PHYSICAL REVIEW D VOLUME 51, NUMBER 2 15 JANUARY 1995

Casimir effect of strongly interacting scalar fields

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(Received 28 April 1994)

The Casimir effect of nontrivial ϕ^4 theory is studied for a rectangular box. The scalar modes satisfy periodic boundary conditions, which correspond to a compactification of space. Nontrivial ϕ^4 theory is obtained by an analytic continuation of the theory to negative quartic coupling. This theory is studied in a renormalization-group-invariant approach. It is found that the Casimir energy exponentially approaches the infinite volume limit, the decay rate given by the scalar condensate. This behavior is very different from the power law of a free theory. This might provide experimental access to properties of the nontrivial vacuum. At small compactification lengths the system can no longer tolerate a scalar condensate, and a first order phase transition to the perturbative phase occurs. The dependence of the vacuum energy density and the scalar condensate on the box dimensions is presented.

PACS number(s): 11.10.Gh, 11.10.Ef

I. INTRODUCTION

The Casimir effect [1–4] in quantum field theory is the change of the vacuum energy density due to constraints on the quantum field induced by boundary conditions in space-time. The contribution to the energy density by the quantum fluctuation of the electromagnetic field was experimentally observed by Sparnaay [5] in 1958, thus verifying its quantum nature. Following this observation the Casimir effect was extensively studied, the renormalization procedure that must be used in order to extract physical numbers out of divergent mode sums being of particular interest. This procedure is most elegantly formulated in a path-integral approach [6], and leads to a full understanding of the Casimir effect for noninteracting quantum fields. Perturbative corrections to the free Casimir effect arising from a weak interaction of the fluctuating fields can also be obtained [7]. It was shown that the net effect of the boundaries is to produce a topological mass for the fluctuating modes [8]. In the recent past there has been a renaissance of the Casimir effect due to its broad span of applications, which range from gravity models [9] to QCD bag models [10] to nonlinear meson theories describing baryons as solitons [11]. A closely related subject is quantum field theory at finite temperature, since it can be described in the path-integral formalism by implementing periodic boundary conditions in the Euclidean time direction [12]. Despite these many different applications it is possible to understand the basic features of the Casimir effect by investigating a scalar theory. It is also of general interest to study ϕ^4, theory due to its important applications, e.g., in the Weinberg-Salam model of weak interactions (see, e.g., [13]), in solid state physics [14], and inflationary models of the early Universe [15]. In these applications a nontrivial vacuum structure is desired for phenomenological reasons. Lattice simulations [16, 17], however, show that for positive quartic coupling ($\lambda > 0$) ϕ^4 theory is trivial (i.e., a free theory due to renormalization). Lattice ϕ^4 theory with a negative bare coupling [18] can be defined by analytic continuation of the theory with positive λ. It was observed that this theory yields the desired nontrivial continuum limit [18]. It might happen that this theory violates the Osterwalder-Schrader axioms [19, 17] indicating problems defining this theory in Minkowski space. However, this question is not settled yet. We will not address this question in this paper but confine ourselves to the Euclidean ϕ^4 theory. Recently, a nonperturbative path-integral approach to nontrivial ϕ^4 theory[1] was proposed which also exhibits a nontrivial phase other than the trivial one [20, 21]. The perturbative (trivial) phase is unstable (at zero temperature) because a second phase with nonvanishing scalar condensate has lower vacuum energy density [20]. It was found that at a critical temperature the energy densities of the nontrivial and perturbative phases are equal, and the nontrivial phase undergoes a first order phase transition to the perturbative one [21].

Using this approach it is possible to study the Casimir effect in a theory with nontrivial vacuum properties. Since the nontrivial phase contains an intrinsic energy scale (i.e., the magnitude of the scalar condensate at zero temperature), one expects deviations of the Casimir energy from the free field law. This presumably provides an access to nonperturbative vacuum properties.

In this paper we investigate the nontrivial phase of four-dimensional ϕ^4 theory (i.e., its version obtained by analytic continuation to negative bare coupling) in a rectangular box. We require periodic boundary conditions in p (< 4) directions of space, which means that in these directions we compactify space to a circle with circumference a_i ($i = 1, \dots, p$). When these compactification lengths a_i are large, the Casimir energy decays exponentially for greater lengths, the decay rate given by the

[1]The theory with an analytic continuation of the bare coupling λ to negative values is called nontrivial ϕ^4 theory throughout this paper.

magnitude of the scalar condensate. At small lengths the field theory no longer tolerates a scalar condensate and the perturbative phase is adopted.

The paper is organized as follows. In the second section we briefly review the Casimir effect of a free theory and the recently proposed nonperturbative approach [20, 21] to ϕ^4 theory. The analytic continuation to negative quartic coupling and the renormalization procedure are discussed. Renormalization-group invariance is shown. In the subsequent section results are presented. The asymptotic behavior of the Casimir energy for large (compared with the scalar condensate) compactification lengths is obtained analytically, and the deviations from the energy in a free-field theory are discussed. The phase transition from the nontrivial vacuum to the perturbative phase at small compactification lengths is studied; the vacuum energy density and the scalar condensate are calculated as functions of the compactification lengths. Discussions and concluding remarks are given in the final section.

II. THE CASIMIR EFFECT OF SCALAR FIELDS

ϕ^4 theory is described by the Euclidean generating functional for Green's functions: i.e.,

$$Z[j] = \int \mathcal{D}\phi \, \exp\left\{ - \int d^4x \, \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 - j(x)\phi^2(x) \right) \right\} , \tag{1}$$

where m denotes the bare mass of the scalar field and λ the bare coupling strength of the ϕ^4 interaction. Provided some regularization prescription is used, the path integral (1) is well defined for $\lambda \geq 0$, which is assumed in Sec. II B where a nonperturbative expansion of (1) will be derived. We will then allow for negative bare couplings and consider the analytic continuation in this expansion. $j(x)$ is an external source for $\phi^2(x)$ which is introduced, so that we can derive the effective potential [22, 23] of the composite field ϕ^2 later on. It was observed in [21] that it is more convenient to use the effective potential of ϕ^2 to study the phase structure of the theory. In particular its minimum value is the vacuum energy density and thus provides access to the Casimir effect, if it is calculated by imposing adequate boundary conditions on the scalar modes. As mentioned in the Introduction, for these initial investigations we adopt the simplest geometry and consider a rectangular box with periodic boundary conditions in p (< 4) directions with distances a_i ($i = 1, \ldots, p$). Therefore the integration over the field ϕ in (1) only includes configurations which satisfy these boundary conditions. As we shall see, the only geometry-dependent divergence is proportional to the volume of the cavity (rectangular box) in the case of periodic boundary conditions, while for Dirichlet or Neumann boundary conditions, divergences depending on the shape of the cavity also occur. Whereas the volume divergence drops out from the renormalized vacuum energy density, the shape-dependent divergences of the latter case enter the effective potential. In that case the results would sensitively depend on the physical structure of the surface [24] (e.g., the scale set by the atomic dimensions of the plates), and such effects are beyond the scope of this paper.

The effective action is defined by a Legendre transformation of the generating functional $Z[j]$: i.e.,

$$\Gamma[\phi_c^2] := - \ln Z[j] + \int d^4x \, \phi_c^2(x) j(x) \, ,$$
$$\phi_c^2(x) := \frac{\delta \ln Z[j]}{\delta j(x)} \, . \tag{2}$$

From here the effective potential $U(\phi_c^2)$ is obtained by restricting ϕ_c to constant classical fields ($\Gamma[\phi_c^2 = \text{const}] = \int d^4x \, U(\phi_c^2)$), which are obtained for a constant external source j. The minimum value of the effective potential $U_{\min}$ is the vacuum energy density and is obtained from (2) at the zero external source: i.e.,

$$\left. \frac{dU}{d\phi_c^2} \right|_{\phi_c^2 = \phi_{c\,0}^2} = j = 0 \, . \tag{3}$$

The minimum classical configuration $\phi_{c\,0}^2$ represents the scalar condensate.

A. Casimir effect for free scalar fields

In this subsection we review the Casimir effect of a *free* scalar theory ($\lambda = 0$) using Schwinger's proper-time regularization. We demonstrate that the minimum of the effective potential U coincides with the mode sum usually considered in the study of the Casimir effect [1–4]. This equivalence was also obtained by using another regularization scheme [2] and previously observed with proper-time regularization in the context of chiral solitons [25].

The minimum of the effective potential of a free scalar theory is in Schwinger's proper-time regularization

$$U_{\min} = \frac{1}{2TV_{d-1}} \, \text{Tr} \, \ln(-\partial^2 + m^2)$$
$$= - \frac{1}{2V_{d-1}} \int \frac{dk_0}{2\pi} \text{Tr}_V \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \, e^{-s(k_0^2 + E^2)} \, , \tag{4}$$

where T is the Euclidean time interval, V_{d-1} is the $(d-1)$-dimensional space volume, and Λ is the proper-time cutoff. For definiteness we use Schwinger's proper-time regularization, but note, however, that the specific choice of the regularization prescription has no influence on the renormalized (finite) result (e.g., compare [23] and [26]). The trace in (4) extends over all modes satisfying periodic boundary conditions. Their eigenvalues are $E^2 = k^2 + \sum_{i=1}^{p} n_i^2 \left(\frac{2\pi}{a_i} \right)^2$, where a_i ($i = 1, \ldots, p$) is the compactification length in the ith direction. After the k_0 integration in (4) has been performed, a partial integration in the s integral yields the mode sum $V_{d-1}U_{\rm min} = \frac{1}{2}{\rm Tr}_V E$ in cutoff regularization with the particular cutoff function $\frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2}, \frac{E^2}{\Lambda^2})$. This establishes the equivalence of the mode sum approach and the approach provided by the effective potential.

In order to introduce in the regularization procedure encountered in the next section we proceed in calculating the Casimir energy for the *massless* case ($m = 0$). The integration over the continuous degrees of freedom can be performed in a straightforward manner, and rewriting

$$\sum_{n=-\infty}^{\infty} e^{-sn^2(\frac{2\pi}{a})^2} = \frac{a}{2\sqrt{\pi s}} \sum_{\nu=-\infty}^{\infty} e^{-\nu^2\frac{a^2}{4s}} , \qquad (5)$$

Eq. (4) becomes

$$TV_{d-1}U_{\rm min} = -\frac{L^{d-p}}{2}\frac{1}{(2\sqrt{\pi})^{d-p}}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s^{1+\frac{d-p}{2}}} \times \prod_{i=1}^{p}\left[\frac{a_i}{2\sqrt{\pi s}}\sum_{\nu_i=-\infty}^{\infty} e^{-\nu_i^2\frac{a_i^2}{4s}}\right] . \qquad (6)$$

The crucial observation is that the ultraviolet behavior is dominated by the integrand at small s and that the only divergences result from the term with all ν_i equal to zero. In the case of a free theory the divergent term is proportional to the d-dimensional space-time volume $V = V_{d-1}\, T$ and a pure constant that can be absorbed by a redefinition of the action. After the substitution $s \to 1/s$ the s integration can be performed in (6) yielding, for the finite part in the limit $\Lambda \to \infty$,

$$-\frac{1}{V}\ln Z = U_{\rm min} = -\frac{1}{2}\frac{1}{\pi^{d/2}}\Gamma\left(\frac{d}{2}\right) Z(a_1\cdots a_p, d) , \qquad (7)$$

where

$$Z(a_1\cdots a_p, d) = \sum_{\{\nu_i\}=-\infty}^{\infty}{}' \frac{1}{(a_1^2\nu_1^2 + \cdots + a_p^2\nu_p^2)^{\frac{d}{2}}} \qquad (8)$$

is the Epstein ζ function (the prime indicates that the term with all $\nu_i = 0$ is excluded from the sum). For $p = 1$ and four space-time dimensions ($d = 4$) one obtains the analytic result for the vacuum energy density and the Casimir energy E_c, respectively, i.e.,

$$U_{\rm min} = -\frac{1}{\pi^2}\frac{1}{a^4}\Gamma(2)\zeta(4) ,$$
$$E_c = V_3 U_{\rm min} = -\frac{\pi^2 L^2}{90a^3} , \qquad (9)$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Riemann's ζ function. This is precisely the result usually obtained by evaluating the mode sum of zero point energies [2].

B. Nontrivial ϕ^4 theory with boundary conditions

In this subsection we describe the nonperturbative approach to ϕ^4 theory provided by the modified loop expansion [20], taking into account the constraints on the scalar field imposed by boundary conditions. We demonstrate that the renormalization procedure is not affected by the presence of a rectangular box implying that the renormalization-group invariance of the infinite volume limit ($a_i \to \infty$) is preserved.

The modified loop expansion [20] is based on a linearization of the ϕ^4 interaction in the path integral (1) by means of an auxiliary field $\chi(x)$: i.e.,

$$Z[j] = \int \mathcal{D}\phi\, \mathcal{D}\chi \exp\left\{-\int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{6}{\lambda}\chi^2(x) + \left(\frac{m^2}{2} - i\chi(x)\right)\phi^2(x) - j(x)\phi^2(x)\right]\right\} . \qquad (10)$$

This linearization was first proposed in [27]. The integral over the fundamental field ϕ is then easily performed, yielding

$$Z[j] = \int \mathcal{D}\chi\, \exp\{-S[\chi, j]\} , \qquad (11)$$

$$S[\chi, j] = \frac{6}{\lambda}\int d^4x\, \chi^2 + \frac{1}{2}{\rm Tr}_{(R)} \ln \mathcal{D}^{-1}[\chi, j] , \qquad (12)$$

$$\mathcal{D}^{-1}[\chi, j]_{xy} = (-\partial^2 + m^2 - 2i\chi(x) - 2j(x))\, \delta_{xy} . \qquad (13)$$

The trace ${\rm Tr}_{(R)}$ extends over all eigenmodes of the operator $\mathcal{D}^{-1}[\chi, j]$, which satisfy the periodic boundary conditions; the subscript (R) indicates that a regularization prescription is required. The approach of [20] is defined by an expansion with respect to the field χ around its mean field value χ_0 defined by

$$\left.\frac{\delta S[\chi, j]}{\delta\chi(x)}\right|_{\chi=\chi_0} = 0 . \qquad (14)$$

This modified loop expansion of the generating functional $Z[j]$ arises from (10) for $\lambda \geq 0$. In order to perform the analytic continuation to negative bare coupling we allow $\lambda < 0$ from now on in this expansion. Note that the modified loop expansion of [20, 21] coincides with an $1/N$ expansion of the O(N)-symmetric ϕ^4 theory [29] for $N = 1$ implying that the convergence of the expansion is doubtful. However, the next to leading order calculation shows that the qualitative behavior of the effective potential is unchanged even for $N = 1$. Quantitative results can be obtained in good approximation from the leading order calculation by a proper redefinition of the renormalization-group-scaling coefficients [30]. We therefore confine ourselves to the leading order of the modified loop expansion, because the Casimir effect of a quantum system with a nontrivial vacuum can be qualitatively investigated already at this stage.

At zeroth order we obtain, from (10),

$$-\ln Z[j](a_1\cdots a_p)=\int d^4x\,\left\{-\frac{3}{2\lambda}(M-m^2+2j)^2\right\}+\frac{1}{2}\mathrm{Tr}_{(R)}\,\ln(-\partial^2+M)\ , \tag{15}$$

where M is related to the mean field value χ_0 by $\chi_0 = i(M-m^2+2j)/2$. The mean field equation for χ_0 (14) can be recast into an equation for M: i.e.,

$$\frac{\delta \ln Z[j]}{\delta M}=0\ . \tag{16}$$

For a constant external source j this equation is satisfied for constant M.

The effect of the rectangular box shows up only in the loop contribution, which is given in Schwinger's proper-time regularization ($d=4$) by

$$\begin{aligned} L &= \frac{1}{2}\mathrm{Tr}_{(R)}\,\ln(-\partial^2+M) \\ &= -\frac{L^{4-p}}{2}\int\frac{d^{4-p}k}{(2\pi)^{4-p}}\sum_{\{n_i\}}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s} \\ &\quad\times\exp\left\{-s\left[\sum_{l=1}^{4-p}k_l^2+M+\sum_{i=1}^{p}\left(\frac{2\pi\,n_i}{a_i}\right)^2\right]\right\}\ , \end{aligned} \tag{17}$$

where L is the extension in the unconstrained directions. The sum over the unconstrained modes (k integration) can be performed easily. Applying Poisson's formula to extract the divergent terms, as we did for the free theory in Sec. I, we obtain

$$\begin{aligned} L &= -\frac{V}{32\pi^2}M^2\,\Gamma\left(-2,\frac{M}{\Lambda^2}\right) \\ &\quad-\frac{V}{32\pi^2}\sum_{\{\nu_i\}}{}'\int_{1/\Lambda^2}^{\infty}\frac{ds}{s^3}\,e^{-sM} \\ &\quad\times\exp\left(-\sum_{i=1}^{p}\frac{a_i^2}{4s}\nu_i^2\right)\ , \end{aligned} \tag{18}$$

where Γ denotes the incomplete Γ function, V the space-time volume, and the prime indicates that the contribution with all $\nu_i=0$ is excluded. This implies that the second term of the right-hand side of (18) is ultraviolet finite, so we can remove the regulator in this term ($\Lambda\to\infty$). Using the asymptotic expression of the incomplete Γ function, we find

$$\begin{aligned} L &= \frac{V}{32\pi^2}\left\{M\Lambda^2+\frac{1}{2}M^2\left(\ln\frac{M}{\Lambda^2}-\frac{3}{2}+\gamma\right)\right\} \\ &\quad-\frac{V}{32\pi^2}\,F_3\,(M,a_1\cdots a_p)\ , \end{aligned} \tag{19}$$

where $\gamma=0.577...$ is Euler's constant and the function F_3 is defined by

$$F_\epsilon\,(M,a_1\cdots a_p)=\sum_{\{\nu_i\}}{}'\int_0^{\infty}\frac{ds}{s^\epsilon}\,e^{-sM}\,e^{-\sum_i\frac{a_i^2}{4s}\nu_i^2} \tag{20}$$

with $\epsilon=3$. The first term on the right-hand side of (19) is the contribution of the loop term L to the effective potential in the infinite volume limit, since the function F_3 vanishes for $a_i\to\infty$. The second term in (19) is thus the modification of the effective potential due to the presence of the Casimir boundary conditions and gives rise to the Casimir effect. Note that this term is finite implying that the boundaries do not affect the renormalization procedure. This is the desired result.

Following the renormalization scheme given in [21] we absorb the divergences in the bare parameters λ, m, j by setting

$$\frac{6}{\lambda}+\frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2}{\mu^2}-\gamma+1\right)=\frac{6}{\lambda_R}\ , \tag{21}$$

$$\frac{6}{\lambda}j-\frac{3m^2}{\lambda}-\frac{1}{32\pi^2}\Lambda^2=\frac{6}{\lambda_R}j_R-\frac{3m_R^2}{\lambda_R}\ , \tag{22}$$

$$j-m^2=0\ , \tag{23}$$

where μ is an arbitrary renormalization point and a subscript R refers to the renormalized quantities. Later we will check that physical quantities do not depend on μ. In the following we consider the massless case $m_R=0$. The coupling strength renormalization in (21) was earlier used by Coleman *et al.* [27] and coincides with the renormalization of the corresponding lattice theory (with negative bare coupling) [18]. Note that the bare coupling becomes infinitesimally negative,[2] if the regulator Λ is taken to infinity. This implies that only the theory with the analytic continuation to negative bare couplings possesses a scaling limit. The lack of a scaling limit of the theory with positive quartic coupling gives rise to triviality of the standard ϕ^4 theory as seen in lattice simulations. The above scaling behavior of the bare coupling is also consistent with perturbation theory [20]. In fact, we have, from (21),

$$\lambda=\frac{\lambda_R}{1-\beta_0\lambda_R(\ln\frac{\Lambda^2}{\mu^2}-\gamma+1)}\ ,\qquad \beta_0=\frac{1}{96\pi^2}\ , \tag{24}$$

implying $\lambda\to 0^-$ for $\Lambda\to\infty$, whereas in contrast an expansion of (24) with respect to the renormalized coupling strength, i.e.,

$$\lambda=\lambda_R(\mu)\left[1+\beta_0\,\lambda_R\left(\ln\frac{\Lambda^2}{\mu^2}-\gamma+1\right)+O(\lambda_R^2)\right]\ , \tag{25}$$

suggests that $\lambda\to+\infty$, if $\Lambda\to\infty$.

Inserting (21)–(23) and L from (19) in (15) one obtains

$$\begin{aligned} -\frac{1}{V}\ln Z[j](a_1\cdots a_p) &= -\frac{3}{2\lambda_R}M^2-\frac{6}{\lambda_R}Mj_R \\ &\quad+\frac{\alpha}{2}M^2\left(\ln\frac{M}{\mu^2}-\frac{1}{2}\right) \\ &\quad-\alpha\,F_3\,(M,a_1\cdots a_p)\ , \end{aligned} \tag{26}$$

[2]This is sometimes called precarious renormalization in the literature [28].

where $\alpha = 1/32\pi^2$ and M is defined by the mean field equation (16): i.e.,

$$-\frac{3}{\lambda_R}M - \frac{6}{\lambda_R}j_R + \alpha M \ln\left(\frac{M}{\mu^2}\right) + \alpha F_2(M, a_1 \cdots a_p) = 0 . \quad (27)$$

It is now straightforward to perform the Legendre transformation (2). The final result for the effective potential is

$$U(\phi_c^2) = \frac{\alpha}{2}M^2\left(\ln\frac{M}{\mu^2} - \frac{1}{2}\right) - \frac{3}{2\lambda_R}M^2 - \alpha F_3(M, a_1 \cdots a_p) , \quad (28)$$

where

$$\phi_c^2 = \frac{1}{V}\frac{\delta \ln Z[j_R]}{\delta j_R} = \frac{6}{\lambda_R}M . \quad (29)$$

The effective potential is renormalization group invariant, since a change in the renormalization point μ can be absorbed by a change of the renormalized coupling strength [20, 21]. Note that due to the renormalization of the composite operator ϕ^2 in (23) $M = \lambda_R\phi_c^2/6$ is renormalization group invariant rather than ϕ_c^2. Thus M is a physical quantity and is referred to as scalar condensate. In the infinite volume limit ($a_i \to \infty$) the effective potential has a global minimum for

$$M = M_0 = \mu^2 \exp\left(\frac{96\pi^2}{\lambda_R}\right) , \quad (30)$$

implying that the ground state has a nonvanishing scalar condensate [21]. Furthermore, the minimum value of the effective potential (vacuum energy density) is related to the scalar condensate by [21]

$$U_0 = -\frac{\alpha}{144}\lambda_R^2(\phi_c^2)^2 \to -\frac{1}{4}\frac{\beta(\lambda_R)}{24}\langle : \phi^4 : \rangle , \quad (31)$$

which yields the correct scale anomaly at this level of approximation.

In order to make renormalization group invariance obvious we remove the renormalization point dependence in (28) by using relation (30). Both the renormalization point μ and the renormalized coupling λ_R drop out, and we obtain

$$U(M) = \frac{\alpha}{2}M^2\left(\ln\frac{M}{M_0} - \frac{1}{2}\right) - \alpha F_3(M, a_1 \cdots a_p) . \quad (32)$$

The effective potential U as a function of the scalar condensate M for periodic boundary conditions in one direction ($p = 1$) is shown in Fig. 1. In this case the results are equivalent to those of finite-temperature field theory, if the inverse compactification length $1/a$ is identified with the temperature (in units of Boltzmann's constant) [12]. Further results of finite-temperature ϕ^4 theory are given in [21]. For a large compactification length a (zero temperature) the effective potential of the infinite volume case is obtained, and this has a minimum at a nonvanishing value of the scalar condensate. At finite

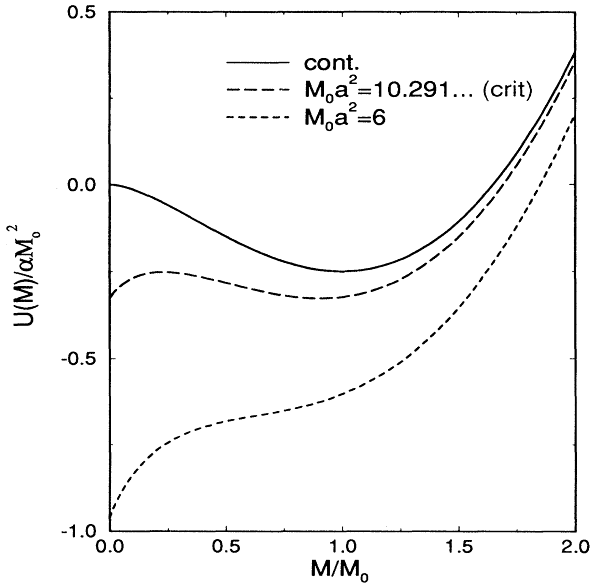

FIG. 1. The effective potential for periodic boundary conditions in one direction ($p = 1$) as a function of the scalar condensate at various compactification lengths.

a, a second minimum at zero condensate M develops, which is referred to as the perturbative phase. At large a, this trivial phase is unstable, because the nonperturbative minimum has a lower vacuum energy density. Decreasing a (increasing temperature) lowers the difference in the energy density between the perturbative and the nonperturbative phase. At a critical length a_c the nontrivial phase becomes degenerate with the perturbative one (at $M = 0$). If a is decreased further, the nontrivial phase becomes metastable and a first order phase transition to the trivial phase at $M = 0$ can occur by quantum fluctuations.

III. RESULTS

In a free massless field theory (with $p = 1$) with periodic boundary conditions there is no intrinsic energy scale in competition with that of the compactification length a. This implies that the vacuum energy density U_0 scales as $1/a^4$ with this length by dimensional arguments. This scaling law was experimentally observed by Sparnaay in the case of the quantum electromagnetic field [5]. In the case of nontrivial ϕ^4 theory an intrinsic energy scale is provided by the scalar condensate. Thus one expects deviations from the $1/a^4$ scaling law of the free theory. Such deviations might provide experimental access to properties of the nontrivial phase.

A. Near the infinite volume limit

The scalar condensate M_v at the minimum of the effective potential is given by the gap equation (27):

$$\left.\frac{dU}{dM}\right|_{M=M_v} = M_v \ln\frac{M_v}{M_0} + F_2(M_v, a_1 \cdots a_p) = 0 . \tag{33}$$

The vacuum energy density U_v is obtained by inserting M_v back into (32). For large compactification length $a_i^2 \gg 1/M_v$ the function $F_\epsilon(M_v, a_1 \cdots a_p)$ can be analytically estimated by noting that only terms with a single $\nu_i \neq 0$ and all others $\nu_j = 0$ contribute to the sum (20), i.e.,

$$F_\epsilon(M, a_1 \cdots a_p) \approx \sum_i \left(\frac{4}{a_i^2}\right)^{\epsilon-1} f_\epsilon\left(\frac{M_v a_i^2}{4}\right) ,$$
$$f_\epsilon(x) = \int_0^\infty \frac{ds}{s^\epsilon} e^{-sx} e^{-\frac{1}{s}} . \tag{34}$$

After some technical manipulations the functions $f_\epsilon(x)$ can be related to the modified Bessel functions of the second kind: i.e.,

$$f_\epsilon(x) = 2 x^{\frac{\epsilon-1}{2}} K_{\epsilon-1}(2\sqrt{x}) \approx \sqrt{\pi}\, x^{\frac{2\epsilon-3}{4}} e^{-2\sqrt{x}} , \tag{35}$$

where the last approximate expression is just the asymptotic form of the Bessel function for $x \to \infty$. There are two contributions to the variation of the vacuum energy density $U_v(M_v(a_i), a_i)$ (32) with the compactification length, one from a change of the scalar condensate and one from the change of the effective potential $U(M)$ via the function F_3. Equation (33) implies that a variation of the condensate does not change U_v in first order, and thus the leading contribution results from a change of F_3. Using the asymptotic form in (35) for F_3 one obtains

$$\frac{1}{\alpha}\Delta U_v = \frac{1}{\alpha}(U_v - U_\infty)$$
$$\approx -2^p \sqrt{\pi} M_0^2 \sum_{i=1}^p \left(\frac{M_0 a_i^2}{4}\right)^{-5/4} e^{-\sqrt{M_0} a_i} , \tag{36}$$

where U_∞ is the vacuum energy density in the infinite volume limit. This is the desired result: Eq. (36) gives the change of the vacuum energy density due to boundary conditions. In free or perturbative field theory (and $p = 1$) this energy decays by the power law $\sim 1/a^4$ [see Eq. (9)]. In contrast, the Casimir energy density (36) of strongly interacting scalar modes decays exponentially (with a power law correction), where the slope is given by the magnitude of the scalar condensate M_0. This implies that at least, in principle, one can decide by observing the qualitative behavior of the Casimir energy, whether the theory is in a perturbative or in a nonperturbative phase. In the latter case it is also possible to extract ground state properties, e.g., the scalar condensate. Since for quantum electromagnetic fields the $1/a^4$ power law was experimentally verified [5], the ground state is trivial, and there is no photon condensate. This is an expected result, since photon self-interactions are absent.

B. At the phase transition

As was seen in Sec. II B for one compactification ($p = 1$), the system undergoes a first order phase transition from the nontrivial vacuum to the perturbative vacuum, if the compactification length becomes small enough. Numerical investigations of the effective potential (32) for various lengths a_i show that the same effect is present for $p > 1$; if the box is small enough, a first order phase transition to the perturbative vacuum occurs.

Equating the energy density U_0 of the perturbative phase at $M = 0$ to that of the nontrivial phase at $M = M_v$ we obtain

$$\frac{M_v^2}{2}\left(\ln\frac{M_v}{M_0} - \frac{1}{2}\right) - F_3(M_v, a_1 \cdots a_p)$$
$$+ F_3(0, a_1 \cdots a_p) = 0 , \tag{37}$$

where the dependence of the scalar condensate $M_v(a_1 \cdots a_p)$ on the compactification lengths a_i is implicitly given by (33). The set of Eqs. (33,37) defines a hypersurface in the space spanned by the lengths a_i, which separates the nontrivial phase from the perturbative one. Note that this transition line is given in terms of renormalization-group-invariant (and therefore physical) quantities.

As we have seen, for *one* compactification ($p = 1$) the formulation is equivalent to finite-temperature ϕ^4 theory (identifying $1/a$ with temperature), and the phase transition at small lengths a has the same structure as in finite-temperature theory at high temperature. Because of this correspondence, the numerical value for the critical length $a_{(c)}$ can be taken from [21]:

$$M_0\, a_{(c)}^2 = 10.29134\ldots . \tag{38}$$

The ratio of the scalar condensate M_v at the transition point and the continuum (zero temperature) condensate M_0 is

$$M_v(a_{(c)}) / M_0 = 0.9041\ldots . \tag{39}$$

Because of the first order nature of the phase transition the scalar condensate has a discontinuity at the transition point $a_{(c)}$ and is zero for smaller lengths.

For compactifications in *two* directions the transition line between the two phases was obtained by solving (33,37) numerically. The result is presented in Fig. 2. Numerical investigations (cf. Fig. 4) suggest that the transition line is approximately given by the equation

$$\frac{1}{a_{c1}^2} + \frac{1}{a_{c2}^2} = \frac{M_0}{9} . \tag{40}$$

For $p = 3$ we have numerically checked that the first order phase transition occurs, if the rectangular box is sufficiently small.

C. Boundary dependence of energy density and scalar condensate

For a given vacuum energy density U_v, Eq. (32) becomes

$$U_v = \frac{M_v^2}{2}\left(\ln\frac{M_v}{M_0} - \frac{1}{2}\right) - F_3(M_v, a_1 \cdots a_p) , \tag{41}$$

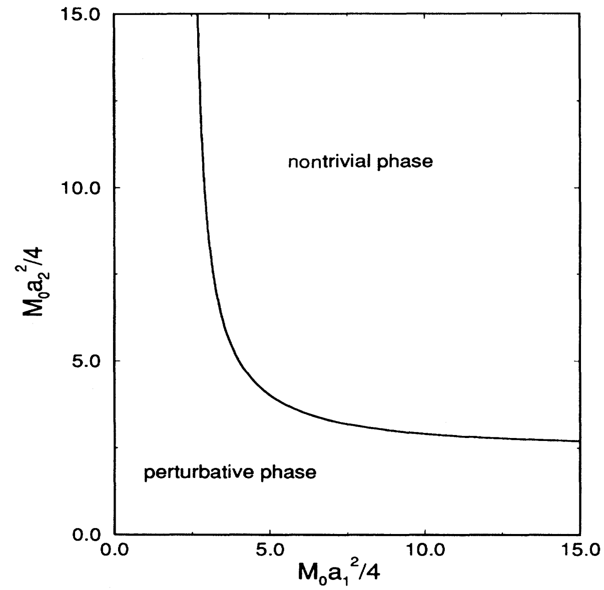

FIG. 2. The transition line separating the nontrivial phase and the perturbative phase, when two compactification lengths a_1 and a_2 are introduced.

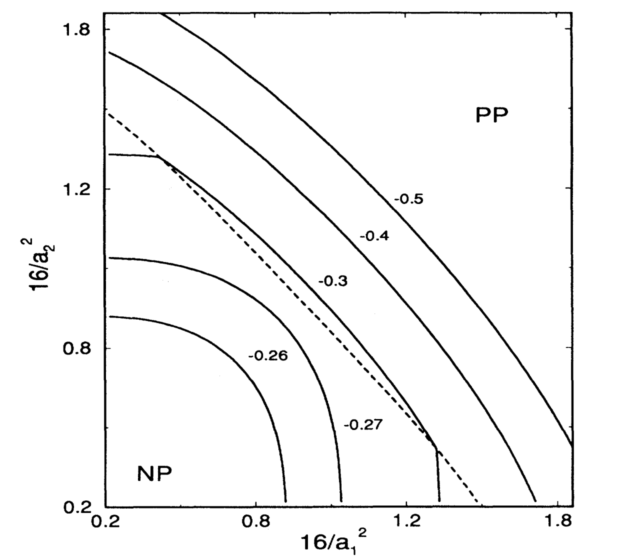

FIG. 4. The lines of constant vacuum energy density $\frac{U_v}{\alpha M_0^2}$ for two compactification lengths; a_i^2 in units of the inverse scalar condensate $1/M_0$.

where M_v is defined by (33). This equation yields a hypersurface of constant energy density in the space spanned by $\{a_i, i = 1, \ldots, p\}$. Comparing (41) with (37), it is easily seen that the phase separating surface is not a surface of constant energy density implying that there are intersections between the two surfaces. We expect that a hypersurface of constant energy density is continuous at the intersection, but not differentiable due to the first order phase transition.

Figure 3 shows the vacuum energy density for periodic boundary conditions in one direction ($p = 1$) as a function of $16/a^2$, where a is the compactification length (or equivalently the inverse temperature). For large values of $16/a^2$ (small a), the perturbative phase is realized, and the $1/a^4$ scaling law is observed. For small values of $16/a^2$ (large a), the scalar theory is in the nontrivial phase, and the energy density exponentially approaches the continuum value [see (36)] given by the scale anomaly (31).

For two compactifications ($p = 2$) Fig. 4 shows lines of constant vacuum energy density in the $16/a_1^2$-$16/a_2^2$ plane. The phase transition line (dashed curve) is also shown. As expected the lines of constant energy density

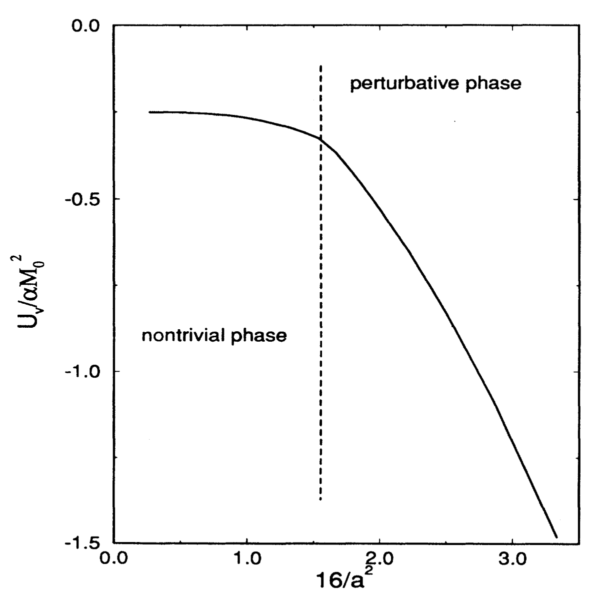

FIG. 3. The vacuum energy density for ($p = 1$) as a function of the compactification length (inverse temperature) in units of $1/\sqrt{M_0}$.

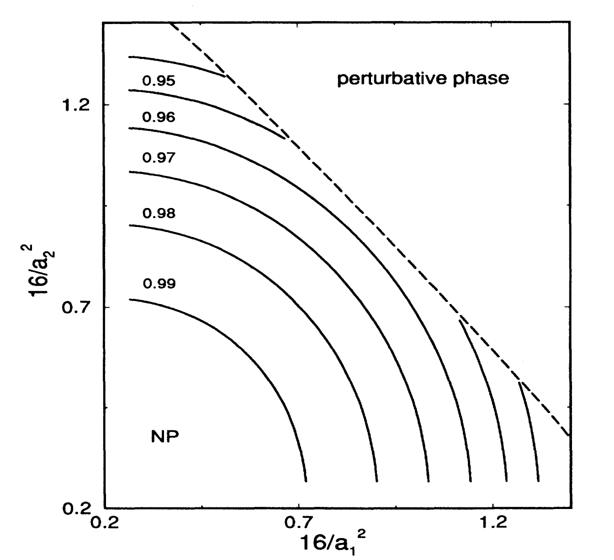

FIG. 5. The lines of constant scalar condensate M/M_0 for two compactification lengths.

are continuous, but have a cusp at the first order phase transition point.

We have also studied the hypersurfaces of constant *scalar condensate* in a_i space. For $p = 1$, this is analogous to the temperature dependence of the scalar condensate, and thus the results are given in [21]. For $p = 2$, the lines of constant condensate (in units of continuum condensate M_0) are presented in Fig. 5. A line of constant condensate is discontinuous at the phase transition line, because the condensate is zero in the perturbative phase. This behavior is again due to the first order phase transition.

IV. DISCUSSION AND CONCLUDING REMARKS

We have studied Euclidean ϕ^4 theory (i.e., its analytic continuation to negative quartic coupling) constrained by a rectangular box. It was shown that the nontrivial ground state undergoes a first order phase transition to the perturbative vacuum, if the extension in at least one space-time direction becomes small enough. For large boxes the finite-size corrections of the energy density in comparison with the infinite volume limit are exponentially small. They are negligible if the compactification lengths are large compared with intrinsic scale provided by the (continuum) scalar condensate (i.e., $a_i\sqrt{M_0} \gg 1$). On the other hand, finite-size effects become important for $a_i\sqrt{M_0} \approx 1$ and induce a phase transition to the trivial vacuum.

We believe that these properties are a common feature of a wide class of quantum field theories. Indeed, an analogous situation is observed in lattice gauge theories. Theoretical investigations show that for high temperatures, pure SU(N) lattice gauge theory has a phase transition from a nontrivial (confining) ground state to a perturbative phase [31]. Numerical simulations of the SU(N) theory use a lattice with size $n_t n^3$, $n \gg n_t$, which corresponds to a system with volume n^3 and inverse temperature n_t. Such a system shows two phases, a nontrivial phase for $\beta < \beta_c(n_t)$ and a deconfined phase for $\beta > \beta_c(n_t)$ [32] [$\beta = 2N/g^2$ with g the SU(N) coupling strength]. This can be related to our considerations as follows. The intrinsic scale of lattice gauge theories is provided by the string tension χ [33] (or equivalently by the gauge field condensate [34] as in the continuum Yang-Mills theory). Our investigations suggest that a finite-size phase transition occurs if

$$n_t n^3 \, a^4 \chi^2 \le 1 , \tag{42}$$

where a is the lattice spacing. The string tension in units of the lattice spacing $a^4\chi^2$ strongly depends on the inverse coupling strength β dictated by the renormalization group. Numerical simulations [33] show that $a^4\chi^2$ decreases with increasing β for fixed n_t and n, implying that (42) is satisfied for $\beta \approx \beta_c$, the coupling strength at which the phase transition occurs.

To conclude, we have studied ϕ^4 theory in a renormalization-group-invariant approach inside a rectangular box with periodic boundary conditions in p directions. We have further investigated the ground state properties of the nontrivial phase affected by the geometrical constraints. The dependence of the vacuum energy density and the scalar condensate on the compactification lengths was studied in some detail. In the nontrivial phase the vacuum energy density exponentially approaches the infinite volume limit, the decay rate given by the magnitude of the scalar condensate. This behavior of the energy density essentially differs from that of a free theory, where it scales according a $1/a^4$ power law. This implies that at least, in principle, one can determine which phase the system has adopted by measuring the Casimir energy. At small compactification lengths the system undergoes a first order phase transition to the perturbative phase. This phase transition is of the same nature as the transition at high temperature.

ACKNOWLEDGMENTS

We want to thank R. F. Langbein for a careful reading of this manuscript and useful remarks. We are also indebted to L. v. Smekal for helpful discussions. This work was supported by DFG under Contract No. Re 856/1-1.

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