Quantum tunneling for the sine-Gordon potential: Energy band structure and Bogomolny-Fateyev relation

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A path-integral calculation of the energy band structure for the sine-Gordon potential with the help of the instanton method is presented. The periodic pseudoparticle configuration obtained recently is seen to be responsible for tunneling at a finite energy which leads to the level splitting of excited states. The lowest band structure due to the tunneling of vacuum instantons is recovered from our result in the low energy limit. The above result is obtained by considering a half period of the classical solution as a kinklike configuration (with nontrivial topological charge). On the other hand if a full period of the solution is treated as a nontopological (bouncelike) pseudoparticle configuration, the imaginary parts of energy eigenvalues can be calculated. We then obtain the Bogomolny-Fateyev relation which is well known in perturbation theory at large order. The agreement of our results with those of WKB calculations also resolves a controversy concerning the method: In fact this method yields the same correct result as the method of complex paths.

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I. INTRODUCTION

The sine-Gordon (SG) potential has attracted considerable attention both in field theory and in condensed-matter physics. The classical SG equation has been extensively studied and the soliton obtained from it satisfying the vacuum boundary condition is perhaps the best known in the literature. It is well known that Coleman [1] established the quantum theory by relating the SG field to the massive Thirring model.

The purpose of the present paper is to present an explicit calculation of quantum tunneling phenomena dominated by so-called nonvacuum instantons at finite energy. The new pseudoparticle configurations [2,3] characterized by nonzero energy satisfy manifestly nonvacuum boundary conditions. We do not repeat the tunneling calculation with vacuum instantons here since this is just a straightforward matter of following the same procedure as for the calculation of the level splitting for the double-well potential [4] or the imaginary part of the energy with bounces for the inverted double-well potential [5].

However, here, as also in our consideration of vacuum bounces [5] and nonvacuum bounces [6] we demonstrate that the pseudoparticle method is, at least as far as the results are concerned, completely equivalent to the method of complex paths [7–10], the results of either method agreeing as expected with those of the leading WKB approximation. The discrepancy which was previously found, pointed out most explicitly in Ref. [7] [see the discussion after Eq. (4.27)], is not one of principle but of approximation (in the evaluation of the damping exponential e^{-W} below). The present investigation therefore not only resolves this previous controversy, but also demonstrates the continued usefulness of the pseudoparticle method in general. In addition to this, the considerations given below are based on the use of nontopological, classically unstable, nonvacuum or periodic instantons which have recently become the subject of extensive investigation under the name of sphalerons [2,3]. We also show explicitly that in the present case (which corresponds to that of the eigenvalue of a self-adjoint operator) the negative eigenvalue of the small fluctuation equation does not represent an instability.

In a recent investigation [11] of the rate of instanton-like transitions at high energies a solution to the classical field equation in the massless (1+1)-dimensional O(3) σ model has been found. The model has been reduced to a mechanical tunneling problem for the SG potential. Therefore the study of tunneling phenomena, especially with nonvacuum instantons for the SG potential, is of considerable interest in the context of efforts towards understanding the subject of baryon- and lepton-number violation at high energy [12].

II. NONVACUUM INSTANTONS AND THEIR SMALL FLUCTUATIONS

Field-theoretic models with solitonic (topologically nontrivial) finite Euclidean action have been discussed for many years. More recently Manton and Samols [2] have discovered a new type of classical configuration of (1+1)-

dimensional ϕ^4 field theory by compacting the space to a circle. The case of the ϕ^4 soliton on a circle was then extended to those of the ϕ^4 bounce and the SG soliton [3], and it was shown that in each of the three cases the fluctuation equation is a Lamé equation. In the present paper the (1+1)-dimensional SG soliton on a circle is considered in 1+0 dimensions for the calculations of tunneling phenomena.

The Lagrangian we consider is

$$\mathcal{L} = \frac{1}{2}\left[\frac{d\phi}{dt}\right]^2 - V(\phi), \tag{2.1a}$$

with the SG potential

$$V(\phi) = \frac{1}{g^2}[1 + \cos(g\phi)] \tag{2.1b}$$

where $g > 0$ denotes the coupling constant. The classical solution which extremizes the action is seen to satisfy the equation of motion

$$\frac{1}{2}\left[\frac{d\phi_c}{d\tau}\right]^2 - V(\phi_c) = -E_{\text{cl}}, \tag{2.2}$$

where we have gone to the Euclidean time $\tau = it$. We take the mass $m_0 = 1$ and natural units $c = \hbar = 1$ throughout. Equation (2.2) can be regarded as the equation of motion of a pseudoparticle with the classical energy $E_{\text{cl}} \geq 0$ which is a constant of integration. Letting E_{cl} be confined to a region $0 \leq E_{\text{cl}} \leq 2/g^2$ the configuration $\phi_c(\tau)$ becomes periodic such that $\phi_c(\tau + \mathcal{T}) = \phi_c(\tau)$, which now corresponds to the periodic boundary condition in the space coordinate of Ref. [3]. The classical solution is [3]

$$\phi_c = \frac{2}{g}\arcsin[k\text{sn}(\tau + \tau_0)] \tag{2.3}$$

where sn, cn, dn, ... are Jacobian elliptic functions with modulus $k = \sqrt{1 - g^2 E/2}$ and τ_0 denotes the position of the pseudoparticle. The elliptic function $\text{sn}(\tau)$ has a period $4\mathcal{K}(k)$. Then there exist critical values of $\mathcal{T}$ such that

$$\mathcal{T} = 4n\mathcal{K}(k), \tag{2.4}$$

where $n = 1, 2, 3, \ldots$ are positive integers and $\mathcal{K}(k)$ is the complete elliptic integral of the first kind and so a quarter of the period of $\text{sn}(\tau)$. We refer to Refs. [2] and [13] for a discussion of the significance of these critical values with regard to the behavior of the classical solution for (1+1)-dimensional ϕ^4 and ϕ^6 field theories. In our case we will see later that n is the number of instanton and anti-instanton pairs or the number of bounces. The pseudoparticle oscillates between two turning points in one of the barriers.

For zero energy $E_{\text{cl}} = 0 (k \to 1)$ the periodic solution (2.3) reduces to the vacuum instanton configuration

$$\phi_c \underset{k \to 1}{\longrightarrow} \frac{2}{g}\arcsin[\tanh(\tau - \tau_0)] \ . \tag{2.5}$$

The small fluctuation equation about the classical solution is a Lamé equation [3]

$$\hat{M}\psi_m = E_m\psi_m \tag{2.6}$$

with the operator

$$\begin{aligned}\hat{M} &= -\frac{d^2}{d\tau^2} - \cos[g\phi_c(\tau)] \\ &= -\frac{d^2}{d\tau^2} - [1 - 2k^2\text{sn}^2(\tau)].\end{aligned} \tag{2.7}$$

The discrete eigenfunctions are [3] $\psi_1 = \text{cn}(\tau)$, $\psi_2 = \text{dn}(\tau)$, $\psi_3 = \text{sn}(\tau)$ with eigenvalues $E_1 = 0$, $E_2 = k^2 - 1$, $E_3 = k^2$, respectively. It is interesting to see the existence of a negative eigenvalue which would result in the imaginary part of energy by the standard instanton method [14]. This reflects the Bogomolny and Fateyev observation [15] that for the large order behavior of a perturbation series, the Green's functions have the imaginary parts at physical values of the coupling constant which are connected with tunneling to the other vacuum states.

III. ENERGY BAND STRUCTURE OF THE SG POTENTIAL AND THE TRANSITION AMPLITUDE FOR QUANTUM TUNNELING

The SG potential has an infinite number of degenerate vacua. Quantum tunneling between neighboring vacua leads to the level splitting while the levels extend to bands due to the translation symmetry expressed by $V(\phi + n\pi/g) = V(\phi)$. In the narrow-band approximation one finds, for the energy, the expression (which is real since the summation extends over positive and negative n),

$$E_i = \mathcal{E}_i + \sum_n J(R_m - R_n)e^{i\kappa(R_n - R_m)}, \tag{3.1}$$

where $\mathcal{E}_i$ denotes the ith eigenvalue of the energy in each well for the harmonic oscillator potential $U(\phi) = \frac{1}{2}(\phi - R_m)^2$ and $R_m = 2m\pi/g$ is the position of the mth minimum. In Eq. (3.1),

$$\begin{aligned}J(R_m - R_n) \equiv \int \psi_i^*(\phi - R_m)[V(\phi) \\ -U(\phi)]\psi_i(\phi - R_n)d\phi\end{aligned} \tag{3.2}$$

is the overlap integral. ψ_i is the eigenfunction corresponding to eigenvalue $\mathcal{E}_i$ and κ is the Floquet parameter associated with the Bloch wave function. If only the contribution from the nearest neighbors is taken into account, i.e., $J(R_m - R_n)$ for $|m - n| > 1$ is taken to be zero, we obtain the energy band formula

$$E = \mathcal{E}_i + 2J\cos\frac{2\pi\kappa}{g}. \tag{3.3}$$

The parameter J is just the level splitting resulting from quantum tunneling (for $\kappa = 0$ the wave functions are periodic). In the following we consider the case of potential wells surrounded by very high potential barriers with correspondingly small tunneling contributions to the eigenvalues. Thus the eigenenergies are almost those of degenerate harmonic oscillators, and in this asymptotic case we are not concerned with the entire bands but only with their edges which correspond to alternately even and odd states. Thus we suppose $|i\rangle_+, |i\rangle_-$ are degenerate eigenstates in neighboring wells (see Fig. 1), respectively with the same energy eigenvalue $\mathcal{E}_i$ such that $\hat{H}^0|i\rangle_\pm = \mathcal{E}_i|i\rangle_\pm$ where $\hat{H}_0$ is the Hamiltonian of the harmonic oscillator as the zero order approximation of our system. The degeneracy will be removed by the small tunneling effect which leads to the level splitting. The eigenstates of the Hamiltonian H become

$$|i\rangle_0 = \frac{1}{\sqrt{2}}(|i\rangle_+ - |i\rangle_-), \ |i\rangle_e = \frac{1}{\sqrt{2}}(|i\rangle_+ + |i\rangle_-) \quad (3.4)$$

with eigenvalues $\mathcal{E}_i + \Delta\mathcal{E}$ and $\mathcal{E}_i - \Delta\mathcal{E}$, respectively. $\Delta\mathcal{E}$ denotes the shift of one oscillator level. It is obvious that

$$_+\langle i|\hat{H} - \hat{H}^0)|i\rangle_- = 2J = \pm\Delta\mathcal{E}. \quad (3.5)$$

In the following we calculate this energy shift $\Delta\mathcal{E}$ as resulting from periodic instantons and instanton–anti-instanton pairs.

The amplitude for a transition from the left-hand well to the right-hand well at the energy $\mathcal{E}$ due to instanton tunneling can be written

$$A_{+,-} =_+ \langle\mathcal{E}|e^{-2\hat{H}T}|\mathcal{E}\rangle_- \simeq \pm e^{-2\mathcal{E}T}\sinh(2\Delta\mathcal{E}T) \ , \quad (3.6)$$

for negligible overlap of the wave functions which dominate over either well. The amplitude (3.6) can also be evaluated with the help of the path-integral method. Then

$$A_{+,-} = \int \psi^*_{\mathcal{E},+}(\phi_f)\psi_{\mathcal{E},-}(\phi_i)K(\phi_f,\tau_f;\phi_i,\tau_i)d\phi_f d\phi_i \ , \quad (3.7)$$

where the Feynman kernel is defined as usual by

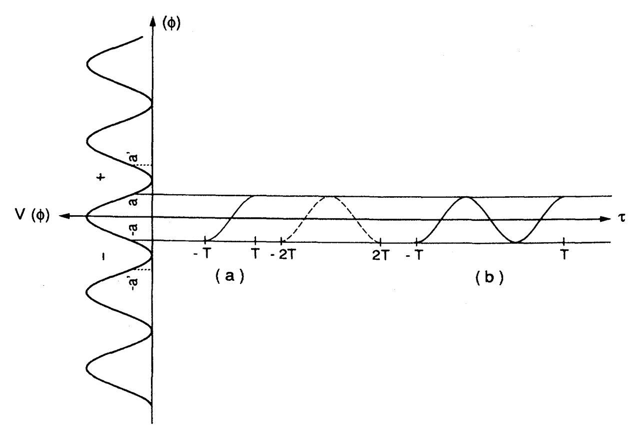

FIG. 1. (a) Trajectories of the nonvacuum instanton (half of period) and a bounce (full period); (b) the case $n = 3$: one instanton plus a pair.

$$K(\phi_f,\tau_f;\phi_i,\tau_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\{\phi\}e^{-S} \quad (3.8)$$

with $\phi_f \equiv \phi(\tau_f)$, $\phi_i \equiv \phi(\tau_i)$ and $\tau_f - \tau_i = 2T$. What we are interested in is this expression in the limits $\phi_i \to -a$, $\phi_f \to a$ ($\pm a$ are the turning points, see Fig. 1), namely, the tunneling propagator through one of the barriers. In Eq. (3.8),

$$S = \int_{\tau_i}^{\tau_f} \left[\frac{1}{2}\left[\frac{d\phi}{d\tau}\right]^2 + V(\phi)\right] d\tau$$

is the Euclidean action of the pseudoparticle, and $\psi_{\mathcal{E},+}$ and $\psi_{\mathcal{E},-}$ are the wave functions of the right- and left-hand wells, respectively, which extend into the domain of the barrier.

IV. EVALUATION OF THE TRANSITION AMPLITUDE

A. Instanton or bounce?

In the vacuum case the instanton is looked at as a stable pseudoparticle configuration (kink) with nontrivial topological charge while the bounce [5,14] is considered to be an unstable nontopological configuration. It has been shown that in the (1+1)-dimensional theory all configurations on a circle with periodic boundary condition [periodic motion with a finite period in our (1+0) dimensional case] are unstable in view of the existence of negative eigenmodes of the small fluctuation equations [2,3]. Instantons and bounces are now only distinguished by the topological charge. We therefore consider the motion through half a period as the trajectory of an instanton while the motion through a full period is considered as the trajectory of a bounce or that of an instanton–anti-instanton pair. Since we consider the tunneling from one well to another, only the instantonlike solution is suitable. However, a problem is the negative eigenmode which should not contribute to the tunneling calculation so that the transitiion amplitude is real. It will be seen how this difficulty is resolved.

B. Removal of the negative eigenmode and the one instanton contribution

As in the standard path-integral calculation we expand $\phi(\tau)$ about the classical trajectory $\phi_c(\tau)$ and thus set

$$\phi(\tau) = \phi_c(\tau) + \chi(\tau), \quad (4.1)$$

where $\chi(\tau)$ is the deviation of $\phi(\tau)$ from $\phi_c(\tau)$ with fixed end points. The boundary condition for $\chi(\tau)$ is (with $\tau_i \to -\mathcal{K}, \tau_f \to +\mathcal{K}$)

$$\chi(\tau_i) = \chi(\tau_f) = 0. \quad (4.2)$$

Substitution of $\phi(\tau)$ of (4.1) into Eq. (3.8) and keeping only terms containing χ up to the second order for the one-loop approximation yields

$$K(\phi_f,\tau_f;\phi_i,\tau_i) = \exp[-S_c(\phi_f,\phi_i;\tau_f,\tau_i)]\int_{\chi(\tau_i)=0}^{\chi(\tau_f)=0} \times\mathcal{D}\{\chi\}\exp[-\delta S] \equiv \exp[-S_c]I, \tag{4.3}$$

where the classical action S_c is evaluated along the trajectory ϕ_c so that

$$S_c=\int_{\tau_i}^{\tau_f}\left[\frac{1}{2}\left[\frac{d\phi_c}{d\tau}\right]^2+V(\phi_c)\right]d\tau \underset{\substack{\phi_i\to -a\\ \phi_f\to a}}{=} \frac{8}{g^2}[E(k)-k'^2\mathcal{K}(k)]+2E_{\text{cl}}T\ . \tag{4.4}$$

Here $E(k)$ denotes the complete elliptic integral of the second kind and $k'=\sqrt{1-k^2}$. The limit of end points tending to the turning points will be taken only after all the integrations have been carried out so that the singularities at the turning points are avoided.

The fluctuation action is seen to be

$$\delta S=\int_{\tau_i}^{\tau_f}\chi\hat{M}\chi d\tau, \tag{4.5}$$

with $\hat{M}$ given by (2.7). Expanding $\chi(\tau)$ in terms of the eigenmodes of $\hat{M}$, the eigenfunctions of the small fluctuation equation (2.6), i.e., setting

$$\chi(\tau)=\sum_m C_m\psi_m \tag{4.6}$$

and changing the integration variables to $\{C_m\}$, the functional intergal I can be formally evaluated to be

$$I=\left|\frac{\partial\chi(\tau)}{\partial C_m}\right|\prod_m\left[\frac{\pi}{E_m}\right]^{1/2} \tag{4.7}$$

if all of the C_m are independent variables. In our case, however, the boundary condition (4.2) leads to constraint equations on $\{C_m\}$, that is,

$$C_1\psi_1(\pm\mathcal{K})+C_2\psi_2(\pm\mathcal{K})+C_3\psi_3(\pm\mathcal{K})=0$$

and so to

$$C_2k'+C_3=0,\quad C_2k'-C_3=0 \tag{4.8}$$

since $\text{cn}\mathcal{K}=0$, $\text{dn}\mathcal{K}=k'$, $\text{sn}\mathcal{K}=1$.

The only solution of these equations is $C_2=C_3=0$ and hence the negative eigenvalue $E_2=k^2-1$ associated with ψ_2 does not contribute to the functional integral I. The transition amplitude as well as the level splitting will therefore be real as expected. For our present purposes we employ an alternative procedure to evaluate the functional integral I. We introduce the transformation (see, e.g., Ref. [16])

$$\chi(\tau)=y(\tau)+N(\tau)\int_{\tau_i}^{\tau}\frac{\dot{N}(\tau')}{N^2(\tau')}y(\tau')d\tau' \tag{4.9}$$

where

$$N(\tau)=\frac{d\phi_c}{d\tau}=\frac{2}{g}k\ \text{cn}(\tau) \tag{4.10}$$

is the zero eigenmode of the small fluctuation equation. The boundary conditions for $y(\tau)$ are obtained from those of $\chi(\tau)$ and are $y(\tau_i)=0$ and

$$y(\tau_f)+N(\tau_f)\int_{\tau_i}^{\tau_f}\frac{\dot{N}(\tau')}{N^2(\tau')}y(\tau')d\tau'\equiv y(\tau_f)+f(\tau_f)=0, \tag{4.11}$$

the latter defining the function $f(\tau_f)$.

With the transformation (4.9) the potential term in Eq. (4.5) is eliminated and the functional integral I of Eq. (4.3) becomes that of a free particle propagator but with the constraint (4.11). Inserting into I the identity

$$\int dy_f\delta(y(\tau_f)+f(\tau_f))=\frac{1}{2\pi}\int_{-\infty}^{\infty}dy_f e^{-i\alpha(y(\tau_f)+f(\tau_f))}d\alpha=1 \tag{4.12}$$

we have

$$\begin{aligned} I&=\frac{1}{2\pi}\left|\frac{\partial\chi}{\partial y}\right|\int dy_f\int_{yi=0}^{y_f}\mathcal{D}\{y\}\exp\left\{-\frac{1}{2}\int_{\tau_i}^{\tau_f}\left[\frac{dy}{d\tau}+i\alpha\frac{N(\tau_f)}{N(\tau)}\right]^2d\tau\right\}\exp\left[-\alpha^2\frac{N^2(\tau_f)}{2}\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]d\alpha\\ &=\frac{1}{\sqrt{2\pi}}\left|\frac{\partial\chi}{\partial y}\right|\left[N^2(\tau_f)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{-1/2}. \end{aligned} \tag{4.13}$$

The Jacobian of transformation (4.9) is [16]

$$\left|\frac{\partial\chi}{\partial y}\right|=\left[\frac{N(\tau_f)}{N(\tau_i)}\right]^{1/2}, \tag{4.14}$$

so that the final result of the functional integration is

$$I=\frac{1}{\sqrt{2\pi}}\left[N(\tau_i)N(\tau_f)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{1/2}. \tag{4.15}$$

The next step in our evaluation of the transition amplitude (3.7) is to find the proper wave functions $\psi_{\mathcal{E}+}$ and $\psi_{\mathcal{E}-}$, and to carry out the end point integration. In the barrier the wave functions have, of course, the WKB form, that is,

$$\psi_{\mathcal{E},-}(\phi_i) = \frac{C\exp(-\int_{-a}^{\phi_i} N(\phi)d\phi)}{\sqrt{N(\phi_i)}},$$

$$\psi_{\mathcal{E},+}(\phi_f) = \frac{C\exp(-\int_{\phi_f}^{a} N(\phi)d\phi)}{\sqrt{N(\phi_f)}} \equiv \frac{Ce^{-\Omega(\phi_f)}}{\sqrt{N(\phi_f)}}, \quad (4.16)$$

where $N(\phi)$ is again the zero mode (which corresponds to the velocity of the pseudoparticle). The renormalization constant C is defined and evaluated as

$$C = \left[\frac{1/2}{\int_a^{a'} \frac{d\phi}{\sqrt{2(\mathcal{E}-V)}}}\right]^{1/2} = \left[\frac{1}{4\mathcal{K}(k')}\right]^{1/2}. \quad (4.17)$$

We expand $S_c(\phi_f, \phi_i, T)$ and $\Omega(\phi_f)$ as power series of $[\phi_f - \phi(T)]$ with $\phi(\tau) \to a$ up to the second order for the Gaussian approximation. Thus

$$S_c = S_c(\phi(T), \phi(-T), T) + \frac{1}{2}\frac{\partial^2 S}{\partial \phi_f^2}\bigg|_{\phi_f=\phi(T)} \times [\phi_f - \phi(T)]^2 + \cdots, \quad (4.18)$$

and, for Ω of (4.16),

$$\Omega(\phi_f) = \frac{1}{2}\frac{\partial^2 \Omega}{\partial \phi_f^2}[\phi_f - \phi(T)]^2 + \cdots.$$

Substituting Eqs. (4.15)–(4.18) into Eq. (3.7), the end point integration $d\phi_i$ along with $N(\phi_i)$ in the denominator becomes a Euclidean time integration and the integration of $d\phi_f$ is seen to become a Gaussian integration. The transition amplitude then is

$$A_{+,-} = \frac{2T}{4\mathcal{K}(k')}e^{-W}e^{-2E_{\rm cl}T} \equiv A_{+,-}^{(1)} \quad (4.19)$$

where we have used the formula (see appendix of Ref. [6])

$$\frac{\partial^2 S_c}{\partial \phi_f^2} = \frac{1}{N(\tau_f)}\left[\dot{N}(\tau_f) + \frac{1}{N(\tau_f)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2}}\right] \quad (4.20)$$

and

$$\frac{\partial^2 \Omega(\phi_f)}{\partial \phi_f^2} = -\frac{\dot{N}(\tau_f)}{N(\phi_f)} \quad (4.21)$$

which follows from the definition of $\Omega(\phi_f)$ in Eq. (4.16).

C. Sum over contributions from instantons plus instanton–anti-instanton pairs

The configuration of one instanton plus one instanton–anti-instanton pair is given by our solution with $n = 3$ in Eq. (2.4) and $T = 3K$. The trajectory for the one particle and one pair is shown in Fig. 1(b). The transition amplitude is seen to be (cf. Refs. [3] and [4])

$$A_{+,-}^{(3)} = \int_{-T}^{T} d\tau_1 \int_{-T}^{T_1} d\tau_2 \int_{-T}^{T_2} d\tau \left[\frac{1}{4\mathcal{K}(k')}\right]^3 e^{-3W}e^{-2E_{\rm cl}T}. \quad (4.22)$$

The total amplitude is given by the sum

$$A_{+,-} = \sum_{m=0}^{\infty} A_{+,-}^{(2m+1)} = e^{-2E_{\rm cl}T}\sinh\left\{\frac{T}{2\mathcal{K}(k')}e^{-W}\right\} \quad (4.23)$$

with [cf. (4.4)]

$$W = S_c - 2E_{\rm cl}T = \frac{8}{g^2}[E(k) - k'^2\mathcal{K}(k)]. \quad (4.24)$$

Comparing (4.23) and (3.6) the level shift is given by

$$\Delta\mathcal{E} = \frac{1}{4\mathcal{K}(k')}e^{-W}. \quad (4.25)$$

V. LOW AND HIGH ENERGY LIMITS

For the energy far below the barrier height $E_{\rm cl} << 2/g^2$, i.e., $k^2 \to 1, k'^2 \to 0$, we expand the complete elliptic integrals $E(k)$ and $\mathcal{K}(k)$ as power series of k'. Up to the second order of k' we find (with formulas given in [17])

$$W = \frac{8}{g^2}\left[1 - \frac{k'^2}{2}\left\{\ln\left[\frac{4}{k'}\right] + \frac{1}{2}\right\} + 0(k'^4)\right]$$
$$= \frac{8}{g^2} - \left(n + \frac{1}{2}\right)\ln\left[\frac{2^5}{(n+\frac{1}{2})g^2}\right]^{n+1/2} - \left(n + \frac{1}{2}\right) \quad (5.1)$$

so that [with $\mathcal{K}(k' = 0) = \pi/2$]

$$\Delta\mathcal{E} = \frac{1}{2\pi}e^{-8/g^2}e^{n+1/2}\left[\frac{2^5}{(n+\frac{1}{2})g^2}\right]^{n+1/2}. \quad (5.2)$$

Here we have used the quantization $E_{\rm cl} = n + \frac{1}{2}$, which follows, for instance, from the oscillator approximation of the periodic potential around one of its minima, and $k'^2 \simeq \frac{1}{2}g^2E_{\rm cl}$. For weak coupling ($g \ll 1$) and also $(n + \frac{1}{2})g^2 \ll 1$ the transition amplitude as well as the band width $2J = \Delta\mathcal{E}$ indeed grow with energy exponentially. With the help of Stirling's formula in the form $n! \simeq \sqrt{2\pi}\left[\frac{n+\frac{1}{2}}{e}\right]^{n+1/2}$ one has

$$\Delta\mathcal{E} = \frac{1}{\sqrt{2\pi}}e^{-8/g^2}\frac{1}{n!}\left[\frac{2^5}{g^2}\right]^{n+1/2}, \quad (5.3)$$

which is in agreement with the result of Ref. [18]. The periodic potential has also been dealt with in Ref. [8] but the level splitting is not given explicitly.

It is more interesting to see the high-energy limit which is related to the instanton transition at high energy in the investigation of baryon- and lepton-number violation. When the energy E_{cl} tends the top of the barrier, i.e., $E_{\mathrm{cl}} \to 2/g^2$ and therefore $k^2 = 2/g^2 - E_{\mathrm{cl}} \to 0$. The complete elliptic integrals now have to be expanded in rising powers of k. Then with

$$E(k) = \frac{\pi}{2}\left[1 - \frac{1}{4}k^2 + O(k^4)\right] ,$$

$$\mathcal{K}(k) = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + O(k^4)\right] ,$$

we obtain

$$W \underset{k\to 0}{\sim} \frac{2\pi}{g^2}k^2 \to 0 . \tag{5.4}$$

Thus in this limit the typical suppression factor of the transition amplitude (which follows already from vacuum instantons) disappears. However, the prefactor $1/4\mathcal{K}(k')$ which is approximately a constant in the low-energy case becomes energy dependent. This reflects the anharmonic nature of the system. We have

$$\frac{1}{4\mathcal{K}(k')} = \frac{1}{4\mathcal{K}\sqrt{1-k^2}} \underset{k\to 0}{\sim} \frac{1}{4\ln(4/k)} \to 0 . \tag{5.5}$$

The transition amplitude is again suppressed, but this time by the prefactor. This is not surprising since the prefactor is proportional to the effective frequency of the real particle in one well. At high energies the number of impacts per unit time at the turning points approaches zero (see also Ref. [19]).

VI. THE BOGOMOLNY-FATEYEV RELATION

For the system under discussion the true vacuum state (or physical vacuum state) is stable. Only the tunneling of pseudoparticles leads to the level splitting. However, if one treats the one well as a perturbation theory vacuum, the tunneling away from the well to infinity would result in an imaginary part of the energy. This phenomenon was observed long ago by Bogomolny and Fateyev in connection with their discussion of the large order behavior of perturbation theory [14]. In the latter case, of course, the potential is no longer the SG potential, but a distorted one which allows the particle to escape to infinity but through the same barrier.

To see how this can happen let us regard the full period of the above solution Eq. (2.3) as the trajectory of a bounce with position $\tau_0 = \mathcal{K}(k)$ as shown in Fig. 1(a) for a period $4T$, since such a full period configuration is characterized by the boundary conditions of a bounce as defined by Coleman [14]. Also, we know from previous calculations in Ref. [5] how a bounce leads to an imaginary part of the energy. Let us calculate the amplitude for the transition from one well and back to the same well due to the pseudoparticle propagator, i.e., the quantity

$$A = \langle E|e^{-4HT}|E\rangle = e^{-4ET} . \tag{6.1}$$

We choose $4T$ here in order to keep the same time scale as for the instanton case. The energy E then becomes a complex number as a result of the tunneling. The transition amplitude A can again be calculated with the help of the path-integral method. We have

$$A = \int \psi_E^*(\phi_f)\psi_E(\phi_i)K(\phi_f,\tau_f;\phi_i,\tau_i)d\phi_f d\phi_i . \tag{6.2}$$

We are interested in the limit $\phi_f = \phi_i \to a$ (where a is a turning point), and $\tau_f - \tau_i = 4T$ denotes a full period of motion of the bounce. With a procedure similar to that used before we have the propagator

$$K(\phi_f,\tau_f;\phi_i,\tau_i) = e^{-S_c(\phi_f,\phi_i,4T)}I \tag{6.3}$$

where S_c is the classical action evaluated along the trajectory of the bounce ϕ_c, i.e.,

$$S_c = \int_{\tau_i}^{\tau_f}\left[\frac{1}{2}\left[\frac{d\phi_c}{d\tau}\right]^2 + V(\phi_c)\right]d\tau \underset{\phi_f=\phi_i\to a}{\longrightarrow} 2W + 4E_{\mathrm{cl}}T . \tag{6.4}$$

It is again the functional integral obtained by taking the fluctuation trajectories $\chi(\tau)$ into account which has to be evaluated. The boundary conditions for $\chi(\tau)$ are now

$$\chi(-2T) = \chi(2T) = 0 \tag{6.5}$$

which lead to a constraint equation on the expansion coefficients of $\chi(\tau)$ with respect to the eigenfunctions of the small fluctuation equation (2.6); i.e., one obtains

$$C_2k' - C_3 = 0 . \tag{6.6}$$

Thus C_2 and C_3 are no longer independent integration variables of two Gaussian integrations. The effective Gaussian integration for C_2 and C_3 becomes

$$\int dC_2 \int dC_3 e^{-E_2C_2^2}e^{-E_3C_3^2}\delta(C_3 - C_2k')$$

$$= \int dC_2 e^{-C_2^2(E_2+k'^2E_3)} = \int dC_2 e^{C_2^2k'^4} . \tag{6.7}$$

By rotating to the imaginary axis the integration gives rise to an imaginary value as expected. Using the same method as in the previous section for the instanton the functional integration is seen to give the same formula as in Eq. (4.15). The WKB wave functions are now defined only in the left-hand well so that [with $N(\phi) = d\phi/d\tau = \dot{\phi}$]

$$\psi_E(\phi_i) = \frac{C\exp\left(-\int_{-a}^{\phi_i}\dot{\phi}d\phi\right)}{\sqrt{N(\phi_i)}},$$
$$\psi_E(\phi_f) = \frac{C\exp\left(-\int_{-a}^{\phi_f}\dot{\phi}d\phi\right)}{\sqrt{N(\phi_f)}}. \tag{6.8}$$

Evaluating the end point integration and suming over the infinite number of bounces we obtain

$$A = e^{-4E_{cl}T}\exp\left[-i\frac{4T}{4\mathcal{K}(k')}e^{-2W}\right], \tag{6.9}$$

where (cf. [6]) the factor $(-i)$ comes from the limit of the factor Δ of the Gaussian integration $d\phi_f$, i.e.,

$$\Delta = \lim_{\pm 2T=\pm 2\mathcal{K}} \frac{1}{\left[1+N(2T)\dot{N}(2T)\int_{-2T}^{2T}\frac{d\tau}{N^2}\right]^{1/2}}$$
$$= \frac{1}{[1-2]^{1/2}} = -i \tag{6.10}$$

where $N(\tau) = d\phi_c/d\tau = \frac{2}{g}k\text{cn}(\tau+\tau_0)$ with $\tau_0 = \mathcal{K}(k)$, in the case of the bounce. The branch cuts associated with this relation are discussed in detail in Ref. [8] in the context of the method of complex paths in the complex time plane.

Comparing Eq. (6.1) and Eq. (6.9) the imaginary part of the energy is seen to be

$$\text{Im}E = \frac{1}{4\mathcal{K}(k')}e^{-2W}. \tag{6.11}$$

We now come to the interesting Bogomolny-Fateyev relation between the level splitting, Eq. (4.25), and the imaginary part of the energy is given by (6.11); i.e., we find

$$\text{Im}E = 4\mathcal{K}(k')(\Delta\mathcal{E})^2. \tag{6.12}$$

For the low energy case $(k' \to 0)$,

$$\text{Im}E = 2\pi(\Delta\mathcal{E})^2, \tag{6.13}$$

or

$$\Delta E = 4\pi i(\Delta\mathcal{E})^2, \tag{6.14}$$

where $\Delta E = 2i\ \text{Im}E$ denotes the discontinuity of the eigenenergy (low-energy case) at the cut $g^2 \geq 0$. We emphasize that the imaginary part of the energy, i.e., Eq. (6.11), is obtained in a natural way by regarding the full period of motion as a bounce configuration. The imaginary part of the ground-state energy follows as a limit of low energy. In the original paper by Bogomolny and Fateyev [15] only the ground state is discussed. It is observed that there are some difficulties in calculating the Taylor expansion of the dispersion integral, since there are no finite action solutions having equal asymptotics at $\pm\infty$ (but there are stable instanton-type solutions). In other words, in our (1+0)-dimensional case, when $T \to \pm\infty (k \to 1)$ the classical configuration (2.3) reduces to the kink solution (2.5). If one starts from the instanton (2.5) it would be impossible to obtain the imaginary part of the energy for the ground state. To overcome the difficulties a special technique was employed in Ref. [15] to obtain the imaginary part of the ground-state energy. We add finally that in Ref. [17] the Bogomolny-Fateyev relation (6.13), (6.14) has been verified from explicit expressions for ImE and $\Delta\mathcal{E}$ obtained by imposing appropriate boundary conditions on matched asymptotic expansions of the solutions of the Schrödinger equation. The same relation holds in the case of the double-well potential as in the case of the SG potential. In fact, the Bogomolny-Fateyev relation applies to theories whose perturbation series are not Borel summable. Thus ImE of the SG theory inserted into the appropriate moment integral relation for the coefficient of the perturbation series yields precisely the large-order behavior determined long ago [18] by a completely different method.

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