Is Compton scattering in magnetic fields really infrared divergent?

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The infrared behavior of QED changes drastically in the presence of a strong magnetic field: the electron self-energy and the vertex function are infrared *finite*, in contrast with field-free QED, while new infrared divergences appear that are absent in free space. One famous example of the latter is the infrared catastrophe of magnetic Compton scattering, where the cross section for scattering of photons from electrons which undergo a transition to the Landau ground state *diverges* as the frequency of the incoming photon goes to zero. We examine this divergence in more detail and prove that the singularity of the cross section is *removed* as soon as proper account is taken of all quantum electrodynamical processes that become indistinguishable from Compton scattering in the limit of vanishing frequency of the incident photon.

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I. INTRODUCTION

Compton scattering is the central mechanism for the redistribution of energy of hot electrons and electromagnetic radiation, and thus for the formation of spectra, in strongly magnetized pulsating x-ray sources [1] and, possibly, certain magnetized γ-ray burst sources [2]. Therefore, this quantum electrodynamical process was among the first to be recalculated for the magnetic field strengths of several 10^8 T which were detected in these objects by way of identification of cyclotron line features. The recalculation was necessitated by the fact that at these field strengths the cyclotron energy becomes of the order of the electron rest energy (equality holds for $B = m_e^2/e = 4.414 \times 10^9$ T), and thus, in calculating quantum electrodynamical processes, the quantization of the electron states into discrete Landau levels has to be fully taken into account [3]. This leads to a drastic change in the structure of cross sections as compared to free space. The relativistic Compton scattering cross section in a strong magnetic field was first derived by Herold [4] for initial and final electrons in the Landau ground state ($n_i = n_f = 0$, where n denotes the Landau quantum number), by Melrose and Parle [5] for $n_i = 0$ and final states $n_f = 0, 1$, by Daugherty and Harding [6] for $n_i = 0$ and arbitrary final states, and Bussard *et al.* [7] for arbitrary initial and final states.

Because of the mathematical complexity of the resulting expressions, the numerical evaluations and implementations into actual radiative transfer calculations were first restricted to considering Compton scattering with electrons in the Landau ground state [8]. It came therefore as a surprise when Brainerd [9] pointed out, for the special case $n_i = 1$, $n_f = 0$, that the cross section for Compton scattering with electrons in *excited* Landau states which during the scattering process undergo a transition to the Landau ground state *diverges* as the energy of the incoming photon goes to zero. Obviously, this process very efficiently turns soft photons into scat-

tered cyclotron photons of several tens of keV, and thus it was argued that the infrared catastrophe of the Compton cross section in a magnetic field could be at the root of the observed deficiency of soft photons in the spectra of magnetized γ-ray burst sources. It is the purpose of this paper to examine the divergence more closely. Our main result is that the singularity of the cross section is removed when proper account is taken of all the quantum electrodynamical processes which degenerate with Compton scattering in the limit of vanishing frequency of the incoming photon.

Let us first briefly recall what is the cause of the infrared catastrophe in ordinary quantum electrodynamics (cf. [10,11] and references therein). It is has been known since the classic paper of Bloch and Nordsieck [12] that infrared divergences appear in theory because, loosely speaking, an accelerated charged particle can emit an infinite number of soft photons with finite total energy. In the real world, any experiment is carried out during a finite time interval, so a finite energy resolution Ω is necessarily inherent in every experiment (a lower bound is given by the energy uncertainty, $h/\Delta T$, although the energy resolution Ω of a real detector will in general be much larger). Therefore, whenever charged particles participate in some reaction, one cannot distinguish experimentally between this reaction and the same one with supplementary (real or virtual) soft photons being emitted or absorbed. Here the term "soft photons" means photons which are undetectable because their energies lie below the detection threshold, $\omega_s < \Omega$ (throughout this paper the subscript s stands for soft photons). For that reason, one way to solve the infrared problem is to take into account in the calculation of transition probabilities the coherent superposition of the S-matrix elements of all indistinguishable processes, in accordance with general principles of quantum theory [13]. Then, as is well known from field-free QED, the infrared divergent contributions to observables should cancel in any order of perturbation theory. However, the mechanism of can-

cellation has to be quite different in QED in magnetic fields: e.g., in Ref. [14] it was proved that the electron self-energy in a magnetic field is *not* infrared divergent. On the other hand, there occurs a new infrared divergence in the magnetic Compton cross section for incident photons. Although this cross section is finite for every finite energy ω_i of the incident photon, its "explosion" for $\omega_i \to 0$ seems unphysical. We note that in a similiar case of infrared divergences for incident photons in the older theory of weak and electromagnetic interactions their cancellations were shown in Ref. [15].

Our line of argument can be illustrated most easily with the help of the Feynman diagrams shown in Fig. 1. In the limit of vanishing frequency ω_i of the incident photon, the second-order process of Compton scattering from a Landau excited electron [Fig. 1(b), $S \propto e^2$] becomes indistinguishable from the process of cyclotron emission [Fig. 1(a), $S \propto e^1$]. Thus, in calculating the transition probability in this limit, the two corresponding S-matrix elements must first be added coherently and subsequently be squared. The squared total S-matrix element then contains terms up to order e^4, which implies that, for the expansion to be consistent, in the coherent superposition all other processes must also be included whose direct or cross terms produce contributions up to order e^4 in the squared total S-matrix element, and degenerate with cyclotron emission in the limit of one or more of the photon frequencies involved going to zero. Obviously these are the following second- and third-order processes: double cyclotron emission with one soft photon [Fig. 1(c)], triple cyclotron emission with two soft photons [Fig. 1(g)], double Compton scattering with two soft photons [Fig. 1(e)], [Fig. 1(f)], and the low-energy part of the vertex correction of cyclotron emission [Fig. 1(d)]. Note that since the electron self-energy in a strong magnetic field is infrared

finite we need not consider soft photon insertions into the external electron lines [16].

In what follows, we will not calculate the infrared finite part of the magnetic Compton cross section but restrict ourselves to the simpler task of investigating the singular terms of the S-matrix elements and demonstrate that a cancellation of their divergences occurs. It is therefore sufficient to consider the S-matrix elements in the limit $\omega_s \to 0$.

We shall prove below that in the limit $\omega_s \to 0$ the contributions of (ordinary) Compton scattering and double cyclotron emission to the total S-matrix element *cancel*, as do the contributions of double Compton scattering, while the term due to triple cyclotron emission vanishes in the cross section, and the vertex correction remains finite. Thus, the infrared catastrophe of the Compton scattering cross section in magnetic fields reported in the literature is *nonexistent*. We note, however, that $(n_i \neq 0) \to 0$ cross sections may still remain large compared to the $0 \to 0$ cross section at low energies, in which case the essence of Brainerd's [9] analysis would remain valid in spite of the absence of a real singularity of the cross section for vanishing photon energies.

II. INFRARED BEHAVIOR OF THE S-MATRIX ELEMENTS

To render our argument quantitative we start from the S-matrix element of Compton scattering in a magnetic field, $S_2^{(2)}$ (in what follows the superscript of S denotes the order in e, and the subscript, the subcaption number given to the process in Fig. 1), which reads (cf. Bussard et al. [7])

$$S_2^{(2)} = \left(\frac{2\pi}{V}\right)^2 \frac{e^2}{(\omega_i\omega_f)^{1/2}} \, \delta(E_f + \omega_f - E_i - \omega_i) \sum_{a,\lambda} \left(\frac{\left(\epsilon_f^* \cdot \mathbf{J}_{f,a}^{(\lambda)}\right)\left(\epsilon_i \cdot \mathbf{J}_{i,a}^{(\lambda)*}\right)}{E_i + \omega_i - \lambda(E_a - i\epsilon_a)} + \frac{\left(\epsilon_f^* \cdot \mathbf{J}_{i,a}^{(\lambda)}\right)\left(\epsilon_i \cdot \mathbf{J}_{f,a}^{(\lambda)*}\right)}{E_i - \omega_f - \lambda(E_a - i\epsilon_a)}\right) . \tag{1}$$

Here, i and f refer to the initial and final states, the sum over a runs over the intermediate Landau states of electrons ($\lambda = +1$) and positrons ($\lambda = -1$), and the quantities $\mathbf{J}$ denote matrix elements whose explicit forms are given in [7]. We note that in the limit $\omega_i \to 0$ these
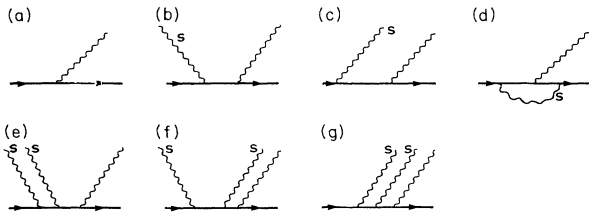
(a) (b) (c) (d)

(e) (f) (g)

FIG. 1. Feynman diagrams of the processes which become experimentally indistinguishable as the energies of the soft photons s go to zero. (Exchange diagrams have been omitted for brevity.)

quantities assume constant, finite values.

The imaginary parts of the energies of the intermediate states, $i\epsilon_a$, in Eq. (1) account [17] for the fact that excited Landau states have nonzero widths, i.e., $i\epsilon_a$ is given by $\frac{1}{2}i\Gamma_n$, where Γ_n is the cyclotron decay rate of an electron in the nth Landau level. Obviously, resonances appear in Eq. (1) at the zeros of the real parts of the energy denominators, which is the case when Compton scattering degenerates into electron cyclotron absorption ($E_i + \omega_i = E_a = E_f$, $\omega_f = 0$) or emission ($E_i - \omega_f = E_a = E_f$, $\omega_i = 0$), i.e., the virtual electron is created "on shell". Because of the nonvanishing widths $\frac{1}{2}i\Gamma_n$ for $n > 0$, these resonances remain finite, with the exception of the case of cyclotron transitions to the Landau ground state ($n_a = n_f = 0$, $\omega_i = 0$, $\omega_f \neq 0$): the latter is stable, viz. $\Gamma_0 = 0$, and thus a genuine singularity occurs in the expression (1). This is the type of infrared divergence pointed out by Brainerd [9]. The infrared divergent part of the S-matrix element is dia-

grammatically shown in Fig. 2. It can be read off Eq. (1) that the divergence of S is of the order $O(\omega_i^{-3/2})$.

We now turn to the S-matrix element of double-cyclotron emission, $S_3^{(2)}$, which can easily be obtained from that of Compton scattering using the crossing symmetry replacements

$$k_{i,s}^\mu \rightarrow -k_{f,s}^\mu . \tag{2}$$

In the limit $\omega_{f,s} \rightarrow 0$ the terms containing $\mathbf{k}_{f,s}$ reduce to nondivergent expressions identical to those of $S_2^{(2)}$. Because of the replacements $\omega_{i,s} \rightarrow -\omega_{f,s}$ in the energy denominators it then follows

$$S_2^{(2)}(\mathbf{k}_s) = -S_3^{(2)}(\mathbf{k}_s) . \tag{3}$$

This implies that in the limit $\omega_s \rightarrow 0$ the divergences of Compton scattering and double cyclotron emission identically cancel.

We now have to show that all other processes which also have to be taken into account produce no new divergences. This is a simple task for the two third-order processes 5 and 6 in Fig. 1 with at least one soft photon, for which an analogous application of the crossing symmetry argument given above yields

$$S_5^{(3)}(\mathbf{k}_s) = -S_6^{(3)}(\mathbf{k}_s) , \tag{4}$$

while in the cross section of the process $S_7^{(3)}$ all possible infrared divergences are canceled by the phase space factors d^3k.

Thus the only critical term that remains is the vertex correction to cyclotron emission [Fig. 1(d)]. Using the
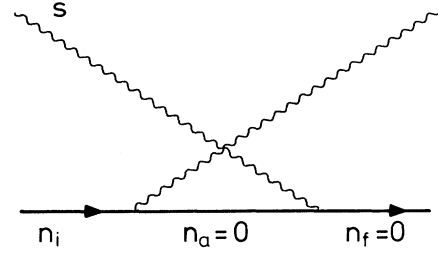

FIG. 2. Infrared divergent part of Compton scattering.

same technique as described in [18] we have derived, to our knowledge for the first time, the vertex correction in a strong magnetic field [19], but will restrict ourselves here to a discussion of the S-matrix element only in so far as is necessary to prove that this process is not infrared divergent. (A full treatment of the vertex correction in a strong magnetic field will be presented elsewhere.) The S-matrix element reads

$$S_4^{(3)} = (ie)^3 \int d^4x \, d^4x' \, d^4x'' \, \bar\psi_f^{(\lambda=+)}(x)\gamma^\mu iS_F(x,x')$$
$$\times \gamma^\nu iS_F(x',x'')\gamma^\varrho \psi_i^{(\lambda=+)}(x'')iD_{\mu\varrho}(x-x'')A_\nu^*(x') , \tag{5}$$

where, as usual, $iD_{\mu\nu}$ denotes the photon propagator, while iS_F describe the electron propagator and $\psi^{(\lambda=\pm)}$ the electron and positron fields in a magnetic field, respectively. Using the temporal gauge for the photon propagator and performing the integrations over time we obtain

$$S_4^{(3)} = (-ie)^3 \, 2\pi \, \delta(E_f + \omega_f - E_i) \int \frac{d^4k}{(2\pi)^4} \int d^3x \, d^3x' \, d^3x'' \, \psi_f^{(\lambda=+)\dagger}(\mathbf{x})\alpha_j$$

$$\times \sum_{a,b} \left\{ \left(\frac{1}{E_i - E_a - \omega - \omega_f + i\epsilon_a} \frac{1}{E_i - E_b - \omega + i\epsilon_b} \psi_a^{(\lambda=+)}(\mathbf{x})\psi_a^{(\lambda=+)\dagger}(\mathbf{x}')\alpha_k\psi_b^{(\lambda=+)}(\mathbf{x}')\psi_b^{(\lambda=+)\dagger}(\mathbf{x}'') \right) \right.$$

$$+ \left(\frac{1}{E_i - E_a - \omega - \omega_f + i\epsilon_a} \frac{1}{E_i + E_b - \omega - i\epsilon_b} \psi_a^{(\lambda=+)}(\mathbf{x})\psi_a^{(\lambda=+)\dagger}(\mathbf{x}')\alpha_k\psi_b^{(\lambda=-)}(\mathbf{x}')\psi_b^{(\lambda=-)\dagger}(\mathbf{x}'') \right)$$

$$+ \left(\frac{1}{E_i + E_a - \omega - \omega_f - i\epsilon_a} \frac{1}{E_i - E_b - \omega + i\epsilon_b} \psi_a^{(\lambda=-)}(\mathbf{x})\psi_a^{(\lambda=-)\dagger}(\mathbf{x}')\alpha_k\psi_b^{(\lambda=+)}(\mathbf{x}')\psi_b^{(\lambda=+)\dagger}(\mathbf{x}'') \right)$$

$$+ \left. \left(\frac{1}{E_i + E_a - \omega - \omega_f - i\epsilon_a} \frac{1}{E_i + E_b - \omega - i\epsilon_b} \psi_a^{(\lambda=-)}(\mathbf{x})\psi_a^{(\lambda=-)\dagger}(\mathbf{x}')\alpha_k\psi_b^{(\lambda=-)}(\mathbf{x}')\psi_b^{(\lambda=-)\dagger}(\mathbf{x}'') \right) \right\}$$

$$\times \alpha_l\psi_i^{(\lambda=+)}(x'')\frac{\epsilon_k^*}{\sqrt{2\omega_f V}} e^{-ik_f x'} iD_{jl}(k)e^{+ik(\mathbf{x}-\mathbf{x}'')} . \tag{6}$$

The spatial integrals lead to complicated expressions consisting essentially of polynomials in the momentum of the virtual photon, they therefore contribute only constant terms when the energy of the virtual photon approaches zero. The potentially infrared divergent part of $S_4^{(3)}$ is

diagrammatically shown in Fig. 3. Replacing $i\epsilon_{a,b}$ with $\frac{1}{2}i\Gamma_n$ as before, restricting the virtual photon momentum to a domain Ω defined by the condition that all processes in point are observationally indistinguishable and taking into account $n_i > 0$, we obtain
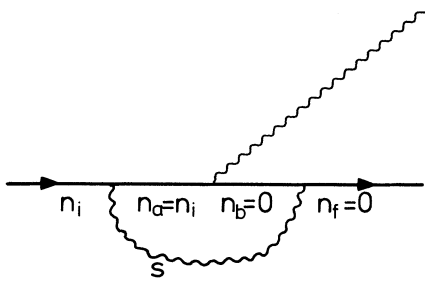
FIG. 3. Potentially infrared divergent part of the vertex correction to cyclotron emission.

$$S_4^{(3)} \propto \int_\Omega d^4k \, \frac{1}{k^2} \, \frac{1}{\Gamma_{n_i}} \, \frac{1}{\omega} \propto \int_{0 \le k \le \Omega} k^3 dk \, \frac{1}{k^3} < \infty \, , \quad (7)$$

from which it follows that the contribution of low-energy virtual photons to $S_4^{(3)}$ is *not infrared divergent*.

III. SUMMARY

In ordinary quantum electrodynamics infrared divergences arise in perturbation theory when corrections by emitted or virtual soft photons are taken into account. Examples for soft photon corrections are the self-energy and the vertex function. By contrast, in QED in strong magnetic fields the self-energy as well as the vertex function are not infrared divergent, but there occur new divergences for *incident* soft photons. For the case of Compton scattering with electrons in excited Landau states which undergo a transition to the Landau ground state we have shown that the divergent parts of the observationally indistinguishable processes cancel for vanishing energy of the soft photons,

$$\lim_{k_s \to 0} \sum_{i \in \{2,3,5,6\}} S_i(k_s) = 0 \, , \quad (8)$$

while the contributions of the remaining processes to the transition probability remain finite or vanish. Thus there is no infrared catastrophe of the Compton cross section for $(n_i \ne 0) \to 0$ transitions and, in contrast with field-free QED, already the total S-matrix element is infrared finite.

We have derived Eq. (8) using the very general argument of crossing symmetry. This symmetry implies that a cancellation of the infrared divergences of emitted and absorbed soft photons will take place in every order of perturbation theory.

Although we have shown that in the limit $\omega_i \to 0$ the Compton cross section tends to a finite value, it is difficult, because of the complexity of the individual S-matrix elements, to answer the question as to the *behavior* of the cross section as a function of photon energy in the vicinity of $\omega_i = 0$. Very fundamental considerations of field-free quantum electrodynamics [10] suggest that for $\omega_i < \Omega \ll m_e$ the basic process without any soft photon—in our case first-order cyclotron emission—will give the dominant contribution to the total transition probability w, viz.,

$$w = \frac{1}{T} \left| S_1^{(1)} \right|^2 \, , \quad (9)$$

in which case soft photons assume the role of spectators. As a consequence, the transition probability will assume the constant value following from Eq. (9) for $\omega_i < \Omega$, which through Ω depends on the actual observational resolution. Therefore, theoretical calculations of spectra (i.e., Monte Carlo simulations) which include transitions $(n_i \ne 0) \to 0$ should be carried out taking into account the finite resolution of the specific detector.

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magnetic Compton cross section is essential, and correct, *only* in the limit $\omega_i \to 0$ (and $\omega_s \to 0$), and not for arbitrary frequencies ω_i and ω_s where the processes can be discriminated experimentally.

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