Dirac versus reduced quantization of the Poincaré symmetry in scalar electrodynamics

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The generators of the Poincaré symmetry of scalar electrodynamics are quantized in the functional Schrödinger representation. We show that the factor ordering which corresponds to (minimal) Dirac quantization preserves the Poincaré algebra, but (minimal) reduced quantization does not. In the latter, there is a van Hove anomaly in the boost-boost commutator, which we evaluate explicitly to lowest order in a heat kernel expansion using ζ-function regularization. We illuminate the crucial role played by the gauge orbit volume element in the analysis. Our results demonstrate that preservation of extra symmetries at the quantum level is sometimes a useful criterion to select between inequivalent, but nevertheless self-consistent, quantization schemes.

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I. INTRODUCTION

There has been a long standing debate in the literature concerning the Dirac versus reduced quantization of gauge theories [1–6]. In Dirac quantization, one constructs quantum operators on the full space of fields prior to reducing to the physical degrees of freedom. The gauge constraints are then realized as operator constraints on physical states. Reduced quantization, on the other hand, as the name suggests, constructs quantum operators for physical observables only. Dirac quantization is simpler in the sense that the full space of fields is usually endowed with a flat configuration space metric. It has the disadvantage, however, of including supposedly unphysical information into the quantization scheme. It is well known that these two approaches to quantization generally lead to distinct quantum systems [1–6], and that the difference can be understood as a factor ordering ambiguity involving the volume element on the gauge orbits [6]. Now it can happen that both approaches are self-consistent, and so even though the respective Hamiltonians may yield different spectra, there is no *internal* criterion with which to select the correct factor ordering. This has been illustrated by Kuchař [3] using a finite-dimensional model, which we will refer to as the helix model.

The purpose of the present paper is to examine in detail the quantization of a field-theoretic version of Kuchař's helix model, namely scalar electrodynamics in flat spacetime. An important distinction between the two models in the present context is that scalar electrodynamics contains a symmetry not present in the helix model: Poincaré covariance. Our principal contribution is to show that Poincaré covariance at the quantum level is sensitive to this factor ordering ambiguity, and provides a suitable internal criterion: minimal[1] Dirac quantization passes, whereas minimal reduced quantization fails.

The paper is organized as follows. In Sec. II we review both the Lagrangian and Hamiltonian analyses of scalar electrodynamics, chiefly to introduce notation. This is followed in the next section by a discussion of the classical Poincaré symmetry: we write down the classical Poincaré charges on the full phase space, and verify that these generate the Poincaré algebra, up to "off-shell" pieces which vanish on the constraint surface. Then in Sec. IV we quantize the Poincaré generators in the functional Schrödinger representation and show that minimal Dirac quantization preserves the Poincaré symmetry when acting on physical states. This involves showing that all potential van Hove anomalies [7–9] vanish, and that the off-shell pieces mentioned above annihilate physical states when quantized. In Sec. V we turn to minimal reduced quantization, and demonstrate that it does *not* preserve the Poincaré symmetry: there exists a van Hove anomaly in, for example, the boost-boost commutator. This calculation involves ζ-function regularization (via heat kernel techniques) [10,11] of (the log of) the gauge orbit volume element. Finally, in Sec. VI we compare both Dirac and reduced quantizations (acting on the same physical state space) to clarify why minimal Dirac quantization succeeds, while minimal reduced does not. It is clear that the volume element on the gauge orbits plays an important role.

II. SCALAR ELECTRODYNAMICS

The Lagrangian density for scalar electrodynamics in flat spacetime is

[1]This term is defined later.

$$\mathcal{L} = \tfrac{1}{2}(D_\mu\varphi)(\overline{D^\mu\varphi}) - U - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{1}$$

where we use spacetime signature $(+---)$, and the indices μ,ν run from 0 to 3. $\varphi := \xi + i\eta$ is a complex scalar field and U is a potential, for example, a mass or self-interaction term, which depends only on $|\varphi|$. The covariant derivative is $D_\mu := \partial_\mu + ieA_\mu$, with corresponding electromagnetic field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

Fixing an inertial frame, and using DeWitt's condensed notation, the Lagrangian $L(t) := \int d^3x\, \mathcal{L}(t,\mathbf{x})$ can be cast into the form

$$L(\lambda, Q, \dot{Q}) = \tfrac{1}{2}G_{AB}(Q)[\dot{Q}^A - \lambda^\alpha\phi^A_\alpha(Q)] \times[\dot{Q}^B - \lambda^\beta\phi^B_\beta(Q)] - V(Q) \ , \tag{2}$$

In the above, the configuration of the system at time instant t is represented by a point in the configuration space, M, with global coordinates

$$Q^A := (A_i(\mathbf{x}), \xi(\mathbf{x}), \eta(\mathbf{x})) \ , \tag{3}$$

where the index A runs over discrete values (including the spatial index $i = 1,2,3$), as well as the continuum $\alpha := \mathbf{x} \in \mathbf{R}^3$. Repeated indices imply summation and/or integration, as appropriate. The overdot on the velocities $\dot{Q}^A$ indicates time derivative, but the time argument has been suppressed. The timelike part of the vector potential plays the role of a Lagrange multiplier: $\lambda^\alpha := -eA_0(\mathbf{x})$. The kinetic energy term in the Lagrangian induces a flat metric on M, with components

$$G_{AB}(Q) := \begin{pmatrix} \delta^{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \delta(\mathbf{x}-\mathbf{y}) \tag{4}$$

in the Cartesian coordinates Q^A. The potential term in L is

$$V(Q) : = \int d^3x\{\tfrac{1}{4}(F_{ij})^2 + \tfrac{1}{2}(\partial_i\xi - eA_i\eta)^2 + \tfrac{1}{2}(\partial_i\eta + eA_i\xi)^2 + U\} \ . \tag{5}$$

Gauge transformations on M are generated by the "gauge vector fields" $\phi_\alpha = \phi^A_\alpha\partial/\partial Q^A$, whose components in the Cartesian coordinates are

$$\phi^A_\beta(Q) = \left(-\frac{1}{e}\partial_{x^i}\delta(\mathbf{x}-\mathbf{y}), -\eta(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \xi(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\right) \ . \tag{6}$$

The gauge vector fields are linearly independent (except on the subspace $\xi = \eta = 0$), and their Lie bracket algebra

$$[\phi_\alpha, \phi_\beta] = f^\gamma_{\alpha\beta}\phi_\gamma = 0 \tag{7}$$

is, of course, Abelian.

The phase space $\Gamma = T^*M$. A straightforward Hamiltonian analysis (see, e.g., [12]) yields the canonical Hamiltonian

$$H(\lambda, Q, P) = \tfrac{1}{2}G^{AB}(Q)P_AP_B + V(Q) + \lambda^\alpha C_\alpha(Q,P) \ , \tag{8}$$

in which G^{AB} denotes the matrix inverse of G_{AB}. The momenta P_A, conjugate to Q^A, are (in uncondensed notation)

$$\Pi_{A_i(\mathbf{x})} = \frac{\delta L}{\delta \dot{A}_i(\mathbf{x})} = \dot{A}_i(\mathbf{x}) - \partial_{x^i}A_0(\mathbf{x}) = F_{0i}(\mathbf{x}) \ , \tag{9}$$

$$\Pi_{\xi(\mathbf{x})} = \frac{\delta L}{\delta\dot{\xi}(\mathbf{x})} = \dot{\xi}(\mathbf{x}) - eA_0(\mathbf{x})\eta(\mathbf{x}) \ , \tag{10}$$

$$\Pi_{\eta(\mathbf{x})} = \frac{\delta L}{\delta\dot{\eta}(\mathbf{x})} = \dot{\eta}(\mathbf{x}) + eA_0(\mathbf{x})\xi(\mathbf{x}) \ , \tag{11}$$

where $\delta/\delta\dot{A}_i(\mathbf{x})$, etc., denotes functional derivative.

The Lagrange multipliers λ^α enforce the Gauss law constraints

$$C_\alpha(Q,P) := \phi^B_\alpha(Q)P_B = \frac{1}{e}\partial_{x^i}\Pi_{A_i(\mathbf{x})} - \eta(\mathbf{x})\Pi_{\xi(\mathbf{x})} + \xi(\mathbf{x})\Pi_{\eta(\mathbf{x})} \approx 0 \ , \tag{12}$$

which defines a constraint surface $\Gamma_C \subset \Gamma$. It turns out that the ϕ_α are Killing vectors and that the potential V is constant along the gauge orbits in M, conditions which, together with (7), are sufficient to guarantee that Γ_C is invariant under time evolution.

III. CLASSICAL POINCARÉ SYMMETRY

Integrating the Lagrangian density $\mathcal{L}$ in (1) over spacetime gives the action, whose functional derivative with respect to the spacetime metric (evaluated at the flat metric $\eta_{\mu\nu}$) yields the symmetric and conserved energy-momentum tensor

$$T^{\mu\nu} = (D^{(\mu}\varphi)(\overline{D^{\nu)}\varphi}) - F^{\rho\mu}F_\rho{}^\nu - \eta^{\mu\nu}\mathcal{L} \ , \tag{13}$$

where indices in parentheses denotes symmetrization. Together with the Poincaré group of isometries, this then leads to the conserved Poincaré charges

$$\mathcal{P}^\mu := \int d^3x T^{0\mu} \ , \tag{14}$$

$$\mathcal{J}^{\mu\nu} := \int d^3x(x^\mu T^{0\nu} - x^\nu T^{0\mu}) \ . \tag{15}$$

As we shall see shortly, as dynamical variables on Γ these constants of the motion canonically generate transformations on the classical states which realize the Poincaré algebra, at least when acting on the constraint surface Γ_C.

We find, in terms of the phase space variables,

$$T^{00} = \tfrac{1}{2}(\Pi^2_{A_i} + \Pi^2_\xi + \Pi^2_\eta) + \mathcal{V}(A_i, \xi, \eta) \ , \tag{16}$$

$$T^{0i} = -\Pi_{A_j} F_{ij} - \Pi_\xi(\partial_i\xi - eA_i\eta) -\Pi_\eta(\partial_i\eta + eA_i\xi) \ , \tag{17}$$

where $\mathcal{V}(A_i, \xi, \eta)$ is the integrand in (5). Thus the generator of time translation

$$\mathcal{P}_0 = \mathcal{C}_2(\tfrac{1}{2}G^{-1}) + \mathcal{C}_0(V) \tag{18}$$

in condensed notation, where

$$\mathcal{C}_s(S) := S^{A_i \cdots A_s}(Q) P_{A_1} \cdots P_{A_s}$$

denotes the homogeneous classical dynamical variable on Γ associated with a symmetric contravariant valence s tensor field S on M [cf. (8)].

After an integration by parts the spatial translation generators turn out to be

$$\mathcal{P}^k = \int d^3x\{-\Pi_{A_l}\partial_k A_l - \Pi_\xi\partial_k\xi - \Pi_\eta\partial_k\eta - eA_k C_\alpha\} =: \mathcal{C}_1({}^kX) \ , \tag{19}$$

where we read off the vector field components

$${}^kX^A(Q) = (-\partial_{x^k}A_i(\mathbf{x}), -\partial_{x^k}\xi(\mathbf{x}), -\partial_{x^k}\eta(\mathbf{x})) -e\int d^3z A_k(\mathbf{z})\phi^A_\gamma(Q) \ . \tag{20}$$

Here $\gamma := \mathbf{z}$ is included in the integration. Similarly, the spatial rotation generators

$$\begin{aligned}\mathcal{J}^k :&= \tfrac{1}{2}[kmn]\mathcal{J}^{mn} \\ &= \int d^3x[kmn]\{x^m(-\Pi_{A_i}\partial_n A_i - \Pi_\xi\partial_n\xi -\Pi_\eta\partial_n\eta - eA_nC_\alpha) - \Pi_{A_m}A_n\} \\ &=: \mathcal{C}_1({}^kY) \ ,\end{aligned} \tag{21}$$

where $[kmn]$ is the completely antisymmetric symbol in three dimensions, with [123]=1, and

$${}^kY^A(Q) = [kmn](-x^m\partial_{x^n}A_i(\mathbf{x}) - \delta^m_i A_n(\mathbf{x}), -x^m\partial_{x^n}\xi(\mathbf{x}), -x^m\partial_{x^n}\eta(\mathbf{x})) -e\int d^3z[kmn]z^m A_n(\mathbf{z})\phi^A_\gamma(Q) \ . \tag{22}$$

Finally, the boost generators

$$\mathcal{K}^k := \mathcal{J}^{0k} = -\mathcal{C}_2(\tfrac{1}{2}{}^kK) - \mathcal{C}_0({}^kV) + t\mathcal{P}^k \ , \tag{23}$$

where the boost tensors

$${}^kK^{AB}(Q) = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tfrac{1}{2}(x^k + y^k)\delta(\mathbf{x}-\mathbf{y}) \ , \tag{24}$$

and the boost potentials

$${}^kV(Q) = \int d^3x\, x^k \mathcal{V}(A_i, \xi, \eta) \ , \tag{25}$$

which are analogous to the potential $V(Q)$ in the Hamiltonian.

For future reference we record here some properties of the various tensors associated with the Poincaré charges. First, the Lie derivative with respect to ϕ_α of every valence zero, one, and two tensor that occurs in $\mathcal{P}^0, \mathcal{P}^k, \mathcal{J}^k, \mathcal{K}^k$ vanishes.[2] This is sufficient (but not necessary) to guarantee that the Poincaré charges are classical observables, that is, gauge invariant on the constraint surface Γ_C. Next, the boost tensors ${}^kK$ have field independent components in the Cartesian coordinates [see (24)], and so are, in fact, covariantly constant [cf. (4)].[3] Finally, the spatial translation and rotation vectors ${}^kX$ and ${}^kY$, while not Killing, are nevertheless divergence-free:

$$\begin{aligned}\nabla\cdot {}^kX &= -\tfrac{1}{2}G_{AB}(\mathcal{L}_{{}^kX}G)^{AB} \\ &= \int d^3x\, d^3y\, \delta(\mathbf{x}-\mathbf{y})\partial_{x^k}\delta(\mathbf{x}-\mathbf{y}) = 0\end{aligned} \tag{26}$$

since $\partial_x\delta(x-y)$ is antisymmetric; similarly for ${}^kY$. As we shall see later, these results considerably simplify the Dirac quantization of the Poincaré algebra.

But first we must work out the algebra at the classical level. In terms of our previous notation the Poisson brackets can be expressed as

$$\{\mathcal{C}_s(S), \mathcal{C}_t(T)\} = \mathcal{C}_{s+t-1}(-[\![S,T]\!]) \ , \tag{27}$$

where $[\![S,T]\!]$ is the Schouten concomitant [14] of S and T. A straightforward but lengthy calculation (see footnote 2) yields

$$\{\mathcal{P}^\mu, \mathcal{P}^\nu\} = 0 - e\int d^3z F^{\mu\nu}C_\gamma \ , \tag{28}$$

$$\{\mathcal{J}^{\mu\nu}, \mathcal{P}^\rho\} = \eta^{\nu\rho}\mathcal{P}^\mu - \eta^{\mu\rho}\mathcal{P}^\nu - e\int d^3z(z^\mu F^{\nu\rho} - z^\nu F^{\mu\rho})C_\gamma \ , \tag{29}$$

[2] For the explicit calculation refer to [13].

[3] This implies they are Killing, and in involution, which, modulo terms that vanish on Γ_C, is necessary for the Poincaré algebra to close.

$$\{\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}\} = \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} - \eta^{\nu\sigma}\mathcal{J}^{\mu\rho} + \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} + e\int d^3z(z^\mu z^\sigma F^{\nu\rho} - z^\nu z^\sigma F^{\mu\rho} + z^\nu z^\rho F^{\mu\sigma} - z^\mu z^\rho F^{\nu\sigma})C_\gamma \ . \tag{30}$$

Since $C_\gamma \approx 0$ defines the constraint surface, we see that we have explicitly verified the classical Poincaré algebra for scalar electrodynamics; we are not aware of any similar calculation in the literature. Notice that since $F^{0j}(\mathbf{z}) = -\Pi_{A_j(\mathbf{z})}$, the "off-shell pieces," which are linear combinations of the constraints, are linear and/or quadratic in the momenta, and these must be dealt with accordingly when we do the Dirac quantization.

IV. DIRAC QUANTIZATION OF THE POINCARÉ SYMMETRY

We now proceed with Dirac quantization of the Poincaré charges, and subsequent verification of the Poincaré symmetry at the quantum level. The phase space $\Gamma = T^*M$, and, since M comes equipped with a (flat) metric G, it is natural to choose Schrödinger picture quantization with state space $\mathcal{F}$ consisting of all smooth complex-valued functions on M, and Hilbert space $\mathcal{H}_{\rm dir} := L^2(M,\mathbf{E})$, the subset of those which are square integrable with respect to the volume form $\mathbf{E}$ associated with G. The quantization map is not unique: here we shall choose the simplest one:

$$\mathcal{C}_0(V) = V \mapsto \mathcal{Q}_0(V) = V \ , \tag{31}$$

$$\mathcal{C}_1(X) = X^A P_A \mapsto \mathcal{Q}_1(X) = -i\hbar\{X^A\nabla_A + \tfrac{1}{2}(\nabla_A X^A)\} \ , \tag{32}$$

$$\begin{aligned}\mathcal{C}_2(K) = K^{AB}P_AP_B &\mapsto \mathcal{Q}_2(K)\\ &= (-i\hbar)^2\{K^{AB}\nabla_A\nabla_B + (\nabla_A K^{AB})\nabla_B\}\\ &= (-i\hbar)^2\nabla_A K^{AB}\nabla_B \ ,\end{aligned} \tag{33}$$

$$\mathcal{C}_3(T) = T^{ABC}P_AP_BP_C \mapsto \mathcal{Q}_3(T) = (-i\hbar)^3\{T^{ABC}\nabla_A\nabla_B\nabla_C + \tfrac{3}{2}(\nabla_A T^{ABC})\nabla_B\nabla_C - \tfrac{1}{4}(\nabla_A\nabla_B\nabla_C T^{ABC})\} \ , \tag{34}$$

where, as before, ∇ is the Levi-Civita connection on M. This will be called "minimal" quantization in that given the leading term (highest order in derivatives) the additional complementary terms are the minimum ones necessary to make the operator self-adjoint.[4]

[4]Of course one may add various additional terms to the right-hand side of (33) or (34) and still preserve self-adjointness. However, as it pertains to (34), this ambiguity is irrelevant to our result, and a specific choice is made just for notational definiteness: The only place a cubic operator appears is in (46) [and also (57)], but there the right-hand side is defined by the left-hand side, so if we change our definition of $\mathcal{Q}_3$ the "van Hove term" simply changes accordingly—the invariant result is that the left-hand side annihilates physical states, as we shall see (in order that the Poincaré algebra close). However, this ambiguity is of great interest in (33): alone, the self-adjoint requirement fixes the quadratic operator only up to the addition of a real scalar "potential" term (which should be linear and homogeneous in the classical observable). Minimal quantization (both Dirac and reduced) sets this term to zero, by fiat, but when acting in the physical Hilbert space the minimal Dirac operator actually looks like the minimal reduced operator, except with a certain additional potential term which depends on the gauge orbit volume element (see, e.g., [15]). The point here is to demonstrate that the Poincaré algebra is sensitive to this difference. Also, notice that the notion of minimality is more natural in Dirac quantization where the (full) configuration space, unlike the reduced configuration space, is flat in our example, and so the usual curvature terms, at least, are absent (see, e.g., [15]).

In the spirit of Dirac [16] we now account for the constraints by quantizing them—on the same footing as any other observable linear in the momenta: $\hat{C}_\alpha = \mathcal{Q}_1(\phi_\alpha)$. The physical state space, $\mathcal{F}_{\rm phys} \subset \mathcal{F}$, is then defined as the collection of states $\Psi_{\rm phys}$ annihilated by the constraint operators:

$$\hat{C}_\alpha\Psi_{\rm phys} = 0 \ \forall\alpha \Leftrightarrow \Psi_{\rm phys} \in \mathcal{F}_{\rm phys} \ . \tag{35}$$

As emphasized by Kuchař [2], the choice of basis for the gauge vectors ϕ_α is arbitrary at the classical level, but that this breaks down at the quantum level, at least if one demands that the constraint operators be self-adjoint. The trouble lies in the complementary divergence term in (32), but can be eliminated by restricting to a preferred basis which is "compatible" with the Hilbert space structure:

$$\mathcal{L}_{\phi_\alpha}\mathbf{E} = 0 \ \forall\alpha \ , \tag{36}$$

i.e., in which the ϕ_α are divergence-free. This restriction is natural in the sense that (35) then implies $\phi_\alpha\Psi_{\rm phys} = 0 \ \forall\alpha$; i.e., $\mathcal{F}_{\rm phys}$ consists of gauge-invariant complex-valued functions on M.

Furthermore, given such a basis one is free to transform to any other basis whose elements are all divergence-free, i.e., taken from the set

$$\mathcal{G} := \{\mu = \mu^\alpha\phi_\alpha | \nabla\cdot\mu = 0 \Leftrightarrow \phi_\alpha\mu^\alpha = 0\} \ . \tag{37}$$

In our case the Lagrangian provides a natural basis of ϕ_α which are Killing, and so certainly satisfy (36). Furthermore, it can be shown that constraint operators constructed from elements of $\mathcal{G}$ will be consistent (first class) iff $\phi_\gamma f^\gamma_{\alpha\beta} = 0$, or, equivalently, $f^\gamma_{\gamma\beta} = \phi_\beta f$ for some scalar f. In our case the structure functions $f^\gamma_{\alpha\beta}$ vanish [see (7)], and so this condition is trivially satisfied.

The Poincaré charges $\mathcal{P}^0, \mathcal{P}^k, \mathcal{J}^k, \mathcal{K}^k$ contain pieces zero-, first-, and second-order in the momenta, and are quantized accordingly using the minimal quantization scheme. In order for the resulting quantum Poincaré charges to be observables they must commute with the constraint operators $\mathcal{Q}_1(\mu)$ (at least on $\mathcal{F}_{\text{phys}}$) for all $\mu \in \mathcal{G}$. Let us check if this is so.

Let the scalar U represent any of the Hamiltonian or boost potentials V or ${}^kV$; we have

$$\frac{1}{i\hbar}[\mathcal{Q}_0(U), \mathcal{Q}_1(\mu)] = \mathcal{Q}_0(-[\![U,\mu]\!]) \ . \qquad (38)$$

But $-[\![U,\mu]\!] = \mathcal{L}_\mu U$ vanishes since the potentials are constant along the gauge orbits. For a vector Z, representing any of the spatial translation or rotation vectors ${}^kX$ or ${}^kY$, we find

$$\frac{1}{i\hbar}[\mathcal{Q}_1(Z), \mathcal{Q}_1(\mu)] = \mathcal{Q}_1(-[\![Z,\mu]\!]) \ , \qquad (39)$$

where $-[\![Z,\mu]\!] = \mathcal{L}_\mu Z$. Using the fact that $\mathcal{L}_{\phi_\alpha} Z = 0$ in our case it is easy to show that $\mathcal{L}_\mu Z \in \mathcal{G}$, so the right-hand side of (39) annihilates Ψ_{phys}. Finally, letting K stand for either the inverse metric G^{-1} or any of the boost tensors ${}^kK$, we have

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_1(\mu)] = \mathcal{Q}_2(-[\![K,\mu]\!]) + \hbar^2 \mathcal{Q}_0(W) \ , \qquad (40)$$

$$W = \tfrac{1}{2}\nabla_A[K^{AB}\nabla_B(\nabla\cdot\mu)] \ . \qquad (41)$$

The $\hbar^2\mathcal{Q}_0(W)$ term is a van Hove anomaly (discussed more fully below), which in this case vanishes precisely because of the restriction (37). Furthermore,

$$-[\![K,\mu]\!] = \mathcal{L}_\mu K = \mu^\alpha \mathcal{L}_{\phi_\alpha} K - 2\psi^\alpha \otimes_S \phi^\alpha \ , \qquad (42)$$

where $\otimes_S$ denotes a symmetrized tensor product. The first term on the right-hand side vanishes, and the Cartesian components of the vector fields ψ^α are $\psi^{\alpha A} = K^{AB}\nabla_B\mu^\alpha$. Quantizing the remaining term yields an operator proportional to

$$\nabla_A \psi^{\alpha A}\phi^B_\alpha \nabla_B + \nabla_A \phi^A_\alpha \psi^{\alpha B}\nabla_B \ . \qquad (43)$$

The first term annihilates Ψ_{phys}, and the second is equivalent to

$$(\nabla\cdot\phi_\alpha)\psi^\alpha + \phi_\alpha\psi^\alpha = [\phi_\alpha, \psi^\alpha] + \psi^\alpha\phi_\alpha \ . \qquad (44)$$

Now the second term on the right-hand side of this expression annihilates Ψ_{phys}, and, furthermore, the commutator vanishes:

$$(\mathcal{L}_{\phi_\alpha}\psi^\alpha)^A = (\mathcal{L}_{\phi_\alpha}K)^{AB}\nabla_B\mu^\alpha + K^{AB}(\mathcal{L}_{\phi_\alpha}\nabla\mu^\alpha)_B = 0 \ . \qquad (45)$$

Thus, the quantum Poincaré charges are observables.

The next question to ask is whether or not they realize the Poincaré algebra when acting on physical states. There are two considerations: van Hove anomalies, and whether or not the minimal quantization of the off-shell pieces in (28)–(30) produces operators which annihilate physical states. We discuss these in turn.

Since the work of Groenewold [8] and van Hove [7] it has been known that no map from classical to quantum observables exists which preserves the entire Poisson algebra.[5] For the minimal quantization map given in (31)–(34) van Hove anomalies appear first in the quadratic-linear commutator: refer to (41), which applies for generic K, and μ replaced by a generic vector field, Z. But the only vector fields occurring in the Poincaré charges are the spatial translation and rotation vectors ${}^kX$ and ${}^kY$, which are both divergence-free [see (26)], and so this particular van Hove anomaly is not present.

For the generic quadratic-quadratic commutator,

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_2(L)] = \mathcal{Q}_3(-[\![K,L]\!]) + \hbar^2\mathcal{Q}_1(Z) \ , \qquad (46)$$

with

$$Z^D = \tfrac{1}{2}\nabla_B\nabla_C[\![K,L]\!]^{BCD} - \nabla_B A^{BD} \ ,$$

$$\begin{aligned} A^{BD} = {} & K^{AB}L^{CD}\mathcal{R}_{AC} + (\nabla_C K^{AB})(\nabla_A L^{CD}) \\ & -\tfrac{1}{3}\nabla_C[K^{AB}(\nabla_A L^{CD}) - K^{AD}(\nabla_A L^{CB})] \\ & -(K \leftrightarrow L) \ . \end{aligned} \qquad (47)$$

For our example, the van Hove term $\hbar^2\mathcal{Q}_1(Z)$ vanishes because the Ricci tensor $\mathcal{R}_{AB}$ is zero (M is flat) and the inverse metric and boost tensors are all covariantly constant. Thus the Poincaré algebra is free of van Hove anomalies under minimal Dirac quantization.

We now come to the quantization of the off-shell pieces in (28)–(30), of which there are essentially only two types. The first type has the form

$$-e\int d^3z F^{ij} C_\gamma =: \mathcal{C}_1(\mu^\gamma\phi_\gamma) \ , \qquad (48)$$

where (with $\gamma := \mathbf{z}$) the scalars

$$\mu^\gamma := -eF^{ij}(\mathbf{z}) \ . \qquad (49)$$

But since the electromagnetic field strength is gauge invariant, we certainly have $\phi_\gamma\mu^\gamma = 0$, so $\mu^\gamma\phi_\gamma \in \mathcal{G}$, in which case $\mathcal{Q}_1(\mu^\gamma\phi_\gamma)\Psi_{\text{phys}} = 0$.

The second type has the form

$$-e\int d^3z\, F^{0j} C_\gamma =: \mathcal{C}_2(\psi^\gamma \otimes_S \phi_\gamma) \ , \qquad (50)$$

where the vector field components

$$\psi^{\gamma A} = (e\delta^i_j\delta(\mathbf{z}-\mathbf{x}), 0, 0) \ . \qquad (51)$$

[5] See, e.g., [9] for a more precise statement.

The quantization of this second type of term is exactly analogous to the discussion following (42), except the Lie derivative corresponding to (45) is

$$(\mathcal{L}_{\phi\gamma}\psi^{\gamma})^A = -\int d^3z\, d^3y \left\{ e\delta(\mathbf{z}-\mathbf{y})\frac{\delta}{\delta A_j(\mathbf{y})} \right\} \phi^A_{\gamma}(Q) \ , \tag{52}$$

which vanishes since $\phi^A_\gamma(Q)$ has no dependence on the field $A_j(\mathbf{y})$ (and note that $\psi^{\gamma A}$ has no field dependence at all). Thus the off-shell pieces do not affect the Poincaré algebra at the quantum level.

In conclusion we see that the minimal Dirac quantization of the classical Poincaré charges realizes the Poincaré algebra when acting on $\mathcal{F}_{\rm phys}$.[6] We now turn to reduced quantization.

V. REDUCED QUANTIZATION OF THE POINCARÉ SYMMETRY

Classical reduction is readily achieved by choosing the complete set of gauge invariant functions:

$$q^a = (B_i(\mathbf{x}), \rho(\mathbf{x})) \ , \tag{53}$$

where $B_i(\mathbf{x}) := A_i(\mathbf{x}) + (1/e)\partial_{x^i}\theta(\mathbf{x})$ and $\varphi(\mathbf{x}) =: \rho(\mathbf{x})\exp[i\theta(\mathbf{x})]$, as coordinates on the reduced configuration space m. Since the constraints are linear in the momenta, the reduced phase space $\gamma = T^*m$, with canonical coordinates (q^a, p_a). An observable $\mathcal{C}_s(S)$ on Γ maps to the corresponding physical variable $c_s(s) := s^{a_1\cdots a_s}(q)p_{a_1}\cdots p_{a_s}$ on γ, where the tensor s on m is the (physical) projection of S.

In particular, the projected inverse metric

$$g^{ab}(q) = \begin{pmatrix} \delta_{ij} - \partial_{x^i}\frac{1}{e^2\rho^2(\mathbf{x})}\partial_{x^j} & 0 \\ 0 & 1 \end{pmatrix} \delta(\mathbf{x}-\mathbf{y}) \ . \tag{54}$$

The other tensors involved in the Poincaré charges can similarly be projected onto m, and the resulting reduced Poincaré charges on γ will obviously realize the Poincaré algebra classically [cf. (28)–(30) with $C_\gamma = 0$], a fact which can be verified by direct calculation. For the purpose of our discussion it is sufficient to know only the projected boost tensors:

$$^m k^{ab}(q) = \begin{pmatrix} \delta_{ij}x^m - \partial_{x^i}\frac{x^m}{e^2\rho^2(\mathbf{x})}\partial_{x^j} & 0 \\ 0 & x^m \end{pmatrix} \delta(\mathbf{x}-\mathbf{y}) \ . \tag{55}$$

We remark that these are Killing tensors, and are in involution with each other: $[\![^m k, {}^n k]\!] = 0$. Also it can be shown that the projected spatial rotation and boost vectors are Killing with respect to the metric g on m, and so necessarily are (Levi-Civita) divergence-free.

We now quantize the reduced Poincaré charges, and attempt to verify the Poincaré symmetry at the quantum level. In analogy with the Dirac quantization considered earlier, we choose the Schrödinger picture quantization with Hilbert space $\mathcal{H}_{\rm red} := L^2(m, \mathbf{e})$, where $\mathbf{e}$ is the volume form associated with the metric g on m. As was the case with Dirac quantization, the choice of quantization map is not unique—especially now that the configuration space is not flat (see [12], and references therein).

But in order to compare Dirac and reduced quantization on an "equal footing," we again choose minimal quantization [the reduced space analogue of (31)–(34)], which is also in keeping with tradition in the Dirac versus reduced quantization debate in the literature [1–6]. In particular, for a physical variable quadratic in the momenta,

$$\begin{aligned} c_2(k) = k^{ab}p_ap_b \mapsto q_2(k) &:= (-i\hbar)^2\tilde{\nabla}_a k^{ab}\tilde{\nabla}_b \\ &= (-i\hbar)^2\{\partial_a k^{ab}\partial_b + k^{ab}(\partial_a \ln\omega)\partial_b\} \ , \end{aligned} \tag{56}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on m, ∂_a is the (functional) derivative with respect to q^a, and $\omega = \sqrt{\det g_{ab}}$ is the measure on m in the coordinates q^a.

Now, as noted above, the classical Poincaré charges realize the Poincaré algebra, and this will automatically extend to the quantum level provided there are no van Hove anomalies. The first place such an anomaly might arise, with quadratic-linear commutators [see (41) with $\nabla \mapsto \tilde{\nabla}$, etc.], it does not, since the projected translation and rotation vectors are divergence-free, as mentioned earlier. This leaves a potential anomaly only with quadratic-quadratic commutators [cf. (47)]. However, instead of dealing with covariant derivatives,[7] and the Ricci tensor of the curved space, m, we calculate directly in terms of ω:

$$\begin{aligned} \frac{1}{(-i\hbar)^4}[q_2(k), q_2(l)] \\ = [\![k,l]\!]^{bcd}\partial_b\partial_c\partial_d \\ +\tfrac{3}{2}\{(\partial_b[\![k,l]\!]^{bcd}) + (\partial_b\ln\omega)[\![k,l]\!]^{bcd}\}\partial_c\partial_d \\ +\{k^{ab}\partial_a(\partial_b(v^d)) + u^b\partial_b(v^d) - (k\leftrightarrow l)\}\partial_d \ , \end{aligned} \tag{57}$$

for generic k and l, where the "vectors"

$$u^b = \partial_a k^{ab} + (\partial_a\ln\omega)k^{ab} \ , \tag{58}$$

$$v^d = \partial_c l^{cd} + (\partial_c\ln\omega)l^{cd} \ . \tag{59}$$

If k and l represent any of either the inverse metric or boost tensors then $[\![k,l]\!]^{bcd} = 0$, as noted earlier—a fact

[6]Here we used the *physical* Poincaré charges to generate the algebra at the classical level; in fact, it can be shown that one can equally well use the *Noether* Poincaré charges, which differ from the physical ones by terms which vanish on shell. Furthermore, the minimal Dirac quantization of both sets of charges are identical when acting on physical states.

[7]Note that even if K is covariantly constant on M, its physical projection k on m need not be.

that must be true for the Poincaré algebra to close. Thus, for the quantum commutator to vanish, as it should, we require all components of the terms in curly brackets in the last line of (57), which we will denote as ζ^d, to be zero.

For instance, the (quadratic-quadratic part of) the boost-boost commutator corresponds to taking $k = {}^m k$, $l = {}^n k$, and we find, using (55), that only the $d = \rho(\mathbf{w})$ component of ζ is potentially nonvanishing:

$$\zeta^{\rho(\mathbf{w})} = \int d^3x\, d^3y\, y^m w^n \delta(\mathbf{x}-\mathbf{y}) \frac{\delta^3 \ln\omega}{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})\delta\rho(\mathbf{w})} + \int d^3y y^m w^n \frac{\delta \ln\omega}{\delta\rho(\mathbf{y})} \frac{\delta^2 \ln\omega}{\delta\rho(\mathbf{y})\delta\rho(\mathbf{w})} - (m \leftrightarrow n) \tag{60}$$

(in uncondensed notation).

In order to evaluate $\delta \ln\omega$ we first observe that [see (54)]

$$\det g^{ab} = \mathrm{Det}\left[\delta_{ij} - \partial_i \frac{1}{e^2\rho^2}\partial_j\right] = \mathrm{Det}\left[\frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)\right] , \tag{61}$$

where Det denotes functional determinant, and $\partial^2 := \partial_i\partial_i$. The last equality follows by decomposing the eigenvectors of the operator $\delta_{ij} - \cdots$ into their transverse and longitudinal parts, and examining the eigenvalues. Hence

$$\delta \ln\omega = -\frac{1}{2}\delta \ln \mathrm{Det}\left[\frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2+\rho^2\right)\right] = -\frac{1}{2}\delta\, \mathrm{Tr}\ln\left[\frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2+\rho^2\right)\right] = -\frac{1}{2}\delta\,\mathrm{Tr}\ln\frac{1}{\rho^2} - \frac{1}{2}\delta\,\mathrm{Tr}\ln\left(-\frac{1}{e^2}\partial^2+\rho^2\right) =: \delta\mathrm{I} + \delta\mathrm{II} , \tag{62}$$

where we have assumed that the functional trace Tr satisfies the usual cyclicity property, and that $\delta\,\mathrm{Tr}\ln A = \mathrm{Tr}A^{-1}\delta A$ for any operator A.

The first term is straightforward to evaluate:[8]

$$\delta\mathrm{I} = \int d^3x \delta\rho(\mathbf{x}) \left\{\frac{\delta(\mathbf{o})}{\rho(\mathbf{x})}\right\} , \tag{63}$$

but the second term is more difficult. Following Hawking's discussion [10] on ζ-function regularization we write

$$\delta\mathrm{II} = \frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int d^3x \int_0^\infty d\tau\, \tau^s \delta D\, K(\mathbf{x},\mathbf{x},\tau)\right]\Bigg|_{s=0} . \tag{64}$$

Here the positive definite operator

$$D := -\frac{1}{e^2}\partial^2 + \rho^2 + \epsilon , \tag{65}$$

where $\epsilon > 0$ is a regulating "mass" parameter. Its associated heat kernel $K(\mathbf{x},\mathbf{y},\tau)$ satisfies

$$\frac{\partial}{\partial\tau}K(\mathbf{x},\mathbf{y},\tau) + DK(\mathbf{x},\mathbf{y},\tau) = 0 , \tag{66}$$

with initial condition $K(\mathbf{x},\mathbf{y},0) = \delta(\mathbf{x}-\mathbf{y})$. D [and δD in (64)] act on the first argument of K. In our case δD is simply $2\rho\,\delta\rho$.

As in, e.g., [11] we factorize

$$K(\mathbf{x},\mathbf{y},\tau) =: K_0(\mathbf{x},\mathbf{y},\tau)\Lambda(\mathbf{x},\mathbf{y},\tau) \tag{67}$$

into a singular piece

$$K_0(\mathbf{x},\mathbf{y},\tau) = \left(\frac{e}{4\pi\tau}\right)^{3/2} \exp\left(-\frac{e^2|\mathbf{x}-\mathbf{y}|^2}{4\tau} - \epsilon\tau\right) , \tag{68}$$

which satisfies (66) with $\rho^2 \equiv 0$ and initial condition $K_0(\mathbf{x},\mathbf{y},0) = \delta(\mathbf{x}-\mathbf{y})$, and a regular piece, $\Lambda(\mathbf{x},\mathbf{y},\tau)$. The latter contains the ρ dependence of K, and satisfies

$$\frac{\partial}{\partial\tau}\Lambda(\mathbf{x},\mathbf{y},\tau) + \frac{1}{\tau}(x^i - y^i)\partial_{x^i}\Lambda(\mathbf{x},\mathbf{y},\tau) = -\left(-\frac{1}{e^2}\partial_x^2 + \rho^2(\mathbf{x})\right)\Lambda(\mathbf{x},\mathbf{y},\tau) , \tag{69}$$

with initial condition $\Lambda(\mathbf{x},\mathbf{x},0) = 1$. We now expand

$$\Lambda(\mathbf{x},\mathbf{y},\tau) = \sum_{n=0}^\infty a_n(\mathbf{x},\mathbf{y})\tau^n , \tag{70}$$

and find that the coefficients a_n satisfy the recursion relation

$$n a_n + (x^i - y^i)\partial_{x^i} a_n + \left(-\frac{1}{e^2}\partial_x^2 + \rho^2(\mathbf{x})\right) a_{n-1} = 0 , \quad n = 1,2,\ldots , \tag{71}$$

with $a_0(\mathbf{x},\mathbf{y}) = 1$. In the coincidence limit we obtain

$$a_1(\mathbf{x},\mathbf{x}) = -\rho^2(\mathbf{x}) , \tag{72}$$

$$a_2(\mathbf{x},\mathbf{x}) = \frac{1}{2}\left(-\frac{1}{3e^2}\partial_x^2 + \rho^2(\mathbf{x})\right)\rho^2(\mathbf{x}) , \tag{73}$$

in agreement with similar work in [17].

Substituting these results into (64) yields an expression valid for $\mathrm{Re}(s) > \frac{1}{2}$, which when analytically continued

[8]We are being somewhat cavalier about regularization, simply because it is difficult to do much better, but we believe that our final conclusions still carry sufficient weight to be of interest, as we shall argue.

to $s = 0$ gives

$$\delta\text{II} = \int d^3x\, \delta\rho(\mathbf{x}) \left\{ -\sum_{n=0}^{\infty} c_n \rho(\mathbf{x}) a_n(\mathbf{x},\mathbf{x}) \right\} , \quad (74)$$

where the coefficients

$$c_n = \left(\frac{e}{4\pi}\right)^{3/2} \frac{\Gamma(n-\frac{1}{2})}{\epsilon^{n-1/2}} . \quad (75)$$

Now although $a_n(\mathbf{x},\mathbf{x})$ effectively goes like $1/n!$ [see (71)], the series in (74) nevertheless diverges as $\epsilon \to 0$, and must therefore be treated as a formal expansion in powers of ϵ. (And note also that powers of e are associated with derivatives of ρ.)

Finally, working out the higher order variations of I and II, and using these results in (60), yields

$$\zeta^{\rho(\mathbf{w})} = \frac{2c_2}{3e^2} w^n \left\{ \frac{\delta \ln\omega}{\delta\rho} \rho(\partial_m \rho) + \left(\partial_m \frac{\delta \ln\omega}{\delta\rho}\right) \rho^2 \right\} - (m \leftrightarrow n) \quad (76)$$

(plus higher order terms). It is instructive to note that if $\delta \ln\omega/\delta\rho \propto \rho^k$, the term in braces vanishes iff $k = -1$, a situation which corresponds precisely to term I of $\ln\omega$ [see (63)]. Term II, on the other hand, contributes a polynomial with higher powers of ρ (and also derivatives of ρ): the leading order (in both ϵ and e) contribution is

$$\zeta^{\rho(\mathbf{w})} = \frac{1}{(12\pi)^2} \frac{e}{\epsilon} (w^n \partial_{w^m} - w^m \partial_{w^n}) \rho^3(\mathbf{w}) , \quad (77)$$

which does not vanish for generic ρ. Also note that higher order terms in the expansion (74) do not contain this combination of e and ϵ, so no cancellation of this piece is possible.

Thus, the boost-boost commutator (and hence the Poincaré algebra as a whole) fails to be realized at the quantum level using minimal reduced quantization.

VI. DISCUSSION

We have thus shown that minimal Dirac quantization acting in $\mathcal{F}_{\text{phys}}$ preserves the Poincaré symmetry of scalar electrodynamics, but that minimal reduced does not. To better understand how this comes about it is instructive to determine what the Dirac-quantized Poincaré charges look like acting on physical states, so they can be compared on the same footing with their reduced counterparts.

For instance, direct calculation using (4), (33), and (53) shows that the kinetic energy operator $\mathcal{Q}_2(\frac{1}{2}G^{-1})$, acting in $\mathcal{F}_{\text{phys}}$, is equivalent to $q_2(\frac{1}{2}g^{-1})$ [see (56)], except with $\partial_a \ln\omega$ replaced by an object we call, similarly, $\partial_a \ln\omega'$, whose only nonvanishing component is

$$\frac{\delta \ln\omega'}{\delta\rho(\mathbf{x})} = \frac{\delta(\mathbf{o})}{\rho(\mathbf{x})} . \quad (78)$$

In fact, the analogous statement applies for the entire set of Poincaré charges: minimal Dirac quantization (acting in $\mathcal{F}_{\text{phys}}$) is identical in *form* with minimal reduced quantization, except with $\partial_a \ln\omega'$ in place of $\partial_a \ln\omega$, a difference which corresponds to retaining only the first term δI in (62), for example. [Compare (78) with (63).]

This means, for instance, that the quadratic-quadratic commutator in minimal Dirac quantization has the same form as (57), but when applied to the boost-boost commutator is easily seen to yield $\zeta = 0$, i.e., no anomaly, as expected from the results of Sec. IV. We remark that, although the term δI in (63) contains $\delta(\mathbf{o})$, and so is not regulated, it is *common* to both the Dirac and reduced approaches, and the (independent) results of Sec. IV support the proposition that this term does not cause a problem with the Poincaré algebra. Rather, it is the additional term δII present in reduced quantization—in particular, those pieces involving *derivatives* of ρ, which begin to appear with the $n = 2$ term in (74)—that causes a van Hove anomaly.[9]

In fact, we observe that exp(−II) is nothing but the volume element, $\sqrt{\det\gamma_{\alpha\beta}}$, on the gauge orbits, where the metric

$$\gamma_{\alpha\beta} := G_{AB}\phi^A_\alpha \phi^B_\beta = \left(-\frac{1}{e^2}\partial_x^2 + \rho^2(\mathbf{x})\right)\delta(\mathbf{x}-\mathbf{y}) . \quad (79)$$

Now it must be emphasized that, in general, the minimal Dirac and minimal reduced quantization schemes are *not* equivalent[10] [1–6]: If one transforms to a common Hilbert space, the inequivalence manifests itself in the quadratic operators as a difference in factor ordering involving precisely the above volume element on the gauge orbits [6]. Minimal Dirac quantization corresponds to a certain "nonminimal" reduced quantization. Furthermore, for a given model it can happen that both the Dirac and reduced factor orderings are self-consistent—the relevant example here being Kuchař's helix model [3] (which is a finite dimensional analogue of scalar electrodynamics). So even though the Hamiltonians might have different spectra, which could, in principle, be measured, there may be no *internal* physical criterion with which to select the correct factor ordering, as happens in the helix model [3].

The significant point here is that scalar electrodynamics has an additional symmetry, the Poincaré symmetry, and, at the quantum level, this symmetry is *sensitive* to this difference in factor ordering (or presence of $\sqrt{\det\gamma_{\alpha\beta}}$), suggesting, in fact, that minimal Dirac quantization is correct, and minimal reduced is not (at least in this case). In other words, in response to [2], for instance, one *cannot* always impose the "principle of minimal coupling" in reduced quantization.

This result also supports previous work [18,15] suggesting a preference for minimal Dirac over minimal reduced because of the natural similarity of the former with several curved-space quantization schemes proposed in the

[9] This additional term δII should not be confused with the additional potential term mentioned in footnote 4.

[10] For example, the respective Hamiltonians have different spectra, in general.

literature.

Thus, demanding the preservation of a sufficiently nontrivial classical symmetry at the quantum level may serve as a useful internal physical criterion with which to select amongst inequivalent factor orderings, as we have demonstrated here. It might be interesting to generalize: to find a large (or the largest) class of symmetries preserved under minimal Dirac quantization, as well as its corresponding "nonminimal" reduced quantization, but *not* necessarily preserved by the minimal reduced quantization, and thus, perhaps, more clearly illuminate the role the gauge orbit volume element plays in this matter.

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[1] A. Ashtekar and G. T. Horowitz, Phys. Rev. D **26**, 3342 (1982).
[2] K. Kuchař, Phys. Rev. D **34**, 3031 (1986); **34**, 3044 (1986).
[3] K. Kuchař, Phys. Rev. D **35**, 596 (1987).
[4] J. Romano and R. Tate, Class. Quantum Grav. **6**, 1487 (1989).
[5] K. Schleich, Class. Quantum Grav. **7**, 1529 (1990).
[6] G. Kunstatter, Class. Quantum Grav. **9**, 1469 (1992).
[7] L. Van Hove, Mem. de l'Acad. Roy. de Belgique (Classe Sci.) **37**, 610 (1951).
[8] H.J. Groenewold, Physica **12**, 405 (1946).
[9] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd ed. (Benjamin/Cummings, New York, 1978).
[10] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[11] B. S. DeWitt, in *General Relativity, an Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[12] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).
[13] R. J. Epp, Ph.D. thesis, University of Manitoba (1993).
[14] A. Nijenhuis, Nederl. Akad. Wetensch. Proc. Ser. A **58**, 390 (1955).
[15] R. J. Epp, Phys. Rev. D **50**, 6578 (1994).
[16] P. A. M. Dirac, Can. J. Math. **2**, 125 (1956); *Lectures in Quantum Mechanics* (Academic, New York, 1956).
[17] R. Nepomechie, Phys. Rev. D **31**, 3291 (1985).
[18] R. J. Epp, Phys. Rev. D **50**, 6569 (1994).