One-loop corrections to the instanton transition in the two-dimensional Abelian Higgs model

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We present an evaluation of the fluctuation determinant which appears as a prefactor in the instanton transition rate for the two-dimensional Abelian Higgs model. The corrections are found to change the rate at most by a factor of 2, for $0.4 < M_H/M_W < 2.0$.

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I. INTRODUCTION

The Abelian Higgs model in 1+1 dimensions has found considerable attention recently since on the one hand it shares certain features with the electroweak theory and on the other hand it is simple enough to serve as a theoretical and numerical laboratory.

In the context of the baryon number violation the high-temperature sphaleron transition has been studied [1]–[6], for which exact classical solutions and an exact expression of the sphaleron determinant [2]–[4] are known, thus providing a complete one-loop semiclassical transition rate which can be studied numerically on the lattice, e.g., by measuring the fluctuations of the Chern-Simons number.

Another prominent feature of the model is the existence of instanton solutions [7] which give rise again to fluctuations in the topological charge of the vacuum and thereby to baryon number violation. It has also been used [8] to study the possibility of baryon number violation in high-energy scattering processes.

If the parameters of the model are chosen appropriately, the instantons are sufficiently rare and the system can be treated as a dilute gas of instantons with Chern-Simons charge $q = \pm 1$. In this approximation the transition rate, or equivalently the density of instantons in the Euclidean plane, is given by [9]

$$\Gamma = \frac{S(\phi_{\rm cl})}{2\pi}\mathcal{D}^{-1/2}\exp[-S(\phi_{\rm cl}) - S_{\rm ct}(\phi_{\rm cl})] \qquad (1.1)$$

to one-loop accuracy. Here $S(\phi_{\rm cl})$ is the instanton action. The coefficient $\mathcal{D}$ represents the effect of quantum fluctuations around the instanton configuration and arises from the Gaussian approximation to the functional integral. This is the object whose computation we will consider here. It is given in general form by

$$\mathcal{D} = \frac{\det'(\mathcal{M})}{\det(\mathcal{M}^0)} = \exp(2S_{\rm eff}^{\rm one\ loop})\ , \qquad (1.2)$$

the second equation relating it to the one-loop effective action. The operators $\mathcal{M}$ are the fluctuation operators obtained by taking the second functional derivative of the action at the instanton and vacuum background field configurations. The prime on the determinant implies omitting the two translation zero modes. The first prefactor $S(\phi_{\rm cl})/2\pi$ takes into account the integration of the translation mode collective coordinates. Finally, the counterterm action $S_{\rm ct}$ in the exponent will absorb the ultraviolet divergences of $\mathcal{D}$. One may also include a corresponding determinant for fermions, which for massless fermions is even known analytically [19] (see below). However, in lattice simulations the instanton rate and therefore fermion number violation can be measured by studying fluctuations of the Chern-Simons number without having to include the fermions explicitly.

For $M_H/M_W \neq 1$ even the classical instanton profiles are known only numerically, so an exact evaluation of the effective action has to be performed numerically. Such computations have been performed recently for the fluctuation determinant of the electroweak sphaleron [10]–[12]. In the first of these [10] the heat kernel definition of the determinant was used, and the heat kernel itself was computed from the eigenvalue spectrum of the fluctuation operator, using a partial wave decomposition. This method was also used in Ref. [12]. Another method for such computations has been proposed recently [13]; it has been applied to the computation of the fluctuation determinant of the electroweak sphaleron [11] and, for the case considered here, in Ref. [14] on which the present work is based.

This paper is organized as follows. In the next section we outline the basic relations of the Abelian Higgs model. The fluctuation operator is derived in Sec. III, its partial wave reduction in Sec. IV. The method of computation is presented in Sec. V. In Sec. VI we consider the renormalization of the effective action and the removal of zero modes. The results are presented and discussed in Sec. VII.

II. BASIC RELATIONS

The Abelian Higgs model in 1+1 dimensions is defined by the Lagrange density (written in the Euclidean form relevant here)

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}|D_\mu\phi|^2 + \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2 + \mathcal{L}_f\ . \qquad (2.1)$$

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Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$
$$D_\mu = \partial_\mu - igA_\mu \ ,$$
$$\mathcal{L}_f = i\sum_{i=1}^{n_f} \overline{\Psi}_L^{(i)} \hat{D}_L \Psi_L^{(i)} + i\sum_{j=1}^{n_f} \overline{\Psi}_R^{(j)} \hat{D}_R \Psi_R^{(j)} \ ,$$
$$\hat{D}_{L(R)} = \gamma_\mu \left(\partial_\mu \mp igA_\mu\right) \ .$$

The particle spectrum consists of Higgs bosons of mass $m_H^2 = 2\lambda v^2$, vector bosons of mass $m_W^2 = g^2 v^2$, and left-right-handed massless fermions of charge g. The anomaly of the gauge-invariant fermionic current

$$J_\mu = \sum_{i=1}^{n_f} \overline{\Psi}_L^{(i)} \gamma_\mu \Psi_L^{(i)} + \sum_{j=1}^{n_f} \overline{\Psi}_R^{(j)} \gamma_\mu \Psi_R^{(j)} \tag{2.2}$$

is given by

$$\partial_\mu J_\mu = 2n_f \left(\frac{g}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu}\right) \ . \tag{2.3}$$

The integral over the divergence of the current which measures the baryon number violation is given by

$$\Delta F = \int d^2x\, \partial_\mu J_\mu \tag{2.4}$$
$$= 2n_f \left(\frac{g}{4\pi}\int d^2x\, \varepsilon_{\mu\nu}F_{\mu\nu}\right) \equiv 2n_f q \ . \tag{2.5}$$

Here q denotes the Chern-Simons term in two dimensions (see, e.g., [15]) and baryon number violation is therefore related to Euclidean gauge field configurations with nonvanishing topological charge q. These are the instanton solutions which mediate tunneling transitions changing the topological charge by q units. We will assume here that the instanton transitions are described sufficiently well by a dilute gas of instantons with Chern-Simons number $q = \pm 1$, a situation for which the rate formula given in the introduction is supposed to hold.

A structure which exhibits such a topological charge and satisfies the Euclidean equations of motion is given by the Nielsen-Olesen vortex [7]. The spherically symmetric ansatz for this solution is given by

$$A_\mu^{\rm cl}(x) = \frac{\varepsilon_{\mu\nu}x_\nu}{gr^2}A(r) \tag{2.6}$$
$$\phi^{\rm cl}(x) = vf(r)e^{i\varphi(x)} \ . \tag{2.7}$$

In order to have a purely real Higgs field one performs a gauge transformation

$$\phi \to e^{-i\varphi}\phi \ ,$$
$$A_\mu \to A_\mu - \partial_\mu\varphi/g \ ,$$
$$\Psi_{L(R)} \to e^{\mp i\varphi}\Psi_{L(R)} \ , \tag{2.8}$$

to obtain the instanton fields in the singular gauge

$$A_\mu^{\rm cl}(x) = \frac{\varepsilon_{\mu\nu}x_\nu}{gr^2}[A(r)+1] \ , \tag{2.9}$$
$$\phi^{\rm cl}(x) = vf(r) \ . \tag{2.10}$$

With this ansatz the Euclidean action takes the form

$$S_{\rm cl} = \pi v^2 \int_0^\infty dr \left[\frac{1}{rm_W^2}\left(\frac{dA(r)}{dr}\right)^2 + r\left(\frac{df(r)}{dr}\right)^2 + \frac{f^2(r)}{r}[A(r)+1]^2 + \frac{rm_H^2}{4}\left(f^2(r)-1\right)^2\right] \ . \tag{2.11}$$

For the case $M_H = M_W$ an exact solution to the variational equation is known [16], for which the classical action takes the value $S_{\rm cl} = \pi v^2$. We will consider here the general case, however, for which the classical equations of motion

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{[A(r)+1]^2}{r^2} - \frac{m_H^2}{2}\left(f^2(r)-1\right)\right)f(r) = 0 \ , \tag{2.12}$$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - m_W^2 f^2(r)\right)A(r) = m_W^2 f^2(r) \tag{2.13}$$

have to be solved numerically.

Imposing the boundary conditions on the profile functions

$$A(r) \sim Cr^2,\ f(r) \sim C'r, \quad \text{as } r \to 0,$$
$$A(r) \to -1,\ f(r) \to 1, \quad \text{as } r \to \infty \ , \tag{2.14}$$

the Chern-Simons number is 1 and the action is finite.

Since we will consider fluctuations around these solutions later on, a good numerical accuracy for the profile functions $f(r)$ and $A(r)$ is required. We have found that the method used previously by Bais and Primack [17] in order to obtain precise profiles for the 't Hooft–Polyakov monopole is very suitable also in this context. The values for the classical action, which determine also the translation mode prefactor, are given in Table I. They agree almost perfectly with the results of Jacobs and Rebbi [18].

The classical action in units of πv^2 is plotted in Fig. 1 for $0.4 < M_H/M_W < 2$. The plot suggests a power law behavior $(M_H/M_W)^\rho$ where $0.40 < \rho < 0.43$; our data and those of Ref. [18] are precise enough to rule out an exact power dependence. While the best fit is obtained with $\rho \simeq 0.41$, a suggestive number in this range

TABLE I. Classical and one-loop actions for various values of M_H/M_W. The fermionic effective action is given for a left- *plus* right-handed fermion. $S_{\rm eff}^{\rm one\ loop}$ is the one-loop bosonic action computed here.

M_H/M_W	$S_{\rm cl}$	$S_{\rm eff}^{\rm ferm}$	$S_{\rm eff}^{\rm one\ loop}$
0.40	0.696196	-0.37853	0.315
0.60	0.813053	-0.38459	0.233
0.80	0.912305	-0.39051	0.102
1.00	1.000000	-0.39616	-0.025
1.25	1.097914	-0.40277	-0.156
1.50	1.186013	-0.40886	-0.276
1.75	1.266416	-0.41445	-0.379
2.00	1.340550	-0.41956	-0.461

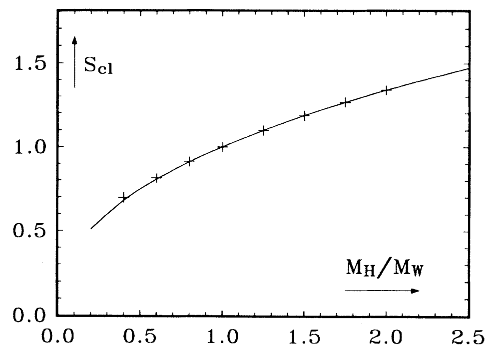

FIG. 1. The classical action. We present the action in (dimensionless) units πv^2. The full circles are the numerical results, the curve displays a possible asymptotic dependence $(M_H/M_W)^{(1-\gamma)}$, where γ is Euler's constant.

is $\rho = 1-\gamma$, where γ is Euler's constant. This could be an asymptotic dependence for large M_H/M_W, it is also displayed in Fig. 1.

Though we are not interested here in the effect of fermionic fluctuations, we could not resist using our profiles to calculate the fermion determinant for massless fermions. This determinant is known exactly [19]; the corresponding effective action, per (left- *plus* right-handed) fermion and after removing the fermionic zero modes, is given by

$$S_{\text{eff}}^{\text{ferm}} = -\frac{1}{2\pi}\int d^2x\alpha\partial^2\alpha\ , \tag{2.15}$$

where, for the instanton,

$$\alpha(r) = \int_0^r dr\frac{A(r)}{r}\ . \tag{2.16}$$

The results are given in Table I and plotted, along with the bosonic effective action, in Fig. 4.

III. THE FLUCTUATION OPERATOR

The fluctuation operator is defined in general form as

$$\mathcal{M} = \frac{\delta^2 S}{\delta\psi_i^*(x)\delta\psi_j(x')}\Big|_{\psi_k=\psi_k^{\text{cl}}}\ , \tag{3.1}$$

where ψ_i denotes the fluctuating fields and ψ_i^{cl} the "classical" background field configuration; here these will be the instanton and the vacuum configurations. If the fields are expanded around the background configuration as $\psi_i = \psi_i^{\text{cl}} + \phi_i$ and if the Lagrange density is expanded accordingly, then the fluctuation operator is related to the second-order Lagrange density via

$$\mathcal{L}^{\text{II}} = \frac{1}{2}\phi_i^*\mathcal{M}_{ij}\phi_j\ . \tag{3.2}$$

In terms of the fluctuation operators $\mathcal{M}$ on the instanton and $\mathcal{M}^0$ on the vacuum backgrounds, the effective action is defined as

$$S_{\text{eff}} = \frac{1}{2}\ln\left\{\frac{\det'\mathcal{M}}{\det\mathcal{M}^0}\right\}\ . \tag{3.3}$$

For our specific model we expand as

$$A_\mu = A_\mu^{\text{cl}} + a_\mu\ , \tag{3.4}$$

$$\phi = \phi^{\text{cl}} + \varphi\ . \tag{3.5}$$

In order to eliminate the gauge degrees of freedom we introduce, as in Ref. [8], the background gauge function

$$\mathcal{F}(A) = \partial_\mu A_\mu + \frac{ig}{2}\left((\phi^{\text{cl}})^*\phi - \phi^{\text{cl}}\phi^*\right)\ , \tag{3.6}$$

which leads in the Feynman background gauge to the gauge-fixing Lagrange density

$$\begin{aligned}\mathcal{L}_{\text{GF}}^{\text{II}} &= \left(\frac{1}{2}\mathcal{F}^2(A)\right)^{\text{II}}\\ &= \frac{1}{2}(\partial_\mu a_\mu)^2 - \frac{ig}{2}a_\mu(\varphi\partial_\mu\phi^{\text{cl}} + \phi^{\text{cl}}\partial_\mu\varphi - \varphi^*\partial_\mu\phi^{\text{cl}}\\ &\quad -\phi^{\text{cl}}\partial_\mu\varphi^*) - \frac{g^2}{8}(\phi^{\text{cl}})^2(\varphi-\varphi^*)^2\ .\end{aligned} \tag{3.7}$$

The associated Faddeev-Popov Lagrangian becomes

$$\mathcal{L}_{\text{FP}} = \frac{1}{2}\eta^*[-\partial^2 + g^2\left(\phi^{\text{cl}}\right)^2]\eta\ . \tag{3.8}$$

In terms of the real components $\varphi = \varphi_1 + i\varphi_2$ and $\eta = (\eta_1 + i\eta_2)/\sqrt{2}$ the second-order Lagrange density now becomes (omitting the superscript from ϕ^{cl} and A_μ^{cl})

$$\begin{aligned}(\mathcal{L}+\mathcal{L}_{\text{GF}}+\mathcal{L}_{\text{FP}})^{\text{II}} &= a_\mu\frac{1}{2}\left(-\partial^2+g^2\phi^2\right)a_\mu + \varphi_1\frac{1}{2}\left[-\partial^2+g^2A_\mu^2+\lambda\left(3\phi^2-v^2\right)\right]\varphi_1\\ &\quad+\varphi_2\frac{1}{2}\left[-\partial^2+g^2A_\mu^2+g^2\phi^2+\lambda\left(\phi^2-v^2\right)\right]\varphi_2\\ &\quad+\varphi_2(gA_\mu\partial_\mu)\varphi_1+\varphi_1(-gA_\mu\partial_\mu)\varphi_2+a_\mu(2g^2A_\mu\phi)\varphi_1+a_\mu(2g\partial_\mu\phi)\varphi_2\\ &\quad+\eta_1\frac{1}{2}\left(-\partial^2+g^2\phi^2\right)\eta_1+\eta_2\frac{1}{2}\left(-\partial^2+g^2\phi^2\right)\eta_2\ .\end{aligned} \tag{3.9}$$

Specifying now the fluctuating fields $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ as $(a_1, a_2, \varphi_1, \varphi_2, \eta_{12})$ the nonvanishing components of $\mathcal{M}$ are

$$\begin{aligned}
&\mathcal{M}_{11} = -\partial^2 + g^2\phi^2 \ , \quad \mathcal{M}_{22} = -\partial^2 + g^2\phi^2 \ , \\
&\mathcal{M}_{13} = 2g^2 A_1 \phi \ , \quad \mathcal{M}_{14} = 2g\partial_1\phi \ , \\
&\mathcal{M}_{23} = 2g^2 A_2 \phi \ , \quad \mathcal{M}_{24} = 2g\partial_2\phi \ , \\
&\mathcal{M}_{33} = -\partial^2 + g^2 A_\mu^2 + \lambda(3\phi^2 - v^2) \ , \\
&\mathcal{M}_{34} = -gA_\mu\partial_\mu \ , \\
&\mathcal{M}_{44} = -\partial^2 + g^2 A_\mu^2 + g^2\phi^2 + \lambda(\phi^2 - v^2) \ , \\
&\mathcal{M}_{43} = gA_\mu\partial_\mu \ , \\
&\mathcal{M}_{55} = -\partial^2 + g^2\phi^2 \ .
\end{aligned}$$

It is understood that the contribution of the Faddeev-Popov operator $\mathcal{M}_{55}$ enters with a negative sign and a factor 2 into the definition of the effective action. The fluctuation operators for the instanton and vacuum background are now obtained by substituting the corresponding classical fields. The vacuum fluctuation operator becomes a diagonal matrix of Klein-Gordon operators with masses $(M_W, M_W, M_W, M_H, M_W)$. It is convenient to introduce a potential $\mathcal{V}$ via

$$\mathcal{M} = \mathcal{M}^0 + \mathcal{V} \ . \tag{3.10}$$

The potential $\mathcal{V}$ will be specified below after partial wave decomposition.

IV. PARTIAL WAVE DECOMPOSITION

The fluctuation operator $\mathcal{M}$ can be decomposed into partial waves and its determinant decomposes accordingly:

$$\ln\det\mathcal{M} = \sum_{n=-\infty}^{+\infty} \ln\det\mathbf{M}_n \ . \tag{4.1}$$

We introduce the following partial wave decomposition for fields:

$$\vec{a} = \sum_{n=-\infty}^{+\infty} b_n(r)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}\frac{e^{in\varphi}}{\sqrt{2\pi}} + ic_n(r)\begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}\frac{e^{in\varphi}}{\sqrt{2\pi}},$$

$$\varphi_1 = \sum_{n=-\infty}^{+\infty} h_n(r)\frac{e^{in\varphi}}{\sqrt{2\pi}} \ ,$$

$$\varphi_2 = \sum_{n=-\infty}^{+\infty} \tilde{h}_n(r)\frac{e^{in\varphi}}{\sqrt{2\pi}} \ ,$$

$$\eta_{12} = \sum_{n=-\infty}^{+\infty} g_n(r)\frac{e^{in\varphi}}{\sqrt{2\pi}} \ .$$

After inserting these expressions into the Lagrange density and using the reality conditions for the fields one finds that the following combinations are real relative to each other and make the fluctuation operators symmetric:

$$\begin{aligned}
&F_1^n(r) = \frac{1}{2}[b_n(r) + c_n(r)] \ , \\
&F_2^n(r) = \frac{1}{2}[b_n(r) - c_n(r)] \ , \\
&F_3^n(r) = \tilde{h}_n(r) \ , \\
&F_4^n(r) = ih_n(r) \ , \\
&F_5^n(r) = g_n(r) \ .
\end{aligned}$$

Writing the partial fluctuation operators, omitting the index n in the following, as

$$\mathbf{M} = \mathbf{M}^0 + \mathbf{V} \ , \tag{4.2}$$

the free operators $\mathbf{M}^0$ become diagonal matrices with elements

$$M_{ii}^0 = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{n_i^2}{r^2} + M_i^2 \ , \tag{4.3}$$

where $(n_i) = (n-1, n+1, n, n, n)$ and $(M_i) = (M_W, M_W, M_W, M_H, M_W)$. The potential $\mathbf{V}$ takes the elements

$$\begin{aligned}
&\mathbf{V}_{11}^n = m_W^2\left(f^2 - 1\right) \ , \quad \mathbf{V}_{12}^n = 0 \ , \\
&\mathbf{V}_{13}^n = \sqrt{2}m_W f' \ , \quad \mathbf{V}_{14}^n = \sqrt{2}m_W f\frac{A+1}{r} \ , \\
&\mathbf{V}_{22}^n = \mathbf{V}_{11}^n \ , \quad \mathbf{V}_{23}^n = \mathbf{V}_{13}^n, \\
&\mathbf{V}_{24}^n = -\mathbf{V}_{14}^n \ , \\
&\mathbf{V}_{33}^n = \frac{(A+1)^2}{r^2} + \frac{m_H^2}{2}\left(f^2-1\right) + m_W^2\left(f^2-1\right) \ , \\
&\mathbf{V}_{34}^n = -2\frac{A+1}{r^2}n \ , \\
&\mathbf{V}_{44}^n = \frac{(A+1)^2}{r^2} + \frac{3}{2}m_H^2\left(f^2-1\right) \ , \\
&\mathbf{V}_{55}^n = m_W^2\left(f^2-1\right) \ , \quad \mathbf{V}_{i5} = 0 \ .
\end{aligned}$$

Choosing the dimensionless variable $M_W r$ one realizes that the fluctuation operator depends only on the ratio M_H/M_W up to an overall factor M_W^2 which cancels in the ratio with the free operator.

V. COMPUTATION OF THE FLUCTUATION DETERMINANT

The method for computing the fluctuation determinant used here is based on the use of the Euclidean Green function of the fluctuation operator. This Green function is defined by

$$(\mathcal{M} + \nu^2)\mathcal{G}(\vec{x}, \vec{x}', \nu) = \mathbf{1}\delta(\vec{x} - \vec{x}') \tag{5.1}$$

and similarly for the operator $\mathcal{M}^0$. It contains the information on the eigenvalues λ_α of the fluctuation operator via

$$\int d^2x\, \mathrm{Tr}\mathcal{G}(\vec{x}, \vec{x}, \nu) = \sum_\alpha \frac{1}{\lambda_\alpha^2 + \nu^2}. \tag{5.2}$$

If we define the function $F(\nu)$ via

$$F(\nu) = \int d^2x\, \mathrm{Tr}[\mathcal{G}(\vec{x}, \vec{x}, \nu) - \mathcal{G}^0(\vec{x}, \vec{x}, \nu)] \ , \tag{5.3}$$

we have

$$-\int_\epsilon^\Lambda d\nu \nu F(\nu) = \sum_\alpha \frac{1}{2} \ln \left\{ \frac{(\lambda_\alpha^2 + \epsilon^2)(\lambda_\alpha^{0\,2} + \Lambda^2)}{(\lambda_\alpha^{0\,2} + \epsilon^2)(\lambda_\alpha^2 + \Lambda^2)} \right\} . \quad (5.4)$$

For $\epsilon \to 0$ this is just the logarithm of the ratio of fluctuation determinants, i.e., the one-loop effective action, regularized with a Pauli-Villars cutoff. The regularization can be removed, the integral can be taken to infinity, after subtracting the one-loop counterterm action (see below). Before taking the limit $\epsilon \to 0$ the two zero eigenvalues have to be removed by subtracting their contribution $\ln \epsilon^2$. Of course ϵ has the dimension of energy. We used here the scale M_W throughout, i.e., by making the radial variable dimensionless. So $S_{\rm eff}$ now contains a term $-\ln M_W^2$, the numerical prefactor $\mathcal{D}^{-1/2}$, and therefore the rate, are computed in units of M_W^2.

After these more formal considerations we have to present a practical way of computing $F(\nu)$. This is done by using the partial wave decomposition to write

$$F(\nu) = \sum_{n=-\infty}^{+\infty} F_n(\nu) , \quad (5.5)$$

where

$$F_n(\nu) = \int dr r \, \mathrm{Tr}[\mathbf{G}_n(r,r,\nu) - \mathbf{G}_n^0(r,r,\nu)] , \quad (5.6)$$

the partial wave Green functions being defined by

$$(\mathbf{M}_n + \nu^2)\mathbf{G}_n(r,r',\nu) = \mathbf{1}\frac{1}{r}\delta(r-r') . \quad (5.7)$$

For $\mathbf{M}_n^0$ the Green function is simply a diagonal matrix with elements

$$\mathbf{G}^0_{n\;ii}(r,r',\nu) = I_{n_i}(\kappa_i r_<) K_{n_i}(\kappa_i r_>) , \quad (5.8)$$

where $\kappa = \sqrt{M_i^2 + \nu^2}$. For the Green function of the operator $\mathbf{M}_n$ the matrix elements similarly become

$$\mathbf{G}_{n\;ij}(r,r',\nu) = f_i^{\alpha-}(r_<) f_j^{\alpha+}(r_>) , \quad (5.9)$$

where the functions $f_i^{\alpha\pm}$ form a fundamental system of solutions of (5.7), regular as $r \to 0$ for the minus sign and as $r \to \infty$ for the plus sign. The correct normalization is obtained by imposing the boundary conditions

$$f_i^{\alpha-}(r) \simeq \delta_i^\alpha I_{n_i}(\kappa_i r) , \qquad f_i^{\alpha+}(r) \simeq \delta_i^\alpha K_{n_i}(\kappa_i r) , \quad (5.10)$$

as $r \to \infty$. Actually we have solved numerically the differential equations for the functions $h_i^{\alpha\pm}$ defined by

$$f_i^{\alpha\pm} = B_{n_i}(\kappa_i r)[\delta_i^\alpha + h_i^\alpha(r)] , \quad (5.11)$$

where B_{n_i} are the appropriate Bessel functions. This way one keeps track of the free contribution $\propto \delta_i^\alpha$ and

$$\mathrm{Tr}\left(\mathbf{G}(r,r,\nu) - \mathbf{G}^0(r,r,\nu)\right) = [h_i^{i-}(r) + h_i^{i+}(r) + h_i^{\alpha-}(r) h_i^{\alpha+}(r)] I_{n_i}(\kappa_i r) K_{n_i}(\kappa_i r) , \quad (5.12)$$

to be inserted into (5.5). The partial wave contributions behave as n^{-3} for large n. The summation implied by Eq. (5.5) has been performed up to maximally $\bar{n} = 25$, the asymptotic tail was appended by fitting the last five terms with an expression $a_n = c_3 n^{-3} + c_4 n^{-4} + c_5 n^{-5}$ and adding the sum over the a_n from $\bar{n}+1$ to ∞. The convergence was monitored by applying this procedure already in each step of the n summation taking $\bar{n} = n$. The convergence was found to be excellent up to values of ν of the order 5. It has to be said, though, that there is considerable cancellation between the negative $n = 0$ contribution and the higher terms. Indeed the ν integration over the $n = 0$ term alone would be divergent even after renormalization. This seems to be an inherent feature for functional determinants for topologically nontrivial configurations, it is related to the fact that the centrifugal barriers of the operator $\mathbf{M}_n$ at $r = 0$ are different from those of $\mathbf{M}_n^0$. This feature also renders impossible a direct application of a theorem on functional determinants as it was used for the faster and more elegant method of Ref. [20]. The deformation of the centrifugal barriers is not related to our using the singular gauge for the classical instanton field. In fact it can be shown by direct calculation that the fluctation equations do not change under the gauge transformation (2.8).

Fortunately, the asymptotic behavior of $F(\nu)$, which is as ν^{-4} after renormalization, sets already in when this function has dropped to values of order 10^{-2} and there the cancellation is not yet delicate.

There is another problem we have to address here which is related to the coupling of fields with different masses in the system of gauge, Higgs, and would-be Goldstone fields. While normally the solutions of the coupled system satisfy vacuum boundary conditions at $r \to \infty$, i.e., the potential decreases sufficiently fast, the cross terms $\mathbf{V}_{i4}$ can cause the Higgs field to change the asymptotic behavior of the gauge and Goldstone components. The solution regular at $r = 0$ behaves normally as $\exp[\kappa_i r]$. If the physical Higgs component is multiplied by $\mathbf{V}_{i4}$ one obtains a behavior $\exp[(\kappa_H - M_W)r]$. This expression enters the right-hand sides of the equations for the Goldstone and gauge fields which themselves behave as $\exp(\kappa_W r)$. So, if $M_H > 2M_W$, these fields change their asymptotic behavior. We find that the radial integral of the trace of the Green function ceases to exist. We think that this is not a shortcoming of the method but a systematical property of the fluctuation determinant. Indeed for $M_H > 2M_W$ the Higgs boson can decay into pairs of gauge particles and also the singularity structure of perturbative graphs changes qualitatively. This subject merits further consideration; here we just restrict our computation to Higgs boson masses smaller than $2M_W$. The gauge fields cannot, on the other hand, decay into Higgs particles, since their coupling joins a physical Higgs and a would-be Goldstone mode; indeed our coupled system has no problems of principle for small Higgs boson masses.

VI. RENORMALIZATION AND ZERO MODES

The Abelian Higgs model is superrenormalizable; all divergences can be removed by a mass counterterm for the Higgs field and a counterterm for the vacuum loops. Expanding around $\phi = v$ and using the corresponding Feynman rules we find divergent tadpole diagrams of the form displayed in Fig. 2, where the internal lines represent the various Higgs, vector, and Faddeev-Popov fields. The various couplings can be read off from the second-order Lagrangian (3.9). For the vertices of the second graph we find $-3i\lambda/2$ for the physical Higgs boson of mass M_H, $-i(g^2+\lambda)/2$ for the would-be Goldstone mode of mass M_W, $ig^2 g_{\mu\nu}/2$ for the gauge field and $-ig^2/2$ for the Faddeev-Popov fields. For the first graph we find the same vertex factors multiplied by $2v$. As a consequence, in summing up the contributions from both graphs the external line factors combine as $(\phi-v)^2+2v(\phi-v) = (\phi^2-v^2)$. The contributions from the gauge field and Faddeev-Popov loops cancel as they should. The tadpole graphs with external gauge field lines (not presented in Fig. 1) cancel against second-order graphs as usual in scalar QED. The counterterm action takes the form

$$S_{\rm ct} = \frac{1}{2}\delta m^2 \int d^2x(\phi^2 - v^2) \ , \tag{6.1}$$

where in unregularized form

$$\delta m^2 = 3\lambda \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M_H^2} + (g^2+\lambda)\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M_W^2} \ . \tag{6.2}$$

In the Pauli-Villars regularization chosen here we rewrite the divergent momentum integrals via

$$\int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{k^2+M_i^2} - \frac{1}{k^2+M_i^2+\Lambda^2}\right) = \int \frac{d^2k}{(2\pi)^2} 2\int_0^\Lambda \nu d\nu \frac{1}{(k^2+M_i^2+\nu^2)^2} = \frac{1}{2\pi}\int_0^\Lambda \frac{1}{M_i^2+\nu^2} \tag{6.3}$$

so that the divergent terms can be rewritten directly as a contribution to a counterterm $F_{\rm ct}(\nu)$ in the integral over ν. We find

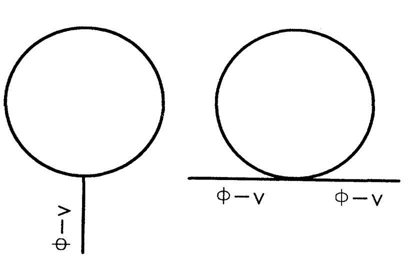

FIG. 2. The divergent tadpole graphs. The vertex factors and internal line masses are given in the text. We do not display the analoguos graphs with external classical gauge field legs since they cancel against second-order contributions.

$$F_{\rm ct}(\nu) = \int_0^\infty dr r[f^2(r)-1]\left(\frac{3M_H^2}{\nu^2+M_H^2} + \frac{M_W^2+M_H^2}{\nu^2+M_W^2}\right) \ . \tag{6.4}$$

If $F_{\rm ct}(\nu)$ is subtracted, $F_{\rm ren}(\nu) = F(\nu) - F_{\rm ct}(\nu)$ behaves as ν^{-4} as $\nu \to \infty$ and the Pauli-Villars cutoff, i.e., the upper limit of integration, can be sent to ∞.

Instead of subtracting the tadpole contributions from $F(\nu)$ this subtraction can be performed already in the partial waves. The tadpole terms can be easily recognized in the potential given at the end of Sec. IV as the diagonal terms proportional to (f^2-1). Denoting these terms by $\mathbf{V}^{\rm tad}_{ii}$ their contribution to the first-order Green function becomes

$$\mathbf{G}^{\rm tad}_{n\ ii}(r,r',\nu) = \int dr'' r'' \mathbf{G}^0_{n\ ii}(r,r'',\nu)\mathbf{V}^{\rm tad}_{ii}(r'') \times \mathbf{G}^0_{n\ ii}(r'',r',\nu) \tag{6.5}$$

(no summation over i). Using some identities for Bessel functions it can be shown that after taking the trace, integrating over r, and summing up the partial waves one obtains $F_{\rm ct}(\nu)$. In the actual computation we have removed the tadpole contributions directly in the partial waves. As an illustration we show, however, in Fig. 3 the function $F(\nu)$ before the subtraction of the tadpole and Faddeev-Popov contributions, both of these contributions separately, and the final $F^{\rm ren}(\nu)$. It follows from perturbation theory that the former behave as ν^{-2} asymptotically, while the latter behaves as ν^{-4}. The numerical integration was performed up to the region where the asymptotic behavior sets in. The remaining integral was performed as $\int d\nu\nu^{-3}$ with a coefficient determined by the last point. The contribution of the integral from $\nu_{\max}$ to ∞ is of the order 0.05 and the error introduced by the extrapolation is certainly one order of magnitude smaller than this value.

One notes in Fig. 3 that $F(\nu)$ behaves for small ν as $2/\nu^2$, a behavior which is due to the translation zero modes and makes the subtraction of $\ln\epsilon^2$ necessary when

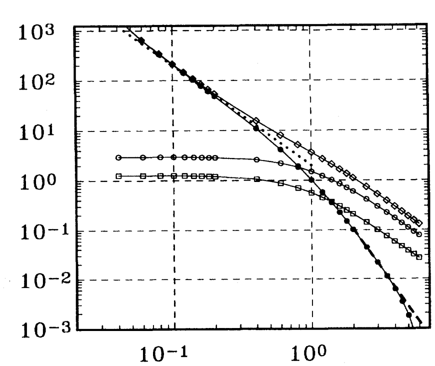

FIG. 3. The integrand $F(\nu)$ for $M_H = M_W$. Empty diamonds, $F(\nu)$ before tadpole and Faddeev-Popov subtraction; empty circles, $F_{\rm ct}(\nu)$; empty squares, the Faddeev-Popov term; full circles, $F_{\rm ren}(\nu)$. The dotted line corresponds to the behavior $2/\nu^2$ due to the zero modes. The dashed line shows the extrapolated asymptotic ν^{-4} behavior.

the lower limit of the integration is taken to 0. In practice, the zero mode pole appears slightly shifted to $\nu = \lambda_0 \approx 0.02$ as can be seen in Fig. 3 from the departure of the expected behavior at values $\nu < 0.1$. So a term $\ln(\epsilon^2 - \lambda_0^2)$ has to be subtracted instead. The integrand was, for $\nu < 1$, decomposed into a pole term and a finite contribution and the former one was integrated analytically. λ_0 can be fixed to at least three significant digits and the finite term turns out to show a smooth behavior $\propto \nu$; we think that this procedure introduces an error of $S_{\rm eff}$ below 0.01. So including the estimate for the error in the asymptotic extrapolation and another 0.05 (i.e., $\simeq 10\%$) for errors in the numerical integration we think that we have determined $S_{\rm eff}$ to within an error of 0.07.

VII. DISCUSSION AND CONCLUSION

The results of our computation of the one-loop effective action

$$S_{\rm eff} = -\lim_{\epsilon\to 0}\left(\int_\epsilon^\infty d\nu\nu F_{\rm ren}(\nu) + \ln\epsilon^2\right) \tag{7.1}$$

are shown in Fig. 4. The fluctuation prefactor $\mathcal{D}^{-1/2}$ (including the counterterm action) is given by $\exp(-S_{\rm eff}^{\rm ren})$. Because of the subtraction of the zero-mode contribution $\ln\epsilon^2$ it has dimension $({\rm length})^{-2}$. Since we have used in our computation the dimensionless variable $M_W r$, the units for the rate are M_W^2 (the action and therefore the zero-mode prefactor being dimensionless). As mentioned in the previous section we estimate the error of our numerical result for $S_{\rm eff}$ to be of the order of 0.07 units.

In contrast to an analoguos computation of the fluctuation prefactor for the sphaleron transition in the electroweak theory the effects of the quantum fluctuations on the transition rate remain quite small here, less than a factor of 2. This could have been expected on the grounds that the number of fluctuating fields is small; effectively, in view of the cancellation between gauge field and Faddeev-Popov degrees of freedom, we have just the physical and the would-be Goldstone part of the Higgs field. Furthermore the dimension of space is reduced from three to two. Nevertheless we think that this expectation had to be checked by a direct computation.

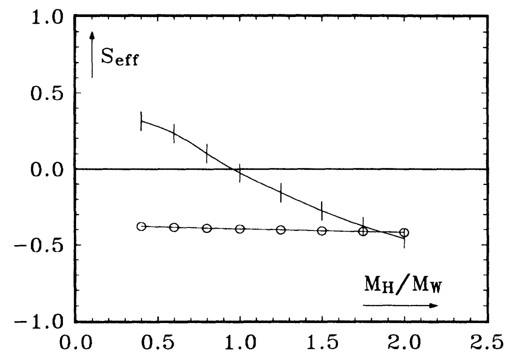

FIG. 4. The one-loop effective action. The vertical lines are the numerical results for the bosonic effective action $S_{\rm eff}^{\rm one\ loop}$, the length of the lines indicate the error; the empty squares are the effective action $S_{\rm eff}^{\rm ferm}$ for massless fermions given by Eq. (2.15). The curves are spline fits.

One cannot compare the classical and quantum action without specifying the dimensionless vacuum expectation value $v = M_W/g^2$. If $v \simeq 1$, the classical action has a value of order π. This has to be considered as an absolute lower limit if one wants to justify the dilute instanton gas approximation. The fact that the one-loop correction is then 1 order of magnitude smaller supports the use of the semiclassical approximation. It would be interesting to compare it to lattice simulations.

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