Exact and approximate fermion Green's functions in QED and QCD

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That special variant of the Fradkin representation, previously defined for scalar Green's functions $G_c(x,y|A)$ in an arbitrary potential $A(z)$, is here extended to the case of vector interactions and spinor Green's functions of QED and QCD. An exact representation is given which may again be approximated by a finite number N of quadratures, with the order of magnitude of the errors generated specified in advance, and decreasing with increasing N. A feature appears for both exact and approximate $G_c[A]$: the possibility of chaotic behavior of a function central to the representation, which in turn generates chaotic behavior in $G_c[A]$ for certain $A(z)$. An example is given to show how the general criterion specified here works for a known case of "quantum chaos," in a potential-theory context of first quantization. When the full, nonperturbative, radiative corrections of quantum field theory are included, such chaotic effects are removed.

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I. INTRODUCTION

This is the third paper of a series [1,2] describing a useful special variant of the exact Fradkin representation [3,4] for vector interactions, with emphasis on QED and QCD. Approximations to the spinor Green's function $G_c(x,y|A)$, defined in the presence of an arbitrary vector potential $A_\mu(z)$ in any number of space-time dimensions, are again given in terms of a finite number N of quadratures; again, the order of magnitude of the errors associated with finite N can be estimated in advance, and decreases with increasing N.

A feature appears for such vector interactions in the form of possible chaotic behavior (specified in terms of a map which depends on proper time) of a central function of the representation, defined in terms of specific potentials $A_\mu(z)$ or field strengths $F_{\mu\nu}$. The equation, or map, which defines this relationship resembles equations of motion, or other forms of mappings, found in general dynamical systems, with solutions which, for appropriate potentials or fields, may display the now-familiar signs of chaos. We state the necessary condition for the possibility of chaos of a general map given in terms of A_μ or $F_{\mu\nu}$, and then give an example of a potentially positive Lyaponov exponent, in terms of which chaos may be defined, for a potential-theory system which is known to contain chaotic behavior.

Any such fermion representation is somewhat complicated by conventional spinorial factors; but the possibility of chaos is an additional component of the construction which appears when dealing with a vector interaction. We here specialize to QED and QCD, but this feature will exist even in a scalar theory with vector interactions; it was not seen in the analysis of Ref. [1] because it was automatically suppressed in the course of an additional, simplifying, infrared (or "eikonal") approximation. In QED it is possible to define this test for chaos in a gauge-independent way, involving only field strengths; in practical terms, the corresponding mapping, and test for chaos, is much simpler within a particular gauge, specified by the potentials.

One profoundly important question arises: do the field fluctuations encountered in the transition from the first quantization of potential theory to the second quantization of quantum field theory have any effect on the former's possibility of chaos? Quantum fluctuations of the electromagnetic field, for example, could enhance, leave unchanged, or suppress the chaotic behavior induced by vectorial interactions in potential theory. We argue in this paper that the third possibility is correct.

These topics are arranged here in the following way. In the next section, a derivation is given for exact and approximate representations of the fermion propagators $G_c(x,y|A)$ in QED. Section III describes an analogous construction of the gauge-independent part of $G_c(x,y|A)$, in which similar forms for the exact and approximate representations appear. Section IV examines the map which can be used, for such vector interactions, to test for chaotic behavior, and we give an example of the construction of a pair of Lyaponov exponents, one of which is almost certainly positive, for a well-known choice of $A_\mu(z)$ in which "quantum chaos" is known to occur. Section V sketches the entire process for QCD, including a map to test for chaos. Section VI explicitly shows how the full, nonperturbative, radiative corrections of quantum field theory remove the quantum chaos of potential theory. The final section contains a very brief summary and discussion of related fundamental questions and specific calculations which can be attempted within the framework of our representation.

II. DERIVATION

As in Ref. [2], our starting point is the exact Fradkin representation [3] for a causal, QED fermion propagator moving in an arbitrary background field $A_\mu(z)$:

$$G_c(x,y|A) = i\int_0^\infty ds\, e^{-ism^2} \exp\left(i\int_0^s ds'\delta^2/\delta v_\mu^2(s')\right)\left[m - \gamma_\mu \frac{\delta}{\delta v_\mu(s)}\right] \times \exp\left[-ig\int_0^s ds' v_\mu(s') A_\mu\left(y - \int_0^{s'} v\right)\right] \times \left(\exp\left[g\int_0^s ds'\sigma_{\mu\nu}F_{\mu\nu}\left(y - \int_0^{s'} v\right)\right]\right)_+ \delta\left(x - y + \int_0^s v\right) , \tag{2.1}$$

where $v_\mu(s')$ is an artificial, proper-time-dependent four-velocity, whose fluctuations produce the causal solution to the differential equation:

$$\{m + \gamma\cdot[\partial - igA(x)]\}G_c(x,y|A) = \delta(x-y) .$$

In (2.1), the $\sigma\cdot F$ exponential must be ordered with respect to s', with those terms carrying the later values of s' standing to the left; this is necessary since any pair of quantities $\sigma\cdot F$ carrying different values of s' will not in general commute. The "linkage operator"

$$e^{\mathcal{D}}\mathcal{F}[v]|_{v\to 0} = \exp\left(i\int_0^s ds'\delta^2/\delta v^2(s')\right)\mathcal{F}[v]|_{v\to 0}$$

acting upon an arbitrary v_μ functional $\mathcal{F}[v]$ may be reexpressed as a Gaussian-weighted functional integral $\mathcal{N}\int d[v]\exp[(i/4)\int_0^s ds' v_\mu^2(s')]\mathcal{F}[v]$ over the same v dependence, with a normalization constant:

$$\mathcal{N}^{-1} = \int d[v]\exp\left(\frac{i}{4}\int_0^s ds' v^2\right) .$$

Inserting a Fourier representation for the δ function of (2.1),

$$\delta\left(x - y + \int_0^s v\right) = \int dp\, \exp\left[ip\cdot\left(x - y + \int_0^s v\right)\right] \quad \text{with } \int dp = \int \frac{d^4p}{(2\pi)^4} ,$$

the operator exp $[\mathcal{D}]$ is required to link in a pairwise fashion, to "factor pair," all the v_μ upon which it acts. The essential difference between the present vector and the previous scalar case, given in terms of a scalar potential $A(y - \int_0^{s'} v)$, is the appearance in (2.1) of the vector forms $v_\mu(s')A_\mu(y - \int_0^{s'} v)$, and the necessity of factor pairing *all* pairs of all the v_μ dependence; in particular, one must retain those linkages between $v_\mu(s')$ and $A_\nu(y - \int_0^{s''} v)$, which will appear upon expansion in powers of g. That is, because $\mathcal{D} = i\int_0^s ds'[\delta/\delta v_\mu(s')]^2$ can only link v dependence at the same value of s', the $O(g)$ factor pairing of $v_\mu(s')$ and its immediate coefficient $A_\mu(y - \int_0^{s'} v)$ must vanish. But in the $O(g^2)$ linkages of

$$\int_0^s ds_1 v_\mu(s_1) A_\mu\left(y - \int_0^{s_1} v\right)\int_0^s ds_2 v_\nu(s_2) A_\nu\left(y - \int_0^{s_2} v\right)$$

there will be contributions coming from pairings of $v_\mu(s_1)$ and $A_\nu(y - \int_0^{s_2} v)$, and from $A_\mu(y - \int_0^{s_1} v)$ and $v_\nu(s_2)$, depending on the relative size of s_1 and s_2. Of course, there are also the relatively simple linkages between the $v_\mu(s_1)$ and $v_\nu(s_2)$, as between $A_\mu(y - \int_0^{s_1} v)$ and $A_\nu(y - \int_0^{s_2} v)$, which were fully analyzed in Ref. [1], but a novel structure now appears when we include pairings between the $v_\mu(s')$ and the $A_\nu(u - \int_0^{s''} v)$. (As mentioned above, this structure was suppressed by an additional, simplifying IR approximation used in Ref. [1].)

To investigate these possibilities, we trivially replace the exponential factor of (2.1):

$$\exp\left[-ig\int_0^s ds' v_\mu(s') A_\mu\left(y - \int_0^{s'} v\right)\right] \tag{2.2}$$

by

$$\int d[\phi]\delta[\phi - v]\exp\left[-ig\int_0^s ds'\phi_\mu(s')A_\mu\left(y - \int_0^{s'} v\right)\right] , \tag{2.3}$$

where $\int d[\phi]\delta[\phi - v]$ denotes a functional integral over the δ functional which replaces $\phi_\mu(s_i)$ by $v_\mu(s_i)$ at every coordinate $0 \le s_i \le s$. That is, for a fixed s, one breaks up the interval 0 to s into R subintervals of width s/R, labeled by the index i; and one subsequently takes the limit $R\to\infty$ of

$$\prod_{\mu=1}^4\prod_{i=1}^R\int_{-\infty}^{+\infty} d^4\phi_i\,\delta\left(\phi_\mu^{(i)} - v_\mu(s_i)\right) ,$$

where $\phi_\mu^{(i)}$ denotes the average value of $\phi_\mu(s')$ in the interval s_i to s_{i+1}.

The next step is to introduce a Fourier representation for the δ functional of (2.3); using continuum notation, this is

$$\mathcal{N}' \int d[\Omega] \exp\left(i \int_0^s ds' [\Omega_\mu(s')\phi_\mu(s') - \Omega_\mu(s') v_\mu(s')] \right) \tag{2.4}$$

given in terms of a functional integral (FI) over $\Omega(s_i)$, $0 \leq s_i \leq s$, with $\mathcal{N}'$ an appropriate normalization. In this way, different $v_\mu(s')$ terms of (2.2) may be isolated and removed from the vicinity of their immediate neighbors, $A_\mu(y - \int_0^{s'} v)$, so that (2.1) may be rewritten as

$$\begin{aligned} G_c(x,y|A) = i \int_0^\infty ds\, e^{-ism^2} \int dp\, e^{ip\cdot(x-y)} e^{\mathcal{D}} \left[m - \gamma \cdot \frac{\delta}{\delta v(s)} \right] \\ \times \left[\exp\left(g \int_0^s ds' \sigma \cdot F \right) \right]_+ \mathcal{N}' \int d\Omega \int d[\phi] \exp\left(i \int_0^s ds' \Omega_\mu(s') \phi_\mu(s') \right) \\ \times \exp\left[-ig \int_0^s ds' \phi_\mu(s') A_\mu\left(y - \int_0^{s'} v \right) \right] \exp\left(i \int_0^s v(s') \cdot [p - \Omega(s')] \right) \Bigg|_{v \to 0} . \end{aligned} \tag{2.5}$$

We can now follow the scalar calculation of Ref. [2]. All steps will be the same, except for the replacement of $p_\mu \int_0^s ds' v_\mu(s')$ by $\int_0^s ds' v_\mu(s') [p_\mu - \Omega_\mu(s')]$, and the appearance of obvious spinor factors; for clarity and completeness this procedure is sketched in the Appendix, and we here quote the result:

$$\begin{aligned} \langle p | G_c[A] | p' \rangle = i \int_0^\infty ds\, e^{-ism^2} \int d^4 z\, e^{-iq\cdot z + \frac{isq^2}{4}} \prod_N{}' \frac{(-i)^2}{(2\pi)^4} \int \int d^4 P_N d^4 Q_N e^{i/2(P_N^2 + Q_N^2)} \\ \times e^{-i \int_0^s ds' [p - \Omega(s')]^2} \exp\left[-g \int_0^s ds' \frac{\partial}{\partial z_\mu} A_\mu\left(\zeta(s') - 2 \int_0^{s'} \Omega \right) \right] \\ \times \{ m - i\gamma \cdot [p - \Omega(s)] \} \left(\exp\left[g \int_0^s ds' \sigma \cdot F\left(\zeta(s') - \int_0^{s'} \Omega \right) \right] \right)_+ , \end{aligned} \tag{2.6}$$

where $\Omega_\mu(s')$ is defined as the solution to the "map" (A12),

$$\Omega_\mu(s') = g A_\mu\left(\zeta(s') - 2 \int_0^{s'} ds'' \Omega(s'') \right) . \tag{2.7}$$

We take the liberty of using the word "map" to describe this equation depending on one continuous s' variable, whose solution is known, in principle, only after an infinite number of iterations.

One small point is worth mentioning. Treating the discontinuous function $\theta(x)$ as the limit of a continuous sequence, it is natural to assign the value $\theta(0) = \frac{1}{2}$ in the evaluation of $-\mathrm{Tr} \ln |\delta f / \delta \Omega|$. That this is correct can be checked by expanding the $G_c[A]$ of (2.20) to first order in g, and explicitly verifying that $\langle p | G_c[A] | p' \rangle \simeq \tilde{S}_c(p) \langle p | p' \rangle + ig \tilde{S}_c(p) \gamma \cdot \tilde{A}(p - p') \tilde{S}_c(p') + \cdots$.

As in the scalar case of Ref. [2], this exact representation will be useful because it has a natural form of the (nonperturbative) approximation defined by retaining only a finite number of integers N in (2.6). The quality of such an approximation may be understood by remembering that it corresponds to the replacement of the variable $|x|$, in (A3), by the right-hand side (RHS) of that equation for a finite set of N values, as pictured in Fig. 1. As the measure Δ of qualitative error, choose as in Ref. [2] the difference in area of each of the curves of Fig. 1, divided by the true area of the straight line x, between 0 and 1. This yields $\Delta_{N=1} = 0.19, \Delta_{N=1,3} = 0.099, \Delta_{N=1,3,5} = 0.067$, etc. Based upon the potential-theory estimates of Ref. [2], these numbers should be regarded as upper bounds. Of course, any quantity which depends critically on the difference of such areas will not be able to be approximated in this way; but one would certainly expect an overwhelming amount of qualitative physics to be correctly described in this straightforward and nonperturbative way. Even for $N = 0$, the "zeroth approximation" corresponding to the neglect of all P_N, Q_N dependence inside the Ω_μ and $\sigma \cdot F$ terms of (2.6), this representation generalizes the scalar "symmetric eikonal" representation of Ref. [1], and should become more and more relevant as momentum transfers (of relevant scattering amplitudes) decrease. One should also note that, as in Ref. [2], there are many other ways of

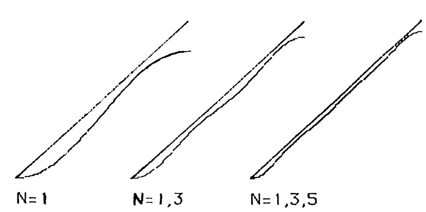

FIG. 1. A plot of $(8/\pi^2) \sum_N' N^{-2} \sin^2(N\pi x/2)$ vs x, for $N = 1$, $N = 1$ and 3, $N = 1$, 3, and 5; and a comparison with x, the exact value of this sum over all odd N, in the range $0 \leq x \leq 1$.

rewriting the exact (2.6), which need not give as accurate an approximation for a finite number of quadratures N, as will (2.6).

III. A GAUGE DIGRESSION

In the next section, we consider nonperturbative solutions of (2.7), and understand their relevance to possibly chaotic behavior. Before that, however, it may be wise to ask if a representation similar to (2.6) can be effected for the gauge-invariant portion of $G_c[A]$ in order to show, in essence, that the map (2.7), while gauge dependent, is not a gauge-dependent fiction. For this, it is useful to define the gauge-invariant part of $G_c(x,y|A)$ as

$$G_c(x,y|F) = \exp\left(-ig\int_y^x d\xi_\mu A_\mu(\xi)\right)G_\mu(x,y|A) \ , \quad (3.1)$$

where $\xi_\mu = \lambda x_\mu + (1-\lambda)y_\mu$, with $0 \le \lambda \le 1$, describes the straight-line path between the points y_μ and x_μ. Under a gauge transformation, $A_\mu \to A_\mu + \partial_\mu\Lambda$, the explicit exponential of (3.1) will generate the factor $\exp\{-ig[\Lambda(x) - \Lambda(y)]\}$, which is the negative of the complete gauge change of the exact Green's function, $G_c(x,y|A+\partial\Lambda) = \exp\{+ig[\Lambda(x)-\Lambda(y)]\}G_c(x,y|A)$.

The gauge-invariant functional $G_c[F]$ has the exact Fradkin representation [4]

$$\begin{aligned}\langle x|G_c[F]|y\rangle = i\int_0^\infty ds\, e^{-ism^2}\left[m - \gamma\cdot\frac{\delta}{\delta v(s)}\right]e^{\mathcal{D}}\delta\left(x-y+\int_0^s v\right) \\ \times\exp\left[-ig\int_0^s ds_1\int_0^{s_1} ds_2\int_0^1 \lambda d\lambda v_\mu(s_1)v_\nu(s_2)F_{\mu\nu}\left(y-\lambda\int_0^{s'} v\right)\right] \\ \times\left(\exp\left[g\int_0^s ds'\sigma\cdot F\left(y-\int_0^{s'} v\right)\right]\right)_+ \end{aligned} \quad (3.2)$$

and we now perform the same sort of analysis for $G_c[F]$ that has been done for $G_c[A]$. Introducing a Fourier representation for $\delta(x-y+\int_0^s v)$, and expanding in powers of g, one writes, for the field strength,

$$F_{\mu\nu}\left(y-\lambda\int_0^{s'} v\right) = \int dk\,\tilde{F}_{\mu\nu}(k)\exp\left(ik\cdot y - i\lambda k\int_0^{s'} v\right)$$

and again replaces the $v_\mu(s_1)$ dependence of (3.2) by $\phi_\mu(s_1)$; for simplicity, we suppress the $\sigma\cdot F$ terms, and add in their effects at the last step. In this way,

$$\begin{aligned}G_c(x,y|F) = i\int_0^\infty ds\, e^{-ism^2}\left[m-\gamma\cdot\frac{\delta}{\delta v(s)}\right]\int dp\, e^{ip\cdot(x-y)}N'\int d[\Omega]\int d[\phi]\exp\left(i\int_0^s \Omega\cdot\phi\right) \\ \times e^{\mathcal{D}}\exp\left(+i\int_0^s ds' v(s')\cdot[p-\Omega(s')]\right) \\ \times\exp\left[-ig\int_0^s ds_1\phi_\mu(s_1)\int_0^{s_1} ds_2 v_\nu(s_2)\int_0^1 \lambda d\lambda\int dk\,\tilde{F}_{\mu\nu}(k)\exp\left(ik\cdot y - i\lambda k\int_0^{s_1} v\right)\right] .\end{aligned} \quad (3.3)$$

One now notes that the argument of the last line of (3.3) may be rewritten as

$$g\int_0^s ds_1\phi_\mu(s_1)\int_0^1 d\lambda\int dk\,\tilde{F}_{\mu\nu}(k)e^{+ik\cdot y}\frac{\partial}{\partial k_\nu}e^{i\lambda k\int_0^{s_1} v} \quad (3.4)$$

any may be integrated by parts to yield

$$ig\int_0^s ds_1\phi_\mu(s_1)\int_0^1 d\lambda\int dk\, e^{+ik\cdot y}\left[i\left(\frac{\partial}{\partial k_\nu}\tilde{F}_{\mu\nu}(k)\right) - y_\nu\tilde{F}_{\mu\nu}(k)\right]e^{-i\lambda k\int_0^{s_1} v} . \quad (3.5)$$

Equation (3.5) has the same form as the $Q(s',k)$ of (2.6), except that the latter's $\tilde{A}(k)$ dependence is replaced by a slightly more complicated dependence on $\tilde{F}(k)$:

$$G_c(x,y|F) = i\int_0^\infty ds\, e^{-ism^2}\int dp\, e^{ip\cdot(x-y)}N'\int d[\Omega]\int d[\phi]e^{i\int_0^s \Omega\cdot\phi}\{m - i\gamma\cdot[p-\Omega(s)]\}$$
$$\times\sum_n \frac{(-ig)^n}{n!}\int_0^s ds_1\cdots\int_0^s ds_n\int_0^s ds_n\int_0^1 d\lambda_1\cdots\int_0^1 d\lambda_n\int dk_1\cdots\int dk_n e^{+iy\sum_l k_l}$$
$$\times R(s_1|k_1)\cdots R(s_n|k_n)e^{\mathcal{D}}\exp\left[i\int_0^s ds' v(s')\Big(p-\Omega(s')-\sum_l \lambda_l k_l\theta(s_l-s')\Big)\right]\Bigg|_{v\to 0}, \quad (3.6)$$

where we have again replaced $\delta/\delta v_\mu(s)$ by $i[p_\mu - \Omega_\mu(s)]$, and where we have used an expansion of an ordinary exponential, with

$$R(s_l|k_l) = -\phi_\mu(s_1)\left[i\left(\frac{\partial\tilde{F}_{\mu\nu}(k_l)}{\partial_{k_\nu}}\right) - y_\nu\tilde{F}_{\mu\nu}(k_l)\right]. \quad (3.7)$$

[Were the $\sigma\cdot F$ term included, one would use instead the OE expansion of (A1), with the term $ig\sigma\cdot\tilde{F}(k_l)\delta(\lambda_l-1)$ added to each $R(s_l|k_l)$ of (3.7).]

The functional operation of (3.6) is again immediate, producing forms similar to (A2), except that each k_i is multiplied by a variable λ_i, and the variables K, z, C_N, P_N reflect that slight complication. All the steps leading to (A4) go through in the same fashion, and one obtains

$$G_c(x,y|F) = i\int_0^\infty ds\, e^{-ism^2}\int dp e^{ip(x-y)}\mathcal{N}'\int d[\Omega]\int d[\phi]\exp\Big(i\int_0^s \Omega\cdot\phi - i\int_0^s ds'[p-\Omega(s')]^2\Big)$$
$$\times\{m - i\gamma\cdot[p-\Omega(s)]\}\prod_N{}'\frac{(-i)^2}{(2\pi)^4}\int d^4P_N d^4Q_N e^{i/2(P_N^2+Q_N^2)}$$
$$\times\int\int\frac{d^4K d^4Q}{(2\pi)^4}e^{iQ\cdot K + i(s/4)K^2}\int\int\frac{d^4z d^4P}{(2\pi)^4}e^{iz\cdot P}$$
$$\times\exp\left[-ig\int_0^s ds'\phi_\mu(s')\int_0^1 d\lambda\int d^4k\Big(-i\frac{\partial}{\partial k_\nu}\tilde{F}\mu\nu(k) + y_\nu\tilde{F}_{\mu\nu}(k)\Big)e^{+ik\cdot\Xi(s',\lambda)}\right]$$
$$\times\left(\exp\left[g\int_0^s ds'\sigma\cdot F(\Xi(s',1))\right]\right)_+, \quad (3.8)$$

where we have added in the $\sigma\cdot F$ dependence, and where, now,

$$\Xi(s',\lambda) = y - \lambda\left[z + Q + s'(P-2p) + 2\int_0^{s'}\Omega + \sum_N{}', R_N(s')\right].$$

It is now appropriate to undo the integration by parts of (3.5), so that the first exponential factor of (3.8) may be rewritten as

$$\exp\left[-ig\int_0^s ds'\phi_\mu(s')\int_0^1 d\lambda[\Xi_\nu(s',\lambda) - y_\nu]F_{\mu\nu}(\Xi(s',\lambda))\right]. \quad (3.9)$$

Then, again reflecting z and P, and shifting to $z\to z-Q$, all Q dependence is removed from Ξ, permitting immediate integration over Q and K. Passage to momentum space is almost as simple as before, except that the dependence on λ prevents integration over y and P from being (trivially) performed. The result is

$$\langle p|G_c[F]|p' = i\int_0^\infty ds\, e^{-ism^2}\prod_N{}'\frac{(-i)^2}{(2\pi)^4}\int\int d^4P_N d^4Q_N e^{(i/2)(P_N^2+Q_N^2)}$$
$$\times\int\int\frac{d^4z d^4P}{(2\pi)^4}\int d^4y\exp\Big(-iq\cdot y + iz\cdot P + i\frac{s}{4}P^2\Big)\{m - i\gamma\cdot[p-\Omega(s)]\}$$
$$\times\exp\Big(-i\int_0^s ds'[p-\Omega(s')]^2 - \text{Tr}\ln\frac{\delta f}{\delta l}\Big)\left(\exp\left[g\int_0^s ds'\sigma\cdot F(\Xi(s',\lambda=1))\right]\right)_+, \quad (3.10)$$

where Ω_μ is a solution of the map expressed in terms of field strengths only:

$$\Omega_\mu(s') = g\int_0^1 \lambda d\lambda W_\nu(s') F_{\mu\nu}(y+\lambda W(s')) \tag{3.11}$$

with $W(s') = z+s'(2p-P)-\sum_N' R_N(s')-2\int_0^{s'}\Omega(s'')ds''$. Although somewhat more complicated than that of (2.6), (3.11) provides an exact representation for the gauge-invariant $G_c[F]$. As for its simpler counterpart $G_c[A]$, $G_c[F]$ may now be nonperturbatively approximated in terms of a finite number, N, of quadratures, and with the same assurance of qualitative accuracy as in Ref. [2].

IV. POSSIBLY CHAOTIC BEHAVIOR

Both (2.6) and (3.10) are exact representations given in terms of (an infinite number of) quadratures over a functional of Ω_μ, defined by the map (2.7) or (3.11). It is the existence of such a map which carries with it the inescapable possibility of chaotic behavior, at least in the present context of vectorial interactions in potential theory. In this section we describe how this appears, and give an example in the context of one well-known case of classical chaos.

All of the analysis used here is given directly in terms of proper time $\tau, x_\mu = x_\mu(\tau)$, rather than in terms of a time, t, as measured in specific Lorentz frame and used as the independent variable of spatial coordinates, $x_k(t)$. In the simplest of the cases considered here, it is possible to calculate the $x_\mu(\tau)$ exactly; and then to express $x_0(\tau) = t(\tau)$ in terms of an elliptic integral whose "amplitude" is directly accessible, so that this relation can be inverted to obtain $\tau = \tau(t)$, thereby solving for the desired $x_k(t)$. In this example no chaos occurs because the appropriate average over Lyapunov exponents vanishes. A discussion of the interplay between the relativistic τ formalism and the noncovariant t formalism will be given elsewhere.

In principle, one might expect nonlinear quantum systems to reflect at least partially the chaotic nature of their classical limits. Indeed, solutions for classical, charged particles moving in a specified Maxwell field $F_{\mu\nu}(x)$ described by

$$\frac{d^2x_\mu}{d\tau^2} = \frac{e}{mc}\sum_\nu \frac{dx_\nu}{d\tau}F_{\mu\nu}(x) \tag{4.1}$$

for appropriately nonlinear x dependence of $F_{\mu\nu}$ must be expected to display the chaotic behavior now well documented in a variety of fields, and by a variety of methods [6,7]. We comment below that, with one important modification, the Green's-function map (2.7), in its semiclassical limit, may be related to a "first integral" of the classical (4.1).

High-energy physics has heretofore escaped the impact of chaotic classical dynamics because of its fortunate ability to rely on perturbative expansions, which destroy the nonperturbative arguments leading to the possibility of chaos. Even certain nonperturbative approximations, such as the standard eikonal approximation [4], remove the possibility of chaos because they, in effect, destroy the needed, exact nature of relevant maps [8]. This can be seen immediately from (2.7), whose perturbative expansion in effect removes $\int_0^{s'}\Omega$ from the argument of A_μ:

$$\Omega_\mu(s') \approx gA_\mu(\zeta(s')) - 2g^2\frac{\partial A_\mu}{\partial z_\nu}(\zeta(s'))\cdot\int_0^{s'} ds'' A_\nu(\zeta(s''))+\cdots , \tag{4.2}$$

so that one sums a g^n expansion for Ω_μ, rather than solving an integral or differential equation, as in (4.2). An eikonal approximation, on the other hand, would replace the term $\int_0^{s'}\Omega(s'')ds''$ in the argument of A_μ by $s'\bar{\Omega}_\mu$, where $\bar{\Omega}_\mu$ is an appropriately chosen, averaged four-velocity suggested by the specific scattering problem. For both cases, the repeated "feedback" obtained from the Ω dependence within A_μ is missing, as the map no longer corresponds to an equation which must be solved, and which can, for suitably nonlinear $A_\mu(x)$, display chaotic behavior. One sees that strict attention to the exact forms of these vector interactions, as well as to those of the nonperturbative, finite-N-quadrature approximations discussed in previous sections, must finally bring the possibility of chaos into the realm of high-energy physics [9].

In a Green's-function context, this possibility appears in a most efficient way, for it is specified not in terms of time, nor of space, nor by a mixed partial differential equation (PDE) formulation, but in terms of proper time. This suggests that chaotic behavior in proper time, when restricted to lie inside the light cone, will correspond to (temporal) chaos; while such behavior outside the light cone may refer to a form of (spatial) turbulence. We shall not elaborate further on this distinction in this paper, but only note that such a Green's-function description in terms of proper time would seem to be an obvious way of simultaneously describing both chaos and turbulence [8].

To the best of the authors' knowledge, Green's functions have previously been used [7] to study the possibility of chaos in semiclassical, nonrelativistic systems. The representations of this paper permit one to understand the appearance of chaotic behavior in the Green's functions describing general, relativistic systems (with vector interactions). It is amusing that the possibility of chaos seems to be enhanced by the true quantum fluctuations of these Green's functions, away from semiclassical limits. As indicated below, for appropriately nonlinear vector potentials, combined with a full relativistic treatment, one must expect to find the possibility of a chaotic comportment of the function Ω_μ, appearing in the integrand of the final, proper-time integration.

For more than a decade, there have been arguments and examples of "quantum chaos" for a classical particle whose motion is in some, semiclassical sense partially governed by Bohr-Sommerfeld quantization. We shall here illustrate our remarks by one of these examples [10], the case of an electron moving in a Coulomb potential upon which is superimposed a magnetic field. Physically, there seems to be no question of chaos when the magnitude of the bound-state electrostatic energy-level differences are either much larger or much smaller than the magnetic energy-level differences; but when the two are comparable, the electron becomes "confused," and its semiclassical motion displays the patterns of irregularity found in chaotic classical systems. By the criterion of the present paper, the Green's function of this example does not appear to be chaotic; but when at least a part of the quantum fluctuations of the true Green's function are retained, one should obtain certifiably chaotic behavior.

We first ask if there is any connection between the map (2.7) and the classical equation of motion (4.1). Let us approximate $G_c[A]$, and hence (2.7), by the neglect of quantum fluctuations of relevant coordinates, defining a semiclassical limit first by dropping all (P_N, Q_N) dependence inside the (2.6) argument of A_μ—this is the QED version of what was termed the "phase-averaged," or $\langle\text{ph}\rangle$, approximation of Refs. [1] and [2]—and secondly by imagining that q is small, so that wave-mechanical fluctuations of the remaining variables are not important. This latter step, the replacement of $G_c^{\langle\text{ph}\rangle}[A]$ by the "no-recoil," or Bloch-Nordsieck approximation $G_{\text{BN}}[A]$, is not essential, but makes for conceptual simplicity.

The length $\zeta(s')$ then becomes $z+s'(p+p') \to z+2s'p$, and we switch to a proper-time variable τ with proper dimensions, by the replacement of s' by $\tau/2m$. With the representation $\Omega_\mu = dX_\mu/d\tau$, the map now reads

$$\frac{dX_\mu}{d\tau} = \frac{g}{2m}A_\mu\left(z+\tau\frac{p}{m}-2\left[X\left(\frac{\tau}{2m}\right)-X(0)\right]\right) .$$

It will be more convenient to denote the argument of A_μ by x and write the equivalent

$$\frac{dx_\mu}{d\tau} = \frac{p_\mu}{m} - \frac{g}{m}A_\mu(x) . \tag{4.3}$$

Then

$$\begin{aligned}\frac{d^2x_\mu}{d\tau^2} &= \frac{-g}{m}\sum_\nu \frac{dx_\nu}{d\tau}\frac{\partial}{\partial x_\nu}A_\mu(x)\\ &= \frac{g}{m}\sum_\nu \frac{dx_\nu}{d\tau}\left[F_{\mu\nu}(x) - \frac{\partial}{\partial x_\mu}A_\nu(x)\right] .\end{aligned} \tag{4.4}$$

Inserting (4.2) into (4.3), there follows

$$\frac{d^2x_\mu}{d\tau^2} = \frac{g}{m}\sum_\nu \frac{dx_\nu}{d\tau}F_{\mu\nu}(x) + 2\sum_\nu \frac{dx_\nu}{d\tau}\frac{\partial}{\partial x_\mu}\left(\frac{dx_\nu}{d\tau}\right) . \tag{4.5}$$

But the last term of (4.5) may be rewritten as $(\partial/\partial x_\mu)\sum_\nu (dx_\nu/d\tau)^2$, which quantity must vanish if the four-velocity $v_\nu - dx_\nu/d\tau$ is to represent a particle on its mass shell (or energy shell, in a nonrelativistic context), an association which is certainly compatible with the remaining terms of (4.5), and which we shall assume temporarily. The result is then just (4.1), showing that in an appropriate semiclassical limit, the map (2.7) is compatible with the standard, classical equation of motion, if only the mass-shell property of the particle were guaranteed (rather than assumed). In fact, the map might be called a "first integral" of (4.1), since it involves terms of one derivative less, although appearing in the decidedly nontrivial form of a map. We shall return to this point below.

For reasons which will become clear immediately, we will allow the magnetic field to vary in two transverse directions, by introducing a function $\phi(x_\perp^2)$ into the expression for the vector potential:

$$A_\mu(x) = \frac{B}{2}[x_1\delta_{\mu 2} - x_2\delta_{\mu 1}]\phi(x_\perp^2) + \frac{iZg}{r}\delta_{\mu 4} , \tag{4.6}$$

where $r = \left[\sum_{i=1}^3 x_i^2\right]^{1/2}$, $x_\perp^2 = x_1^2 + x_2^2$, and the four-vector notation used here is $a_\mu = (\mathbf{a}, ia_0)$.

Substitution of (4.6) into the map (2.7) leads to

$$\frac{dx_\mu}{d\tau} = v_\mu - \frac{g}{m}A_\mu(x) \equiv f_\mu(x)$$

or, in component from, with $v_\mu = (1/2m)(p+p')$, $x_4 = ix_0 = it$,

$$\begin{aligned}&x_\mu(\tau) = z_\mu + \tau v_\mu - 2[X_\mu(\tau) - X_\mu(0)], \quad \omega = gB/2m ,\\ &\frac{dx_1}{d\tau} = v_1 + \omega x_2\phi(x_\perp^2) = f_1(x) ,\\ &\frac{dx_2}{d\tau} = v_2 - \omega x_1\phi(x_\perp^2) = f_2(x) ,\\ &\frac{dx_3}{d\tau} = v_3 = f_3, \quad \frac{dt}{d\tau} = v_0 + \frac{Zg^2}{mr} = f_0 .\end{aligned} \tag{4.7}$$

We have neglected the quantum fluctuations specified by variations of the (P_N, Q_N), so that $x_\mu(\tau)$ acts as effective position and time coordinates of a particle with "average" momentum $(p+p')/2$. Note, however, that the four-velocities calculated from (4.7) will not, with constant v_μ, satisfy the mass-shell condition.

To test for chaos, one is instructed [6,7] to calculate the local, or instantaneous Lyapunov exponents λ^α as the eigenvalues of the Jacobian of the continuous transformation: $\det|\delta f/\delta x - \lambda|$, and then to average the λ^α—there are a variety of ways to do this—over a sufficiently long τ interval (such as a period, for periodic orbits), to be able to see whether any $\langle\lambda^\alpha\rangle$ may be considered a positive constant over that interval; if so, the system is expected to be chaotic. In this case, the calculation of the local exponents is straightforward, yielding two zero exponents and a pair which satisfy

$$\lambda = \pm i\omega\phi[1 + 2x_\perp^2\phi'/\phi]^{1/2} . \tag{4.8}$$

From (4.8) it is clear that a constant magnetic field, $B \to B_0$ for $\phi \to 1$, corresponds to imaginary roots, and hence to pure oscillations in the distance between neighboring trajectories. Hence, this Green's function, in its semiclassical limit, does not display the chaotic behavior found in the classical motion; and the reason for the difference is precisely that the "motion" to which (4.7) corresponds does not contain the needed mass-shell restriction. If the map (4.6) is altered so that the mass-shell condition is maintained, one finds equations equivalent to (4.1), reminiscent of Hamilton-Jacobi theory, in which the same analysis leads to the possibility of chaos.

In the present context, the only chance of chaos (trajectories that diverge exponentially with increasing proper time) is to have ϕ'/ϕ sufficiently negative to convert the square root of (4.8) to imaginary values, in which case the possibility of one positive eigenvalue will exist. However, if the magnetic field is allowed to fall away to zero, or even to change sign, the "recurrence" of the motion will be lost, as the particle moves out to larger and larger $x_\perp$ values. What is needed, then, is another augmentation of the field, that is, of ϕ, so that the particle is bound in a narrow $x_\perp$ range, and where the possibility exists that the average value of one exponent will be positive.

Unfortunately, however, this motion appears to be integrable [11], with explicit, oscillatory solutions possible in relevant energy ranges; and one may expect that either the average values of both exponents will vanish, or that even if one average is real and positive the appellation "chaotic" is not appropriate. In order to achieve motion in the present context which is truly chaotic, one must retain at least a part of the oscillatory τ dependence contained in the P_N, Q_N terms, which were neglected in taking the semiclassical limit. Then the problem is no longer integrable, overlapping frequencies will appear, and chaotic motion always occurs. Examples of this are well known in the mathematical literature of chaos [7], and in the present context signify that there is a fundamental uncertainty built into these exact Green's functions of first quantization.

This can be of disastrous practical significance because chaos in the construction of $\Omega_\mu(s')$ means ultrasensitive dependence upon the initial conditions for the differential equation for $dX_\mu(s')/ds'$, analogous to (4.7). Those $s' = 0$ initial conditions depend upon $z + (2\sqrt{s}/\pi)\sum'_N (1/N)P_N$; and if machine integration is going to be necessary for the evaluation of integrals over the latter variables, unavoidably small errors will be introduced into the initial conditions for $X_\mu(s')$. Thus, if the map (2.7) defining $\Omega_\mu(s')$ is chaotic, the results of such numerical integration can lead to wildly differing answers.

Very similar remarks may be made for the map (3.11) of the gauge-invariant $G_c[F]$, and for the somewhat more involved Green's function of QCD obtained in the next section. Apparently, all of these Green's functions, as well as the appropriate closed-fermion-loop functionals $L[A] = \mathrm{Tr}\ \ln(G_c[A]/G_c[0]^{-1})$, display in their proper-time representation the basic uncertainty which follows from the possibility of a chaotic comportment of Ω_μ for sufficiently nonlinear $A_\mu(z)$. In practical terms this may not seem particularly important, since those few potential-theory problems which can be solved exactly display no chaos, and if all the rest which cannot be solved exactly are approximated in such a way that the basic chaotic behavior exhibited above is suppressed.

In theoretical terms, however, this discovery can be a profound shock—it was to the authors—for the existence of these Green's functions forms an underpinning to all of conventional, quantum field theory, in the sense that all of the latter's correlation functions, or n-point functions, may be obtained in terms of Gaussian-weighted, functional integrations over all possible fluctuations of the fields $A_\mu(z)$ appearing in products of the $G_c[A]$ and $\exp\{L[A]\}$. What will now happen, in principle, to exact, field-theory expressions for various physical quantities? Does the nonperturbative sum of all the Lamb shift terms, were it ever possible to calculate them, have a numerical value subject to chaotic fluctuations? Fortunately, the answer to these questions is no, as explained in Sec. VI, where "second quantization" comes to the aid of "first."

V. EXTENSION TO QCD

Return to (2.1) and make the replacement $A_\mu(z) \to A^a_\mu(z)\lambda_a$, where the λ_a denote matrices of the fundamental representation of SU(N), for $N = 3$ the Gell-Mann matrices. An immediate difference here is that all exponential factors of exact and approximate representations are now replaced by ordered exponentials (OEs). But all steps of the analysis of Sec. II will go through effectively unchanged if one makes the replacement

$$\left(\exp\left[-ig\int_0^s ds'\phi_\mu(s')A^a_\mu(s')\cdot\lambda_a\right]\right)_+ = \mathcal{N}'\int d[\alpha]\int d[\beta]\exp\left(i\int_0^s ds'\alpha_a(s')\beta_a(s')\right)\left(\exp\left[i\int_0^s ds'\lambda_a\beta_a(s')\right]\right)_+ \times\exp\left(-ig\int_0^s \phi_\mu(s')\alpha_a(s')A^a_\mu(s')\right), \quad (5.1)$$

where α_a, β_a, are s'-dependent vectors [in the sense of this fundamental representation of SU(N)], whose functional integration on the RHS of (5.1) reproduces the LHS OE; as in Sec. II, $\mathcal{N}''$ is a normalization constant defined so that

$$\mathcal{N}''\int d[\alpha]\exp\left(i\int_0^s ds'\vec{\alpha}(s')[\vec{\beta}(s') - g\phi_\mu(s')\vec{A}_\mu(s')]\right) = \delta\left[\vec{\beta} - g\phi_\mu\vec{A}_\mu\right] \quad (5.2)$$

and we have suppressed in (5.2) all but the s' dependence of the argument of A^a_μ. All steps leading to (2.6) are unchanged, except for the additional $\mathcal{N}'' \int d[\alpha] \int d[\beta]$ and the replacement of the map (2.7) by

$$\Omega_\mu(s'|\alpha) = g\alpha_a(s') A^a_\mu\left(\zeta(s') - 2\int_0^{s'} \Omega\right) . \tag{5.3}$$

Now, Ω is an implicit functional of α, as well as of the P_N, Q_N, and subsequent integration over these variables must be understood:

$$\begin{aligned}\langle p|G_c[A\cdot\lambda]|p'\rangle = i\int_0^\infty ds\, e^{-ism^2}\cdot \mathcal{N}''\int d[\alpha]\int d[\beta] e^{i\int_0^s \vec{\alpha}\cdot\vec{\beta}}\left(e^{i\int_0^s \vec{\lambda}\cdot\vec{\beta}}\right)_+ \\ \times \prod_N{}' \frac{(-i)^2}{(2\pi)^4}\int\int d^4P_N d^4Q_N e^{(i/2)(P_N^2+Q_N^2)}\int d^4z\, e^{-iq\cdot z+isq^2/4} \\ \times\{m - i\gamma\cdot[p-\Omega(s'|\alpha)\}\exp\left(-i\int_0^s ds'[p-\Omega(s'|\alpha)]^2\right) \\ \times\left(\exp\left[g\int_0^s ds'\sigma\cdot F\left(\zeta(s') - 2\int_0^{s'} ds''\Omega(s''|\alpha)\right)\right]\right)_+ \\ \times\exp\left[-g\int_0^s ds'\alpha_a(s')\partial_\mu A^a_\mu\left(\zeta(s') - 2\int_0^{s'} ds''\Omega(s'|\alpha)\right)\right]\end{aligned} \tag{5.4}$$

with (5.3) and $F^a_{\mu\nu}(x) = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu)$.

Inasmuch as the Ω_μ of (5.3) resembles that of (2.7), with $\vec{\alpha}(s')\cdot\vec{A}_\mu(s')$ replacing $A_\mu(s')$, all of the analysis of Sec. IV goes through, and we conclude that the map (5.3) can be used, for arbitrary $\alpha_a(s')$, to determine if and when chaotic behavior exists for Ω_μ, and hence for the exact representation (5.4). Again, one feels more confident of the analysis for a semiclassical, or $\langle\text{ph}\rangle$ approximation, where the P_N, Q_N fluctuations are missing in $\zeta(s')$, and one has other knowledge that chaotic effects may occur in first-quantized QCD.

Apart from the possibility of chaos, (5.4) satisfies a basic need for an exact representation which can be approximated in a nonperturbative and reasonable straightforward way, and in which one again has at least an idea of the order of magnitude of the errors introduced by the approximation. As for QED and the previous scalar case, use of a finite number N of quadratures should generate sensible, accurate, and not-too-complicated approximations for quark Green's functions in the presence of an effective gluon background. This is briefly discussed in Sec. VII.

VI. FULL RADIATIVE CORRECTIONS AND QUANTUM CHAOS

In the preceding sections we have seen how the exact and approximate representations for $G_c[A]$ and $G_c[F]$ can lead to a natural version of quantum chaos. In this section we try to understand what happens when the full radiative corrections of quantum field theory are included.

We will apply the exact representation of this paper to the calculation of the simplest, two-point function of QED, the dressed fermion propagator in an external field:

$$\begin{aligned}\langle p|S'_c[A^{\text{ext}}]|p'\rangle = e^{\mathcal{D}}_A\langle p|G_c[A^{\text{ext}}+A]|p'\rangle \\ \times\left.\frac{e^{L[A^{\text{ext}}+A]}}{\langle S[A^{\text{ext}}]\rangle}\right|_{A\to 0} ,\end{aligned} \tag{6.1}$$

where $\mathcal{D}_A = (i/2)\int(\delta/\delta A_\mu)D_{c,\mu\nu}(\delta/\delta A_\nu), D_{c,\mu\nu}$ is the photon propagator, $L[A] = \text{Tr}\ln[1+ig\gamma\cdot AS_c] = L\{G_c[A]\}\to L\{G_c[F]\}$ is the fermion closed-loop functional, with the vacuum-to-vacuum normalization factor

$$\langle S[A^{\text{ext}}]\rangle = e^{\mathcal{D}} e^{L[A^{\text{ext}}+A]}|_{A\to 0} .$$

The functional linkage operations here are exactly equivalent to functional integration over a Gaussian weight:

$$e^{\mathcal{D}_A}\mathcal{F}[A]|_{A\to 0} = \mathcal{N}\int d[A] e^{-\frac{1}{2}\int A(\mu^2-\partial^2)A}\mathcal{F}[A] ,$$

where $(\mu^2-\partial^2) = D_c^{-1}$, and $\mathcal{N}^{-1}$ is the same functional integral but with $\mathcal{F}[A] = 1$.

It will be simplest to work in the "quenched" approximation, neglecting the A fluctuations of $L[A^{\text{ext}}+A]$; this means the replacement of $L[A^{\text{ext}}+A]$ by $L[A^{\text{ext}}]$, and $\langle S[A^{\text{ext}}]\rangle$ by $\exp\{L[A^{\text{ext}}]\}$ in (6.1). This is not essential to subsequent arguments, for one can include arbitrary powers of $L[A^{\text{ext}}+A]$ fluctuations with unchanged conclusions. The same remark is true for the calculation of all other n-point functions of the theory, and this will become clear below.

We shall work with the simpler, $G_c[A]$ formulation, and now ask the reader to return to the exact representation of (A10) before the final $\mathcal{N}'\int d[\Omega]\int d[\phi]$ integrations leading to (2.6) were performed. In compact notation this may be written as

$$\langle p|G_c[A]|p'\rangle = i\int_0^\infty ds\, e^{-ism^2}\int d^4z\, e^{-iq\cdot z+isq^2/4}\prod_N{}'\frac{(-i)^2}{(2\pi)^4}\int\int d^4P_N d^4Q_N e^{i/2(P_N^2+Q_N^2)}$$
$$\times\mathcal{N}'\int d[\phi]\int d[\Omega]\mathcal{F}[\Omega]\exp\left[i\int_0^s ds'\phi_\mu(s')\left\{\Omega_\mu(s')-gA_\mu\left(\zeta(s')-2\int_0^{s'}\Omega\right)\right\}\right], \tag{6.2}$$

where $\mathcal{F}[\Omega]$ represents all the remaining Ω dependence visible in (A10). For simplicity, we suppress the $\sigma\cdot F$ term of (2.15); but this A dependence can be incorporated without difficulty, and the conclusions are unaltered. The effect of all the radiative corrections corresponding to nonperturbative fluctuations of the quantized electromagnetic field can now be seen by inserting (6.2) into the quenched version of (6.1):

$$\langle p|S_c'[A^{\rm ext}]|p\rangle = e^{\mathcal{D}_A}\langle p|G_c[A+A^{\rm ext}]|p'\rangle|_{A\to 0}\,. \tag{6.3}$$

The essential operation is the linkage operator acting on the second line of (6.2), which yields

$$\mathcal{N}'\int d[\Omega]\mathcal{F}[\Omega]\int d[\phi]e^{i\int_0^s ds'\phi_\mu f_\mu}\exp\left(+\frac{ig^2}{2}\int_0^s ds_1\int_0^s ds_2\phi_\mu(s_1)D_{c,\mu\nu}(\Xi)\phi_\nu(s_2)\right), \tag{6.4}$$

where $f_\mu(s')=\Omega_\mu(s')-gA_\mu^{\rm ext}[\zeta(s')-2\int_0^{s'}\Omega]$ and $\Xi(s_1,s_2)=(s_1-s_2)\mathcal{P}-2\int_{s_2}^{s_1}ds''\Omega(s'')$.

The difference between (A10) and (6.4) is that the functional integral over ϕ is now Gaussian, and produces

$$[\mathcal{N}']^{1/2}\int d[\Omega]\mathcal{F}[\Omega]e^{-\frac{1}{2}{\rm Tr}\,\ln K}\exp\left[-\frac{i}{2}\int_0^s ds_1\int_0^s ds_2 f_\mu(s_1)\langle s_1|(K^{-1})_{\mu\nu}|s_2\rangle f_\nu(s_2)\right]$$

or

$$[\mathcal{N}']^{1/2}\int d[f]\mathcal{F}[f+gA^{\rm ext}]e^{-\frac{1}{2}\int f\cdot K^{-1}\cdot f}\exp\left(-{\rm tr}\,\ln\left[1-g\frac{\delta A^{\rm ext}}{\delta\Omega}\right]\right)e^{-\frac{1}{2}{\rm Tr}\,\ln K}, \tag{6.5}$$

where $\langle s_1|K_{\mu\nu}|s_2\rangle = g^2D_{c,\mu\nu}(\Xi)$. Equation (6.5) is a Gaussian-weighted, functional integral over $\int d[\Omega]$, or over $\int d[f]$, which is sufficiently complicated so that it cannot be evaluated explicitly. However, the map of (2.7) no longer appears, nor will the possibility of chaos which results from that map. Here, the sharp δ functional of (A11) has been replaced by a smoother, Gaussian-weighted integrand over a kernel defined by the radiative corrections; and however complicated the final, nonperturbative results may be, it is not the chaos of Sec. IV. Rather, it is a clear example of what has been termed [12] "environment-induced decoherence," as the special coherence underlying the δ functional is removed by the radiative corrections, along with the map which can lead to chaos.

The same functional operations acting upon any pair of $G_c[A]$, each with their own representation (6.2), will lead to a linked combination of terms, of form analogous to (6.4):

$$\mathcal{N}_a'\int d[\Omega_a]\mathcal{F}_a[\Omega_a]\mathcal{N}_b'\int d[\Omega_b]\mathcal{F}_b[\Omega_b]\int d[\phi_a]\int d[\phi_b]\exp\left[i\int_0^{s_a}\phi_a f_a+i\int_0^{s_b}\phi_b f_b\right]$$
$$\times\exp\left[\frac{ig^2}{2}\int d^4u\int d^4w\, Q_\mu(u)D_{c,\mu\nu}(u-w)Q_\nu(w)\right] \tag{6.6}$$

with

$$Q_\mu[u]=\int_0^{s_a}ds_a'\phi_\mu^{(a)}(s_a')\delta^{(4)}\left[u-\zeta_a(s_a')+2\int_0^{s_a'}\Omega_a\right]+\int_0^{s_b}ds_b'\phi_\mu^{(b)}(s_b')\delta^{(4)}\left[u-\zeta_b(s_b')+2\int_0^{s_b'}\Omega_b\right]$$

and where we have used the (a,b) indices to distinguish terms coming from the two $G_a[A]$. Again, the essential point is that (6.6) is Gaussian in both ϕ_a and ϕ_b, and hence the "coherent" maps of form (2.7) are no longer present. If any number of such $G_c[A^{\rm ext}+A]$ factors are included, the conclusion will be the same, while the same remark is true if any of the $G[A^{\rm ext}+A]$ factors are replaced by corresponding $L[A^{\rm ext}+A]$ (each of which has a representation very similar to that of $G_c[A^{\rm ext}+A]$).

VII. SUMMARY AND DISCUSSION

In this paper we have extended the exact representation for a scalar Green's function with scalar interaction to the cases of vectorial interactions of QED and QCD, and we have seen how the possibility of chaos can appear in this potential-theory context. We have derived maps for a central function of each exact representation whose nonperturbative and exact solutions may display the chaotic behavior so familiar in so many other branches of nonlinear science, a behavior here displayed in terms of the most basic Green's functions $G_c[A]$. These representations may be approximated in a nonperturbative way, by retaining only a finite number N of quadratures of the exact description, and as in the scalar case, one knows in advance the order of magnitude of the errors as a $f(N)$.

Many questions and applications may now be phrased and attempted, and at least partially resolved. For example, it is now possible to express, in terms of a few quadratures, an approximate but reasonable "generalized eikonal" or $\langle \mathrm{ph} \rangle$ representation for a quark propagator in a specified gluon background field. This quantity will be most useful in extending the gluon-sector analysis of QCD given [13] in terms of a modified field-strength formalism; in fact, the entire analysis of Refs. [1] and [2], and of this paper, originated in attempts to find decent, nonperturbative approximations for such quark propagators.

Defined in the presence of "condensed" gluon fields in the form of thin flux tubes, such quark Green's functions will not be necessary for an eikonal quark-quark scattering picture at extremely high energies; but they will most assuredly be necessary to "anchor" the ends of such thin, flux tubes to quarks, and antiquarks, before a precise discussion of confinement can be achieved.

Of course, expressing an answer in terms of a "few quadratures" does not mean that no further approximation need be made when evaluating those integrals; rather, this is an independent question which may well require some numerical analysis. However, it would seem to be worthwhile to have even reached this point.

Finally, we have applied our representations to determine whether quantum chaos, as known in a potential-theory context, can exist in the full quantum field theory, when all sectors of a quantum system are susceptible to quantum fluctuations. One might naively expect that a summation all over relevant classes of potentials A_μ will surely contain at least one subclass of potentials for which chaotic behavior must appear. But it turns out that the sum over all quantum fluctuations removes the map which gives a potential-theory Green's function (with a vector interaction) its distinctive possibility of quantum chaos, in a manner which has elsewhere been suggested and termed "induced decoherence." In fact, the full quantum field theory does not contain such quantum chaos. The argument presented for QED can be easily generalized to QCD, and to any other theory (containing a vector interaction).

Various directions for further inquiry are suggested by the analysis of this paper.

(i) The application of (5.4) and its finite-N approximations to nonperturbative, quark-structure problems of QCD. Here, one will need a reasonable way of approximating the map (5.3) and of performing the subsequent functional integrations $\int d[\alpha] \int d[\beta]$, and for this it may be useful to employ an "eikonal" approximation which simplifies the argument of A^a_μ, in the map (5.3).

(ii) The application of these representations to the calculation of the logarithms of the fermion determinant, $L[A]$. Certainly, a reasonably simple, finite-N approximation could be useful in many circumstances which require a nonperturbative estimate of $L[A]$.

(iii) The development of simple numerical procedures for estimating finite-N approximations to $G_c[A]$ and $L[A]$.

(iv) What form of map occurs in a first-quantization theory with arbitrary tensorial interaction? Must such a theory contain the possibility of chaos, and if so, is such chaos suppressed in the full quantum theory?

These are some of the questions, and possibilities, suggested by the representations of this paper.

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APPENDIX

In order to develop a useful nonperturbative result, one first expands (2.5) in powers of g, with the assurance that its resummation will be performed in a manner analogous to that of Ref. [2]:

$$G_c(x,y|A) = i\int_0^\infty ds\, e^{-ism^2}\int dp\, e^{ip\cdot(x-y)}\{m - i\gamma\cdot[p-\Omega(s)]\}\sum_n(-ig)^n\int dk_1\cdots\int dk_n \exp\left(iy\sum_{l=1}^n k_l\right)$$
$$\times\int_0^s ds_1 Q(s_1|k_1)\int_0^{s_1} ds_2 Q(s_2|k_2)\cdots\int_0^{s_{n-1}} ds_n Q(s_n|k_n)$$
$$\times e^{\mathcal{D}}\exp\left[i\int_0^s ds' v(s')\left(p-\Omega(s')-\sum_{l=1}^n k_l\theta(s_l-s')\right)\Bigg|_{v\to 0}\right], \tag{A1}$$

where $Q(s_l|k_l) = \phi(s_l)\tilde{A}(k_l) + i\sigma\tilde{F}(k_l)$, and where we have used the form of expansion appropriate to an ordered exponential (OE). We have also replaced in (2.3) the operator $(m - \gamma[\delta/\delta v(s)])$ by $(m - i\gamma[p - \Omega(s)])$, valid since $s > s_i, s'$.

The functional operation of the last line of (A1) can be performed immediately, and yields

$$\exp\left[-i\int_0^s ds'\left(p - \Omega(s') - \sum_{l=1}^{n} k_l\theta(s_l - s')\right)^2\right]$$

or

$$\exp\left(-i\int_0^s ds'[p-\Omega]^2\right)\exp\left(+2i\sum_l s_l p k_l - 2i\sum_l k_l\int_0^{s_l} ds'\Omega(s')\right)\exp\left(-i\sum_{l,m} k_l k_m h(s_l, s_m)\right), \quad \text{(A2)}$$

where $h(s_l, s_m) = \int_0^s ds'\theta(s_l - s')\theta(s_m - s') = \frac{1}{2}(s_l + s_m - |s_l - s_m|)$. As in Ref. [2], the heart of this representation is the replacement of $\frac{1}{2}|s_l - s_m|$ by the "sawtooth" Fourier series formula below. In the exact representation of $|x|$,

$$|x| = \frac{8}{\pi^2}\sum_N{}' \frac{1}{N^2}\sin^2\left(\frac{N\pi x}{2}\right), \quad \text{(A3)}$$

valid for $0 \le |x| \le 1$, set $x = (s_l - s_m)/s$. Because $0 < s_i < s$, the expansion is valid and we obtain the required exact representation

$$|s_l - s_m| = \frac{s}{2} - \frac{4s}{\pi^2}\sum_N{}' \frac{1}{N^2}\cos\left(\frac{N\pi(s_l - s_m)}{s}\right), \quad \text{(A4)}$$

where the prime indicates summation over all positive, *odd* integers, N.

The contribution of $-i\sum_{l,m} k_l \cdot k_m h(s_l, s_n)$ may then be written as

$$-i\left[\sum_l k_l s_l\right]\left[\sum_m k_m\right] + \frac{is}{4}\left[\sum_l k_l\right]^2$$

$$-\frac{2is}{\pi^2}\sum_N{}' \frac{1}{N^2}\left\{\left[\sum_l k_l\cos\left(\frac{N\pi s_l}{s}\right)\right]^2 + \left[\sum_l k_l\sin\left(\frac{N\pi s_l}{s}\right)\right]^2\right\}. \quad \text{(A5)}$$

Again, it is convenient to introduce the variables $K = \sum_l k_l$, $z = \sum_l k_l s_l$,

$$C_N = \sum_l k_l\cos\left(\frac{N\pi s_l}{s}\right), \quad S_N = \sum_l k_l\sin\left(\frac{N\pi s_l}{s}\right), \quad \text{(A6)}$$

and to rewrite the exponential of (A5), under the $\int ds_l$ of (A2), in the form

$$\int\frac{d^4K d^4Q}{(2\pi)^4}\exp\left[iQ\cdot\left(K - \sum_l k_l\right) + i\frac{s}{4}K^2\right]\int\frac{d^4z d^4P}{(2\pi)^4}\exp\left[iP\cdot\left(z - \sum_l s_l k_l\right) - iz\sum_l k_l\right]$$

$$\times\prod_N{}'\int\int\frac{d^4C_N d^4P_N}{(2\pi)^4}\exp\left\{iP_N\left[C_N - \sum_l k_l\cos\left(\frac{N\pi s_l}{s}\right)\right] - \left(\frac{2is}{\pi^2 N^2}\right)C_N^2\right\}$$

$$\times\prod_N{}'\int\int\frac{d^4S_N d^4Q_N}{(2\pi)^4}\exp\left\{iQ_N\left[S_N - \sum_l k_l\sin\left(\frac{N\pi s_l}{s}\right)\right] - \left(\frac{2is}{\pi^2 N^2}\right)S_N^2\right\}. \quad \text{(A7)}$$

No perturbative index n is needed for any of these auxiliary integrals $K, Q, z, P, C_N, Q_N, S_N, P_N$, for exactly the same integrals, and their weights are needed for every n. The first line of (A7) reproduces the contribution $-(1/2)\sum k_l k_m(s_l + s_m)$ of what was called in Refs. [1] and [2] the "phase-averaged" ⟨ph⟩ approximation, while the remaining terms of (A7) generate all the corrections.

With (A7) inserted under the integrals of (A1), the "factorized" sum over n may be performed, yielding

$$G_c(x,y|A) = i\int_0^\infty ds\, e^{-ism^2}\int dp\, e^{ip\cdot(x-y)}\cdot\mathcal{N}'\int d[\Omega]\int d[\phi]\{m - i\gamma\cdot[p-\Omega(s)]\}$$
$$\times\exp\left(i\int_0^s \Omega\cdot\phi\right)\exp\left(-i\int_0^s ds'[p-\Omega(s')]^2\right)$$
$$\times\int\int\frac{d^4K d^4Q}{(2\pi)^4}\exp\left(iQ\cdot K + \frac{is}{4}K^2\right)\int\int\frac{d^4z d^4P}{(2\pi)^4}e^{iz\cdot P}$$
$$\times\prod_N{}'\int\int\frac{d^4C_N d^4P_N}{(2\pi)^4}\exp\left(iP_N\cdot C_N - \frac{2is}{\pi^2}\frac{C_N^2}{N^2}\right)\int\int\frac{d^4S_N d^4Q_N}{(2\pi)^6}\exp\left(iQ_N\cdot S_N - \frac{2is}{\pi^2}\frac{S_N^2}{N^2}\right)$$
$$\times\exp\left(-ig\int_0^s ds'\phi_\mu(s')A_\mu(\Xi)\right)\left(\exp\left[+g\int_0^s ds'\sigma\cdot F(\Xi)\right]\right)_+ , \qquad \text{(A8)}$$

where $\Xi = y - z - Q + s'(2p-P) - 2\int_0^{s'} ds''\Omega(s'') - \sum_N' \rho_N(s')$, and $\rho_N(s') = P_N\cos(N\pi s'/s) + Q_N\sin(N\pi s'/s)$. Now, the C_N and S_N integrations may be performed, yielding, for each N,

$$-\left(\frac{i\pi^6 N^4}{4s^2}\right)\exp\left[\frac{i\pi^2N^2}{8s}(P_N^2+Q_N^2)\right]$$

which suggests that a rescaling of all eight components is appropriate

$$P_N \to \left(\frac{2\sqrt{s}}{\pi N}\right)P_N, \quad Q_N \to \left(\frac{2\sqrt{s}}{\pi N}\right)Q_N$$

and leads to

$$G_c(x,y|A) = i\int_0^\infty ds\, e^{-ism^2}\int dp\, e^{ip(x-y)}\mathcal{N}'\int d[\Omega]\int d[\phi]\exp\left(i\int_0^s\Omega\cdot\phi - i\int_0^s[p-\Omega]^2\right)$$
$$\times\{m - i\gamma\cdot[p-\Omega(s)]\}\int\int\frac{d^4Kd^4Q}{(2\pi)^4}\exp\left(iQ\cdot K+\frac{is}{4}K^2\right)\int\int d^4z\, d^4P\, e^{iz\cdot P}$$
$$\times\prod_N{}'\frac{(-i)^2}{(2\pi)^4}\int\int d^4P_N d^4Q_N\exp\left(\frac{i}{2}(P_N^2+Q_N^2)\right)\exp\left(-ig\int_0^s ds'\phi_\mu(s')A_\mu(\Xi)\right)$$
$$\times\left[\exp\left(g\int_0^s ds'\sigma\cdot F(\Xi)\right)\right]_+ , \qquad \text{(A9)}$$

where $\Xi = y - z - Q + s'(2p-P) - \sum_N' R_N(s') - 2\int_0^{s'}\Omega$, with $R_N = (2\sqrt{s}/\pi N)\rho_N$. Note that if any of the P_N, Q_N dependence inside A is dropped, the normalization of those P_N, Q_N integrals gives exactly multiplicative factors of $+1$.

It is convenient to reflect the variables $z\to -z$, $P\to -P$, and then to make the translation $z \to z - y + Q$, so that all y and Q dependence is removed from Ξ. Integrations over Q and K are then easily performed. In momentum space, using

$$\langle p|G_c[A]|p'\rangle = \int dx\, e^{-ip\cdot x}\int dy\, e^{+ip'\cdot y}G_c(x,y|A)$$

one finds

$$\langle p|G_c[A]|p'\rangle = i\int_0^\infty ds\, e^{-ism^2}\mathcal{N}'\int d[\Omega]\int d[\phi]\exp\left(i\int_0^s\Omega\cdot\phi\right)\{m - i\gamma\cdot[p-\Omega(s)]\}$$
$$\times\prod_N{}'\frac{(-i)^2}{(2\pi)^4}\int\int d^4P_N d^4Q_N\exp\left(\frac{i}{2}(P_N^2+Q_N^2)\right)\exp\left(\frac{isq^2}{4}\right)\int d^4z\, e^{-iq\cdot z}$$
$$\times\exp\left(-ig\int_0^s ds'\phi_\mu(s')A_\mu(\Xi)\right)\left(\exp\left[g\int_0^s ds'\sigma\cdot F(\Xi)\right]\right)_+ , \qquad \text{(A10)}$$

where, now, $\Xi_\mu(s') = \zeta_\mu(s') - 2\int_0^{s'} ds''\Omega_\mu(s'')$, with $\zeta(s') = z + s'\mathcal{P} - \sum_N' R_N(s')$, $\mathcal{P} = p + p', q = p - p'$, and $R_N(s') = (2\sqrt{s}/\pi N)[P_N\cos(N\pi s'/s) + Q_N\sin(N\pi s'/s)]$.

Finally, we return to the remaining s-dependent functional integrals, and note that $\mathcal{N}'\int d[\phi]$ can be easily performed, generating

$$\delta\left[\Omega(s') - gA\left(\zeta(s') - 2\int_0^{s'} ds''\Omega\right)\right] . \tag{A11}$$

We evaluate this δ functional as a product of $\delta(f(\Omega_i))$, each defined at the mesh point s_i, with $\Omega_i = \Omega(s_i)$ and $f_i = f(\Omega_i) = \Omega_i - gA[\zeta(s_i) - 2\sum_{l=0}^{i} \Delta s \Omega_l]$:

$$\prod_i \int d\Omega_i \delta(f_i) = \prod_i \int df_i \delta(f_i) \left[\det\left(\frac{\delta f}{\delta \Omega}\right)\right]^{-1}\Bigg|_{\Omega=\Omega^{(0)}}$$

and where $\Omega^{(0)}$ is the solution of the equation

$$\Omega_\mu^{(0)}(s') = gA_\mu\left(\zeta(s') - 2\int_0^{s'} ds''\Omega^{(0)}(s'')\right) . \tag{A12}$$

If there is more than one solution $\Omega^{(0)}$ to (A12), a summation must be made over all such solutions. The product of all such δ functions then generates

$$\exp\left[-\mathrm{Tr}\ \ln\left(\frac{\delta f}{\delta\Omega}\right)\right] \tag{A13}$$

in which we suppress the superscript $^{(0)}$ of $\Omega_\mu^{(0)}$, and where

$$\frac{\delta f_\mu(s')}{\delta\Omega_\nu(s'')} = \delta_{\mu\nu}\delta(s'-s'') + 2g\theta(s'-s'')\frac{\partial}{\partial z_\nu}A_\mu\left(\zeta(s') - 2\int_0^{s'} d\bar{s}''\Omega(\bar{s}'')\right) . \tag{A14}$$

Because of the $\theta(s' - s'')$ factors of (A14), and the corresponding requirement of "retardedness," the $\mathrm{Tr}\ln[\delta f/\delta\Omega]$ factor of (A13) may be replaced by its lowest-order term (which vanishes in the Lorentz gauge), $\exp\left(-2g\theta(0)\int_0^s ds'(\partial/\partial z_\mu)A_\mu[\zeta(s') - 2\int_0^{s'}\Omega]\right)$, leading to the final result stated in (2.6) of the text.

[1] H. M. Fried and Y. Gabellini, Phys. Rev. D **51**, 890 (1995).

[2] H. M. Fried and Y. Gabellini, Phys. Rev. D **51**, 906 (1995).

[3] E. S. Fradkin, Nucl. Phys. **76**, 588 (1966).

[4] H. M. Fried, *Functional Methods and Eikonal Models* (Editions Frontières, Gif-sur-Yvette, France, 1990). Chapter 5 of this book contains derivations of our starting equations, (2.1) and (3.2).

[5] See, for example, in *Chaos and Quantum Physics*, Proceedings of the Les Houches Workshop, edited by M-J Grannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991); *Information Dynamics*, NATO Advanced Study Institute Series B: Physics, edited by H. Atmanspacker and H. Scheingraker (Plenum, New York and London, 1990).

[6] See, for example, *Engineering Applications of Dynamics of Chaos*, edited by W. Szemplinska-Stupnicka and H. Troger (Springer-Verlag, Wien, 1991).

[7] See, for example, L. E. Reichl, *The Transition to Chaos* (Springer-Verlag, Berlin, 1992), and the very many references cited therein.

[8] H. M. Fried, in *Proceedings of the Vth Blois Conference on Chaos and Complexity,* Blois, France, edited by J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, 1993).

[9] An interesting variant to the particle-field decomposition of the present paper is provided by recent work of the Duke group, who solve the classical Yang-Mills field equation on a lattice, and find chaotic behavior illustrated by an array of positive Lyaponuv exponents: B. Müller and A. Trayonov, Phys Rev. Lett. **23**, 3387 (1992); C. Gong, Phys. Lett. B **298**, 257 (1993); Phys. Rev. D **49**, 2642 (1994); C. Gong, S. G. Matinyan, B. Müller, and A. Trayanov, *ibid.* **49**, R607 (1994); B. Müller, in *Quantum Infrared Physics*, Proceedings of the Workshop, Paris, France, 1994, edited by H. M. Fried and B. Müller (World Scientific, Singapore, 1994).

[10] J. B. Delos, S. K. Knudsen, and D. W. Noid, Phys. Rev. A **30**, 1208 (1984), and references quoted therein.

[11] We are indebted to Walter Craig for this discussion and demonstration.

[12] H-T Elze, CERN-TH Report No. 7297/94; Nucl. Phys. **B436**, 213 (1995); W. Zurek and J. P. Paz, Los Alamos Report No. LA-UR 94-927 (unpublished); and other references quoted therein.

[13] H. M. Fried, Phys. Rev. D **46**, 5574 (1992).