Fermion determinants in static, inhomogeneous magnetic fields

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The renormalized fermionic determinant of QED in 3 + 1 dimensions, $\det_{\rm ren}$, in a static, unidirectional, inhomogeneous magnetic field with finite flux can be calculated from the massive Euclidean Schwinger model's determinant $\det_{\rm Sch}$ in the same field by integrating $\det_{\rm Sch}$ over the fermion's mass. Since $\det_{\rm ren}$ for general fields is central to QED, it is desirable to have nonperturbative information on this determinant, even for the restricted magnetic fields considered here. To this end we continue our study of the physically relevant determinant $\det_{\rm Sch}$. It is shown that the contribution of the massless Schwinger model to $\det_{\rm Sch}$ is canceled by a contribution from the massive sector of QED in 1 + 1 dimensions and that zero modes are suppressed in $\det_{\rm Sch}$. We then calculate $\det_{\rm sch}$ analytically in the presence of a finite flux, cylindrical magnetic field. Its behavior for large flux and small fermion mass suggests that the zero-energy bound states of the two-dimensional Pauli Hamiltonian are the controlling factor in the growth of $\ln \det_{\rm Sch}$. Evidence is presented that $\det_{\rm Sch}$ does not converge to the determinant of the massless Schwinger model in the small mass limit for finite, nonzero flux magnetic fields.

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I. INTRODUCTION

Fermion determinants produce an effective measure for the boson fields of a Euclidean field theory when the fermions are integrated out, as shown long ago by Matthews and Salam [1]. Such determinants in an external boson field, or a random boson field with a cutoff, are infinite dimensional and need to be defined [2] before the boson fields can be integrated. Once defined, their analysis is notoriously difficult, especially if they possess a symmetry that should be preserved. For example, most classical estimates of fermion determinants [3] invariant under a local U(1) transformation violate this invariance. The lack of nonperturbative information on fermion determinants is reflected in the necessity to make loop expansions or the more extreme quenched (valence) approximation. As a result physical effects predicted by the theory may be lost.

Nonperturbative information on the fermion determinant, such as its growth in the complex coupling plane, is central to an analysis of the nature of the perturbation series of the associated field theory [4]. Intuition tells one that Fermi statistics, visible in the alternating signs of the determinant's loop expansion, ought to slow down the growth of a perturbation series with order.

There is also the question of stability. Specifically, in the case of an Abelian gauge field, the measure is Gaussian, so that if the fermion determinant grows faster than an inverted Gaussian, it is doubtful that it is integrable with respect to the gauge field's measure.

Given the sparseness of nonperturbative information on physically relevant fermion determinants in four dimensions we thought it useful to try to find a solvable example for a broad class of boson fields. An obvious choice is the fermion determinant of quantum electrodynamics in time-independent, unidirectional, inhomogeneous magnetic fields. Although of physical interest, this choice of fields suffers from the fact that they are a set of measure zero as far as the functional integral over the vector potential is concerned. Nevertheless, the fermion determinant for such magnetic fields remains unsolved except for the special case of a homogeneous magnetic field that was dealt with over half a century ago by Euler, Heisenberg, and Weisskopf [5] and later on again by Schwinger [6]. A thorough understanding of this problem would be helpful for a more general understanding of the physical content of fermion determinants in quantum electrodynamics. As we will see below, there are some significant simplifications that recommend this problem for analysis.

This paper is organized as follows. In Sec. II, we review previous relevant results. Section III justifies our restricted choice of magnetic fields on which the results of Sec. II rely. We then go on in Sec. IV to discuss the suppression of zero modes in the fermion determinant of the massive Schwinger model. As Sec. II makes clear, the four-dimensional determinant is obtained by integrating this determinant over the fermion's mass. Section IV also illustrates the profound change in the two-dimensional determinant when the fermions are given a mass. In Sec. V, we calculate the massive Schwinger model's determinant for a finite, nonzero flux magnetic field. Section VI contains a discussion of the zero-mass limit of the massive Schwinger model's determinant, and Sec. VII summarizes our results.

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II. REVIEW OF PREVIOUS RESULTS

In a previous paper [7] it was shown that the renormalized fermion determinant $\det_{\rm ren}$ of quantum electrodynamics in four dimensions (QED_4) in a smooth, static, unidirectional magnetic field with fast decrease at infinity is related to the Euclidean, massive Schwinger model's (Euclidean QED_2's) determinant $\det_{\rm Sch}$ in the same magnetic field. Specifically,

$$\ln \det_{\rm ren} = \frac{e^2 V_\parallel}{4\pi^2} \int \frac{d^2k}{(2\pi)^2} |\hat{B}(k)|^2 \int_0^1 dz z(1-z) \ln\left[\frac{k^2 z(1-z)+m^2}{m^2}\right] + \frac{V_\parallel}{2\pi}\int_{m^2}^\infty dM^2 \ln \det_3(M^2), \tag{2.1}$$

where the gauge-invariant $\det_3$ is related to $\det_{\rm Sch}$ by

$$\ln \det_{\rm Sch} = -\frac{e^2}{2\pi}\int \frac{d^2k}{(2\pi)^2}|\hat{B}(k)|^2 \int_0^1 dz \frac{z(1-z)}{k^2 z(1-z)+m^2} + \ln \det_3(m^2). \tag{2.2}$$

Since the first term in (2.2) is the contribution to the effective action from the second-order vacuum polarization graph, $\ln \det_3$ may be viewed as the sum of all one-loop fermion graphs in two dimensions beginning in fourth order. Our designation of this sum by $\ln\det_3$ follows the definition of Seiler [2] and Simon [3], with the third-order graph vanishing by C invariance. Equation (2.1) states that the sum of the corresponding graphs in QED_4 is obtained by integrating $\ln \det_3$ over the fermion mass m. The first term in (2.1) is the contribution to the effective action of QED_4 from the on-shell renormalized second-order vacuum polarization graph. The function $\hat{B}$ is the Fourier transform of the magnetic field, which may be assumed to point along the z axis, in which case $V_\parallel$ is the volume of the zt box. In both (2.1) and (2.2) the determinants are defined by Schwinger's proper time definition [6]. Note that the charge e will always occur in the combination eB in position space, which has the invariant dimension L^{-2}. Note also that in Euclidean QED_2 potentials associated with unidirectional magnetic fields are a set of measure one.

The lesson of (2.1) is that the massive Schwinger model [8] is more than a model in view of its direct bearing on physics in four dimensions. Unlike the determinant of the massless Schwinger model [9], $\det_{\rm Sch}$ in (2.2) is not known explicitly. Nevertheless there are some important results. One of these is an expression of the paramagnetism of fermions in an external magnetic field as summarized by the "diamagnetic bound":

$$\det_{\rm Sch} \le 1, \tag{2.3}$$

for $m^2 \ge 0$ [2,10,11]. This bound also expresses the positivity of the one-loop effective action for Euclidean QED_2. As a consequence of (2.3) we were able to put an upper bound on the growth of $\det_{\rm ren}$ in QED_4 for strong, unidirectional, inhomogeneous magnetic fields. Performing the dilation $A_\mu \to \lambda A_\mu$ on the vector potential and letting λ become large we obtained

$$\ln \det_{\rm ren} \underset{\lambda \gg 1}{\le} \frac{\lambda^2 e^2 V_\parallel \|B\|^2}{24\pi^2} \ln\left(\frac{\lambda^2 e^2 \|B\|^2}{m^2}\right) + O(\lambda^0), \tag{2.4}$$

where $\|B\|^2 = \int d^2r B^2(\mathbf{r})$ [7].

There are other results for $\det_{\rm Sch}$ implicit in the literature. For example, if A_μ falls off sufficiently rapidly so that $\int d^2r |A_\mu|^q < \infty$ for all $q \ge \frac{1}{2}$, then one can relate $\det_{\rm Sch}(e)$ to the zeros of the determinant, considered as a function of a complex coupling e [12]. It is known [13] that for $m \ne 0$ these zeros occur in quartets e_n, $-e_n$, e_n^*, and $-e_n^*$, and therefore $\det_{\rm Sch}$ cannot vanish for real e for these potentials [14]. We will defer the discussion of the result of Haba [15] for $\det_{\rm ren}$ until Sec. III.

In view of the direct connection between QED_4 and QED_2 in the case of unidirectional magnetic fields it was thought worthwhile to obtain more specific information of $\det_{\rm Sch}$ that would enable one to make use of (2.1). In a second paper [16] it was shown that the exact calculation of $\det_{\rm Sch}$ reduces to a problem in nonrelativistic, supersymmetric quantum mechanics. That is to say,

$$\ln \det_{\rm Sch} = \frac{e^2}{2\pi}\int d^2r \phi \partial^2 \phi + 2m^2 \int_0^e d\lambda\, \mathrm{Tr}\{[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}]\phi\}, \tag{2.5}$$

where the supersymmetric operator pair $H_\pm = (\mathbf{p} - \lambda \mathbf{A})^2 \mp \lambda B$ are obtained from the two-dimensional Pauli Hamiltonian in (3.3) below. The auxiliary potential ϕ is related to the vector potential by $A_\mu = \epsilon_{\mu\nu}\partial_\nu \phi$ and to the magnetic field by $B = -\partial^2\phi$ or

$$\phi(\mathbf{r}) = -\frac{1}{2\pi}\int d^2r' \ln|\mathbf{r}-\mathbf{r}'| B(\mathbf{r}'). \tag{2.6}$$

The antisymmetric tensor $\epsilon_{\mu\nu}$ is normalized to $\epsilon_{12} = 1$. Again, the starting point for the derivation of (2.5) was the proper time definition of $\det_{\rm Sch}$, which respects gauge invariance and allows one to select the Lorentz gauge. Our representation of $\det_{\rm Sch}$ makes a sharp separation between the contribution from the massless Schwinger model, the first term in (2.5), and its massive counterpart. We have not integrated the first term in (2.5) by parts as is usually done. For nonzero flux fields, A_μ falls off like $1/r$, and therefore an integration by parts is invalid in this case.

Within the Lorentz gauge there is still the restricted gauge freedom of $\phi \to \phi + c$, where c is a constant. Since $\det_{\rm Sch}$ is gauge invariant, the term proportional to c in

(2.5) must vanish, in which case

$$\frac{e^2\Phi}{2\pi} = 2m^2 \int_0^e d\lambda \, \mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}], \tag{2.7}$$

since $-\partial^2\phi = B$ and $\Phi = \int d^2rB$. Differentiating (2.7) with respect to e gives the index theorem on a two-dimensional Euclidean manifold [17]:

$$\begin{aligned}\frac{e\Phi}{2\pi} &= m^2 \mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] \\ &= n_+ - n_- + \frac{1}{\pi}\sum_l [\delta_+^l(0) - \delta_-^l(0)],\end{aligned} \tag{2.8}$$

where $n_\pm$ denote the number of positive and negative chirality zero-energy bound states of $H_\pm$, and $\delta_\pm^l(0)$ are the zero-energy phase shifts for scattering by the Hamiltonians $H_\pm$ in a suitable angular-momentum basis l. Thus, the index theorem in QED_2 follows from gauge invariance.

Now suppose (2.5) is written as

$$\begin{aligned}\ln \det{}_{\mathrm{Sch}} = \frac{e^2}{2\pi}\int d^2r\phi\partial^2\phi \\ +2\int_0^e d\lambda \, \mathrm{Tr}\{[(P_+^{(\lambda)}(0) - P_-^{(\lambda)}(0)]\phi\} \\ +\text{nonzero modes},\end{aligned} \tag{2.9}$$

where $P_\pm^{(\lambda)}(0)$ are the zero-mode spectral measures associated with $H_\pm$. Noting that $H_+ - H_- \sim B$ and assuming B sufficiently weak, we calculated the second term in (2.9) in first Born approximation and showed that it canceled the massless Schwinger model term. We conjectured that this was true in general, and we will show that it is in Sec. IV.

Finally, we wish to retract a claim in [16]. By the Aharonov-Casher theorem [18], $n_+(n_-)$ are given by $[|e\Phi|/2\pi]$, depending on whether $e\Phi > 0$ ($e\Phi < 0$), where $[x]$ stands for the nearest integer less than x and $[0] = 0$. Thus, if $|e\Phi|/2\pi > 1$ there are zero-energy bound states, and it was stated that when these are included in the second term of (2.9) the logarithmic growth of ϕ and the slow, algebraic fall off of the bound-state wave functions for large r would cause $\det_{\mathrm{Sch}}$ to vanish. This is false. In fact, zero modes are suppressed in $\det_{\mathrm{Sch}}$ as we will see in Sec. IV.

III. CHOICE OF FIELDS

The fermion determinant is part of a functional integral whose measure, $d\mu(A)$, is that of the gauge-fixed, free Maxwell field. As $d\mu(A)$ may be realized on $\mathcal{S}'$, the space of tempered distributions, we seem to be stuck with rough potentials that are hard to analyze. A way out is to realize that there is a logarithmic ultraviolet divergence in QED_2 due to the vacuum energy graph shown in Fig. 1 that has to be regularized and subtracted out. One way to regularize [19] is to smooth A_μ by convoluting it with an ultraviolet cutoff function $h_\Lambda \in \mathcal{S}$, the functions of

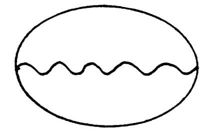

FIG. 1. The logarithmically divergent contribution to the vacuum energy in QED_2.

rapid decrease. That is, let $A_\mu^\Lambda = A_\mu * h_\Lambda$ so that A_μ^Λ is a polynomial bounded C^∞ function, meaning that for each $\alpha = (m,n)$ there is an $N(\alpha)$ and a $C(\alpha)$ with

$$\left|\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}A_\mu^\Lambda(x,y)\right| \le C(1+x^2+y^2)^N. \tag{3.1}$$

The regularized free-photon propagator is

$$\int d\mu(A) A_\mu^\Lambda(x) A_\nu^\Lambda(y) = D_{\mu\nu}^\Lambda(x-y), \tag{3.2}$$

whose Fourier transform $\hat{D}_{\mu\nu}^\Lambda(q)$ is proportional to $|\hat{h}_\Lambda(q)|^2$, where $\hat{h}_\Lambda$ is the Fourier transform of h_Λ. One possibility is to choose $\hat{h}_\Lambda = C_0^\infty$ with $\hat{h}_\Lambda(q) = 1$ for $q^2 \le \Lambda^2$ and $\hat{h}_\Lambda(q) = 0$ for $q^2 > 2\Lambda^2$. Note that the measure $d\mu(A)$ is not regularized. Thus, without loss of generality we may assume that our potentials are smooth and polynomial bounded. Hereafter we will drop the superscript Λ and denote these potentials simply as A_μ.

Now in order that the fermion determinant exist it seems necessary to require the magnetic fields derived from the potentials to have finite flux. The reason for this restriction is connected with the degeneracy of the ground state associated with the two-dimensional Pauli Hamiltonian

$$H = (\mathbf{p} - \mathbf{A})^2 - \sigma^3 B \ge 0. \tag{3.3}$$

Specifically, Avron and Seiler [20] have considered the class of polynomial, infinite-flux magnetic fields

$$B(\mathbf{r}) = \sum_{n=0}^{N} \lambda_n(\mathbf{r} - \mathbf{C}_n)^{2k_n}, \tag{3.4}$$

where $\{\lambda_n\}$ and $\{\mathbf{C}_n\}$ are arbitrary real numbers and $\{k_n\}$ and N are non-negative integers. They have shown that the ground state of H is infinitely degenerate and that the manifold of zero-energy bound-state wave functions is parametrized by a point in $\mathbf{R}^{2(2k_{\max}+1)}$, irrespective of the translational invariance of the magnetic field. In the constant field case, $k_{\max} = 0$, one has point spectrum, and the vector space is a plane whose points specify the center of rotation of Landau orbits, corresponding to the known degeneracy $|eB|L_x, L_y/2\pi$, with $L_xL_y \to \infty$. This degeneracy persists for all excited states in the constant field case, and we suspect that the excited states for fields with $N > 0$ are at least as degenerate as $\mathbf{R}^{2(2k_{\max}+1)}$. Although we will show in Sec. IV that zero modes are suppressed in our representation (2.5) of $\det_{\mathrm{Sch}}$, it cannot make sense out of the volumelike divergences associated with the degeneracy of the excited modes. Nor are we aware of any definition of determinant that can. Accordingly, we will confine our attention here to finite-flux fields. In our view, a determinant of an infinite-flux field should be considered as the limit, if it exists, of a determinant calculated from a field confined to local planar regions.

Restriction to finite flux is also consistent with the additional need to regulate volume divergences before taking the thermodynamic limit of the Euclidean Green's functions. Such divergences appear in the vacuum energy graphs, including the one in Fig. 1, when the determinant is integrated with respect to $d\mu(A)$. A volume cutoff may be introduced via the determinant, assumed previously calculated for a generic class of fields, by replacing $B(\mathbf{r})$ with $g(\mathbf{r})B(\mathbf{r})$, where g is a suitable volume cutoff such as $g \in C_0^\infty$ or $g = \chi_\Lambda$, the characteristic function of a bounded region $\Lambda \subset \mathbf{R}^2$.

At this point we mention the result of Haba [15], who studied $\det_{\text{ren}}$ in the field $F_{\mu\nu}(x) = G_{\mu\nu} + F'_{\mu\nu}(x)$, where $G_{\mu\nu}$ is constant, and $F'_{\mu\nu}$ is sufficiently smooth and bounded. It was concluded that with the introduction of a space-time volume cutoff Λ on the otherwise infinite flux field $F_{\mu\nu}$ that

$$\begin{aligned}\ln \det{}_{\text{ren}} = \frac{|\Lambda|}{48\pi^2}(G^2 - \tfrac{3}{2}|G^*G|)\ln(1+G^2)\\ -G^2\int_\Lambda d^4x b(G,F'),\end{aligned} \tag{3.5}$$

where b is bounded in G. This saturates our upper bound (2.4) for the special case $F'_{\mu\nu} = 0$ and $G_{12} = B$. Haba then goes on to use this result to obtain evidence suggesting the instability (triviality) of QED_4. In view of the potential importance of this result and of our foregoing remarks concerning infinite-flux fields, it would be worthwhile to repeat the calculation with $F_{\mu\nu}$ confined to a finite region $\Lambda \subset \mathbf{R}^4$ from the beginning and repeating Haba's estimates with Λ held fixed.

As a consequence of the need to regularize and the finite-flux condition we may confine our attention to smooth, polynomial bounded potentials that, in the Lorentz gauge, have the asymptotic form

$$A_\mu(\mathbf{r}) = -\frac{\Phi}{2\pi}\frac{\epsilon_{\mu\nu}x_\nu}{r^2} + O\left(\frac{1}{r^2}\right), \tag{3.6}$$

where Φ is the flux associated with B and $r \gg a$, the range of B. Our long-term goal is to study $\det_{\text{Sch}}$ under the scaling $B \to \lambda B$, $\lambda \to \infty$ as well as the determinant's mass dependence. This latter point, as we now know, is especially relevant to determining $\ln\det_{\text{ren}}$ in QED_4 for unidirectional, static, inhomogeneous magnetic fields.

We know of no previous calculation of a finite-flux determinant associated with massive fermions in two or more dimensions. In Sec. V, we will calculate such a determinant analytically for a magnetic field confined to a thin cylindrical shell with radius a:

$$B(r) = \frac{\Phi}{2\pi}\frac{\delta(r-a)}{a}. \tag{3.7}$$

Although B is not derivable from a polynomial bounded potential it has the virtue that the $\delta(r)/r$-type singularity of a magnetic flux string is absent and that $\det_{\text{Sch}}$ exists for this field. It is an instructive example, especially as we believe it gives an insight to the matters raised above.

IV. SUPPRESSION OF ZERO MODES IN $\det_{\text{Sch}}$

A. Conventions

Consider a Dirac fermion in a static, unidirectional magnetic field directed along the z axis. Its Hamiltonian in the xy plane is

$$H = \gamma^0\boldsymbol{\gamma}\cdot(\mathbf{p} - e\mathbf{A}) + \gamma^0 m, \tag{4.1}$$

with the γ matrices[1] $\gamma^0 = \sigma^3$, $\gamma^k = -i\sigma^k$, $k = 1,2$. It has the structure

$$H = \begin{pmatrix} m & L \\ L^\dagger & -m \end{pmatrix}, \tag{4.2}$$

where L is a linear differential operator, and $L^\dagger$ is the Hermitian conjugate of L. The positive- and negative-energy eigenfunctions of H,

$$H\psi_{E,\lambda}(\mathbf{r}) = E\psi_{E,\lambda}(\mathbf{r}), \tag{4.3}$$

are normalized to

$$\int d^2r\psi^\dagger_{E,\lambda}(\mathbf{r})\psi_{E',\lambda'}(\mathbf{r}) = \delta_{\lambda\lambda'}\delta(E-E'), \tag{4.4}$$

in the energy continuum, where λ is a degeneracy parameter. As a consequence of (4.2) the eigenvalues satisfy $E^2 \geq m^2$. Since

$$H^2 = (\mathbf{p} - e\mathbf{A})^2 - eB\sigma^3 + m^2, \tag{4.5}$$

we have

$$\left.\begin{matrix} LL^\dagger \\ L^\dagger L \end{matrix}\right\} = (\mathbf{p} - e\mathbf{A})^2 \mp eB, \tag{4.6}$$

so that the supersymmetric operator pair $H_\pm$ in Sec. III is given by $H_+ = LL^\dagger$, $H_- = L^\dagger L$. The eigenfunctions of H_+ and H_- will be denoted by $u_{w,l}$ and $v_{w,l}$, respectively:

$$\begin{aligned} H_+u_{w,l}(\mathbf{r}) &= wu_{w,l}(\mathbf{r}),\\ H_-v_{w,l}(\mathbf{r}) &= wv_{w,l}(\mathbf{r}),\end{aligned} \tag{4.7}$$

where l is a degeneracy parameter and $w = E^2 - m^2$. Their normalization in the energy continuum, $w > 0$, is

[1]Where possible we adopt the notation and conventions of Jaroszewicz [21] whose analysis of the chiral anomaly associated with the Hamiltonian (4.1) in a solenoidal magnetic field is relevant to the work presented here.

$$\int d^2 r u^*_{w,l}(\mathbf{r}) u_{w',l'}(\mathbf{r}) = \delta(w-w')\delta_{ll'}. \tag{4.8}$$

Since the continuum extends down to $E = \pm m$ or $w = 0$ we expect the states on the edge of the continuum to show up as unbounded resonances [22]. When $|e\Phi|/2\pi > 1$, bound states will also appear at the bottom of the continuum [18], and care must be taken to include these in the completeness relations for $\psi_{E,\lambda}$ and $u_{w,l}, v_{w,l}$.

B. Suppression of zero modes

Referring back to (2.5) consider

$$\partial \ln \det_{\mathrm{Sch}}/\partial e = \frac{e}{\pi}\int d^2 r \phi \partial^2 \phi + 2m^2 \int d^2 r \phi(\mathbf{r}) \langle \mathbf{r}|(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}|\mathbf{r}\rangle. \tag{4.9}$$

Now consider the large mass limit of

$$\begin{aligned} m^2\langle \mathbf{r}|(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}|\mathbf{r}\rangle &= m^2 \int_0^\infty dt e^{-tm^2}\langle \mathbf{r}|e^{-tH_+} - e^{-tH_-}|\mathbf{r}\rangle \\ &= m^2 \int_0^\infty dt e^{-tm^2}\left(\frac{e}{2\pi}B(\mathbf{r}) + O(t)\right) \\ &= \frac{e}{2\pi}B(\mathbf{r}) + O\left(\frac{1}{m^2}\right), \end{aligned} \tag{4.10}$$

where we used the heat kernel asymptotic expansion [16]

$$\langle \mathbf{r}|e^{-tH_\pm}|\mathbf{r}\rangle = \frac{1}{4\pi t}[1 \pm teB(\mathbf{r}) + O(t^2)]. \tag{4.11}$$

Combining (4.7) and (4.10) we get

$$\begin{aligned} \lim_{m^2\to 0}\left\langle \mathbf{r}\left|\frac{m^2}{H_+ + m^2} - \frac{m^2}{H_- + m^2}\right|\mathbf{r}\right\rangle &= \lim_{m^2\to\infty}\int_0^\infty dw \frac{m^2}{w+m^2}\sum_l [|u_{w,l}(\mathbf{r})|^2 - |v_{w,l}(\mathbf{r})|^2] \\ &= \int_0^\infty dw \sum_l [|u_{w,l}(\mathbf{r})|^2 - |v_{w,l}(\mathbf{r})|^2] \\ &= \int_0^\infty \langle \mathbf{r}|dP_+(w) - dP_-(w)|\mathbf{r}\rangle = \frac{eB(\mathbf{r})}{2\pi}, \end{aligned} \tag{4.12}$$

where $P_+(w) - P_-(w)$ is the difference of spectral measures associated with $H_\pm$ and where $H_\pm = \int_0^\infty w dP_\pm(w)$.

Equation (4.12) has the following physical interpretation. The charge density induced in the vacuum by the background magnetic field is

$$\langle j^0(\mathbf{r})\rangle = -\frac{e}{2}\int_m^\infty dE \sum_\lambda (|\psi_{E,\lambda}(\mathbf{r})|^2 - |\psi_{-E,\lambda}(\mathbf{r})|^2), \tag{4.13}$$

which is determined by the spectral asymmetry density

$$\begin{aligned} \eta(E,\mathbf{r}) &= \frac{i}{2\pi}\mathrm{disc}_E \mathrm{tr}\langle \mathbf{r}|(E-H)^{-1} - (E+H)^{-1}|\mathbf{r}\rangle \\ &= \sum_\lambda [|\psi_{E,\lambda}(\mathbf{r})|^2 - |\psi_{-E,\lambda}(\mathbf{r})|^2], \end{aligned} \tag{4.14}$$

where $\mathrm{disc}_E h(E) \equiv h(E+i\epsilon) - h(E-i\epsilon)$. Because of the structure of H in (4.2), η may also be expressed in terms of the eigenfunctions of $H_\pm$ in (4.7) [21,23] as

$$\eta(E,\mathbf{r}) = 2m\sum_l [|u_{E^2-m^2,l}(\mathbf{r})|^2 - |v_{E^2-m^2,l}(\mathbf{r})|^2], \tag{4.15}$$

so that

$$\langle j^0(\mathbf{r})\rangle = -\frac{e}{2}\int_0^\infty dw \frac{m}{\sqrt{w+m^2}} \times \sum_l [|u_{w,l}(\mathbf{r})|^2 - |v_{w,l}(\mathbf{r})|^2]. \tag{4.16}$$

Comparing (4.16) with (4.12) we see that the limit in (4.12) is equivalent to

$$\lim_{m\to\infty}\langle j^0(\mathbf{r})\rangle = -\frac{e^2 B(\mathbf{r})}{4\pi}. \tag{4.17}$$

The natural length scale here is the range of the magnetic field, and so (4.17) states that the vacuum change density induced by a magnetic field whose range is large compared to the fermion's Compton wavelength is $-e^2 B(\mathbf{r})/4\pi$, in agreement with a remark by Jaroszewicz [21].

Now consider (4.9) again in the form

$$\partial \ln \det_{\rm Sch}/\partial e = \frac{e}{\pi}\int d^2r\phi\partial^2\phi + 2\int d^2r\phi(\mathbf{r})\int_0^\infty dw\frac{m^2}{m^2+w}\sum_l(|u_{w,l}(\mathbf{r})|^2-|v_{w,l}(\mathbf{r})|^2)$$
$$= \frac{e}{\pi}\int d^2r\phi\partial^2\phi + 2\int d^2r\phi(\mathbf{r})\int_0^\infty\left(1-\frac{w}{w+m^2}\right)\langle\mathbf{r}|dP_+(w)-dP_-(w)|\mathbf{r}\rangle. \quad (4.18)$$

From the last line of (4.12) we finally obtain

$$\partial \ln \det_{\rm Sch}/\partial e = -\frac{e}{\pi}\int d^2r\phi B + 2\int d^2r\phi(eB/2\pi) - 2\int d^2r\phi\int_0^\infty \frac{w}{w+m^2}\langle\mathbf{r}|dP_+(w)-dP_-(w)|\mathbf{r}\rangle$$
$$= -2\,\mathrm{Tr}\left[\phi\left(\frac{H_+}{H_++m^2}-\frac{H_-}{H_-+m^2}\right)\right], \quad (4.19)$$

which shows explicitly that zero modes are suppressed in $\det_{\rm Sch}$. It also shows how substantially mass has altered the fermionic determinant of QED_2: the contribution of the massless Schwinger model to $\det_{\rm Sch}$ has been canceled by a contribution from the massive sector of QED_2.

C. Gauge invariance of (4.19)

If we revert back to the definitions (2.5) or (4.9) of $\det_{\rm Sch}$, then the invariance of $\det_{\rm Sch}$ under the restricted gauge transformation $\phi \to \phi + c$ results in the index theorem (2.8). Recall that the massless Schwinger model's contribution to $\det_{\rm Sch}$ was critical to establishing this result. Now, in (4.19), we see that the massless Schwinger model's contribution has been canceled by a contribution from the massive sector of $\det_{\rm Sch}$ that contains the zero modes of $H_\pm$. Invariance of (4.19) under $\phi \to \phi + c$ now requires that

$$\int d^2r\int_0^\infty dw\frac{w}{w+m^2}\sum_l(|u_{w,l}(\mathbf{r})|^2-|v_{w,l}(\mathbf{r})|^2) = 0\ . \quad (4.20)$$

In the remainder of this section we will recall the known result that the integrand in (4.20) can be expressed as the divergence of a current [21,23] and then go on to show explicitly that (4.20) is true in the case of radial symmetry.

Differentiating (4.7) with respect to w, denoted here by an overdot, one easily obtains [21]

$$|u_{w,l}(\mathbf{r})|^2-|v_{w,l}(\mathbf{r})|^2 = \nabla\cdot\mathbf{S}_{w,l}(\mathbf{r}) \ \text{ for } w>0, \quad (4.21)$$

where

$$\mathbf{S}_{w,l} = -[u^*_{w,l}\nabla\dot u_{w,l} - (\nabla u^*_{w,l})\dot u_{w,l}$$
$$-2iu^*_{w,l}\mathbf{A}\dot u_{w,l} - (u\to v)]. \quad (4.22)$$

To make further progress we will specialize to the case of radial symmetry so that the degeneracy parameter l is identified with angular momentum. We would then like to interchange the space integral with the sum over partial waves in (4.20) in order to convert the space integral to an integral over a circle at infinity. Jaroszewicz has already discussed this interchange in another context [21], and we repeat his reasoning here. Since $H_+ - H_- \sim B$, and B has finite range, a, the difference between the wave functions $u_{w,l}$ and $v_{w,l}$, for fixed w, decreases with increasing l due to the rising centrifugal barrier that excludes them from the region where $B(\mathbf{r}) \neq 0$. The energy required for the wave function to penetrate the region $r < a$ is of the order of $w > l^2/a^2$, suggesting that the partial-wave sum in (4.20) is effectively cut off at $|l| \sim \sqrt{w}a$. Accepting this reasoning, we get

$$\mathrm{Tr}\left(\frac{H_+}{H_++m^2}-\frac{H_-}{H_-+m^2}\right)$$
$$= \int_0^\infty dw\frac{w}{w+m^2}\sum_l\lim_{R\to\infty}\int_{S^1_R}\mathbf{S}_{w,l}(\mathbf{R})\cdot d\boldsymbol{l}, \quad (4.23)$$

where $d\boldsymbol{l} = \hat{\mathbf{R}}Rd\theta$ in the r,θ plane. Since the potential $\mathbf{A}$ may be assumed to be a pure gauge field at infinity tangential to S^1_∞, the $\mathbf{A}$-dependent terms in (4.22) may be dropped. Indeed our potentials (3.5) approach vortex fields that manifestly satisfy this assumption. Using the asymptotic form of $u_{w,l}$,

$$u_{w,l}(r,\theta) \underset{r\gg a}{\sim} 2^{-1/2}\pi^{-1}w^{-1/4}r^{-1/2}e^{-il\theta}\cos\left[\sqrt{w}r - \frac{\pi l}{w} - \frac{\pi}{4} + \delta^u_l(w)\right], \quad (4.24)$$

and similarly for $v_{w,l}$, where $\delta^u_l(w)$ and $\delta^v_l(w)$ are the scattering phase shifts, one gets [21]

$$\lim_{R\to\infty}\int_{S^1_R}\mathbf{S}_{w,l}(\mathbf{R})\cdot d\boldsymbol{l} = \frac{1}{\pi}\sum_l[\dot\delta^u_l(w)-\dot\delta^v_l(w)] \ \text{ for } w>0, \quad (4.25)$$

where the overdot continues to denote differentiation with respect to w. The factor $e^{-il\theta}$ instead of $e^{il\theta}$ in (4.24) is for later notational convenience. Because of the supersymmetry of the operator pair $H_\pm$ we have, from (4.7),

$$L^\dagger H_+ u_{w,l}(\mathbf{r}) = H_- L^\dagger u_{w,l}(\mathbf{r}) = w L^\dagger u_{w,l}(\mathbf{r}), \tag{4.26}$$

which indicates that $L^\dagger u_{w,l} \propto v_{w,l}$ and hence that

$$\delta_l^u(w) = \delta_{l-1}^v(w) \mod \pi \text{ for } w > 0, \tag{4.27}$$

in agreement with Jaroszewicz [21]. In deriving (4.27) we used

$$\begin{aligned} L^\dagger &= e^{i\theta}\left(\frac{\partial}{\partial r} - ieA_r + \frac{i}{r}\frac{\partial}{\partial\theta} + eA_\theta\right) \\ &\underset{r\gg a}{\sim} e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} + \frac{e\Phi}{2\pi r}\right), \end{aligned} \tag{4.28}$$

where $\mathbf{A} = (A_r, A_\theta)$. The condition for the invariance of (4.19) under the restricted gauge transformation $\phi \to \phi + c$ is now reduced to

$$\begin{aligned} \sum_{l=-\infty}^{\infty} [\dot\delta_l^u(w) - \dot\delta_l^v(w)] &\equiv \lim_{L\to\infty} \sum_{l=-L}^{L} [\dot\delta_l^u(w) - \dot\delta_l^v(w)] \\ &= \lim_{L\to\infty} [\dot\delta_{-L}^u(w) - \dot\delta_{L+1}^u(w)] = 0 \quad \text{for } w > 0. \end{aligned} \tag{4.29}$$

This is physically reasonable since the wave equations for $u_{w,l}$ and $v_{w,l}$ become scale invariant outside the range of B, where the potentials $V_\pm$ defined by

$$H_\pm = -\partial^2 + V_\pm, \tag{4.30}$$

have a $1/r^2$ behavior and to where $u_{w,l}$ and $v_{w,l}$ are mainly confined due to the enormous centrifugal barrier building up as $|l| \to \infty$. Hence, we do indeed expect $\lim_{L\to\infty}\dot\delta_{|L|}^{u,v}(w) = 0$.

V. $\det_{\rm Sch}$ IN A CYLINDRICAL MAGNETIC FIELD

A. The Green's functions

In this section we will use the representation (4.9) of $\det_{\rm Sch}$, as it is simpler than (2.5) and more instructive than (4.19). For the magnetic field in (3.7) we have, for the auxiliary potential ϕ given by (2.6),

$$\phi(r) = -\frac{1}{2\pi}\int d^2r\, \ln|\mathbf{r} - \mathbf{r}'| B(r') = \begin{cases} -\dfrac{\Phi}{2\pi}\ln r, & r > a, \\ -\dfrac{\Phi}{2\pi}\ln a, & r < a. \end{cases} \tag{5.1}$$

Since $\det_{\rm Sch}$ is invariant under $\phi \to \phi + c$ we may let $c = (\Phi/2\pi)\ln a$ and use the potential

$$\phi(r) = -\frac{\Phi}{2\pi}\ln\left(\frac{r}{a}\right)\theta(r-a), \tag{5.2}$$

in which case the contribution of the massless Schwinger model is eliminated straightaway:

$$\begin{aligned} \int d^2r\phi\partial^2\phi &= -\int d^2r\phi B \\ &= -\frac{\Phi^2}{2\pi}\phi(a) \\ &= 0. \end{aligned} \tag{5.3}$$

The associated vector potential is

$$\begin{aligned} \mathbf{A} &= \frac{\Phi}{2\pi r}\theta(r-a)\hat{\boldsymbol\theta} \\ &\equiv \frac{\Phi(r)}{r}\hat{\boldsymbol\theta}. \end{aligned} \tag{5.4}$$

Referring to (4.9) we see that the Green's functions $G_{\pm,l}$ defined by

$$\begin{aligned} \langle r,\theta|(k^2 - H_\pm)^{-1}|r',\theta'\rangle &= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} \langle r|(k^2 - H_{\pm,l})^{-1}|r'\rangle e^{-il(\theta-\theta')} \\ &= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} G_{\pm,l}(k;r,r')e^{-il(\theta-\theta')} \end{aligned} \tag{5.5}$$

are central to the calculation of $\det_{\rm Sch}$. The radial wave functions of $H_{+,l}$ and $H_{-,l}$, denoted by $u_{k^2,l}(r)$ and $v_{k^2,l}(r)$, respectively, satisfy

$$\left(-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{[l+e\Phi(r)]^2}{r^2}-\frac{e\Phi'(r)}{r}\right)u_{k^2,l}(r)=k^2u_{k^2,l}(r),$$
$$\left(-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{[l+e\Phi(r)]^2}{r^2}+\frac{e\Phi'(r)}{r}\right)v_{k^2,l}(r)=k^2v_{k^2,l}(r). \tag{5.6}$$

These equations have linearly independent solutions $H_l^{\pm}(kr)$ for $r<a$ and $H_{l+\Phi}^{\pm}(kr)$ for $r>a$, where H_ν^+ and H_ν^- denote the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$, respectively. The dimensionless constant $e\Phi/2\pi$ has been denoted simply by Φ.

The calculation is simplified by introducing the Green's functions

$$\mathcal{G}_{\pm,l}(k;r,r')=\sqrt{rr'}G_{\pm,l}(k;r,r'), \tag{5.7}$$

where

$$\mathcal{G}_{\pm,l}(k;r,r')=\langle r|(k^2-\mathcal{H}_{\pm,l})^{-1}|r'\rangle, \tag{5.8}$$

and

$$\mathcal{H}_{\pm,l}=-\frac{d^2}{dr^2}+\frac{[l+\Phi(r)]^2-\frac{1}{4}}{r^2}\mp\frac{\Phi'(r)}{r}. \tag{5.9}$$

The outgoing-wave Green's functions $\mathcal{G}_{\xi,l}^+$ ($\xi=\pm\equiv\pm1$) are constructed from [24]

$$\mathcal{G}_{\xi,l}^+(k;r,r')=-\frac{\phi(k,r_<)f_+(k,r_>)}{\mathcal{J}_+(k)}, \tag{5.10}$$

where ϕ and f_+ are regular and irregular solutions, respectively, of

$$\mathcal{H}_{\xi,l}\psi_\xi=k^2\psi_\xi, \tag{5.11}$$

$\mathcal{J}_+$ for $\xi=\pm$ are the associated Jost functions, and $r_<, r_>$ denote the lesser and larger values of r, r', respectively. The indices l and ξ have been omitted on the right-hand side of (5.10) to reduce notational clutter.

Regular solutions of (5.11) are

$$\phi(k,r)=\begin{cases}k^{-|l|}\sqrt{r}J_l(kr), & r<a,\\ k^{-|l|}\sqrt{r}[\alpha H_{l+\Phi}^+(kr)+\beta H_{l+\Phi}^-(kr)], & r>a,\end{cases} \tag{5.12}$$

and the irregular, outgoing wave solutions are

$$f_+(k,r)=\begin{cases}\sqrt{kr}[AH_l^+(kr)+BH_l^-(kr)], & r<a,\\ \sqrt{kr}H_{l+\Phi}^+(kr), & r>a.\end{cases} \tag{5.13}$$

The constants α, β, A, B are determined by the joining conditions at $r=a$ obtained from (5.11) and (5.9):

$$\phi(k,a-)=\phi(k,a+),$$
$$\phi'(k,a+)-\phi'(k,a-)=-\xi\Phi\phi(k,a+)/a, \tag{5.14}$$

and similarly for f_+. These give

$$\alpha=\frac{i}{4}\pi\xi ka[J_l(ka)H_{l+\Phi-\xi}^-(ka)-J_{l-\xi}(ka)H_{l+\Phi}^-(ka)],$$
$$\beta=\frac{i}{4}\pi\xi ka[J_{l-\xi}(ka)H_{l+\Phi}^+(ka)-J_l(ka)H_{l+\Phi-\xi}^+(ka)], \tag{5.15}$$
$$A=\frac{i}{4}\pi\xi ka[H_{l+\Phi}^+(ka)H_{l-\xi}^-(ka)-H_l^-(ka)H_{l+\Phi-\xi}^+(ka)],$$
$$B=\frac{i}{4}\pi\xi ka[H_l^+(ka)H_{l+\Phi-\xi}^+(ka)-H_{l-\xi}^+(ka)H_{l+\Phi}^+(ka)].$$

Finally, the Jost functions are

$$\begin{aligned}\mathcal{J}_+&=W(f_+,\phi)\\ &=k^{-1/2-|l|}\beta rW[H_{l+\Phi}^+(kr),H_{l+\Phi}^-(kr)]\\ &=-\frac{4i}{\pi}\beta k^{1/2-|l|},\end{aligned} \tag{5.16}$$

where W is the Wronskian, and β is given in (5.15).

At this point $\mathcal{G}_{\pm,l}^+$ is fully determined by (5.10), (5.12), (5.13), (5.15), and (5.16). We could now continue $\mathcal{G}_{\pm,l}^+$ into the upper half of the k plane by letting $k=me^{i\theta}$, $\theta\to\pi/2$ with $m>0$, thereby making contact with the Green's functions $\langle\mathbf{r}|(H_\pm+m^2)^{-1}|\mathbf{r}\rangle$ in (4.9). We will do this later. But first we want to demonstrate that when the flux is sufficiently large for fixed l, $\mathcal{G}_{\pm,l}^+$ does indeed contain a zero-energy bound state. This will be done by deriving a completeness relation.

B. Completeness

The proof of the completeness of the bound and scattering wave functions associated with $\mathcal{H}_{\pm,l}$ will follow the general procedure outlined by Newton [24], taking into account that $\mathcal{H}_{\pm,l}$ contains a $1/r^2$-type potential.

Consider the integral

$$\begin{aligned}I(r)&=\int_C dkk\int_0^\infty dr'h(r')\mathcal{G}_{\xi,l}^+(k;r,r')\\ &=-\int_C dkk\int_0^r dr'h(r')\frac{\phi(k,r')f_+(k,r)}{\mathcal{J}_+(k)}\\ &\quad-\int_C dkk\int_r^\infty dr'h(r')\frac{\phi(k,r)f_+(k,r')}{\mathcal{J}_+(k)}\\ &=I_1+I_2,\end{aligned} \tag{5.17}$$

where $h(r)$ is square integrable, and C is the contour in Fig. 2. The contribution of $I_{1\Gamma}$ to I_1 from the large semicircle Γ is evaluated by using the asymptotic forms of ϕ, f_+, and $\mathcal{J}_+$ in the upper k plane. From (5.15) and (5.16),

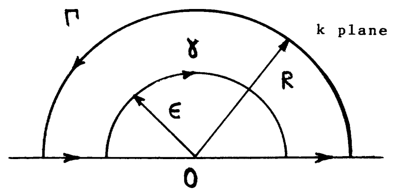

FIG. 2. The contour of integration C in the k plane for the integral in (5.17).

$$\mathcal{J}_+ = \xi a k^{3/2-|l|}[J_{l-\xi}(ka)H^+_{l+\Phi}(ka) - J_l(ka)H^+_{l+\Phi-\xi}(ka)] \underset{|k|\to\infty}{\sim} -\frac{2i}{\pi}k^{1/2-|l|}e^{-i\pi\Phi/2}. \tag{5.18}$$

From (5.12), (5.13), and (5.15) one gets, in the upper k plane,

$$\phi(k,r) \underset{|k|\to\infty}{\sim} \sqrt{2/\pi}k^{-|l|-1/2}\cos(kr - \tfrac{1}{4}\pi l - \tfrac{1}{4}\pi), \tag{5.19}$$
$$f_+(k,r) \underset{|k|\to\infty}{\sim} \sqrt{2/\pi}\exp\{i[kr - \tfrac{1}{2}\pi(l+\Phi) - \tfrac{1}{4}\pi]\},$$

and hence, for $|R| \to \infty$,

$$\begin{aligned} I_{1\Gamma} &\sim -i\int_\Gamma dk \int_0^r dr' h(r')\exp[i(kr - \tfrac{1}{2}\pi l - \tfrac{1}{4}\pi)] \\ &\quad \times \cos(kr' - \tfrac{1}{2}\pi l - \tfrac{1}{4}\pi) \\ &\sim -\frac{i}{2}h(r)\int_\Gamma dk \int_0^r dr' e^{ik(r-r')} \\ &\sim \tfrac{1}{2}i\pi h(r). \end{aligned} \tag{5.20}$$

The contribution of $I_{2\Gamma}$ to I_2 from the contour Γ may be defined by replacing the upper limit of the r' integration in (5.17) by $r+\mu$, $\mu > 0$ and letting $\mu \to \infty$ later [24]. Then for $|R| \to \infty$,

$$\begin{aligned} I_{2\Gamma} &\sim -i\int_\Gamma dk \int_r^{r+\mu} dr' h(r')\exp[i(kr' - \tfrac{1}{2}\pi l - \tfrac{1}{4}\pi)] \\ &\quad \times \cos(kr - \tfrac{1}{2}\pi l - \tfrac{1}{4}\pi) \\ &\sim -\frac{i}{2}h(r)\int_\Gamma dk \int_r^{r+\mu} dr' e^{ik(r'-r)} \\ &\sim \tfrac{1}{2}i\pi h(r). \end{aligned} \tag{5.21}$$

Because of the analytic properties of $H^+_\nu(ka)$ and $J_l(ka)$ in the upper half of the k plane, $\mathcal{J}_+(k)$ has no zeros for $\mathrm{Im}k > 0$. For example, on the positive imaginary k axis we have, from (5.18),

$$\mathcal{J}_+(e^{i\pi/2}ka) = \frac{2iak}{\pi}e^{-i\pi(|l|+\Phi)/2}[I_{l-\xi}(ka)K_{l+\Phi}(ka) + I_l(ka)K_{l+\Phi-\xi}(ka)], \tag{5.22}$$

where K_ν and I_l are modified Bessel functions. Since $K_\nu(x)$, $I_l(x)$ are positive for $x > 0$, $\mathcal{J}_+(e^{i\pi/2}ka)$ is manifestly free of zeros for $k > 0$. Therefore,

$$\int_C dkk \int_0^\infty dr' h(r')\mathcal{G}^+_{\xi,l}(k;r,r') = 0. \tag{5.23}$$

Combining (5.17) with (5.20), (5.21), and (5.23) we get

$$h(r) = -\frac{i}{\pi}\int_0^{r+\mu} dr' h(r')\left(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty\right) dkk\frac{\phi(k,r_<)f_+(k,r_>)}{\mathcal{J}_+(k)} + \frac{i}{\pi}\int_0^{r+\mu} dr' h(r')\int_\gamma dkk\mathcal{G}^+_{\xi,l}(k;r,r'). \tag{5.24}$$

Equation (5.24) may be simplified by the following relations, valid for positive k:

$$\begin{aligned} \phi(ke^{i\pi},r) &= \phi(k,r), \\ f_+(ke^{i\pi},r) &= -e^{i\pi/2}e^{-i\pi(l+\Phi)}f_+^*(k,r), \\ \mathcal{J}_+(ke^{i\pi}) &= e^{3i(\pi/2)}e^{-i\pi|l|}e^{-i\pi\Phi}\mathcal{J}_-(k), \end{aligned} \tag{5.25}$$

where

$$\mathcal{J}_-(k) = W(f_-,\phi) = \frac{4i\alpha}{\pi}k^{1/2-|l|}, \tag{5.26}$$

$$f_-(k,r) = \sqrt{kr}H^-_{l+\Phi}(kr), \quad r > a, \tag{5.27}$$

and where α is given by (5.15). For real k, $\mathcal{J}_+^*(k) = \mathcal{J}_-(k)$. Equations (5.25), (5.26) and the result, valid for real k,

$$\mathrm{Im}[\mathcal{J}_-(k)f_+(k,r)] = \frac{2k}{\pi}\phi(k,r), \quad r < a, \tag{5.28}$$

reduce (5.24) to

$$h(r) = \frac{4}{\pi^2}\int_0^{r+\mu} dr' h(r')\int_\epsilon^\infty dk\frac{k^2}{|\mathcal{J}_+(k)|^2}\phi(k,r)\phi(k,r') + \frac{i}{\pi}\int_0^{r+\mu} dr' h(r')\int_\gamma dkk\mathcal{G}^+_{\xi,l}(k;r,r'). \tag{5.29}$$

The integral around the semicircle γ will contribute only if $\mathcal{G}^+_{\xi,l}$ develops a second-order pole at $k=0$. A tedious calculation confirms that this happens for $\xi = +$, $l \le 0$, $l+\Phi > 1$ and for $\xi = -1$, $l \ge 0$, $l+\Phi < -1$. This result is in accord with a remark by Jaroszewicz [21], who stated this would happen for $\Phi \ne$ integer; our result holds for all admissible values of Φ. The residue of the double pole is the zero-energy bound state ψ_l of $\mathcal{H}_{\pm,l}$:

$$\lim_{\epsilon\downarrow 0}\int_\gamma dkk\mathcal{G}^+_{\xi,l}(k;r;r') = -i\pi\psi_l(r)\psi_l(r'), \tag{5.30}$$

for $\xi(l+\Phi) > 1$, $\xi l \le 0$, and where

$$\psi_l(r) = \left[\frac{2(1+|l|)(|l+\Phi|-1)}{a^2|\Phi|}\right]^{1/2} \sqrt{r} \times \begin{cases} \left(\frac{r}{a}\right)^{|l|}, & r < a \\ \left(\frac{a}{r}\right)^{|l+\Phi|}, & r > a. \end{cases} \tag{5.31}$$

The bound states are normalized:

$$\int_0^\infty dr \psi_l^2(r) = 1. \tag{5.32}$$

Combining (5.30) with (5.29) and letting $\mu \to \infty$ gives the completeness relation

$$\frac{4}{\pi^2}\int_{0+}^\infty dk \frac{k^2}{|\mathcal{J}_+(k)|^2}\phi(k,r)\phi(k,r') + \psi_l(r)\psi_l(r') = \delta(r-r'). \tag{5.33}$$

We have gone through this calculation to show that $\mathcal{G}^+_{\pm,l}$ does indeed contain a bound state for sufficiently large flux and how it manifests itself. Later, when we consider the small mass limit of $\det_{\rm Sch}$, one should keep in mind that this is being controlled by the bound states, since the $m \to 0$ limit is approaching the second-order pole of $\mathcal{G}^+_{\pm,l}$ along the positive imaginary k axis.

C. Calculation of $\det_{\rm Sch}$

The outgoing-wave Green's function $\mathcal{G}^+_{\pm,l}(k,r,r')$ in (5.10) may be continued to $k = im$, $m > 0$. According to the definition of $\det_{\rm Sch}$ in (4.9) we will only need the difference of $\mathcal{G}^+_{+,l}$ and $\mathcal{G}^+_{-,l}$, which simplifies the calculation. Because of (5.2) we may confine our attention to $r = r' > a$. Then from (5.10), (5.12), (5.13), (5.16), and (5.26),

$$\mathcal{G}^+_{+,l}(kr,r) - \mathcal{G}^+_{-,l}(k;r,r) = \frac{i\pi r}{4}\left(\left.\frac{\mathcal{J}_-(k)}{\mathcal{J}_+(k)}\right|_{\xi=+} - \left.\frac{\mathcal{J}_-(k)}{\mathcal{J}_+(k)}\right|_{\xi=-}\right)[H^+_{l+\Phi}(kr)]^2 , \tag{5.34}$$

which is related to the S matrix for the lth partial wave by

$$S_l(k) = -\frac{\mathcal{J}_-(k)}{\mathcal{J}_+(k)}e^{-i\pi\Phi} = e^{2i\delta_l} . \tag{5.35}$$

In (5.34) the Jost ratios can be rearranged to give

$$\frac{\mathcal{J}_-}{\mathcal{J}_+} = 2\frac{J_{l-\xi}(ka)J_{l+\Phi}(ka) - J_l(ka)J_{l+\Phi-\xi}(ka)}{J_{l-\xi}(ka)H^+_{l+\Phi}(ka) - J_l(ka)H^+_{l+\Phi-\xi}(ka)} - 1 . \tag{5.36}$$

We now let $k \to me^{i(\pi/2)}$, $m > 0$, in (5.34), (5.36), and (5.8), using [25]

$$J_\nu(ame^{i\pi/2}) = e^{i\pi\nu/2}I_\nu(ma) ,$$
$$H^+_\nu(ame^{i\pi/2}) = \frac{2}{\pi i}e^{-i\pi\nu/2}K_\nu(ma) , \tag{5.37}$$

to obtain

$$\langle r|(\mathcal{H}_{+,l}+m^2)^{-1}|r\rangle - \langle r|(\mathcal{H}_{-,l}+m^2)^{-1}|r\rangle = -r\left(\frac{I_{l-1}I_{l+\Phi} - I_lI_{l+\Phi-1}}{I_{l-1}K_{l+\Phi} + I_lK_{l+\Phi-1}} - \frac{I_{l+1}I_{l+\Phi} - I_lI_{l+\Phi+1}}{I_{l+1}K_{l+\Phi} + I_lK_{l+\Phi+1}}\right)K^2_{l+\Phi}(mr) , \tag{5.38}$$

where all modified Bessel functions in the brackets have argument ma. If the two terms in (5.38) are combined by a common denominator then a remarkable simplification occurs to give

$$\langle r|(\mathcal{H}_{+,l}+m^2)^{-1}|r\rangle - \langle r|(\mathcal{H}_{-,l}+m^2)^{-1}|r\rangle = 2\Phi r\left\{\left[ma\frac{d}{dma}\ln\left(\frac{I_l(ma)}{K_{l+\Phi}(ma)}\right)\right]^2 - \Phi^2\right\}^{-1}$$
$$\times\begin{cases} \dfrac{K^2_{l+\Phi}(mr)}{K^2_{l+\Phi}(ma)}, & r > a \\ \dfrac{I^2_l(mr)}{I^2_l(ma)}, & r < a , \end{cases} \tag{5.39}$$

where we have put on record the result for $r < a$. As a check on our results we verified that

$$m^2 \int_0^\infty dr \sum_{l=-\infty}^{\infty} [\langle r|(\mathcal{H}_{+,l} + m^2)^{-1}|r\rangle - \langle r|(\mathcal{H}_{-,l} + m^2)^{-1}|r\rangle] = \Phi \ , \tag{5.40}$$

for $m^2 > 0$, in accordance with (2.8). This was done by interchanging the integral and sum, which is allowed since the series in (5.40) is uniformly convergent for $r \geq 0$, and then defining $\sum_{l=-\infty}^{\infty}$ as $\lim_{L\to\infty} \sum_{-L}^{L}$. The integrals of the modified Bessel functions are known [26]. Again, the reader is reminded that Φ denotes $e\Phi/2\pi$, where Φ is the flux, and that this symbol is active from (5.6) onward.

Finally, combining (4.9), (5.2), (5.5), (5.7), and (5.39), gives

$$\frac{\partial \ln \det_{\rm Sch}}{\partial \Phi} = -4(ma)^2 \Phi \int_1^\infty dr r \ln r \sum_{l=-\infty}^{\infty} \frac{K_{l+\Phi}^2(mar)}{K_{l+\Phi}^2(ma)} \left\{ \left[ma \frac{d}{dma} \ln \left(\frac{I_l(ma)}{K_{l+\Phi}(ma)} \right) \right]^2 - \Phi^2 \right\}^{-1} . \tag{5.41}$$

The term in the curly brackets is positive for all Φ. Figure 3 displays the numerical calculation of the right-hand side of (5.41) for the cases $ma = 1$ and 10^{-2} for $0 \leq \Phi \leq 999.5$. The plots were generated for half-integral values of Φ. The data in both cases are consistent with a logarithmic growth of $\partial \ln \det_{\rm Sch} / \partial \Phi$ with Φ given by $-\ln \Phi$+const, where the constant is about 0.3 for $ma = 1$ and -2.5 for $ma = 10^{-2}$, consistent with the "diamagnetic bound" in Eq. (2.3).

The integral in (5.41) can be calculated explicitly for integer Φ. Assume this is the case, and let $\Phi = N$. Since the sum in (5.41) is uniformly convergent for $r \geq 1$, we may let $l \to l - N$ and interchange sum and integral. Since $K_{-l} = K_l$ and $I_{-l} = I_l$, we need only consider $l \geq 0$. Then from [27] one can derive the result, valid for $l = 0, 1, \ldots$,

$$\begin{aligned} a_l &\equiv \int_1^\infty dr r \ \ln r K_l^2(mar)/K_l^2(ma) \\ &= -\frac{1}{2} \frac{d}{dma} \left(\frac{K_{l+1}}{K_l} \right) - \frac{l}{2(ma)^2} \left((-1)^l \frac{K_0^2}{K_l^2} + 2 \sum_{n=1}^{l} (-1)^{l-n} \frac{K_n^2}{K_l^2} \right) , \end{aligned} \tag{5.42}$$

where the modified Bessel functions on the right-hand side have argument ma. This gives

$$\begin{aligned} \left. \frac{\partial \ln \det_{\rm Sch}}{\partial \Phi} \right|_{\Phi=N} &= -2(ma)^2 a_0 \left\{ \left[ma \frac{d}{dma} \ln \left(\frac{I_N}{K_0} \right) - N \right]^{-1} - \left[ma \frac{d}{dma} \ln \left(\frac{I_N}{K_0} \right) + N \right]^{-1} \right\} \\ &\quad -2(ma)^2 \sum_{l=1}^{\infty} a_l \left\{ \left[ma \frac{d}{dma} \ln \left(\frac{I_{l-N}}{K_l} \right) - N \right]^{-1} + \left[ma \frac{d}{dma} \ln \left(\frac{I_{l+N}}{K_l} \right) - N \right]^{-1} \right. \\ &\quad \left. - \left[ma \frac{d}{dma} \ln \left(\frac{I_{l-N}}{K_l} \right) + N \right]^{-1} - \left[ma \frac{d}{dma} \ln \left(\frac{I_{l+N}}{K_l} \right) + N \right]^{-1} \right\} . \end{aligned} \tag{5.43}$$

The remainder of this section will be confined to an analysis of the asymptotic behavior of $\det_{\rm Sch}$ for large flux. The case of small fermion mass is also of interest as this limit will be controlled by the zero-energy bound states as noted in Sec. V B. In addition, we conjecture that the low mass end of the integral in (2.1) will control the large flux growth of $\ln \det_{\rm ren}$. Hence we are led to consider the limit $N \gg 1 \gg ma$ of $\det_{\rm Sch}$.

D. $\det_{\rm Sch}$ for $N \gg 1 \gg ma$

The calculation of the above limit requires the large-l behavior of a_l in (5.42). Letting $x = ma$ and using

$$K_{l+1} = \frac{2l}{x} K_l + K_{l-1} \ , \tag{5.44}$$

we get

$$a_l = -\frac{1}{2} \frac{d}{dx} \left(\frac{K_{l+1}}{K_l} \right) + \frac{l}{x^2} \left[\left(\frac{K_{l-1}}{K_l} \right)^2 - \left(\frac{K_{l-2}}{K_l} \right)^2 + \cdots + \frac{1}{2} (-1)^{l+1} \frac{K_0^2}{K_l^2} \right] . \tag{5.45}$$

From [28] one finds, for $l \gg 1 \gg x$,

$$K_l(x) = \left(\frac{\pi}{2l}\right)^{1/2}\left(\frac{2l}{ex}\right)^l\left[1+\left(\frac{1}{12}-\frac{1}{4}x^2\right)\frac{1}{l}+\left(\frac{1}{288}-\frac{13}{48}x^2+\frac{1}{32}x^4\right)\frac{1}{l^2}+O\left(\frac{1}{l^3}\right)\right] , \quad (5.46)$$

and hence

$$\frac{K_{l-1}(x)}{K_l(x)} = \frac{x}{2l}\left[1+\frac{1}{l}+\left(1-\frac{1}{4}x^2\right)\frac{1}{l^2}+O\left(\frac{1}{l^3}\right)\right] , \quad (5.47)$$

so that

$$a_l = \frac{1}{4l^2}+O\left(\frac{1}{l^3}\right) . \quad (5.48)$$

Combining (5.48) with

$$x\frac{d}{dx}\ln\left(\frac{I_{l\pm N}}{K_l}\right) > N , \quad (5.49)$$

valid for l, $x \geq 0$, we may conclude that

$$\lim_{N\to\infty}\sum_{l=1}^{\infty} a_l\left[x\frac{d}{dx}\ln\left(\frac{I_{l\pm N}}{K_l}\right)+N\right]^{-1} = 0 , \quad (5.50)$$

in which case the second, fifth, and last terms in (5.43) vanish in the limit $N \to \infty$.

Considering the first term in (5.43) one finds that, for $x \ll 1$,

$$x\frac{d}{dx}\ln\left(\frac{I_N}{K_0}\right) - N$$
$$= -\left[\ln\left(\frac{x}{2}\right)\right]^{-1} + O\left(\left[\ln\left(\frac{x}{2}\right)\right]^{-2}\right) , \quad (5.51)$$

and

$$x^2a_0 = -\frac{1}{2}\left[\ln\left(\frac{x}{2}\right)\right]^{-1} + O\left(\left[\ln\left(\frac{x}{2}\right)\right]^{-2}\right) , \quad (5.52)$$

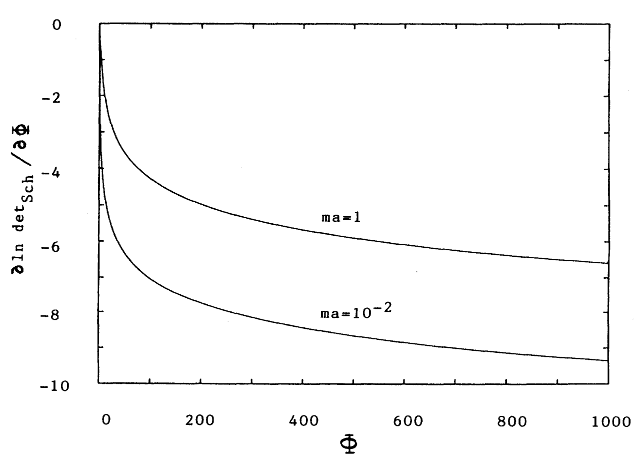

FIG. 3. Numerical calculation of the right-hand side of Eq. (5.41) for half-integral values of Φ with $ma = 1$ and 10^{-2}.

so that the first term in (5.43) tends to -1 for $N \gg 1 \gg x$.

Now consider the fourth term in (5.43). Combining [28]

$$I_l(x) = (2\pi l)^{-1/2}\left(\frac{xe}{2l}\right)^l\left[1+O\left(\frac{1}{l}\right)\right] , \quad (5.53)$$

for $l \gg 1 \gg x$ with (5.46) one finds

$$x\frac{d}{dx}\ln\left(\frac{I_{l+N}}{K_l}\right) = 2l + N + O\left(\frac{1}{l}\right) , \quad (5.54)$$

which together with (5.48) gives

$$\sum_{l=N+1}^{\infty} a_l\left[x\frac{d}{dx}\ln\left(\frac{I_{l+N}}{K_l}\right)-N\right]^{-1} \sim \frac{1}{8}\sum_{l=N+1}^{\infty} l^{-3} . \quad (5.55)$$

For the range $1 \leq l \leq N$, use (5.53) and $K_l'/K_l < 0$ for $l, x \geq 0$ to conclude, for $N \gg 1$, $l \leq 1$,

$$x\frac{d}{dx}\ln\left(\frac{I_{l+N}}{K_l}\right) - N \sim l - x\frac{K_l'}{K_l} > l . \quad (5.56)$$

Therefore, for $x \ll 1$

$$\lim_{N\to\infty}\sum_{l=1}^{\infty} a_l\left[x\frac{d}{dx}\ln\left(\frac{I_{l+N}}{K_l}\right)-N\right]^{-1} = \text{finite} . \quad (5.57)$$

Finally, consider the remaining third term in (5.43). For the range $l \geq N+1$ the third term may be written

$$\sum_{N+1} \equiv \sum_{l=1}^{\infty} a_{l+N}\left[x\frac{d}{dx}\ln\left(\frac{I_l}{K_{l+N}}\right)-N\right]^{-1} . \quad (5.58)$$

From (5.46) for $N \gg 1 \gg x$, $l \geq 1$

$$x\frac{d}{dx}\ln\left(\frac{I_l}{K_{l+N}}\right) = x\frac{I_l'}{I_l} + l + N + O\left(\frac{1}{N+l}\right)$$
$$> l + N + O\left(\frac{1}{N+l}\right) , \quad (5.59)$$

since $I_l'/I_l > 0$ for $l, x \geq 0$. Equation (5.59) together with (5.48) ensures that $\lim_{N\to\infty}\sum_{N+1}$ =finite.

For the range $1 \leq l \leq N$ let

$$b_l = x^2a_2\left[x\frac{d}{dx}\ln\left(\frac{I_{N-l}}{K_l}\right)-N\right]^{-1} , \quad (5.60)$$

where we used $I_{-l} = I_l$. We find, for $x \ll 1$,

$$b_1 = -\tfrac{1}{2}\ln x + \text{const} , \quad (5.61)$$

$$b_2 = \frac{1}{2}\left(1-\frac{1}{N}\right) + O([x\,\ln x]^2) . \quad (5.62)$$

The terms in (5.60) for $3 \le l \le N$ and $x \ll 1$ behave as

$$a_l = \tfrac{1}{4}(l-2)^{-2} - \frac{x^2}{8}\frac{2l-3}{(l-1)^3(l-2)^3} + O(x^4) , \tag{5.63}$$

$$x\frac{d}{dx}\ln\left(\frac{I_{N-l}}{K_l}\right) - N = \frac{Nx^2}{2(N-l+1)(l-1)} + \frac{x^4}{8}\frac{3N^2l - N^3 - 4N^2 - 3l^2N + 8Nl - 5N}{(l-1)^2(N-l+1)^2(l-2)(N-l+2)} + O(x^6) , \tag{5.64}$$

so that

$$\sum_{l=3}^{N} b_l = \tfrac{1}{2}\ln N + \text{const} . \tag{5.65}$$

Combining the results obtained above on the sums in (5.43) together with (5.61) and (5.65) give

$$\left.\frac{\partial \ln \det_{\text{Sch}}}{\partial \Phi}\right|_{\Phi=N} = -\ln\left(\frac{N}{ma}\right) + \text{const} . \tag{5.66}$$

The $-\ln N$ growth is consistent with the data in Fig. 3. Recall that Φ denotes $e\Phi/2\pi$, the two-dimensional chiral anomaly, and that $[e\Phi/2\pi]$ is the number of zero-energy bound states of the two-dimensional Pauli Hamiltonian (3.3). Assuming a smooth variation of $\det_{\text{Sch}}$ with Φ, we get, for $ma \ll 1 \ll e\Phi/2\pi$,

$$\ln \det_{\text{Sch}} = -(\text{no. bound states}) \times \left[\ln\left(\frac{\text{no. bound states}}{ma}\right) + \text{const}\right] . \tag{5.67}$$

We suspect that (5.67) holds more generally than for the finite-flux magnetic field in (3.6).

VI. ZERO-MASS LIMIT OF $\det_{\text{Sch}}$

Consider either of the representations (2.5) or (4.19) for $\det_{\text{Sch}}$. Does the $m = 0$ limit of $\det_{\text{Sch}}$ exist? If it does, is it continuous in the sense that the massless Schwinger model's determinant is regained? We do not have any definite answers to these questions, but we suspect that they probably depend on whether the magnetic fields have zero or nonzero flux as we will now explain.

Already at the level of second-order perturbation theory one runs into trouble for fields with $\Phi \neq 0$. The determinant to order e^2 is

$$\ln \det_{\text{Sch}} = -\frac{e^2}{2\pi}\int \frac{d^2q}{(2\pi)^2}|\hat{B}(q)|^2 \int_0^1 dz \frac{z(1-z)}{z(1-z)q^2 + m^2} . \tag{6.1}$$

The finite-flux field (3.7) gives

$$\begin{aligned}\ln \det_{\text{Sch}} &= -\left(\frac{e\Phi}{2\pi}\right)^2 \int_0^1 dz I_0\left(\frac{a|m|}{\sqrt{z(1-z)}}\right) \times K_0\left(\frac{a|m|}{\sqrt{z(1-z)}}\right) \\ &\underset{m\to 0}{=} -\left(\frac{e\Phi}{2\pi}\right)^2 \ln(a|m|) + \text{finite} ,\end{aligned} \tag{6.2}$$

while the zero flux field

$$B(r) = \frac{B}{r}[\delta(r-a) - \delta(r-b)] , \tag{6.3}$$

gives

$$\ln \det_{\text{Sch}} = -(eB)^2 \int_0^1 dz z(1-z) \times \int_0^\infty dq q \frac{[J_0(qa) - J_0(qb)]^2}{z(1-z)q^2 + m^2} , \tag{6.4}$$

which converges to the massless Schwinger model's determinant in the limit $m = 0$:

$$\begin{aligned}\ln \det_{\text{Sch}} &= -\frac{e^2}{2\pi}\int \frac{d^2q}{(2\pi)^2}\frac{|\hat{B}(q)|^2}{q^2} \\ &= -(eB)^2 \ln(a/b) \quad \text{for } a > b > 0 .\end{aligned} \tag{6.5}$$

Going beyond perturbation theory we have from (5.67) for the finite-flux field (3.7)

$$\ln \det_{\text{Sch}} \underset{m\to 0}{=} \frac{|e\Phi|}{2\pi}\ln(|m|a) + \text{finite at } m = 0, |e\Phi| \gg 1 . \tag{6.6}$$

This result agrees with intuition, that is

$$\ln\left[\frac{\det(H+m^2)}{\det(p^2+m^2)}\right]^{1/2} \underset{m\to 0}{=} (\text{no zero modes of} H) \times \ln(|m|R) + \text{finite} , \tag{6.7}$$

where H is the Pauli Hamiltonian (3.3), and R is some natural length scale, such as the range of B. The massless Schwinger model's determinant is not regained.

Finally, Seiler [12] defined the massless Schwinger model's determinant, $\det[1 - e(1/\not{p})\not{A}]$, by a renormalized determinant. This work was done by making a formal similarity transformation and defining the determinant as $\det_{\text{ren}}(1 - eK)$ with

$$K(A) = \frac{\not{p}}{|p|^{3/2}}\not{A}\frac{1}{|p|^{1/2}} , \tag{6.8}$$

where K is considered as an operator on two-component square-integrable functions. The determinant $\det_{\text{ren}}$ excludes the linearly divergent graph $\text{Tr}K$ but retains the graph $\text{Tr}K^2$, which is defined in some gauge-invariant way. Assuming that the magnetic fields have zero flux and in particular that $\int d^2r|A_\mu|^q < \infty$ for all $q \geq \frac{1}{2}$, Seiler was able to show that $K \in \mathcal{C}_n$ [the space of operators for which $\text{Tr}(K^\dagger K)^{n/2} < \infty$] for all $n > 2$ and that the spectrum of K consists only of the origin. This latter result is an expression of the triviality of the massless Schwinger model and implies that all single-loop "photon-photon" scattering graphs of order e^4 and higher vanish [29]. These results allowed Seiler to obtain the well-known result for the massless Schwinger model's determinant:

$$\ln \det_{\rm ren}(1 - eK) = -\frac{e^2}{2\pi} \int d^2r A_\mu^2 \ , \qquad (6.9)$$

where A_μ is in the Lorentz gauge.

In the nonzero flux case A_μ falls off as $1/r$, and $K(A)$ ceases to be even a compact operator: the eigenvalues of K fill an open disc and $K \notin \mathcal{C}_n$ for any $n \geq 1$. Consequently, it is no longer possible to define the massless Schwinger model's determinant in terms of a renormalized determinant whose zeros reflect the eigenvalues of $K(A)$. The lesson here is that the transition from a zero-flux magnetic field to one with nonzero flux is not a smooth one and that the zero-mass limit of $\det_{\rm Sch}$ will be flux sensitive.

VII. NET RESULT

Fermionic determinants are at the heart of fermionic field theories. In the case of QED the determinant in 3+1 dimensions in a static, undirected, finite (including zero) flux magnetic field can be calculated from the determinant of the massive Euclidean Schwinger model, $\det_{\rm Sch}$, in the same magnetic field by integrating over the fermion's mass. Therefore, the massive Schwinger model is physically relevant. The calculation of $\det_{\rm Sch}$ reduces to a problem in nonrelativistic supersymmetric quantum mechanics, and the gauge invariance of $\det_{\rm Sch}$ is closely linked to the index theorem on a two-dimensional Euclidean manifold. The inclusion of mass qualitatively changes the determinant in 1+1 dimensions to the extent that the massless Schwinger model's contribution to $\det_{\rm Sch}$ is canceled by a contribution from the massive sector. Evidence was given that the zero-mass limit of $\det_{\rm Sch}$ is not continuous in the sense that the massless Schwinger model's determinant is not regained for nonzero flux magnetic fields.

It is believed that the first calculation of $\det_{\rm Sch}$ for a finite-flux magnetic field is given in Sec. V. The behavior of the determinant for large flux and small mass suggests that the zero-energy bound states of the two-dimensional Pauli Hamiltonian are the controlling factor in the growth of $\ln \det_{\rm Sch}$. If this is the case then the implication of this fact on the still unknown growth of the renormalized determinant of QED_4 in the same magnetic field, which is determined by (2.1), remains to be seen.

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