp\left(-ig\int_{-\infty}^0 ds\, V(y+2sp')\right) . \tag{B5}$$

One next neglects the difference between p and p' inside the arguments of V, and so writes

$$\xi \frac{\partial f}{\partial g} = -\int d^3y\, V(y) e^{-i(p-p')\cdot y} \exp\left(-ig\int_{-\infty}^{+\infty} ds\, V(y+2sp)\right). \tag{B6}$$

The exponential factor here is

$$i\chi(\mathbf{b}) = -\frac{ig}{2p_\parallel}\int \frac{d^2k_T}{(2\pi)^2} \tilde{V}(\mathbf{k}_T, k_\parallel = 0) e^{i\mathbf{k}_T\cdot\mathbf{b}} \ ,$$

so that

$$\xi \frac{\partial f}{\partial g} = -\int d^2b \int dy_\parallel \int \frac{d^3k}{(2\pi)^3} \tilde{V}(\mathbf{k}_T, k_\parallel) e^{i\mathbf{q}\cdot\mathbf{b}} e^{i(p'_\parallel - p_\parallel)\cdot y_\parallel + i\mathbf{k}_T\cdot\mathbf{b} + ik_\parallel y_\parallel} e^{i\chi(b)} \ . \tag{B7}$$

Because the energy-shell conditions for p and p' mean that $p_\parallel = p'_\parallel$, the longitudinal integration of (B7) is trivial, and one can then perform the integration over coupling to obtain

$$f = \frac{2ik_0}{\xi}\int d^2b\, e^{i\mathbf{q}_T\cdot\mathbf{b}}[1-e^{i\chi(b)}] \ , \tag{B8}$$

where ξ is the constant of proportionality between $\tilde{V}(q)$ and the first Born approximation.

APPENDIX C

Other versions [24] of the eikonal approximation have been constructed and used directly in configuration space, in a way which retains the (x,y) interchange symmetry of the exact Green's function; it may be useful to sketch their similarities to, and differencs from, the no-recoil and $\langle$ph$\rangle$ momentum-space approximations; here, however, the construction is given starting from the exact Fradkin representation.

Because of the δ function of (10), the argument of its A dependence may be increased by $[x-y+\int_0^s ds'' v(s'')]\alpha(s')$, that is by zero, where α is completely arbitrary. In this way, (10) may be replaced by

$$\langle x|G_c[A]|y\rangle = i\int_0^\infty ds\, e^{-ism^2} \exp\left(i\int_0^s ds'\delta^2/\delta v^2\right)\delta\left(x-y+\int_0^s v\right)$$
$$\times \exp\left[-ig\int_0^s ds' A\left(x\alpha(s')+[1-\alpha(s')]y+\alpha\int_0^s v-\int_0^{s'} v\right)\right]_{v\to 0} . \tag{C1}$$

Now introduce a Fourier representation for the δ function of (C1), so that the functional operations of (C1) take the form

$$e^{\mathcal{D}} \exp\left(i\int_0^s v\cdot p\right)\mathcal{F}[v]|_{v\to 0} \ , \tag{C2}$$

with $\mathcal{D} = i\int_0^s ds'\delta^2/\delta v^2$ and

$$\mathcal{F}[v] = \exp\left[-ig\int_0^s ds' A\left(x\alpha+(1-\alpha)y + \alpha\int_0^s v - \int_0^{s'} v\right)\right] .$$

Using the easily proven linkage relation

$$e^{\mathcal{D}}F[v]H[v] = e^{\mathcal{D}_{12}}\left(e^{\mathcal{D}}F[v_1]\right)\left(e^{\mathcal{D}}H[v_2]\right) \ ,$$

where $\mathcal{D}_{12} = 2i\int_0^s ds' \frac{\delta}{\delta v_\mu^{(1)}(s')}\frac{\delta}{\delta v_\mu^{(2)}(s')}$, (C2) becomes

$$e^{-isp^2} \exp\left(-2p\cdot\int_0^s ds' \frac{\delta}{\delta v(s')}\right) e^{\mathcal{D}}\mathcal{F}[v]|_{v\to 0} \ , \tag{C3}$$

which changes (C1) to

$$e^{-isp^2}e^{\mathcal{D}}\exp\left[-ig\int_0^s ds' A\left(x\alpha+(1-\alpha)y+2p(s'-s\alpha)\right.\right.$$

$$\left.\left.+\alpha\int_0^s v-\int_0^{s'} v\right)\right]_{|v\to 0}. \qquad \text{(C4)}$$

For the functional computation yet to be performed, imagine the choice of parametrization: $u_\mu(s') = dZ_\mu(s')/ds'$, with $Z_\mu(s')$ an s'-dependent coordinate, and $\Delta Z_\mu(s') = Z_\mu(s') - Z_\mu(0)$ as s'-dependent path difference. The δ-function of (C1) requires $\Delta Z(s)+x-y=0$, while the Z-dependent terms of the argument in A in (C4) are $\alpha(s')\Delta Z(s) - \Delta Z(s')$. Since $e^{\mathcal{D}}F[v]|_{v\to 0} \equiv N\int d[v]e^{i/4\int_0^s ds' v^2(s')}F[v]$, with $N^{-1} = \int d[v]e^{\frac{i}{4}\int_0^s ds' v^2}$, this passage from $v(s')$ to $Z(s')$ corresponds to a change from a functional integral $\int d[v]$ to a sum over paths specified by $\Delta Z(s')$.

Let us, in the eikonal spirit, now restrict that sum over paths $\Delta Z(s')$ to straight-line paths between 0 and $y-x$: $\Delta Z(s') = (s'/s)\Delta Z(s)$. All Z dependence in the (C4) argument of A will then vanish if the choice is made: $\alpha(s') = s'/s$; and further, the p dependence of that argument will also disappear. If contributions from no other paths are allowed, which means no fluctuations permitted of $v(s')$ about that constant slope, then the remaining, normalized functional integral collapses to unity, and one obtains

$$\langle x|G^{\rm Eik}[A]|y\rangle = i\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\int_0^\infty ds\, e^{-is[m^2+p^2]}\exp\left(-ig\int_0^s ds' A\left\{x\frac{s'}{s}+y\left(1-\frac{s'}{s}\right)\right\}\right), \qquad \text{(C5)}$$

from which it is clear that the x, y interchange symmetry is preserved. Note that $\xi_\mu = x_\mu(s'/s) + y_\mu(1-s'/s)$ represents the straight line path between y_μ and x_μ.

Under the variable change $u = s'/s$, the s integration of (C5) is easily performed:

$$\int\frac{d^4p}{2\pi^4}e^{ip\cdot(x-y)}\left[m^2+p^2+g\int_0^1 du\, A(xu+(1-u)y)\right]^{-1}, \qquad \text{(C6)}$$

a deceptively simple form which one might interpret as a free-field propagator with a coordinate-dependent mass, a form reminiscent of exact solutions for propagation in a single laser field. In momentum space, however, (C6) is difficult to use. For example, consider the simplest, nontrivial term, linearly proportional to gA. This part of the exact propagator is proportional to the factors

$$g\tilde{A}(p-p')[m^2+p^2]^{-1}[m^2+p'^2]^{-1}, p = p'+q\,, \qquad \text{(C7)}$$

whereas the contribution of (C6) yields, instead,

$$g\tilde{A}(p-p')\int_0^1 du\{m^2+[up+(1-u)p']^2\}^{-2}\,. \qquad \text{(C8)}$$

Mass-shell amputation applied to (C7) produces an obviously correct result, while the same operation upon (C8) vanishes unless $q=0$. Clearly, mass-shell amputation does not commute with the usual eikonal, or no-recoil limit $p \gg q \neq 0$ and the problem is acerbated in higher orders; even though (C8) displays the proper TR symmetry, it is quite awkward to use for realistic scattering situations.

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